

# ANALYTIC STACKS

Lectures by Dustin Clausen and Peter Scholze

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These are crowd-sourced lecture notes for the Clausen-Scholze course on Analytic Stacks,

[https://www.youtube.com/playlist?list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/playlist?list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

The repository for these notes is

<https://github.com/ysulyma/analytic-stacks>

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## 1. INTRODUCTION (CLAUSEN)

[https://www.youtube.com/watch?v=YxSZ1mTIpAA&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=YxSZ1mTIpAA&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

Thank you everyone for your patience. Sorry about the technical stuff and welcome. This is going to be a course about analytic geometry. The title is "Analytic Stacks", and we're going to be trying to explain the foundations for analytic geometry that we've been trying to set up for the past few years. In this first lecture, I want to give an introduction, or maybe more properly, an introduction to the first half of the course because otherwise, I'd be talking about too many concepts in one single lecture. Let me set the stage by giving some motivation.

Classically, there are several different theories of analytic geometry. I'll just list the ones that I may be familiar with. Maybe the gold standard is the usual theory of complex analytic spaces. In the smooth case, these are the complex manifolds. So, there are things you get by gluing together open subsets of  $C$  to the  $N$  along biholomorphisms. So, maps that locally admit a power series expansion. Then, there's the nonsmooth case as well where you locally also allow the zero locus of some finite collection of analytic functions on those open subsets. That's one thing.

There's a generalization of this which was presented in Sarah's book on Lie groups and Lie algebras, which is the locally analytic manifolds or generalization of the smooth case at least. Here, you start with a complete normed field, and it can be archimedean or non-archimedean. Then, again, you glue open subsets of some  $K$  to the  $D$  along locally analytic maps or locally analytic isomorphisms. That means locally around every point you have a convergent power series expansion with coefficients in the field  $K$ . When  $K$  is the complex numbers, it does just recover the smooth case of one, and that's a nice reasonable theory. When  $K$  is the real numbers, it's a sort of a version of the theory of real manifolds, and that's also kind of a nice reasonable theory. But when  $K$  is non-archimedean, it's not particularly geometrically rich unless you have some extra structure like a group structure or something like this, which gives it more geometry.

In the non-archimedean case, for example, when  $K$  equals  $\mathbb{Q}_p$ , there's the structure is not rich enough because the topology on  $\mathbb{Q}_p$  or  $\mathbb{Q}_p$  to the  $D$  is totally disconnected. So, it's not geometrically rich because the topology is too disconnected. Here, every unit ball will break up into  $p$  many other unit balls, and those break up into  $p$  many unit balls. For example, in Sarah's book, you can find a discussion of the classification of compact  $p$ -adic manifolds, and it's just a very simple combinatorial classification. There's the dimension, another invariant, and there's just really not much going on there. This led then, well, this and some examples coming from uniformization of elliptic curves led John Tate to introduce the rigid analytic geometry, and this is over a non-archimedean field. So, it's a geometrically rich theory which works also in the non-archimedean case.

In contrast to the theories above, you don't kind of really think of it in terms of specifying some ways of gluing open subsets of  $K$  to the  $D$ , or you don't even necessarily think about it in topological terms at all. You more actually think about it in algebraic terms. So, instead of focusing on a local model, which in the classical case might be something like an open polydisk, you instead concentrate on a class of locally allowed functions. So, there's a turn here. So, focus on local rings of functions instead of local topology.

So, and you kind of let the ring of functions tell you sort of what the topology is supposed to be. And the other turn is that the local rings of function, the local models that he uses, are not functions convergent on an open disk, but functions convergent on a closed disk, which is something that makes sense to use in the non-archimedean context. The local rings of functions are well quotients of functions convergent on a closed polydisk, and those are the so-called Tate algebras. Then, the manner in which you're allowed to glue these local models to get global rigid analytic spaces is slight. It's kind of halfway between algebraic geometry and usual analytic geometry. It's not clear if Tate algebra is just the functions convergent on the open disk or quotient of. Oh, okay, it's a point about terminology. Yeah, it's not. Sometimes, in some references, they, but I'm not sure if it's a mistake or another. I mean, because I'm not, because depends on the reference, right? Okay, that's in the two references they say, but all the other one avoid algebra. Okay, let's say it's an apoid algebra then. Yeah, I don't. Okay. And this was generalized, whoops, by Huber to the theory of adic spaces, and this is a generalization. So, note that rigid analytic geometry takes place over a base field, and a very loose analogy would be that adic spaces are to rigid analytic spaces as general schemes are to varieties over a field. So, it's kind of, yeah, some useful generalization where you don't have to have a fixed

base field. And again, you focus on, you don't think necessarily of your local models topologically, kind of think of them as being algebraically specified in terms of a ring of functions. But here also there's a new twist. So, still have local ring of functions. By the way, I don't mean local ring in the technical sense from commutative algebra. I mean like ring of functions on a local model, yeah, just to just to be clear. Um, uh, say called  $A$ , but you also include extra data of a certain subring  $A_+$  inside  $A$ , and what this  $A_+$  does is, well, to a, you attach some space of valuations, uh, and then  $A_+$  will single out those valuations which view this subring  $A_+$  as being consisting of integral elements. Um, so, uh, and it's actually a very nice extra flexibility you have in Huber's theory that you can consider different choices of  $A_+$  on a given appropriate topological ring  $A$ , um, and we'll kind of see from a different perspective what this choice of  $A_+$  is really doing, uh, later in this lecture, um, okay.

So, maybe I'll go over here. So, when you say local ring of functions, yeah, do you mean really a sheaf on that open set? I just, no, I just mean it's something like you have, you specify a certain kind of ring, and then you say that's my functions on my basic geometric model, you know, like for you have to have transition right when you go from one open set and intersect with other open, yeah, so I what I didn't discuss gluing, so that's that you need to say that kind of thing when you discuss gluing. We'll touch on it a little bit later, but I'm actually going to, for this lecture, I'm mostly going to stick to this sort of adic context where you just look at a single  $A$ , but of course, that does need to be discussed and it will be discussed, yeah. Then there's Berkovich's theory, so, these rings of functions you see both in the rigid analytic context and in the more general adic space context, the, you know, the basic examples are things like you have a ring and it's complete with respect to a finitely generated ideal and you give it the inverse limit topology where all the quotients are discrete and then maybe you're also allowed to invert something, provided that the inversion is well, provided that that inversion is kind of inverting everything you completed along, so to speak. So, for example, in the Tate case, you can write these rings of functions. So, the most basic example of these things is you can take, like,  $\mathbb{Z}_p$  or let's say  $\mathbb{Z}$ , just a usual ring of polynomials in one variable, and you can complete it in the  $p$ -adic topology and then you can invert  $p$  and that's the functions on the closed unit disc in the  $\mathbb{A}^1$  over  $\mathbb{Q}_p$ . So, complete along something and then invert it, and these are also examples of  $p$ -adic rings, and what Berkovich does is he says let's just work with arbitrary Banach rings to start with. So here, the local models are given by Banach rings, so that again, you're allowing, but now again, you're... while this theory is confined to the non-archimedean case, this theory over here actually allows archimedean phenomena as well because the real numbers and the complex numbers also count as Banach rings, and the global theory here is not quite as smoothly functioning as... as... as... as... so... to this ring, Berkovich attaches this space of multiplicative seminorms, so this is just a compact Hausdorff space, and the... the kind of gluing you're allowed to do is in some sense organized by this Banach... by this Berkovich space, um, yeah. And I don't necessarily want to get into too much detail about that right now, um, but for example, you can see the complex analytic spaces as a special case of this, and you can also see rigid analytic geometry in some sense as a special case of this, okay? So, this is my very, very brief uh review of classical theories of analytic geometry, and please feel free to ask questions if you have questions. Um, some certainties lost over like the shiness in the Uber Theory, correct in has this different gluing like in one you can glue along open sets to get to get things like he has to work only over a field condition, then something quite artificial to glue it like you do in but in his language, in some sense it just transports yes to his language with some extra things which are difficult to remember, some conditions under which the theories are equivalent, yes, exactly, exactly like the notion allows you to understand, yes, exactly, exactly, yeah, so yeah, so the relation with original geometry is indeed artificial, it's kind of not the correct, okay, but anyway, um, so what do I want to say now? I want to exp... Okay, so we have all these theories, that's great, but now what again was the motivation behind coming up with a new theory? Is it just... is it just going to be point six on the list? Well, not exactly. So, uh, so why introduce a new theory? Well, these are all three or four, or depending on how you count these two five, theories of analytic geometry, and kind of the relationships between them are more or less well known. Like I said, that one is a special case of two and three, this one's a special case of this one and it's also a special case of that one, that one's a special case of that, you can formulate comparisons and the web of these things is kind of well understood, in spite of the subtleties sometimes involved in the comparisons, it's fairly well understood, um, but so far, there's no common framework in which you can put all of these examples, they all have their own flavors, and while you can formulate comparisons, it's not that those comparisons are taking place in some larger category that you consider, it's kind of done by

hand in every situation, formulating the comparisons between these things, so, uh, want to accommodate all examples, I never remember how to spell accommodate, uh, uh, so that's one thing, um, you'd like a general theory of analytic spaces which can be specialized to whichever context you might uh be interested in, um, the second reason would be that well so of all these of all the above uh the only Rich Theory um allowing both uh archimedean and non-archimedean uh geometry uh is Burkovich's but the gluing uh uh is not so well worked out and in particular I want to say that in so the gluing was investigated by Burkovich he he basically restricted to the non-archimedean case almost from the start first of all um and then more general gluings were investigated by Poo for example um but in that they always do they always do the same thing they fix Bon offerings as the base ring and then they define a pine space of some dimension over that base ring and then they glue along sub some kind of subsets of a pine space that they that they pick out that's sort of how the gluing works in Burkovich's theory so it's not like so it's all take and then the Bon rings they take as their base rings often

have restrictive hypotheses on them so it's not like you're really understanding how you can glue two Bonac rings together to get some more global object and indeed the things you're gluing along in these a pine spaces are oftentimes not even controlled by Bon rings themselves they're just some other kind of object so the nature of the gluing is constrained to a finite type situation and it's a little artificial uh uh well not not no I mean it's not necessarily artificial but it doesn't quite fit the mold when you think of how you go from like a pine schemes to general schemes for example just by gluing those local models in some naive way um and so let me say and let's say only what you could say finite type um so maybe I go over here uh so another reason is that even individually in their own context in which they're supposed to operate uh these theories are less flexible uh than uh for example the theory of schemes and one major reason has to do with issues of descent.

Uh, so for example, one of the main constructions when you have a scheme is the category of quasi-coherent sheaves. Um, and that's a big fancy name, but it's really something simple. When you have a commutative ring, you look at the category of modules over that ring, and then it just glues to a general scheme, and that's what a quasi-coherent sheaf is. Um, but you don't have that in analytic geometry in any of the classical theories, and the reason is, okay, I said you always have some local ring, um, which is describing the local geometry, so to speak, and of course, to that ring, you can certainly assign the category of all R-modules, but then it doesn't glue. So, it's just not the case that if you have, you know, in any of the allowed gluings that people choose, it's just not the case that an R-module on two open subsets or closed subsets or anything that have gluing data on the intersection globalizes to a general R-module. The view point was in analytic that coherent sheaves are the basic tool, yes, this is the classical person, yes, and I was going to say this, um, so you only can glue, uh, so maybe finite type or finally presented modules, and this does give rise to the theory of coherent sheaves, which is a beautiful and extremely useful theory. It's one of the main tools you have in analytic geometry, but it's still constrained by the finiteness hypotheses that come into it. So, for example, if you have a map of analytic spaces, you can't consider like the push-forward of the structure sheath as a coherent sheath, but that should be that's one of the main examples of quasi-coherent sheaths you like to play with in algebraic geometry. I mean, unless the map is finite or something, um, so the theory of coherent sheaths is really nice and it works well in analytic geometry and basically all of these contexts, but it's still not as general and flexible as we're used to from algebraic geometry with quasi-coherent sheaves which have no inherent finiteness conditions, um, so, so these are all kind of, you could say, theoretical reasons why we might want to new theory, um, but there's also potentially a practical reason, so this is much more speculative, um, so coming from the Langlands program, so, uh, far Fargan Schulza, they famously geometrized a local Lang lens and that led to kind of a clarification of the local Lang lens program, and what was this geometric geometrization based on, it was based on replacing QP, uh, by some more exotic object, uh, far Fanten curve or really you have to let the curve vary in families in some sense you could say far Fen curves, um, and this was produced in the language of attic spaces so it was attic space over QP not at all of finite type, um, so quite a somewhat exotic Beast which thankfully this theory of attic spaces existed to accommodate it, um, and um, again quite speculatively one might hope that not just the local uh Langlands program but the global Lang L program can also be geometrized, um, very okay Peter's always very optimistic but I'm always uh of global Lang length but this would involve replacing say Q or Z by some family of exotic analytic spaces, um, and whatever such a thing is it's going to have to have both archimedean and non-archimedean aspects, for example, there should also be a version over the real

numbers which I believe Peter is working out, um, and it also is not going to be finite type in any sense so there is simply no existing theory which could possibly give the language to describe such an object if such an object even exists, yeah, but it's good to have a theory a precise theory to guide exploration of the possibility of such exotic things, um, so that's another motivation um, so that is the end of my um motivation section, uh, so now's a good time for questions if people have them, so you you are going to introduce this B SP I mean theory what what is are more general theory encompasses all of this yeah I'm going to we're going to introduce a new theory and explain the relation to the previous theories, yeah, yeah, so that yeah that's good to say yeah so our our goal is in this course is to introduce a new theory of analytic geometry and to explain the relation with the previous theories, yeah, so you will let not only the basic analytic ring but also some analytic spaces analytic yeah yeah but today I'm only going to give an introduction to the apoid situations kind of analytic Rings because it'll already be enough, uh, already be enough there, yes, why is the theory of BL spaces insufficient, oh, it doesn't have any archimedean, it doesn't, there are no, it's non-archimedean by Design, yeah, yeah, Al, some I remember forgot now how it was called some kind of spectrum that they combines and forgot the name of this someone consider but they didn't develop it so uhhuh yeah there could be other theories than the ones I

listed I just listed the ones that I knew were well studied and yeah, okay, so then let me move on uh, so continuing the introduction um, I actually had a question, yes please, so um, in the previous uh, sort of theories like uh, I think like at the birk ofage setting there were like Rich uh, theories of chology itology and stuff that were Vari that yeah but I mean um yeah sure bur deeply studied atal kology in in setting of burkich spaces yeah okay and okay um so the next section is called condensed math so the I mean the cond condensed math yes uh the the F this this issue here the issue with the scent um it can be attributed to the fact that these in all these different theories these local rings that you have describing the local models they're not just abstract rings they're topological rings and for many purposes for example uh for uh well yeah so the local models classically are in fact topological rings and it's important to remember the topology so for example okay an algebraic geometry the uh polinomial ring in two variables which is functions on Aline and two space is the tensor product of polinomial ring in one variable and polinomial ring in one variable okay but in say rigid analytic geometry if you take the T algebra in dimension one and tensor it with a tate algebra in dimension one again you're going to get some crazy thing because you took an algebraic tensor product and you forgot the ptic topology but if you do a a Pally complete tensor product you get the ring of functions the correct geometric ring of functions the two variable case so you need to in performing constructions such as tensor products which are basically calculating fiber products geometrically you need to remember the topology that much is clear um so and that's at the basis for the reason why you don't have a naive theory of quasi-coherent sheaves and you have

Quasi-coherent sheaves and you have naive problems with gluing outside the finite type case. I mean, finally generated, uh, case, but topological rings, uh, and topological modules over them, which would be the kind of natural thing to do if you're thinking about quasi-coherent sheaves, are not suitable, uh, for a general theory, and the basic reason there's, there's many ways of saying it, but the basic reason is that, uh, you know, if you form this category, it's, it's not going to be a billion, the category of topological modules over a topological ring, and you can dress it up however you like, you can make it much more specific or whatever, it's just not going to be a billion, and the phenomenon is that if you have a dense inclusion of modules which happens all the time when you have infinite-dimensional things, uh, then it's going to be both an epimorphism and a monomorphism, uh, generally speaking, uh, in your reasonable categories, but it's not going to be an isomorphism, so you separated, yeah, yeah, I would have to say separated to make that literally a true claim, so you have a non-strict map, then the image is two topologies, and this is not yeah, CEG world, yes, yes, yes, yes, um, right, so, uh, so what we do is we kind of go very back to the start, um, so we, uh, go back to basics and Define a replacement for the category of topological spaces, and we do that in such a way that it's then very easy to pile algebraic structure on top of those things and talk about the analogues of topological rings and topological modules over them and that such that we will get an aelion category, uh, in the end, um, and the basic idea is one that is very old and I'm not sure, so certainly it was, uh, certainly, you know, Gro deck used this idea many times and I don't but I don't, I think it might even be older than Gr, topological space is kind of funny because you have second-order data, you have a set of points and then you have a set of subsets of that, and that's fundamentally what makes it difficult to mix with uh, algebraic structures, so instead you'd want to stick with kind of first-order data just points and the



basic idea is you single out uh, a collection of nice let's say test spaces, uh,  $S$ , and then instead of uh encoding a topological Space  $X$  as traditionally so a set and some set of open subsets um, we just record the data uh of what should be continuous Maps uh, from your test space to that topological space, and kind of atati the structure and properties you see in that situation so we were only we we're going to choose a nice collection of test spaces and then we're going to say we only we only care about a topological space in so far as uh, the phenomena are seen by maps from these nice test objects, um, so you still have  $X$  set I put quotation marks around this so so I'll become a little more formal in a second and then you can ask your question, no, but traditionally like they wanted to doop Theory with then so one way was to have ACC a set and to fix a collection of  $S$  to  $X$  some people some work with this yes, yeah, yeah, exactly substitute usual on you can Define homotopy and ology and do something yes EX exactly yes, yeah, um, so yeah, maybe spanner is a person and yeah, I don't know, yeah, um, so formally, uh, so formally, uh, our test spaces, uh, will be profinite sets so-called profinite sets, uh, which is which is the same thing as a totally disconnected uh, compact house door spaces, um, it's also just the same thing as inverse limits of finite sets where the finite sets have a discrete topology and the inverse limit has the inverse limit topology, um, and so this is what we we uh used in a previous iteration of this kind of course, um, so the first course Peter taught on condensed math was uh with this class of test spaces but it does cause some troubles because uh uh, it's a it's a large category so there's no if there's no Cardinal bound on the profinite sets then when you encode all of this data you're encoding more than a set's worth of data and it it does cause some technical troubles we more or less worked around them but so we're actually going to take a slight variant of this for the purposes of this course um and we'll explain in more detail in the first few lectures I think um why we make this precise choice but so so a we'll say a light profinite set uh is a accountable inverse limit of finite sets it's also the same thing as requiring it to be metrizable um and then so uh a light condensed set uh is a sheath of sets on the category of light uh, profinite sets uh with respect to the groi topology uh, and now I'll explain the groi topology so so it covers our finite collections uh uh of jointly surjective Maps continuous Maps yeah um um so finite dint unions are covers and a surjection gives a cover so this is like theology because you could add you can add also okay a covering seeve is one which contains a finite collection of jointly surjective Maps Okay um so okay this is we're using the language of gr toies here uh to be sure and possibly not everyone is familiar with uh the language of Gro to and kind of the general how you play with categories of sheaves and so on let me make it more explicit uh so more explicitly uh a light condensed set is a functor okay I'm not going to avoid the language of categories and functors uh so light profinite set up to the category of sets uh such that so the first thing is is that well  $X$  of empty set equals Point second thing is that  $X$  of a disjoint Union of two things is the product uh and the third thing is that if you have a surjection  $t$  to  $S$  then uh  $XS$  is The Equalizer of uh  $XT$  and then the two different pullback Maps you have to the fiber product um and and the example example is any topological Space  $X$  gives a condensed set where the funter of points is just given by The Continuous maps from  $s$  to  $X$  so it's easy to see well the the functor

iality is just that you if you have a map from  $T$  to  $S$  and a map from  $s$  to  $X$  you get a map from  $T$  to  $X$  and it's easy to verify all of these properties uh for continuous Maps out to an arbitrary topological space so groi topology I mean you can just unwind it to this but um actually it's kind of a bit of a bit elusory the elementary nature of this definition.

Because um we are going to fairly seriously make use of the theory of Gro IND topology and sheaves in this course, it's just not worth it to avoid that theory, especially since it comes up both here in the definition of light condensed set and also later in the way in which you GL glue uh the apine case of analytic spaces to General analytic spaces. Um, it's not going to be we're just going to use the theory, so if you're not familiar with the theory of Gro deque topologies and sheaves I suggest and you want to follow this course I suggest you read up on it, um, okay, yes, question, so what is a morphism between two uh, two profile sets like Prof set, yeah, just a continuous map, require some filtration it's just a continuous map but it's also a map of pro systems if you're thinking of it as a accountable inverse limit it's equivalent proem it's yeah, yeah, it's the same same thing yes, do we know what are the points in the category of FL content sets points in the c ah oh oh in the topos thetic sense yeah, I think so yeah they gean um debatable so we we'll discuss more such things in the in the coming lectures um isance of topological space in is that skul no not not for General topologic IAL spes but for a large class of topological spaces it is yeah um right so so some there's one thing that's clear right I started with a general idea or you take some collection of test spaces and then you okay then you say what you axiomatize the properties of maps from test spaces to a given topological

space and you arrive at this axiomatics but okay it's actually not so simple because there's many possible choices of test spaces and beyond that there's many possible choices of which properties you want to put in your act which properties you see mapping out from those test spaces to  $X$  that you put in the axioms uh for your general objects like condensed sets actually there are other properties Satisfied by  $X$  of  $s$  when  $X$  is a topological space that I didn't put in the axiomatics um so there's kind of a little bit of a delicate balance here and we will discuss more about how we arrived at precisely this this Choice both of the test category of light profinite sets and this for now I just want to make a couple of remarks which will maybe give a a sense for why we make this precise definition so so first of all uh two maybe the two most important examples of light profinite sets are the point uh and then there's this uh the one point compactification of the natural numbers um and with this you kind of get the underlying set so if you take  $X$  of  $s$  then you think of that as the underlying set of your condensed set and with this you get kind of a notion of a convergent sequences uh in your your condens set you get a set of conver set of convergent sequences again it's abstract so it's not literally well in the case of a topological space it literally is the set of convergent sequences so also those things were considered in this topology of oh yes yes yes that's correct yeah and but they had this they use this projective things projective covers which are not light that's correct so this will be discussed this will be discussed in in in due time yeah so but yeah so yeah it's true bot and schultza had this definition but that doesn't mean that it was necessarily the correct thing to do for the purposes I'm about to discuss but yeah it turned out it was um okay and then but then um uh so then another remark is that allowing all surjections uh to count as covers gives a nice simplification of the structure of the category and in particular it gives some good homological algebra properties when you pass to light condensed aylan groups um this you have lot of flexibility of working locally when you allow arbitrary surjections to count as covers but on the other hand uh restricting the topology or requiring uh the topology the gro de topology to be finitary uh gives good categorical compactness properties for light profinite sets uh sitting by the on meding inside all uh light condensed sets and and moreover and even and even for uh all metrizable compact house D spaces so for example uh the unit interval famously is a has a surjection from the canra set given by decimal expansions uh and this is a light profinite set and the fact that you have a finitary Gro deque topology and on the other hand this this guy is covered by this guy which is one of the basic test objects it means that the compactness of these compact hous door spaces which kind of we know from General topology actually translates into a nice categorical compactness property inside this larger category um and for this it's actually important to have a these larger light profinite sets than just the sets of convergent sequences okay um let's see okay questions Prof at light it's countable no no that's not countable I mean you you could ask the collection of clo and subsets to be countable so count yeah or yeah something like or it's the same as metrizable for sure so yeah no no they adjust every point of the accountable neighborhood  $B$  is weaker than the second count yeah some way yeah yeah you need you need a accountable basis for the topology in total yeah so anyway okay yeah it is the same thing as opposite category of the countable yes I guess yeah exactly yeah it's the same saying that the set of clo and subsets is is countable yeah okay further questions um so I remind you that this is this is just an introduction we will go into much more detail in the in the coming lectures okay so so what do we have now we have uh oh maybe I give myself a Blackboard um so what do you have so far when now I've explained the light Prof finite sets and so um we're going to move on to analytic rings and I'll start with a point which is that these okay from now on sorry so I'm going to drop the light okay so just so I don't have to write it and say it all the time from now on condensed set means light condensed set and profinite set means light profinite set I always have this cabil uh hypothesis floating around um so condensed sets uh gives rise to the notion of condensed ring and condensed module over condensed ring and it's really uh if you're familiar with the gr topologies and so on it's completely immediate so it's just a sheath of rings and then a sheath of module over that sheath of rings on this uh on this site here it's also just a ring object in this category

and a module object over the ring that ring ring object in this category yeah a question Zoom to write a little bit bigger oh a question to write a little bit bigger that sounds like more of a comment or a request okay uh I will do my best and please hassle me again if I don't live up to it yeah

Um they didn't ask me to write more clearly. Well anyway, Tech, sorry, the formulas more clearly, thanks. Yeah, okay, um. And that's all well and good and you might think naively, okay, so now we have a category of condensed rings, for example. Why can't we just use that as our uh local models for our analytic geometry kind of by analogy with schemes, where schemes are based on discret  $H$ . By the way, ring means commutative

ring. Um, schemes, you start with discrete rings and then you figure out a way to glue them and then you get schemes. Um, but it's not enough just condensed rings to get a good theory of analytic geometry.

But no, please,  $\text{cont}$  string is the same object where  $\text{SE}$  is replaced by  $\mathbf{a}$ . Yes, that's correct. So, uh, but there's no additional structure on ring, just an abstract ring, right? But condensed ring, it's just an abstract ring with those properties. No, no, but no, no, no, it's a collection of abstract Rings, one for each  $s$ , yeah, yeah, okay, right, yeah, but it should satisfy all those properties and  $\text{Xs cross XT}$ , what does it become, the tensent product of the Rings? No, the cartesian product of the Rings, yes, I see, okay, um, yeah,  $\text{U where was I H}$ , but are not enough uh to give a good geometry. And the basic reason is as follows, so maybe I should say condens ringings alone. So, the category of condensed rings has pushouts given by relative tensor products, just like in classical commutative Rings, um, and those relative tensor products are what geometrically speaking should calculating fiber products for you, um, and they're the things that I said should correspond to completed tensor products, right, um, but if you have condensed strings,  $\mathbf{a}$  and  $\mathbf{b}$ , over a condensed string  $\mathbf{K}$ , and you form this relative tensor product in this category, you can ask well, what is the underlying ring of this and it turns out it's not actually hard to see from the nature of the grot topology that this is the same as the abstract tensor product of the underlying ring rings of all the individual things. So, this condensed ring here is just, just gives a condensed structure just on the yeah, yeah, just gives a an well non-trivial to be sure but just gives a condensed structure on the on the abstract tensor product, so it's not in particular is not giving a completed tensor product the completion procedure does change the underlying set, right, so this is not not yet doing the correct thing, uh, okay, so to fix this uh, we put additional structure on a condensed string, um, so we record some class of modules, I mean condensed modules, uh, which are to be considered as complete in some sense, complete. So, the basic um, yeah, and that will uh, oh, I forgot to write larger, oh, that would give the notion of analytic ring.

So, an analytic ring will be a condensed ring together with some extra structure which will tell you which of the which of the condensed modules over that condensed ring you should consider as complete with respect to the theory that's being described by the analytic ring, um, but before I make the definition more precise uh, I have to scare more people away. I already said you should know Gro de apologies, get ready, um, so I kind of I have to say more precisely what I mean by a ring, so but I'm going to scare you but then I'm going to say well you shouldn't be too scared, uh, so why is this why is this a question I just I already told you ring means commutative ring right, but um, You didn't say no, I I did you need to pay better attention um, so what kind of ring um, okay so experience in algebraic geometry shows that the generally correct notion of a fiber product of schemes is actually the derived fiber product which on the apine level corresponds to derived relative tensor product of rings. Now the reason more people don't do it that way is because uh, it it's a technical hassle to talk about these things, these derived tensor products and derived rings and so on, but actually that that's not true anymore we have Jacob L's works it's not a technical hassle anymore you just have to you just have to do it and it's no problem so we're going to do it because it's the the correct uh it gives you the correct relative tensor products with giving a good theory in general.

Now that being said basically all of the basic examples that we discuss almost all of the basic examples that come up will not have any derived structure they'll just be ordinary rings so you can comfortably follow the course even if you're not very familiar with derived rings but you should bear in mind that for you know for the for the general claims that we're making it won't necessarily be true if you imagine everything to be an ordinary ring although in examples many things will indeed be  $\text{Ord Ary Rings}$  um, but now we go down a rabbit hole because once you decide to work with some notion of derived Rings there's actually several inequivalent choices of what you could mean by that um, so so should be derived uh, but which kind there's there's two basic options and that's kind of uh,  $\mathbf{e}$  infinity algebras and what some people call animated commutative rings. I'm one of those people uh, which

Are the things that are presented by simplicial commutative rings, and then there's also the choice of whether you want it to want to allow negative homotopy. In both cases, um, we won't allow negative homotopy, and we're going to, for the purposes of this course, we'll choose this one here. It's more directly tied to classical algebraic geometry. When you start with ordinary schemes and you take derived tensor products, the things you get always have this extra structure, so it makes sense to remember that and not think about these more general things. But actually, the whole theory that we're developing works perfectly fine in any of the different variants, and in fact, it's even less technical to set up in this seemingly more complicated setting here, for reasons which I think we'll get into. Um, but so algebra in the sense of Spectra



exact, there's also that choice. Yeah, you could do you mean infinity algebras over  $\mathbb{Z}$  or over the sphere Spectrum. Yeah, um, so okay, so formally then, so just to get it on the so formally, uh, a light animated ring, oh, sorry, condensed animated ring, is a hypersheaf of animated rings on the uh, site of light profinite sets, but again, I'm not saying light. Um, okay, and now I'm going to make another convention that I'm probably just going to say ring when I mean animated ring, and if I'm want to stress that it actually just lives in degree zero, I'll say, uh, classical maybe or static. Um, static being kind of the opposite of animated, um, right, and that should hope also help those of you who are not familiar with the theory to just pretend that everything is an ordinary ring because that's that's pretty much okay. Um, all right, um, and the basic invariant for us of such a condensed animated ring is its derived category, so the, oh, I need to write bigger of such, uh, such an  $R$  is its full derived category. Uh, uh, is this the in  $L$  in somewh, yeah, so yeah, you look at just at hypersheaves of modules over this, you know, up unbounded modules over this sheaf of rings, are yeah, just a sec, do simpli and coal or does it do it, I don't know, coal no if you want to, you have simplicial modules of simplicial ring, this is not enough to have unbounded in the, no yeah, you don't I mean yeah, you don't really set it up like that, you don't talk about simplicial modules over simplicial ring because then that would be the connective part, you could do that and then just say you kind of formally add in the negative things by by filtered coits or something, I mean by yeah, you do it by I mean the way he does it is he forgets the infinity algebras and then that's just a commutative algebra object in  $D$  of  $\mathbb{Z}$ , and then that has a natural notion of module in the infinity category Theory, so so I mean the theory of modules factors over the underlying infinity algebra and that's just and this is in in which in the in higher algebra maybe or or  $SAG$  probably discussed in more detail spectral algebraic geometry, so but there is no uh, I'm going to say something to help Orient yeah, so if  $R$  is static so again that means it's just an ordinary  $T$  string uh, then this is the usual or the infinity category enhancement of the usual uh, unbounded derived category uh, of the aelon category of condensed  $R$  modules again in the in the totally naive sense of you have a sheaf of rings and you take a sheaf of modules over that sheaf of rings yes, can I ask you to comment on specifically why you want hypers sheaves everywhere it's because we want things like convergence of posnov Tower and we know we can prove that in the world of hypers sheaves but we can't prove that in the world of sheaves so we don't know that sheaves and hypers sheaves are the same thing and also we can always prove everything is hypers Chief and we can always I mean hyper covers never give us more trouble in practice than ordinary covers okay so we're not losing anything by requiring that yes just have a questiony like yes like we can Define schemes in general like just Co limits of representable sheets uhhuh what happens if we do the same here like we take like we glue representable shav over condensed strings no with just condensed strings you're never going to get a good theory you need you need this extra structure yeah I mean if you take sheet over condens strink Sal doesn't I mean the same reason will hold yeah I mean this will be this will be your pullback in pre- sheeps and it's just not the right thing so yeah uh other questions what oh yes hi Matthew yes is not admitted a static animated ring is a condensed anim in the new sense seems like you need some Vanishing of chology because was she condition in just a one categorical s it works I'm not going to get into it right now we'll talk after yeah yeah this is not the time for such a technical question apologies but don't worry if it's correct yeah um okay so uh okay so now I can give the the formal definition um wait did you define are theyre uh yes yes I don't want to say discreet because we also have this condensed stuff and then you discreet could mean yeah so that's the reason for changing the terminology yeah okay so what  $M$  no for no no right exactly exactly um okay so the definition is so an analytic ring uh is a pair  $R$  uh and then I'm going to use funny notation uh uh where this triangle thing uh is a condensed ring and uh the derived category of the analytic ring is supposed to be a full subcategory of the derived category of this condensed ring which is sort of its envelope um is such that and then we're going to demand some rather strong closure properties remember the idea was this was supposed to be singling out a collection of complete modules um and the first condition is that uh this full subcategory is closed under so inside this ambient category here is closed under all uh limits and colimits uh the second property is that if let's say  $n$   $L_i$  in here and  $M$  lies in here uh then kind of the internal Ram uh from  $M$  to  $n$  still lies in the smaller thing um the third condition uh is kind of technical so I said we uh we wanted our rings to be connective so no negative homotopy in some sense we also want to require that our analytic Rings be connective and we say it like this so if uh if uh I'd say

This uh denotes the left ad joint to the inclusion. Um, so again that's some kind of completion functor. Then uh this completion uh sends uh the connective subcategory here to the connective subcategory again

so it preserves connective objects. I should have said I'm sorry I meant to remind over here so I said if  $R$  is static this is the usual unbounded derived category of this ailen category. In particular, it has a  $t$  structure, has a notion of connective objects and anti- $c$  connective objects but even in general for a general  $R$  you still have a  $t$  structure but it's not the derived category of its hard anymore. When your ring is not static, say which kind of structure you have in just in terms of the vanishing of cohomology, yeah, exactly, and you use homological notation. I always use homological notation. Yes, yes. Do we need object  $uh$   $H$  why do we require this? There's some statements  $uh$  for some statements it's convenient to have a kind of reduction from a general animated ring to the static case, namely, it's  $Pi\ 0$  and for this kind of reduction, it's  $um$ , it's important to have this kind of control on connectivity. So once you assume the limits and  $C$  this is under usual categories or Infinity categories, the same that is well if it's a triangulated subcategory closed under products and direct sums then yeah then to have the int you need. Sometimes some Cal condition is it automatic yes yes so like being generated by a set yeah it's automatic it's automatic yeah Qui one question yes  $uh$  with condens ring you mean condens animated ring yeah I I made that convention maybe only in words but yes exactly uhhuh so from now on a ring is a static ring and an animated ring is a ring  $um$  okay  $uh$  over let's let another one have a chance first  $uh$  yes the analog of the ICL inside that indeed it is indeed it is yes  $uh$  over no just to make sure so one and two implies that the left agent exists right okay  $um$  ah right I forgot to say what a map is so a map of analytic rings sorry I I I I I need to hear what you said so the cond ring is a animated for always anim anim yeah uhh is just a map of condensed rings such that so it's just a condition  $uh$  if  $M$  lies in  $D$  of  $s$   $uh$  then the Restriction of scalars of  $M$   $uh$  should lie in  $D$  of  $R$  so so  $uh$  along  $R$  to  $S$  yeah I wrote it a little funny but I hope I hope the meaning is clear so if you have an object in  $D$  of  $s$  triangle  $uh$  which happens to lie in  $D$  of  $s$  then when you restrict to an our triang our triangle module it should line  $D$  of  $R$   $um$  okay so I'll make some remarks so  $uh$  there's always a  $t$  structure on  $D$  of  $R$  and in fact it's quite naive so the connective part is just  $uh$  the intersection of the connective part for the enveloping ring  $um$  and same with the anti- $c$  connected part so you can check everything in this potentially more familiar category here so it's actually  $zg$  graded not just positively graded the modules are  $z$ -graded yes the ring is positively graded that's what I thought okay but the modules you allowed to be  $zrad$  indeed indeed  $um$  and in particular you get an aelan category so  $D$  of  $R$  the heart of the  $T$  structure which again is just a  $d$  of  $R$  intersect this  $um$  and actually this ailen category also determines the the analytic ring structure so I can also so I can also say that  $uh$   $D$  of  $R$  is giving an analytic ring structure on our triangle and  $um$  there's actually an equivalent axiomatics  $uh$  just at the aelan level so I could instead say that I give a a anim a condensed ring and an aelan subcategory of you know the heart of  $D$  of  $r$  or  $D$  of  $R$  triangle satisfying certain axioms yes sorry  $um$  just maybe kindly of slow it down but for the definition an analytic  $R$  yeah I'm just trying to like understand why like  $uh$  why in what sense is it analytic like like why yeah so that's something I can't answer right now it'll come when I when we discuss examples and so but the motivation was the simple thing I said that we want relative tensor products to be completed and okay  $uh$  yes Robert can I drop hard triangle if I want and just remember the category as a I forgot an axium sorry you just reminded me that because the answer to your question is no because of this axium  $um$  sorry I forgot to require that the the unit should be complete so yeah the ring itself should be complete then why can't I drop the Top our triangle entirely well you yeah and then you need some extra structure on  $D$  of  $r$  i remember as a condensed category yeah yeah that's still not enough because I'm doing the animated context and not the infinity context but if I remember yeah then that's enough yeah yeah yeah yep  $um$  okay so that's another perspective is you were just giving an Aon category of complete modules there's also a third perspective which is useful so so to to understand

What an analytic ring structure is, this is very abstract, and it's talking about big categories and stuff. But for a light profite set, I wasn't, I said I wasn't going to say light  $S$ . We can consider the free  $S$ , the free module. So it's denoted  $R$  bracket  $S$ . And what is it by definition? You take the free module over the condensed ring, and then you just complete it. So put in there via the left joint. And these generate  $D$  of  $R$  greater than or equal to zero under co-limits. So those are kind of your basic basic building blocks, your basic generating objects.

And again, there's another equivalent axiomatics which takes as the second data in the pair, not the category full subcategory or  $Dr$ , but just the collection of free modules on profinite sets, objects in  $D$  of  $R$  triangle. So to gain intuition about what this thing can kind of look like, it's useful to think. So this is not

in the heart in general. In general, it's not in the heart. In basically all examples, it is, but in the general theory, it's not.

So think, so intuition, so this  $R$  triangle bracket  $S$  is kind of, well, it's just  $R$  linear combinations of points in  $S$ , kind of completely intuitively, finite  $R$  linear combinations of points in  $S$ . But you can think of it as a space of  $R$  linear combinations of derac measures. Hey Robert, I did something for you. And then, then this  $RS$  is some completion. So that's a bigger space of measures.

So again, an analytic ring structure can also be thought in terms of as specifying some space of measures on a profinite set. And what is the role of this space of measures? So an  $M$ , let's say in the heart for simplicity, lies in  $\mathrm{Dr} \text{ heart}$  if and only if for all maps  $F$  from our triangle  $S$  to  $M$  of our triangle modules, there exists a unique extension along  $R$  bracket  $S$ .

Or in other words, if  $F$  is kind of a function from your profinite set to your module, which is some kind of linear algebra object with a topology, and  $\mu$  is one of these kinds of measures that you're allowing, then we get a well-defined integral of this function along the integral over  $S$  of the function. I don't know  $D \mu$  or whatever you want.

You can pair them to get a value in the target module. So this is explaining some sense in which this is this behaves like a completeness condition. It's complete enough that you can do non-trivial integrals against certain classes of measures, which you specify as part of the data. You could say yes, is your  $D$  the full subcategory? Yes. And is it compa... no, even this one isn't. They're all presentable, but not, yeah, yeah.

And that's because we did this light restriction, yeah. So we'll have to get into that. Yes, how is the light restriction? Is it  $\Omega$  one compactly generated? It is, it is  $\Omega$  one compactly generated. Yes. Is it dualizable? No, I don't think so. Yeah, yes. Well, I wasn't claiming there exists a unique extension. I was claiming this condition is equivalent to this condition.

So were you saying why if this lies here, does there exist this unique extension? Yeah, so that's because basically just by left adjointness, but it comes from unraveling the definitions. What is the question? Was which one is only one comply generated? The big one or the small one? Neither. Oh no, both of them, sorry, both. Both. I marred the  $\Omega$  one, yeah. Both are, oh, both, yes, okay.

So I don't think I'm going to have time to get to examples, which is rather unfortunate, but I don't know, maybe Peter will, I don't know, we'll see what Peter plans for Friday. Be nice to talk about some examples. Um, but instead, I think I'll probably finish, well, yeah, I'll probably finish with a discussion of co-limits in the category of analytic rings, and in particular, I want to talk about pushouts because this is the crucial thing which is supposed to give completed tensor products which correspond to geometrically good fiber products.

So where am I? Here, so your anima structure on... sorry, anity, oh yeah. So one of the things we prove is that it comes for free, okay, yeah. I mean you have, I mean this is a, I mean no make no mistake in the maps. You have a map of animated rings from the  $r$  triangle to  $S$  triangle, but then it turns out whatever the linear algebra operations you might expect to like symmetric powers and so on will sort of automatically go through.

Yeah, it's not obvious by any means. So your definition of you back left that, yeah, that's right, that. So that's the most important functor, is the left adjoint to the thing that was in the definition, that's a very good point, that's yeah. So there's some things I'm not mentioning like that  $D$  of  $R$  is actually symmetric monoidal and this completion functor is a symmetric monoidal functor and these pullbacks, the left adjoints we were talking about, are symmetric monoidal.

So these are the things that  $\mathrm{um}$  and preserve. Subat, well by definition it's defined to be the left adjoint to this restricted functor, yeah, or you take the but but is not true that if you do the left joint on the level of the these envelopes that it necessarily preserves the category, you have to complete at the end.

So again it's kind of a completed tensor product in this base change here,  $\mathrm{um}$ , thanks for the question, yeah. Okay so so co-limits in analytic rings,  $\mathrm{um}$

so so so  $\mathrm{um}$  filtered co-limits or more generally sifted co-limits. I'm sorry, oh, the usual notion. So based on finiteness, I mean yes, oh Alf not filtered, yeah. I thought it might be important because you have, I mean indeed one could imagine it might be important. But when I say this then there's no discrepancy, so it's a, I mean there's no ambiguity, so  $\mathrm{um}$  so if you have filtered co-limit of  $RI$ , then the underlying animated  $R$  is

just the filtered co-limit of the underlying things, um, and also the free modules are similarly described is just the filtered cimit of the the free modules on the um so that's rather rather naive, um.

And what's kind of left is pushouts and this is more interesting so pushouts so if again we have maps of analytic rings I'll call them  $K \rightarrow A$  and  $B \rightarrow A$  um I'll write the push out as a relative tensor product just because um then the derived category of the pushout can be more or less immediately described so I'm not talking about the first point in the data but just the second Point um uh so abstractly this category will be the same thing will

Actually, be a full subcategory of. You take the push out in condensed rings, and then it's the full subcategory such that the underlying  $A$  triangle module lies in  $DA$  and the underlying  $B$  triangle module lies in  $DB$ .

But, but this caution. A triangle tensor  $K$  triangle  $B$  triangle  $D \rightarrow A$   $\mathrm{Tor} K B$  is not an analytic ring. So, it satisfies one through three but not four. So, it's almost an analytic ring. The only thing is that the unit object, the underlying ring, is not complete. But then you fix that by applying a completion procedure to fix this. You can still prove there's a left adjoint to  $DA \rightarrow \mathrm{Tor} K B$  sitting inside  $DA$  triangle tensor  $K$  triangle  $B$  triangle.

So, I think you're using  $D$  in two ways here. Simp, you mean the category of  $A$  is a ring. This when you're writing it, it's also it's also part of the definition of anal. That's true, that's true. So, let me make a remark to reconcile this that there's a trivial example of an analytic ring structure on any condensed ring, which is that you take  $D$  of  $A$  to be equal to all  $D$  of  $A$  triangle. And with this interpretation, the notations are completely consistent. So, you could call that analytic ring. You could call that analytic ring a triangle. It's just the um. So, every condensed ring can be viewed as an analytic ring with kind of trivial analytic ring structure or maximal analytic ring structure. Everything is complete. And with that in mind, then there's actually no conflict in the a different thing you could have done here, which is take so you have a an antic  $R$ . So, it has a subcategory is part the data. You could take  $DA \rightarrow \mathrm{Tor} DP$  as the categor  $DK$ . And that would be a different thing then doing this. Oh no, actually, it's the same. Yeah, it's the same.

Yeah, so um okay. What next? Um, yeah, question. Oh, question. Oh, I'm going to call on the person who raised his hand first. Sor yeah, that one just the same as the tensor product of the categories. Oh, that was already asked and answered. Yeah, over okay, is it, do you change the ring when you apply completion? You change, you change the, because the simpli  $R$  is  $D$  anyway, you apply the completion in the category  $D$  and then you must get another simpli ring. Yes, we have to prove that it's not obvious but yes. So, there's a so yeah, so then when you apply that completion procedure, the category stays the same but then this becomes completed. And also, I should make a remark that in complete generality, it can be rather difficult to understand this completion process. You kind of have to iterate applying the completion for  $A$  and the completion for  $B$  sandwiching them between each other take account do it countably many times take a co-limit like abstractly that's the formula for this completion procedure here. Now in practice, it turns out you can calculate it and this is one of the points uh in practice I don't think I have time to discuss examples today but this completion procedure which replaces this by the true underlying ring of uh the the the pushout in uh analytic Rings it produces the geometrically correct completed tensor products in analytic geometry.

Um okay so maybe um there's a question from Zoom. Yes, is the condition on the fenus still necessary or did you show that it is always satisfying it's always satisfied in this that's one of the reasons for choosing this light condensed set framework it's actually always satisfied it doesn't matter um so maybe okay maybe I'll start to talk about examples so I risk kind of getting cut off in the middle of an explanation but I feel like it's just too dry without well I won't get very far okay let's do let's try let's try um so I want to talk about uh maybe solid analytic Rings um and this will relate to attic spaces or hu pairs um so uh it's kind of going to be a non-archimedian addition so so I mentioned that if you if you have an analytic ring and we haven't talked about how to produce them yet but if you have one um then it's nice to look at these free modules and profinite sets to get an idea about what what's going on what what spaces of measures do you are you actually looking at here and I also said that the basic example of a profinite set well besides the point was this  $n$  Union Infinity classifying convergent sequences so but um in this linear case it's it's natural to consider the following so given an analytic ring  $R$  it's natural to consider what you could call I guess space of measures on the natural numbers and I don't mean the free module on this discret thing but what I mean is you take uh the free module on this sequence space and then you mod out by Infinity so

this in some sense classifies null sequences in our modules oh I have a backboard up there too and it turns out it's not hard to show that um addition on  $n$  induces a ring structure on  $uh$  on this  $MRN$  um and as a ring it kind of sits in between two rather extreme options uh uh so maybe maybe you want to think of this as  $t$  to the  $N$  for the purpose of this kind of discussion um so it's sitting somewhere in between the polynomial algebra and the power series algebra over your ring  $R$  um as you could imagine for something like a space of null sequences right uh or sequences with some gross growth condition I mean it's really dual to null sequences it's maybe some kind of summability condition um so geometrically speaking we have the apine line and we have some version of the formal neighborhood of the origin and then we have something that sits somewhere in between right um and now I'm going to single out a condition which is kind of a non-archimedean condition that morally speaking will mean that this guy uh lies inside the open unit disc of radius one um but formally so let's say definition uh  $R$  is solid uh if um if you if you do this you get zero but since you qu by the constant  $R$  Infinity is just the ah okay this is the home yeah that's just  $R$  and then but you know and then it's mapping into  $r$   $n$  Union Infinity by the inclusion of the point Infinity so  $R$  infinity is the free yeah in this the free  $R$  module on

this finite yes view as a as a as a condensed object or no no it's not cond what what is that like solid over  $Z$  or uh just a sec I mean this is the  $i$ ' so far I've just said this definition right so you I mean in this sense but

Just a second.

Okay, so there's an interpretation of this which is well if you have this and it's actually equivalent then you get a measure so to speak. So,  $T - 1$  has to, you know, multiplication by  $T - 1$  has to kill. So, if you have anything here there has to be a pre-image under multiplication by  $T$  minus 1. So, this means you get some measure here such that  $T - 1 \times \mu$  is equal to the unit object in this ring  $MRN$ , which is kind of the sequence  $1, 0, 0, \dots$ . And if you think about what this means, thinking about this measure space sitting between polynomial and power series, this corresponds to kind of sum over  $n$ ,  $t$  to the  $N$ , that's at the very least what it maps to in the formal power series ring. But on the other hand, this measure space, as I said, classifies null sequences.

And you know, so and this measure pairs with a null sequence to if you have a null sequence in an  $R$  module  $M$  and a measure, I said you can pair the two things to get a function. And the way it works is you take your null sequence, you put it as coefficients here. And yeah, you set  $t$  equal to one. So, what this, the interpretation of this is that every null sequence, you kind of have to work it out but the interpretation is that every null sequence is summable, which is kind of classic non-archimedean condition. So, solid is kind of one way of saying non-archimedean in this context but it's kind of fun that geometrically you can think of it as constraining the location of something between zero and the whole real line.

Maybe I will state the theorem. Yes, no. Okay, so theorem, there's a question from chat, the multiplication closed in  $MR$ , multiplication closed, I mean, it's a ring. I don't know what may I have sort  $S$  as you construct it. I guess it's ConEd as some cone it's not the best. Okay, okay, yeah, yeah, yeah, yeah, yeah, yeah, yeah, yeah, but in the Universal case  $Z$  with the trivial analytic ring structure, it lives in degree zero, there's no, I mean and also that's a summand, it's really but it's still not it's not in the heart it is in the Universal case it is and to produce a ring structure I can work in it's a ring yeah in the Universal case it's a ring and then by base change it's a ring in whatever sense you want in whatever other context I okay, yeah, okay, um, right.

So theorem, so there exists a solid analytic ring, so it's called "zolid" and well, the underlying condensed ring is just the usual integer  $Z$  kind of discrete topology. And then well, the derived category is something which I'll discuss in more detail such that an analytic ring is solid if and only if there exists necessarily unique map from zolid to  $R$  and moreover, you can actually understand this analytic ring very very explicitly. So, there are some nice results on linear algebra in this basic category.

So, the first thing is that you, well, the first thing you want to ask is what are the free modules on profinite sets. So, let's say  $S$  is some countable inverse limit of finite sets then this is just the inverse limit of the free module on the finite set which is just a finite direct sum of copies of  $Z$ . And also, this is abstractly isomorphic to some countable product of copies of  $Z$  countably infinite unless of course  $S$  is itself a finite set. What does that say next to the there exist is that say unique? It says necessarily unique, yeah, yes.

So, the second thing is that, oops. All right, so these, so these. These are remember I said that these guys always generate the category so in this case these products generate the category but moreover these guys here are Compact and projective generators of the well let's say the of the heart um but they live in degree



zero so another thing you have is that the derived category here is just the  $D$  usual derived category of its heart so it's enough to talk about the bilan category um and also these are flat with respect to the tensor product the completed tensor product uh which I'm kind of mentioned exists um so here's another point where we use the lightness because Sasha fimale proved that this is not hold if you increase the cardinalities on the profinite sets um and moreover you can calculate tensor products rather easily so tensor product of this with this uh over  $Z$  solid is just have the infinite distributive law so to speak um that makes for very easy calculation and uh sorry so just asking the the regular is not enough it is really yeah, it's really accountable really yeah, yeah um right so the and the the collection of finitely presented objects in  $D$  of  $Z$  heart uh which is it generates it under filtered Co limits um is a bilan and closed under extensions and every every finitely presented  $M$  has a resolution a free resolution you could say by product of copies of  $Z$ 's of length at most two meaning a complex where there's three non-trivial terms and two non-trivial maps uh so this kind of gives you a very good hold on calculations in this category very very explicit um so note that you can interpret this sort of as saying that uh so zolid behaves like a regular ring of Dimension two uh so  $Z$  is a regular ring of Dimension one we somehow picked up an extra dimmension and that can be attributed to the nonous dorf phenomena that you see in uh in solid ailon groups um but in in all things told you get a very good handle on this category so I think that's the the only example I have time to discuss and thank you for your attention last actually there's only isable there's only one uh right there's a single generator actually this is kind of a general phenomenon because the the free module on the canra set will always will always generate yeah, yes when when when we taking the pushup uh there's a completion fter AI gives you a DED category object why is it that it's a ring yeah that's something you have to prove and

it's not obvious thank you yeah mhm yes Chad ask question if there will be lecture notes and video recordings video recordings yes lecture notes we're kind of trying to write a book at the same time as we give these lectures and it's not clear to what extent we'll be releasing things sequentially or all at once at some point so in Uh, uh, uh. I mean, keep me motivated to. No, I'm sorry. . What can I say? Yeah, I don't know. Or do you like Reman Rock? Of course, I like Reman Rock. Okay then, it proves the most general possible Reman Rock theorems in analytic geometry. Yeah, so he crucially uses these derived categories and the fluidity of the formalism and so on.

Yes, what's the notion of the right-hand triangle? Um, Peter told me Huber uses it. Uh, had to find something. Yes, is there technical advantage to using a light condens set instead of the general condens? I mean for this antic. Yes, advantage, but for just topology. Oh, for general topology, well, I don't, yeah, I mean, there are some more subtle properties that tend only to hold for general topology maybe, not, but once you start talking about topological groups and so on maybe, yeah, yeah, nice. The same question but backward, yes, go ahead.

I just strictly prefer the light setup to the other one, unboundedness and yeah. Is there any case where I would actually want to go back to that one? Well, it's, at the very least, psychologically comforting when you have a strong limit cardinal that you get like compactly generated derived categories and so on. Turns out it's not so important necessarily in practice, but it's kind of maybe quite important psychologically. But then okay, that's just a larger cardinality bound why you would go all the way, it's just to avoid choosing a cardinality bound and so you can say all compact Hausdorff spaces are profinite I mean or condensed sets but there's no real reason necessarily.

What is dist use Infinity algebras? What are those? The structure, can we compare? But I don't, I don't understand the question but what is this term infinite? Oh, e-Infinity, ah, okay, uh, uh, everything works the same except it's a bit easier if you use e-Infinity algebras, yeah, the people who are comfortable with the infinity algebras are not laughing, why, why not, why not even better? Why not E1? I what is the commutativity, oh yeah, so you could, you could do a version of this theory of course with E1 and E2 but but there's something very special about e infinity or animated commutative which is that co-products are the same as relative tensor products that's very nice and moving to realms where that's broken can be a real pain.

Okay, so that's it, thank you.

## 2. LIGHT CONDENSED SETS (SCHOLZE)

[https://www.youtube.com/watch?v=\\_4G582SIo28&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=_4G582SIo28&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

All right, so welcome back to the second lecture. I guess I should make an announcement about the schedule for the next few weeks, which is a little bit different than usual. Usually, it should be the case that Dustin lectures Wednesdays at PS and Fridays I lecture here. But due to some traveling, I will give the next few lectures all at MPI, but note that there will be a gap of two lectures. So there are no lectures next Friday and the Wednesday after.

So the goal of today's lecture is to recall—partly this will be much repetition of a very similar lecture I gave four years ago, also five years ago. But something I want to stress throughout is why we switch to the Huber setting and which properties you gain when you make this restriction.

Before I go there, let me just say a few introductory words from my perspective about what was said last time. So what's our goal with all of this analytic geometry? For me personally, one goal since 10 years or so has always been to find—there's all this fancy geometry that has been developed  $p$ -adically, like perfectoid spaces, prismatic cohomology,  $p$ -adic shtukas, and the geometrization of local Langlands. This all works quite beautifully over the  $p$ -adic numbers. But it has to use some quite fancy  $p$ -adic geometry, like these highly non-noetherian perfectoid spaces and so on.

I mean, I'm always hoping that similar ideas or techniques could also be applied not just over  $p$ -adic local fields, but also real local fields. That's actually something that I think really is possible now. But then also over the whole integers, globally.

So in some sense, what I want is some kind of notion of a global  $\mathrm{Spec}(\mathbf{Z})$ , something like that. And this guy, this should definitely be some kind of analytic space, some kind of adic space, some non-affinoid—I mean, some model by some rings with a non-archimedean norm. Well, it should definitely include those archimedean and non-archimedean parts,  $\mathbf{R}$  and  $\mathbf{Z}_p$ . You know, hopefully a kind of uniform language to talk about these parts. But it should also be extremely non-noetherian—you can't hope for any of the usual finiteness properties often imposed.

So it's clear that you need some new language to talk about these things. And so basically, the goal of this course is to develop a language in which it is at least conceivable that such objects exist.

I mean, here's some kind of very vague idea that's possibly completely misguided. So when you do these perfectoid things, you have some kind of perfect field, maybe  $\mathbb{C}_p$  or something like this. But then when you have an atlas over this, you're always doing the thing where you adjoin a variable, but with it also all its  $p$ -power roots. So I mean, this is a prototypical example.

Well, if you think about how you would do something similar, like not a fixed prime but kind of globally—well, then you say you definitely want to get rid of the choice of the fixed prime here. So okay, so maybe you have no idea what the base is, but at least you can try to understand what happens relatively. And then relatively, I think you at least certainly want to adjoin some variable with all of its rational powers. Because each prime  $p$  is forced in some way.

But then you're only taking care of the finite primes, and then you maybe wonder, what could be the analog of this at an infinite prime? I mean, something that suggests itself—and again, maybe it's completely misguided, but something that you tend to wonder is whether you should try to form some ring where you adjoin all real powers of a variable  $T$ . But then you begin to wonder, what kind of object should this even be?

I mean, so you can definitely just treat the real numbers as a discrete thing and then adjoin here and all real powers, and then this would be some kind of algebra. But I mean, this feels somewhat artificial—that's the real, I mean, and sometimes then the real numbers are just some uncountable dimensional Cantor space, and you don't feel like you really made the situation any better.

So you definitely want to kind of keep track of the topology on the reals here. But then it's mixing the topology, like the profinite topology you usually have appearing, in a very strange way with the real

Correspondences,  $F$ -schemes and similar—this should be some algebra modeling for some local model of the space  $\mathrm{St}$  I called this last time. But even I'm more unclear what kind of geometry that should be.

So maybe these are not at all the correct objects. But maybe we could at least, if these would turn out to be the correct objects, want to have a language we can talk about those. So some such things will be involved.

Okay, and so when Dustin came to Bonn in 2018, he somehow made the suggestion that one should really everywhere kind of replace topological spaces by condensed sets. Since 2018, we've been pursuing this path of replacing topological spaces by condensed sets.

Something that was definitely clear from the very start is that this switch does resolve a lot of the foundational issues you usually have when you work with some analytic geometry. For example, it makes it possible to remove Noetherian hypotheses that are quite pervasive, for example for adic spaces or formal schemes. I'll talk about abelian categories of complete modules and so on, and derived categories.

It was clear that a lot could be gained by doing the switch. Simultaneously, something that I always liked is that suddenly such a thing as  $\mathbf{Z}[[T]]$  does make sense. This is something I will in some form also discuss today. Maybe this is not really the correct thing to consider, maybe some other thing, but I was quite happy that this kind of object at least exists in this framework.

So our attitude then was that maybe we have really no idea what we're really looking for, but let's just try to develop the foundations of some kind of analytic geometry from the perspective that you should start from some kind of condensed rings. Then try to build as natural as possible a framework so as to at least accommodate all the known examples and possibly extend them quite a bit further, for example allowing Schemes over  $\mathbb{F}_1$  and  $\mathbb{F}_{1^2}$ .

Then just see where you're led to, and maybe someday one can hope to understand how such more exotic examples might be related to some kind of interesting geometry or number theory. Basically, try it out.

Also, in the known formalisms, there are sometimes things that are slightly beyond the traditional categories. For example, if you work with adic spaces, there's the usual things built on Banach algebras. But Grosse-Klönne for example has a theory where you have some overconvergent functions instead, and usually this is yet another category. You would also like to accommodate all these variants.

Peter, can you hear me? Great. There's a request to write a bit larger if possible. Oh, we cannot. Okay. Maybe we can increase the— [Camera discussion]

Okay, so this course will not really touch on anything fancy like that. It will just try to really lay out what the formalism should be and how it accommodates the known examples.

Okay, let's start the course. Let me start by talking about condensed sets. The starting point for the whole theory is profinite sets, which will for the moment be restricted to these small things. First of all, I want to recall the following proposition. Sorry, I'm not writing bigger.

Always, the following categories are equivalent: The pro-category of finite sets. Sometimes this is a purely combinatorial kind of category, where the objects are just certain diagrams of finite sets, formal inverse limits, where the  $S_i$  are just some finite sets and  $I$  can be taken to be cofiltered.

Recall that this means that  $I$  is not empty, and whenever you have two elements in  $I$ , you can find one that is strictly smaller.

The morphisms, well, this is an inverse limit, so you can pull that out first. But then there are two traditions about writing limits and colimits, either with an arrow and just a "lim" or writing "lim" and "colim". I'm kind of combining the two for maximum clarity.

I might at some point just write lim and colim, but for the moment I'll use the directional arrows. But then when you have a formal inverse limit, nothing just means one in TJ. This—

One can also think of this  $S$  more concretely, but now using—in some sense this whole theory could be developed without mentioning topological spaces, but let's not try to do so. It's also equivalent to totally disconnected spaces, where here you send—let me write the functor.

A third possibility is to take the category of Boolean algebras. So recall what a Boolean algebra is: it's a commutative ring  $R$  such that for all  $x \in R$ ,  $x^2 = x$ . In particular,  $(-1)^2 = -1$ , meaning that  $2 = 0$ . So these are—let me write down some functors.

So if you have a formal inverse limit of  $S_i$ , you can map this to  $S$ , which is the limit of  $S_i$  but with the inverse limit topology. And then this would map to the continuous functions on  $S$  valued in  $\{0, 1\}$ , which then also, if you compose the two things, is just the colimit of all  $\{0, 1\}$ -valued functions.

And one can also go back. If you have a Boolean algebra  $A$ , this maps to  $\text{Spec}(A)$ , which is also—you want a map from  $A$  to—so actually all points of the spectrum are just  $\text{Hom}(A, \{0, 1\})$ . And actually, we can

also write this  $\text{Hom}$  as a colimit, a limit over all finite subalgebras  $A_i \subset A$  of  $\text{Hom}(A_i, \{0, 1\})$ . Alright, let me not say anything about the proof. I mean, one can actually quite easily check these functors.

So most often, I will actually think in terms of this presentation. Whenever I have a profinite set, I will usually just present it in some way as such a limit. I'll most often think about it that way.

And now that I want to come to the main things, I need to tell you two ways to measure how big—well, let's say  $S$ . When I write this, I implicitly mean that the  $S_i$  are finite sets and the index category is filtered.

The size of  $S$  is just the cardinality of the underlying set. And then there is something that's traditionally called the weight. Maybe this has a different name, let's just—so this is the cardinality of the corresponding Boolean algebra.

Let me make this definition. Here,  $S$  is light if it has kind of the smallest possible weight, except it could be finite. We definitely want infinite things.  $S$  is light if the weight is at most  $|S|$ . It's countable, yeah, so this is also equivalent to—actually, if  $\lambda$  is infinite, then it's also equal to the smallest possible cardinality for a small, possible—

In general, there are many ways to present a profinite set. For example, a point you could present as an index limit of a large profinite set of just always a one-element set, which is kind of silly. But there's always a minimal possible choice. And if it's infinite, then this is unique.

So in other words, a profinite set is light if and only if it's at most countable. Alright, so let me give some examples.

Just a quick question: "How does it compare to having countably many non-trivial closed subsets?" How do the properties compare to having countably many non-trivial closed subsets? If you take the maximum—I mean, the closed subsets are basically the ideals, right? How many of them—what does it mean?

Yeah, I think that's also the same thing. Because any such collection is given by countably many finite collections of such subsets. And I think this is still the same.

So let's do some examples of profinite sets and of their size and weight. Okay, so for finite sets, you can figure out what the size and weight is.

And then maybe the first, the smallest kind of infinite profinite set is the one-point compactification of the integers. And one way to think about this is that it's a limit of all maps  $\mathbf{N} \rightarrow [n] \cup \{\infty\}$  counting up to Limit. So it's still light, but obviously it's much bigger than this one if you just think in terms of the number of points. Number of points is, in general, these are really different.

Let me give one more example that has some relevance in the theory, although excluded. So that one very important thing: any locally compact space has two compactifications—one compactification, and the Stone-Ćech compactification. This one is small, this one is really huge. So let me not try to present this as a limit.

Well, let me instead of presenting it as a limit, let me really say what are the continuous functions from  $S$  to  $\mathbf{F}_2$ . These are the continuous functions from the Stone-Ćech compactification. But by the universal property of Stone-Ćech compactification, this is really just a continuous map from the natural numbers to  $\mathbf{F}_2$ . But these are discrete, so this is really just the set of giving such  $\mathbf{N} \rightarrow \mathbf{F}_2$ . You just have to specify the image of zero, and this is just any subset of the integers. So this is just the set of all subsets of the numbers.

So we see that this actually has weight  $2^{\aleph_0}$ , and it's not super clear, but you can show that it has size  $2^{2^{\aleph_0}}$ . It's pretty large. So in all those examples, it turns out that, well, in the example that you gave, the size in the infinite case—the size is either equal to the weight, or it is bigger.

For some reason, I only hear you very softly. We need to get CL. Okay, okay. So I just... There is something that I don't remember the answer to. Do you hear me now? Yes, now I hear you better.

Okay, so in the examples, of course there are trivial estimates that the size is at most  $2^{\text{weight}}$  and at most  $2^{\text{size}}$ . And so you can ask whether there is about inequalities between them. So in the case where you consider, like you have the countable set and the Stone-Ćech, the size is  $2^{\text{weight}}$ . So the question is whether it can be the other way in the infinite case, that is, that you have actually... When I prepared, because I wanted the same question and I didn't, but I think I know an example, but anyway it's a bit complicated. But that's also what I suspected.

So let me tell you what I know is easy and sufficient for us to know. Here's a proposition. As Ofer mentioned, you have trivial estimates, and I will tell you they are trivial, that  $\lambda$  is at most  $2^\kappa$  and  $\kappa$  is at most  $2^\lambda$ . And you see that this one can be attained, and actually in quite large generality you can make

examples where this becomes as large as this. But it's actually quite hard to give examples where  $\lambda$  becomes bigger than  $\kappa$ .

So let me just note that in the case that's most relevant to us, if you have a profinite set that's countable, then actually it can also be written as a countable limit of finite sets. So all the ones, these are like, I mean, this wouldn't follow from this inequality, right?

Why are these called "trivial"?  $\lambda$ , I said, was the continuous functions. Certainly bounded by all the maps from, this is obviously  $2^\kappa$ . And on the other  $\kappa$  is the homomorphisms from the Boolean algebra, right? Because I said you can recover the profinite set as a map from Boolean algebra to  $F_2$ . And again, this is bounded by all the maps from  $\lambda$ .

So what's quite nice about this is that these estimates even hold true in the finite case. In the finite case, actually this one becomes  $\kappa = \lambda$ .

Let me do this one case where  $\kappa$  is equal to  $\aleph_0$ . So then you can enumerate  $S$ , we can enumerate the elements. And then for each  $n$ , inductively choose a quotient  $S_n$  compatible, so that first three elements  $z_n$  inject any finite set. You can always find such a quotient.

Then you see that the map from  $S$  to the inverse limit of  $S_n$

As Scholze mentioned, you build a finite quotient where these elements are distinguished and then take the limit.

Alright, okay. So because this SL condensate will be so important for us, let me just rephrase the first proposition for light condensates. The following categories:

First, let's call it  $\text{Pro}^N$ , the sequential pro-category of finite sets. The objects here are not some fancy limit along some cofiltered poset, but really just some limit along the integers of some finite sets. And the morphisms are as before, but you are not allowed to change the  $m$ , so that's an important thing. So the  $\text{Hom}$  from  $\varprojlim F_m$  to  $\varprojlim G_m$  is again, you can first pull out the limit and then take the other thing to be the colimit. Let me actually note that there's a different way to think about this. Basically, something you are always allowed to do in a pro-object is to pass to any cofinal subsequence. And then, if you want to give a morphism from here to here, you think that first you extract a subsequence of the  $n$ s, and then you really just give compatible maps. So one way to express it is, it's really some colimit over all possible strictly increasing functions  $\varphi$  of compatible maps for all  $m$  towards  $G_{\varphi(m)}$ , but from the three-scale version.

You can also phrase this pairability condition in terms of totally disconnected compact Hausdorff spaces, and it's precisely the condition that they are metizable. And the last one was Banach algebras, and for Banach algebras, we precisely made the condition so there we made it.

So let me just note one very simple proposition that will be used throughout. If you work in  $\text{Prof}_{\text{fin}}$ -sets, and this says you have all limits, but if you restrict to light  $\text{Prof}_{\text{fin}}$ -sets, then you still have all countable limits and sequential limits of surjections are surjective. Maybe I should say surjectivity is just meant in terms of the underlying sets, like taking the actual limit. So in other words, surjectivity on the point set of the component processes. And I mean, something similar is true, that any limits of surjections are surjective in all  $\text{Prof}$ -sets, but there it uses some rather high-powered compactness result, like the Tychonoff theorem. In the sequential case, it's really kind of stupid, you just successively lift.

Okay, so the first interesting thing I want to mention is that the natural numbers actually play a bit of a universal role within the light profinite sets. That is, if you have any light profinite set  $S$ , then there exists a surjection from  $\hat{\mathbb{N}}$  onto  $S$ . I mean, the proof is really simple induction. It just writes  $S$  as a sequential limit and then at each point, just pick a large enough finite quotient of  $\hat{\mathbb{N}}$  to accommodate everything you already have.

Alright, so now I want to come to two properties that light profinite sets have and that fail in general, and that will play a technical role in what we're doing. These are some of the key reasons that we make the switch. Now, two properties that are special to light profinite sets, they may also work for some other  $\text{Prof}$ -sets, but that would be hard to single out there.

First, open subsets of light profinite sets can be too wild. If  $U$  is an open subset of a light profinite set  $S$ , then  $U$  is actually a countable disjoint union of profinite subsets of  $S$ . So this is to be contrasted with the following: In general, there exist open subsets  $U$  in a profinite set  $S$  that are not disjoint unions of profinite subsets. One way in which this can fail is, for example, you could take a product of profinite sets. I mean, disjoint unions take coproducts to products, and on profinite sets, these are totally disconnected. So you can't have non-trivial open coverings in this sense. But in general, the structure of these open subsets



Let  $U$  be an open subset of a profinite set  $S$ . Then  $U$  is a union of clopen subsets, so we can write  $U = \bigcup_n S_n$  where each  $S_n$  is clopen in  $S$ . More precisely, choose a presentation  $S = \varprojlim S_n$  where the transition maps  $S_{n+1} \rightarrow S_n$  are surjective. Then  $U$  is the union over all  $n$  of the images of  $S_n \setminus f_n^{-1}(S \setminus U)$ .

So now you've at least written it as a sequential union of closed subsets. Maybe I should have said, one way to think about open subsets purely in this language of pro-categories and profinite sets is to think about the closed things instead. Closed subsets should themselves be profinite sets, and then the closed subsets are precisely the injective maps of profinite sets.

The closed complement should be a profinite set. Someone injects, you take the preimage of this. Then if you want, you can write  $U$  as a union over all  $n$ . Now these are subsets and are themselves profinite sets. Okay, so that's one nice thing.

Here's another one. Then  $S$  is an injective object in the category of profinite sets. I will spell out what I mean by that. This means that whenever you have an injection of profinite sets  $Z \rightarrow X$  and a map  $Z \rightarrow S$ , you can always find an extension  $X \rightarrow S$ .

Assume that these don't characterize all injective objects. Or are these exactly injective objects? So there are more injective objects. For example, any injective object in general is closed under taking products, so any product of profinite sets would also be allowed. But the profinite sets in particular are profinite.

So let me prove this. Actually, the first thing one should check is the case where  $S$  is just  $\mathbb{F}_2$ . In this case, it means that the continuous maps from  $X$  to  $\mathbb{F}_2$  are in bijection with the clopen subsets of  $X$ . Or equivalently, any clopen subset of  $Z$  can be extended to a clopen subset of  $X$ . Consequently, I don't know what an exercise to do.

Okay, let's assume we know this case. Then in general, just write  $S$  as a limit of finite sets  $S_n$ . In general, the transition maps are not required to be surjective, but I will assume that all the maps are surjective. You can always assume that.

In this case, you will argue by induction on  $n$ . If you want to extend the map to  $S$ , you need to extend it compatibly to all the  $S_n$ 's. But if you've already extended the map to  $S_n$ , then extending further to  $S_{n+1}$  means the whole situation decomposes into a disjoint union over all the fibers over  $S_n$ .

So you can assume that  $S_n$  is just a point, but then  $S_{n+1}$  is just some finite set. Then it's a very easy exercise to extend, maybe just have to extend a bunch of clopen subsets. Okay.

Exercise: Figure out why this argument doesn't apply to a general profinite set. You might still think that you can similarly extend as a limit along surjective maps. You can always do that and then try to inductively extend.

Just a small remark. If  $Z$  and  $S$  are both empty, do you need something? No,  $Z$  is empty and  $X$  is not empty. Okay, if  $S$  is not empty, it's okay.

Okay, I think that's it for my general preparations about profinite sets. So let's now finally come to the definition of profinite condensed sets on a site with the following covers. We always take a profinite set  $S$  and a cover of  $S$  by other profinite subsets. This is the cover, and also surjectivity.

So let me, as last time, spell out what this really means. A profinite condensed set  $X$  is a functor from profinite sets to sets satisfying the following conditions:

1 Here is my attempt at correcting the transcript with punctuation, capitalization, and paragraph breaks:

Just need to  $M$  both of them individually, and then there is this funny condition. This comes from allowing all the surjective maps. This means that whenever you have any surjective map of profinite sets, then to give a map from  $S$  to  $X$ , it's sufficient to give a map from  $T$  to  $X$ .

At least as a first approximation, you want that any map from  $S$  to  $X$  is determined by what it does on  $T$ . But actually, you also want to characterize which maps from  $T$  to  $X$  actually come from  $S$ , and these should be the ones that agree on fibers of this map, so to speak. The good way to say this is that the two ways you can make a map out of the fiber product, either first projecting to the first coordinate or the second coordinate, this should agree. This is what the sheaf condition for this surjection says.

Before expanding machinery, let me just tell you this key example to have in mind. Let's say  $A$  is a compact Hausdorff topological space. Then we can define  $\underline{A}$ , and this is the thing that takes any  $S$  to the continuous maps from  $S$  to  $A$ . Here the presheaf on profinite sets precisely remembers how profinite sets map continuously into your topological space. This has all the fun properties.

Whenever you have a map between profinite sets, that remembers how a continuous map from one gives one to the other. It turns out that this condition is always satisfied. There's actually not a concrete topology,

so if you would omit continuity then this would be clear. If you want a map from  $S$  into  $A$ , it's sufficient to give one from  $T$  into  $A$ , and then it factors over  $S$  if and only if on the fibers the map is constant, which is kind of expressed by this.

But you're saying more than that here, because you ask that the maps have the continuous property. In other words, if you have just any map from  $S$  into  $X$  and you know that if you restrict to  $T$  it becomes continuous, then it was actually continuous to start with. Equivalently,  $S$  is actually a quotient and has a quotient topology from  $T$ . This is actually a general property for profinite compact Hausdorff spaces - they are actually quotients. In particular,  $|A|$  is just  $A$  as a set.

In general, for any condensed set  $X$ , we think of  $|X|$  as the underlying set. But you can also evaluate on some of our other favorite profinite sets, and maybe the most important one is this one-point compactification of the integers. What does this correspond to?

It's a continuous map from  $\mathbf{N} \cup \{\infty\}$  to  $A$ . In other words, it's a sequence in  $A$  together with a limit point. So in other words, it's a convergent sequence, generally with the choice of a limit point, though most often there's at most one limit point.

Similarly, when evaluating  $X$  on this, where giving such a convergent sequence, we also have to give a witness for what the limit is.

Okay, and then you can do more wild things. You can evaluate this at the Cantor set, and well, this is what it is. One thing to note however is that it's just a set, but it comes equipped with all the continuous endomorphisms of the Cantor set.

Let me just give one remark here and then forget about this forever. If you have a condensed set, it's completely determined by what I will describe as a functor from profinite sets to sets. By  $|X|$ , the Cantor set together with the actions, where this is really just an abstract set. This guy here is just, so you could think of a condensed set purely algebraically as a set equipped with an action of the profinite monoid. But I think this is a strictly worse way to think about this. Don't do that.

But I mean, it makes the point that in some sense such a set is a very algebraic kind of thing. But why, maybe I should at least say why this is the case. This is precisely the case because if you want to know what the value on any other  $S$  is, you can cover it by Cantor sets. And then for the fiber product

You're saying it into the category of what, exactly? Well, I think just this functor from sets to... No, sorry, this presheaf on this abstract monoid. Right, just consider sets equipped with an action of this monoid, this abstract monoid. I think this is a fully faithful embedding.

Maybe put a topology on this that... Actually, no, because you need to ensure that this funny covering condition... Here, I mean that for any surjection in the category, in particular, you need to ask the sheaf condition. And this amounts to some conditions that are possible.

Right, so this functor that takes a topological space to... At least for today, I want to stress the other direction, because I'm also discussing something related.

So this has a left adjoint that takes any condensed set  $X$  and maps it to the underlying set (what you want to think of as the underlying set) and canonically equips it with a certain quotient topology.

So whenever you have anything that you want to think of as a continuous map from  $S$  to  $X$ , you in particular get a map from  $S$  (or rather, just the underlying set of  $S$ ) to this  $X^*$ . This gives us a map from  $S^*$  to  $X^*$ . Take this union here and endow all of these with their natural topology as a condensed space, and this was an isomorphism.

Okay, so the image actually lands in... This will always be like "mildly compactly generated". What is this? This is for fixed  $S$ , and all  $X$ . Or if you want, you could just take the full subcategory, because everything is a presheaf on this.

So for topological spaces, there's a notion of a compactly generated topological space, which is one where, when you want to test whether a map is continuous, it's enough to test it on compact spaces mapping to it. And this is by definition the case for this  $X^*$ , because if you want to test continuity from here to somewhere, by definition of the quotient topology, you only have to test it from here.

So you only have to test continuity on these condensed sets, but these are actually metrizable. So it's actually in this sense "mildly compactly generated". I hope you can imagine what this should mean.

And conversely, if you start with a mildly compactly generated space, first treat it as a condensed set and then go back, you're precisely recovering the correct topological space, because basically exactly this condition here, and because the condensed objects anyway will come back.

In other words, there is any... Basically, any underlying... So in other words, the kind of unit of the adjunction map in this case.

So they have the... I mean generally, any  $X$  maps to this  $X^{**}$ , the unit of the adjunction. And on these guys, it's an isomorphism.

And so in particular, this means that these guys will form a full subcategory of all condensed sets. And it's... This is a very weak condition to be in this subcategory. I mean, virtually all the topological spaces that ever arise in nature... I mean, for example, we're mostly using these condensed sets in our, like, functional analysis, so to say, for topological modules. And any kind of Banach spaces, Fréchet spaces, whatever, they all have this property. They are all usable. So it's not so bad.

I wanted to make one more remark here about the relation to other similar tools that have been considered in the literature.

One is something that I think was quite influential. There's a paper of Johnstone called "On a topological topos", where he has a similar idea that, because topological spaces are not such a very well-behaved category, you should rather try to find a topos that is very well-behaved, which is very close to topological spaces.

So this is something that's achieved by this condensed sets. They form a topos, and one that is extremely closely related to topological spaces. And such things have been done before. One is Johnstone's topological topos. This is based on just the sequence spaces, so no larger profinite sets appear, only the sequence spaces.

And he uses a canonical topology. So on any category, there's a so-called "canonical topology", which is the finest one for which all the representable presheaves are sheaves. In general Infinity that are really just infinite covers, so it's an infinite collection of them that covers, but no finite subcollection will cover. But this actually leads to a Banach-Alaoglu property.

You could also just use a finitary one and then get a version of astosch where I think most of what does also works. And then actually there's one that extremely close to what we're doing. I think a paper by es, if I remember right. Basically what they're doing is they take like profinite sets, but only finite. This is a finitary topos, so that's nice, but they don't allow all continuous maps.

I'm not sure if I will come to it today, but it's really important for the good algebraic properties for the functors we want to do to allow all here. I might come to such a point today.

I think they make explicit what their stuff is in terms of this picture, but I think when you only allow the discrete, it is slightly better to make this exclusive.

So you said that being finitely generated is a really weak property. It gives some wellability, for example, in schwartz, or even weak sequential implies sequential. Sequential just means that this sequence space is enough to check, and even that is basically always satisfied. That's why I mean Johnson I think came up with the idea to just use this one, because usually it's actually too small to around with a version that uses even smaller spaces like this.

But in the end, it's actually quite important for us to keep the countable sets in, because if you want like a countable set that surjects onto any metric compact space, for example any like closed manifold or something, you can always cover it by countable. And this is actually important for us, that you can always find from like one profinite the whole thing. Otherwise we couldn't control the other at all, it would become infinite.

Right, so I have here something about this, but look not now.

Right, so maybe this is all I want to say right now about condensed sets, and as I said, for us their main importance is as a home for doing homological algebra. So let's talk about like abelian groups.

Recall like on any topos, abelian groups always form an abelian category. In particular, colimits exist, and there's a set of generators. In particular, this applies to condensed sets.

And so in particular, it's definitely abelian, and now something must have happened that like in topological abelian groups, we run into this issue that they are not at all abelian categories. But now we have abelian categories, so let me briefly discuss how that's possible.

Dustin mentioned the inclusions can be problematic. So for example, you might take  $\mathbb{Q}$  inside  $\mathbf{R}$ , which is a natural topology, or even more drastically, you could have  $\mathbf{R}$  a discrete topology inside  $\mathbf{R}$  with its natural topology. So these are maps of topological abelian groups, perfectly nice ones, but where the cokernel is kind of problematic.

So let me briefly just compute these cokernels and abelian groups, or like what happens if you take underline. Well, first of all, a point definitely just quotients.

But more interestingly, what happens to give a condenser, we also have to give the values at any set  $S$ . And so like, you have to be slightly careful when you take quotients, because now the sheaf condition actually becomes important. And like the naive answer you might guess is that this should be continuous maps to the reals mod continuous maps from  $\mathbb{Q}$ , where  $\mathbb{Q}$  is discrete. These are just locally constant.

And you can actually prove that you don't have to sheafify in this case, and this is already a sheaf, and this is the true answer. Okay, so this means that this quotient still kind of remembers something about how there was a topology on this guy, even if you can't really phrase it in terms of the topology itself in this guy. It still remembers that on this part you should take all the continuous maps, but on this part you should only allow the locally constant ones.

And so now let's even do the more drastic thing where you modify all of  $\mathbf{R}$  by a discrete guy. Well, then the underline set is just zero, right? It's  $\mathbf{R}/\mathbf{R}$ , which is zero.

But there should be a value at Very much nonzero. So here's some controlling this funny thing by observing that it has nontrivial maps on general. So, say again... Andity, it's not... Sorry, I again meant to... The, yeah, think... I mean, the rational numbers are not at all embedded in the real numbers. Okay, so let me just state the theorem and then maybe prove it next.

So, part of this I already said. Condensed light condensed... In particular, the exact... But even better, countable limits. Countable products. And for the people that know this funny XM, that's Soal 86 XM inen Le's hope paper, which is about some funny way that products can affect the co-limits. And this is satisfied for products.

This is worse than in all condensed being GS, where all products are exact. Here it's just the countable ones, but I don't know, I mean, most of them really only take countable limits, so it's not so bad. But it's one reason that we were at first a bit hesitant to make this switch. It's also now not anymore... It doesn't have enough compact projective objects.

But one thing that's extremely nice: there is a free guy guys stop. So you can take the light profile set convergent sequence, and then you can always build a free condensed group on there. This turns out to be internally projective. And this is really a property that's extremely specific to the light setting.

Within all condensed being groups, we have plenty of projective objects, but none of them are internally projective, except trivial cases like  $\mathbf{Z}$ . But also, all of them are really, really big. I mean, they all come from extremely disconnected sets, so they are kind of impractical.

This one wouldn't be projective in condensed groups, but within light condensed groups, it just so happens to be projective and even internally projective. And so this is the only setting of any variant of condense theory in groups where I'm aware of any non-trivial object that's internally projective. And like, the free on a convergent sequence is actually kind of important as a very basic object in the theory. So it's really nice that within this setting, it has its good categorical properties. And it's one of the main reasons we made the switch to the light setting.

The forgetful functor, say, from light condensed groups to the underlying light cond... This has a left adjoint, the free functors on objects. Any light profile sets with free being group on that. I will discuss it more next time. In particular, inside here you have light profile sets while... And in particular, you can take on the light profile set and infinity fre, okay?

So I have this and then discuss some other things on... One questions? Can I ask a question about the... So you said you stated two facts about light case that are not true in the general case. This is a technical question to understand the contra in fact.

I found... So the one can give cont example using the interval from zero to the first uncountable ordinal, which is a forfinite set. And so for the second statement, I take this cross itself and then the closed subset, which is the first uncountable ordinal cross the set union, the set cross this last element, and then the whole thing, the overall product. This is not the retract of the whole product. And for the first thing, I take the complement of the last element, and this is a big open which is not disjoint union of  $\text{Clen}$ . This is not difficult to see.

On the other hand, you state that there are cases where the shift topology is nonzero. So just for a general interest, I want to know if this is true in this case, and what is the reference for this fact? The shift topology can be... No, if you take the first uncountable order, treat it as a profile set, and remove the limit point, this is a case where the open is not a disjoint union of profile sets. Doesn't high... I think it has.

But you said that people know that sometimes it has... I think you can definitely put examples where there is... You can also take the Stone compactification of integers and remove a part of the boundary. Also pretty, maybe less computable in general.

Is it still true that light top is... You take category stop this... Yeah, you know, the underlying set, right? So that's enough for face.

There no further questions, and let's stop here and we resume on Wednesday.



### 3. LIGHT CONDENSED SETS II (SCHOLZE)

[https://www.youtube.com/watch?v=me1KNo3WJHE&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=me1KNo3WJHE&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

#### Unfinished starting from 0:00

Okay, so let me recall a little bit where we were last time. Last time, we had this category of profinite sets. There are several ways to think about this: either as sequential limits of finite sets, or as metrizable totally disconnected compact Hausdorff spaces, or as countable Boolean algebras. We equip this with a Grothendieck topology where covers are generated by the following two families: finite disjoint unions and all surjective maps.

As I kind of stated last time, this has a very important consequence that I want to stress again: sequential limits of covers are still covers. This property will actually be extremely crucial in order to have a good resulting theory of something like topological abelian groups that we want to use to have good homological algebra properties. This is something I maybe want to stress today a little bit, where this appears.

Then we had this category of condensed sets, which were these sheaves for the Grothendieck topology on this category of profinite sets. In general, whenever you have sheaves on some site and the generating site embeds (well, at least if it's a subcanonical site), you have a Yoneda embedding. You have the profinite sets sitting inside there. This sends any profinite set  $S$  to the functor that takes any  $T$  to the maps from  $T$  to  $S$ , maps in profinite sets. If you prefer to think more concretely in terms of compact Hausdorff spaces, this is continuous maps.

One important property, which is also completely general, is that the image generates under colimits. You can think of general condensed sets as being built out of profinite sets by some kind of gluing procedure.

Then we also compared this to the category of topological spaces. In particular, we can embed this here as certain compact Hausdorff spaces. In fact, more generally, for any topological space  $A$ , you can build such a thing. I think I called a topological space  $A$  last time  $\underline{A}$ , which is given by the same procedure. You send any  $S$  to the continuous maps from  $S$  to  $A$ . So this diagram commutes.

As I also said last time, and I kind of want to improve on something I said last time, this  $A \mapsto \underline{A}$  is left adjoint. It takes any condensed set  $X$  to, well, if you have a condensed set, in particular you have the value on a point, which we think of as the underlying set of this condensed set. Then we were equipping this with a certain topology.

Here we're taking the disjoint union over all possible maps from the countable set into  $X$  of the resulting map from the countable set onto  $X_{top}$ . The reason for the countable set being that it's any profinite set that admits a section from the countable set. So it's enough to allow the countable set here.

But something which I kind of forgot about and that Yok Morita reminded me after my lecture is that actually, you can describe this quotient topology differently. Because the countable set, I mean, convergence of the countable set can be detected by sequences. Or more precisely, if you map all possible convergent sequences to the countable set, it's still a quotient map.

So actually, you can also consider all, let's call this  $\beta$ , all convergent sequences in  $X$  of just the convergent sequence. This is also a quotient map in topological spaces.

So if you want, the countable set can actually be thought of as the colimit over all, say for example, countable closed subsets. So for example, finite unions of convergent sequences. This is in **Top**. It will not be true in condensed sets, and I will discuss this in a second. But it is true in **Top**.

This means that equivalently, you can describe this corresponding topological space as just the one where you just remember what the convergent sequences are. That's enough to determine this topology. In particular, this means that this funny notion that I was talking about of being compactly generated is actually just the same thing as being sequential. So that continuity can be checked by checking whether convergent sequences go to convergent sequences.

And so the thing I said last time, someone, more succinctly, is saying that sequential topological spaces embed into light condensed sets. Okay.

Okay, so from this perspective it seems that allowing the countable set didn't really help at all. But now I want to say why we allow the countable set. So is it the same or more general than first countable topological spaces? I forgot all my general topology. Does somebody know, is sequential the same as first countable? It could be, discrete on huge, but if the greatness of first countable... Because in the previous version of the

theory it was first countable, I forgot, I think first countable with all points closed. But now we don't need all points closed because we don't have the set theory, right?

I mean, why allow the countable set? We're frozen, I'm frozen. I hope I'm un-frozen.

So the reason is that in any topos, for example, whenever you have sheaves on any site, there is an extrinsic notion of being compact and of being Hausdorff. So these are the following notions that I want to recall. Frozen again, frozen again, why? Oh my God.

So this general topos notion is what's called quasicompact. If any cover admits a finite subcover, here this general notion would amount to saying that there exists a surjection from the countable set onto  $X$ , or  $X$  is empty. Yes, thanks.

And then there is this intrinsic notion of being quasiseparated, of being Hausdorff. And this notion was introduced by Grothendieck in SGA4. And of course, Grothendieck was using algebraic geometry lingo, so instead of Hausdorff he said separated. And because it's just a general topos notion, and not the specific separated notion in algebraic geometry, he called this quasiseparated.

So in general this would say that for all quasicompact, well maybe I'm assuming here something about enough quasicompact objects, but there are enough quasicompact objects. And for all quasicompact  $Y$  and  $Z$  mapping to  $X$ , so we hope that the technical problems got fixed, let's see, the fiber product between the two is quasicompact.

And again, let me simplify this here again, because one can always surject onto any quasicompact guy by something by the countable set. For all two maps from the countable set onto  $X$ , the fiber product, I should have chosen notation for the countable set, is still quasicompact.

So this I'm basically trying to say that, sorry, need not be surjective, just a map, that if you map a countable set into  $X$ , then it may not be injective. So this might factor over some quotient of the countable set. But you're somehow declaring here that it's the quotient by a closed equivalence relation on the countable set, which is one way to express some kind of Hausdorffness.

Okay, so in any topos you can talk about these quasicompact and the quasiseparated objects. And at least if your topology is finitary, the quasicompact guys are exactly those where like a finite union of the generating objects maps onto it.

So if we only allowed, for example, the convergent sequence as our basic guy in the test category, and the quasicompact objects would all be quotients of this guy, or at least be countable. So in particular, the countable set would not at all be quasicompact. And if you want to access the countable set, it would be precisely by such a colimit, that would be this huge colimit of smaller objects.

And this would mean that the intuition for what a compact object is, like the countable set is very much a compact object and you want that, this would be destroyed if you passed to such a topos.

And in fact, I mean, with our notion of what the generating objects are, you in fact get this really nice proposition that, by the way, quasicompact I very often abbreviate to "QC". I mean, in principle I hate abbreviations, but okay, this is QC and this is QS. And if both of them are assumed, then these are called "QQS".

So you can wonder, what are the compact Hausdorff, so to say, light condensed sets?

So this means that the topos, abstract topos theoretic notion of quasicompactness really very closely mirrors the idea of what a compact space is. And similarly, the general topos theoretic notion of what Hausdorff means mirrors precisely what you think Hausdorff should mean. And so this forces our Grothendieck topology to be finitary, because otherwise the basic objects wouldn't be quasicompact. And it forces us to include the countable set in the formalism.

Then you can also wonder more generally, if it drops the compactness hypothesis but still requires a Hausdorff condition, what are those things? So what are the quasiseparated condensed sets? Those also can be described in more standard terms.

So we know that the metrizable compact Hausdorff spaces are allowed. And in general, you should think of these guys as being rising unions of quasicompact subsets. In fact, they are exactly the ind-category, where all the transition maps are closed immersions of metrizable compact spaces. Let me write injections—I really mean all maps are injective, in which case they're automatically closed immersions of metrizable compact spaces. In the ind-category. So last time I was talking about the pro-category, which were formal inverse systems, and these are formal directed systems. So there is a category, where the objects are functors from

a filtered poset towards metrizable compact Hausdorff spaces with all the maps injective. And functors, I mean maps, are the usual thing on ind-objects.

Yeah, so you could also drop the Hausdorff here and then drop this here. And let me just point out that within this category you have what topologists call the compactly generated spaces. Probably meaning sequential. But note that they are not the same, because in this category, as I said previously, you can write the countable set as a huge colimit of countable closed subsets, but not in the category of quasiseparated condensed sets.

I mean, these are both condensed sets. This is the prototypical example of an ind-system of metrizable compact Hausdorff spaces. This is also an ind-system where you just have one term, it's actually some quasicompact quasiseparated guy. But as condensed sets, they are very different. Because this is really just a formal colimit, and this is just one object. In particular, this guy is quasicompact and this one is not.

But actually, if you ask for ind-systems that are countable, then on such things, these all come from here. So some of the difference between these two categories only comes when the colimit category is pretty large, as it is in this case.

So I don't understand the subtlety. Is there a condition on—so if you take the condensed set which is the direct limit of the injections of the countable unions of images of the sequence space, does this give something by the equivalence of categories which is in this category MCG? It's not an equivalence of categories, right? This is just a full inclusion.

What is included in that? Ah okay, this one is included. Okay, I did not realize that this is what you were saying.

All right. So let me just say something that's implicit. We said that something very nice, like the interval, is definitely a nice metrizable compact Hausdorff space. We said it should be quasicompact as a condensed set. And I said that quasicompact means that there must be a surjection from the countable set. In fact, that is true. You can find a surjection from the countable set, and in fact some kind of canonical one.

I mean, if you have a sequence  $a_0, a_1, a_2, \dots$  of either zeros or ones, you can send this to the number in binary  $0.a_0a_1a_2\dots$ , the binary expansion, which is an element in the interval. And any point in the interval admits such a representation. So you get the surjective map.

But we definitely need the countable set, or something as large as a countable set, to surject onto here. And so you can then recover this whole guy as a certain closed equivalence relation here, where the equivalence relation is precisely this nasty thing that  $0.111\dots = 1.000\dots$ . All right Okay, so we're mainly interested in this formalism of condensed sets as a framework for doing some kind of algebra where all the objects have something like a topology.

To get this started, let's talk about condensed abelian groups. There are actually two ways to think about this, which I didn't say last time. Either these are abelian group objects in condensed sets, or these are sheaves on this category of condensed sets with values in abelian groups. Let me just write in symbols: sheaves on  $\mathbf{Cond}$  with values in  $\mathbf{Ab}$ . Similar remarks apply to any kind of algebraic structure. If you have rings, for example, you can view them as ring objects in condensed sets or as sheaves of rings.

From the general theory of sheaves and Grothendieck topologies, we know that this is an abelian category, in fact a Grothendieck abelian category. So it has filtered colimits, etc. It also has a tensor product. Let me describe a little bit what the properties are.

The unit object is just the condensed set  $\underline{\mathbb{Z}}$  associated to the integers as a discrete set. If you have two condensed abelian groups  $M$  and  $N$ , then  $M \otimes N$  is the sheafification of the presheaf that sends a condensed set  $S$  to the tensor product  $M(S) \otimes N(S)$ . This is just a functor from condensed sets to abelian groups, a "presheaf", and then you can always sheafify.

There's a further property that I want to stress, which is also completely general. If you have a condensed abelian group, then you can forget its abelian group structure and just have an underlying condensed set. But this has a left adjoint, a kind of free construction. A "free condensed abelian group"—let me drop the "condensed" when I say that now—takes any condensed set  $X$  to the free abelian group on  $X$ . Here, as in general for these colimit-type constructions like tensor products and left adjoints, you always have to sheafify. So it's the sheafification of the presheaf that takes any  $S$  to the free abelian group on  $X(S)$ .

There is already some structure here that you don't often think about in topological abelian groups. You don't really often take the free group on a topological space. Here it exists, and it's actually a completely fundamental structure.

The idea is that this free abelian group on  $X$  is some topological abelian group. What is its underlying group? It's just the free abelian group on the underlying set  $X(*)$ . So if you have any kind of topological space, you can just take its free abelian group. Of course, this machinery will then put some kind of topology on the free abelian group.

Let me actually discuss this in an example. Sheafification won't change the value on the point, because any cover of the point is split, so the sheaf condition is kind of vacuous on the point.

Here's an example, which is kind of related to the introduction I gave last time. We can take the real numbers  $\mathbf{R}$ . Very soon I will forget to write the underline all the time, because implicitly everything has become a condensed set. But okay, so we have the real numbers as a condensed set, and then we can take the free condensed abelian group on that.

What kind of object is this? It's sums, if you want, of real numbers  $x$  weighted by integers  $n_x$ , where the  $n_x$  are almost all zero. You might think of these points  $x$  as measures, and then these are finite sums of measures.

But now it also has some topology, where you kind of remember that these  $x$ 's are allowed to move continuously. But then, if you have  $x + y$  in general, that's a non-zero element. However, when  $x$  and  $y$  become the same, then suddenly this collapses.

It's maybe not so clear how you would actually describe this topologically. If you wanted to describe this as a topological abelian group, you would have to declare what the open subsets are, and I think that's a little bit tricky to visualize.

But you can actually say what it is as a condensed set. First Here is my attempt at correcting the transcript:

Again, the  $C_0(\mathbf{Z})$  itself can be written as a rising union over all integers  $n$  of subsets where you are only allowing sums  $\sum_{i \in I} n_{x_i} [x_i]$  where the sum of the absolute values of the  $n_{x_i}$ 's is at most  $n$ . Everything is contained in something like this, as condensed sets.

So whenever you have a profinite set mapping into here, it will actually factor over one of these subsets. And these guys, they are compact Hausdorff and metrizable. It's a kind of fun exercise to figure out how to describe such a compact Hausdorff space.

I claim that whenever you have any compact Hausdorff space and you look at finite integral sums of points of them where the sum of the coefficients is at most  $n$ , there is a canonical compact Hausdorff topology on that. Classically, this takes a little bit of thinking. But in this formalism, this free construction just produces it for you automatically.

The subtle part is that there are some kind of non-trivial identifications you have to make. In general,  $x + y$  is a non-zero element, but when they become equal, you have to collapse this to zero. To make this true, you actually have to use that in our Grothendieck topology, we didn't just allow finite disjoint unions but also effective epimorphisms. Otherwise, this wouldn't come out right.

Also, by general nonsense, if you start with something that already had a group structure, then if you pass to the free ring, this now has a ring structure where the multiplication comes from the addition. In particular, this is actually a condensed ring completely naturally. This is kind of related to the question I had in my first lecture, like how do you draw an element in all its real powers? It's just done by this construction.

These are general features. Now I want to mention a few features that are quite specific to this light condensed setting. The first thing is that countable products are exact. This might seem like an operation you're maybe not so often doing, but something you are certainly very often doing when you do some kind of functional analysis is to take sequential limits of surjective maps, and they are still surjective.

For example, in functional analysis, maybe you have some kind of Fréchet space and it's a sequential limit of Banach spaces. Maybe in that case it's not even surjective, but it will also come out right. You definitely want sequential limits to behave nicely. For example, if all the transition maps are surjective, you definitely want the limit to still be surjective. As I will argue in a second, this more or less forces you to go where your basic objects that define your site must be some kind of totally disconnected things.

You have these, and there is this other property that I will explain in a second. You have the sequence space  $\mathbf{Z}^{\mathbf{N}}$ , it's a profinite set or light condensed set, and you take the free ring on that. This turns out to be internally projective.

Recall that one way to say what projective means is that in any abelian category, you can define Ext groups. It means that  $\text{Ext}^i(P, -) = 0$  for  $i > 0$ . Internally projective makes sense when your category also has a tensor product, because if you have a tensor product, then you can define an internal Ext, and internally projective means that the internal  $\text{Ext}^i(P, -) = 0$  for  $i > 0$ .

The first two properties are actually things that are better in all condensed abelian groups, because all products are exact. Well okay, the third property is also solely still true. However, the third property is something that's only true in the light setting.

Let me just define the internal Ext. Basically, if you have a tensor product, then you can also ask for an internal Hom, which is some kind of adjoint of the tensor product. Similarly, the internal Ext will be some version of the same thing on Ext groups.

Okay, so let me prove this actually. Let me first note that one reduces to showing that taking products of surjections is surjective. The only thing that's not true for general products is that a product of injective maps is always injective and so on. The only thing that's not clear is that the countable product of surjections

For all  $M$ , you can take the product over all  $n$  at most  $M$  of  $M_n$ , and then the product over  $N$  bigger than  $M$  of  $N_n$ , and this surjects onto the product of the  $N_n$ 's. Now because a finite product is the same thing as a finite direct sum, they always preserve surjective maps. They're always exact, and so you can always, like, for finitely many coordinates, kind of do the lift. But then this guy here, I mean, this map is the limit now over  $M$  of these things.

And so if we're asking whether a countable limit of surjections is still surjective, let's assume you have such a diagram. We have  $M_0, M_1, M_2$ , and  $M_\infty$  is the limit of these, which certainly maps to  $M_0$ . We ask ourselves whether this is surjective. So what does it mean to be surjective in the sense of sheaves?

This means that whenever you have one of your objects generating your site, so any light profinite set, and a map from here, then, well, if it was a surjection as presheaves, you should immediately be able to lift that to here. But in fact, it's enough to find a surjective map from some—let's call it  $S_\infty$ —and a lift to here. So does there exist some other light profinite set surjecting onto  $S$  and a lift to  $M_\infty$ ? That's the question. That's what surjectivity on coverings amounts to.

But let's just see what happens. We definitely, well, right, so this is this map to  $M_0$ . But we know, because this map is surjective, we know that there is a light profinite set and a lift to here. And then again, because this map is surjective, there exists some further light profinite set and a lift to here.

And so now you can just take  $S$  to be the limit of this diagram. So inductively, you construct light profinite sets with a map to here, and then take the countable limit of these surjective maps. As I said last time, and I think I recalled today, countable limits of surjections are still surjections in light profinite sets. Hence, this guy is still allowed as a cover in our Grothendieck topology.

So in the definition of your  $X$  group in the sheafification, you drop the underline and the left  $p$ . So  $\mathbf{Z}_p$   $s$ , you mean  $\mathbf{Z}_p$   $s$  underline? Yeah, let me drop the underlines. Okay, so  $S$  is for me a light profinite set, and they sit inside of light condensed sets. So yeah, if you feel better, make an underline, I mean, yeah. And like, the Yoneda embedding has no decoration for me, it's just the same thing. But yeah, also in three, like, this threefold guy, it's like, maybe I should also underline 10 here.

Okay, so before I go on to the proof of three, let me reflect a little bit on what happened here. This finishes one and two, and let me come back to that in a second.

So here, the critical thing—we definitely, for doing good homological algebra, like a functional analysis kind of homological algebra, you definitely want property two. But critical for two was exactly this property that countable limits of surjective maps are still surjective, that limits of covers in your Grothendieck topology are covers.

And I claim that this basically forces you to use totally disconnected spaces as building blocks. Why? You might also have the idea, and I think people are doing that, that if you want some kind of nice category of something like topological real vector spaces, you might work in this kind of smooth setting where you take your defining site to be like smooth manifolds, and the Grothendieck topology just the usual one of open covers of smooth manifolds.

But in that case, this property that countable limits of covers are covers is just not true. Because if you have, like, maybe just the real line, then you can cover it by two intervals. Each one of them you can again cover by two intervals, and then keep doing that. But then the intervals shr



Disconnected compact Hausdorff spaces. And initially, we took all of them. We realized it's slightly better to restrict to symmetrizable ones and to have all surjective maps as covers. Because you can actually show that any surjective map of topological spaces can be written as a sequential limit of actually open covers. So if you want open covers and their sequential limits, you need all of them.

Actually, I should maybe say that this is not some kind of ahistorical comment. This pro-étale topology was, in fact, first—I mean, so this stuff about condensed sets, this comes from something called the pro-étale topology, like originally in Bhatt-Scholze for schemes. And there, the wish was precisely that limits of surjective maps should still be surjective. Because this was—we wanted to have certain sheaves which were naturally certain inverse limits, and we want them to be well-behaved. And for this reason, we allowed these countable limits, or all limits of covers, to still be covers. So this is really the origin of this whole theory.

All right, so this was a small interlude. Excuse me. So if you have a surjective map of topological spaces, for example, you take the  $\mathbf{N} \cup \{\infty\}$ , two copies of  $\mathbf{N} \cup \{\infty\}$ . So all of the two sequences will converge to infinity. And then this is covered by two copies of  $\mathbf{N} \cup \{\infty\}$ , and they intersect at infinity.

You made a statement to the effect that—yeah, well, okay, not all transition maps are surjective. But actually, you know, I mean, actually something slightly stronger is true. I don't need all the transition maps to be surjective. I only need that the maps down to  $S$  are surjective, and the limit is still okay. And in that sense, you can realize  $S$  as okay in this other sense.

Okay, something slightly weaker would be—yeah, I'm not sure. I would have to think how much difference it makes.

Okay, so I want to prove this thing that this is internally projective. But before I do that, let me make a warning that this is a phenomenon that's only true once you pass to groups. So  $\mathbf{N} \cup \{\infty\}$  is not at all projective in condensed sets.

And I actually kind of expected Gab wanted to go there with this remark. Because this is precisely the example that I need to do. So you can have a surjective map of topological spaces and the convergent sequence downstairs, so that there does not exist a lift. And one example for this would be to just take this to be the convergent sequence itself, and this to be like, I don't know,  $2\mathbf{N} \cup \{\infty\}$  disjoint union  $2\mathbf{N} + 1 \cup \{\infty\}$ . Some breakup of  $\{\infty\}$  as like the limit of the even guys and the odd guys.

Then like on the integers, you only have one possible lift. But the even guards converge to something else than the odd guards. But miraculously, once you pass to groups—it so happens that if these were condensed abelian groups, in fact you don't—then any convergent sequence can be lifted.

All right, let me actually use a slightly different guy. So let  $M$  be the free group on the null sequence. So you can take  $\mathbf{N} \cup \{\infty\}$  and mod out by  $\{\infty\}$ , which is actually a direct factor, right? Because you have  $\{\infty\}$  mapping here and then projecting to a point. So this splits, actually has a direct summand.

And so, and of course, the integers themselves, they are projective. That's okay. So the question is really whether this other part—so this classifies null sequences mapping out of  $M$ , is the same thing as giving a null sequence in the other guy. Or in other words, it's a free condensed abelian group on a null sequence. And free group on a convergent sequence, but then the limit point should be zero.

So if we want that this guy is internally projective—and let me actually focus on the projectivity, and then just The property that  $\infty$  goes to 0... Because this is a surjective map of condensed groups, this means that there is some surjective map from a profinite set that lifts to  $\mathbf{N}$ . That's just what the surjectivity means.

Okay, now how do covers of  $\mathbf{N} \cup \{\infty\}$  look like?  $\mathbf{N} \cup \{\infty\}$  is like you have a discrete set and then it accumulates towards infinity. Now, in the pullback, you have some complicated  $S$  upstairs here. But you can always make that smaller because, for each of the discrete points here, you can just pick any lift. I don't care which one, and then all these lifts together with keeping everything at  $\infty$ , this is a closed subspace. So there exists  $S'$ , a closed subspace, so that over the integers, it's just always a point.

You can assume that this  $S$  here is somehow a different compactification of  $\mathbf{N}$ . Without loss of generality, we can always make  $S$  smaller here, as long as it's surjective. So we can assume that  $S$ , the part over any finite thing, is just a point. Okay, but there are many compactifications of the natural numbers. For example, this one, so we can't expect that we can directly split that.

But now we actually use a property of profinite sets that I mentioned last time. Okay, so let  $S_\infty$  be the fiber over  $\infty$ . This might be profinite, so there's a certain subspace in this, but everything here is profinite. In particular,  $S_\infty$  is profinite. I stress it here because here it's really the critical point where profiniteness is

used. This means that it's injective in profinite sets. In all profinite sets, it's also non-empty. And so there exists a retraction  $r$  from  $S$  to  $S_\infty$ .

Okay, so let me call this inclusion here  $i$ . Maybe this was my original map  $f$  and this was a map  $g$ . Now we can just write down the lift. Consider the following map from  $S$  to  $\mathbf{N}^\sim$ , which is the map  $g$  that we had. But then from it, you subtract what you would get if you first use the retraction and then re-embed into  $S$  and then apply  $g$ .

So if you look at  $S_\infty$ , then on  $S_\infty$ , these two maps are just the same because this was a retraction. Am I frozen again? So on  $S_\infty$ , it's actually just the constant map 0. Was that a question? Am I frozen again? Do you see me now? Yeah, okay.

But  $S$  surjects onto  $\mathbf{N} \cup \{\infty\}$  and this map to  $\infty$ . Because this is actually a surjective map here, you can show that this is a pushout in profinite condensed sets. That's basically a version of this gluing condition that I gave. If I want to give a continuous map from here to somewhere, it's enough to give one here and here which agree on the overlap. But it's definitely enough to give one here, and then the question is just when does it descend down to  $\mathbf{N} \cup \{\infty\}$ ? This means that on fibers it must be constant, but there's only one fiber in some sense where there's something to check, which is the fiber at infinity. And there we precisely assume that it all just comes from a point.

Okay, but now you're precisely in this situation. You have a map from  $S$ , you have the zero map, and they agree on  $S_\infty$ . So this means that actually this whole map factors over a map from  $\mathbf{N} \cup \{\infty\}$  to here. Let me call that map  $f$ .

Okay, and now we basically have all the structure we need, and we just need to check that we actually produce the lift. So what do we have now? Now we have a map from  $\mathbf{N} \cup \{\infty\}$  to  $\mathbf{N}^\sim$  which vanishes on  $\infty$ . So in other words Because this was a retraction, and so this means that, because of this pushout property, you really get an exact sequence. You have a map from the convergent sequence modulo infinity to  $N$ .

Now the question is whether we've actually lifted  $f$ , and the claim is that we did. To check that we did, note that this guy here is still surjected on by the free guy on  $S$ . So to see that this map here is  $f$ , it's enough to check for the composite.

The point is that if I take this thing that I used here as a correction term and I project that down to  $N$ , then this map from  $S$  to  $N$  vanishes on all of  $S_\infty$ . This term projects to zero in  $N$ , because its composite map from  $S$  and  $N$  vanishes on  $S_\infty$ . So the correction term indeed vanishes, and  $g$  was a lift, so we're fine.

For internal projectivity, let me just not do it. It's essentially the same argument, you just have an auxiliary profinite set floating around that you also need to cover a bit. Again, it's critical that sequential limits of surjections are surjective.

Here's just one remark about the comparison to all condensed abelian groups. This is something that I already said a bit earlier, but just to say it more explicitly:

In all condensed abelian groups, products are exact and you have projective generators  $\mathbf{Z}[S]$  where  $S$  is what's called an extremely disconnected set. For example, it's a Stone-Ćech compactification of some discrete set. These are huge things, they have cardinality  $2^{2^{\aleph_0}}$ .

One thing that's better there is that you really have projective generators. You don't have projective generators in light condensed groups. You have this one guy which is the free guy on a null sequence, but the free guy on a countable set cannot be covered by a projective object. This is slightly unfortunate, but actually not that bad in practice.

The good thing is that in light condensed groups, you have this really explicit projective object which is the free guy on the  $\mathbf{N}$  sequence. Whereas here, these projective guys exist by the axiom of choice, but they are completely inexplicit and their precise structure depends on which model of set theory you're working with.

But one thing I want to point out is that none of those projective generators there are internally projective. Basically, if you take a product of two Stone-Ćech compactifications, and  $I$  and  $J$  are infinite, this product is never projective. That's actually not so easy to see, but we can prove it. This is a rather severe technical issue in the category of all condensed abelian groups. It's extremely nice that you have these projective guys, but for many arguments you would really like to know that they are internally projective, and they are not. That's bad.

In light condensed groups, at least we have this one really nice internally projective object. That's good.

The other thing I wanted to point out explicitly is that the free guy on the  $\mathbf{N}$  sequence is not projective in all condensed abelian groups. The light ones embed into the category of all condensed abelian groups, and you could ask if it's still projective there. But there, precisely the Stone-Ćech nonsense gives the obstruction. When some universal compactification of the integers, the biggest one, is the Stone-Ćech compactification, this rejects. You can show that there does not exist a splitting here.

Maybe another thing I should say is that this is also related to certain questions about Banach spaces that people have studied in literature. There, the only known injective Banach spaces—there's some kind of duality that what used to be projective in this condensed stuff will become injective in the category of Banach spaces—are continuous functions from some  $S$  into  $\mathbf{R}$ , where  $S$  is one of these extremely disconnected guys which I could have also allowed. Basically, such a Stone-Ćech compactification or a retract of it.

It's known that all injective Banach spaces are retracts of continuous functions on a Stone-Ćech compactification.

What about injectives? In your case, it is for light condensed abelian groups. Is it enough injective? For all condensed abelian groups, there are no nonzero injectives. In all condensed abelian groups, there are no non-zero injective objects.

Okay, yeah, they exist for set-theoretic reasons. I mean, for like general nonsense reasons in light condensed abelian groups. But I don't think you can write any of them down.

And what corresponds to this free abelian group sequence is the Banach space of null sequences. This is not injective as a Banach space. But it is what's known as separably injective, where you only test this injectivity against separable Banach spaces. This is very much related to the thing that this guy is not projective in condensed abelian groups, but it is in light condensed abelian groups.

Actually, I think I made this realization that it is projective in light condensed abelian groups after looking up this proof that this guy is separably injective. It was an open question in the Banach space literature whether the continuous functions on this guy is a separably injected Banach space. In the notes on complex geometry, Kan kind of proves that it's not separably injective. In particular, this guy also doesn't behave like a projective object, even when you test against light condensed abelian groups.

Anyways, maybe one thing to take away here is that you might think that all this totally disconnected nonsense shouldn't really appear when you do function analysis over the reals. But actually, people that studied Banach spaces intensively, they are very much studying continuous functions on totally disconnected things.

Okay, right. Yeah, maybe let me finish by talking about homological algebra. You also said something, I forgot in which context, about the axioms AB5 and AB6 of abelian categories.

In Grothendieck's Tohoku paper, you can find a lot of axioms that an abelian category might or might not satisfy. AB5 is the question of whether all products are exact. This is true in all condensed abelian groups. It's not true in light ones, but at least the countable ones are okay. When AB5 is satisfied, you can ask about AB6, which is a certain question about the commutation of infinite products with filtered colimits. This always sounds confusing, but you can actually make a statement that's true.

This property, AB6, is always true if you have enough projective generators. In particular, it's true in condensed abelian groups. If you restrict this question to countable products, then it's also true in light condensed abelian groups. So AB6 holds for countable products. One way to see this is that you have this fully faithful embedding of light condensed abelian groups into all condensed abelian groups. This commutes with all colimits and countable limits. In particular, these countable products are some of the correct products there, the ones that you would also compute in condensed abelian groups. But you can also just check it by hand.

In the last bit, I want to talk about cohomology. When you study topological spaces, you probably also care about the cohomology, in particular for manifolds where you would want that to be the singular cohomology. When you think about a topological space, we already have an idea of what the cohomology should be, just as we already had an idea of what a complex thing should be like, and so forth. But on the other hand, whenever you work in a topos, that topos somehow comes with its preconceived notions of not only what compact objects are, but also what cohomology is.

If  $X$  is any condensed set and  $M$  is any abelian group (it could even be a condensed abelian group, but let's restrict to discrete ones for the moment), you can define the cohomology of  $X$  with coefficients in  $M$ . This is just something that is there whenever you work in the topos.

One way to define the cohomology of  $X$  with integral coefficients, in terms of the formalism that I now introduced: You take  $\mathrm{R} \mathrm{Hom}$  in the category of  $X$ -groups in light condensed abelian groups (actually they would also be the same as in all condensed abelian groups, oh well, I'm doing light stuff so let me stick

So, um, if say  $X$  is a CW complex, this thing is  $X_\bullet$  or it's underlined. It's exactly the singular cohomology. This might seem a little bit weird because we didn't really put any geometry into the definition of these condensed sets, right? We were just using totally disconnected things, and suddenly using totally disconnected things, we are still able to probe whether that circle is not contractible. Um, but it comes out right.

Um, and, um, right, uh, so let me actually recall that. I mean, yeah, let me make one remark about this and then stop. So as I said, this is here the  $\mathbf{Z}$  of these  $X_\bullet$  groups, and I told you how to think about this, right? So this was, uh, the thing where you take finite sums of  $X$ -points of  $X$  valued by integers. Um, and so this is actually—there's a Dold-Thom theorem which basically tells you that this guy is something like a model for the homology of  $X$  once you, once you pass, once you treat this up to homotopy equivalence. This kind of guy is a model for the homology of  $X$ . Uh, and, well, for us the interval in condensed sets is like an actual interval, it's not a point. So we didn't yet pass to homotopy, uh, but one way in which you're doing that is by taking  $X^s$  out of it, because an interval cannot map to something, uh, this being discrete. And so, uh, yeah, so this dualizing, I mean this is like homology, so the dual should be like cohomology, which fits the picture. Uh, and yeah, let me just, uh, stop here.

Other questions? Uh, all right, so we can also consider the internal Ext-functor. Um, in this case, this would just unravel to an adjunction to, uh, also replacing  $M$  by the continuous functions from  $S$  to  $M$ , which is still an abelian group, so it doesn't really do that much more. Um, uh, maybe one comment to make is that, and maybe I won't talk much about it, but, um, for all sorts of things like continuous group cohomology and so on, sometimes it's not quite clear, like, what is a continuous representation of a group, what is the right notion of continuous group cohomology? Because often this is just represented by some explicit cochain complex, uh, but if you just treat your topological groups as condensed abelian groups, uh, then it's clear what an action on, like, a condensed abelian group should be, and it's clear what cohomology should be. There's some kind of general 2-categorical answer to what it should be, and this always gives you the expected answer. It's not always the same thing as the thing computed by continuous cochains, but when it's not, it's a better answer.

So, so here, when you, when you have a CW complex, if you have more generally a local system, I think it will give a sheaf on condensed  $X_\bullet$  over  $X$ -underline, and then you can say that the usual cohomology, it's like singular sheaf cohomology. It satisfies cohomological descent, for, for at least for compact Hausdorff spaces, it satisfies cohomological descent of surjections of compact Hausdorff spaces. So to compare the two sides, it looks like using the theory of cohomological descent to, to, well, essentially to do the, should give, to do it with local systems. By, yeah, you can, you can use any coefficient system, really, on  $X$ -underline. This is, yeah, it's a really robust result.

Um, but if you, for example, work in, um, like, this topos of sequential spaces, where you only allow  $\mathbf{N} \cup \{\infty\}$  as a generating object, then if you would try to compute this cohomology of, like, the interval, then you would first have to express your interval in terms of your generating object. So we would write this as this huge colimit of all these, uh, small countable sub-closed subsets in the interval, and then this  $C$

#### 4. EXT COMPUTATIONS IN (LIGHT) CONDENSED ABELIAN GROUPS (SCHOLZE)

[https://www.youtube.com/watch?v=EW39K0J7Hqo&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=EW39K0J7Hqo&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

##### Unfinished starting from 0:00

All right, I think that's a signal to start. Good morning, welcome back. So today I want to talk about some EXT computations in condensed mathematics, and then some...

So I guess the basic claim is that this category of condensed abelian groups is a very convenient framework for doing homological algebra. It's a very nice abelian category. You can, without any problem, form a derived category. And because topological spaces essentially embed faithfully into condensed sets, topological abelian groups also basically embed faithfully (under some mild assumptions).

So now you have a nice home to play with them, but then because you have an abelian category, you can ask about EXT groups, and then you can wonder what they actually are. And for this theory to be somewhat useful, you want all the answers of these EXT computations to give you reasonable answers that are interpretable.

So the first theorem that I stated last time is the following theorem. One thing you can do is start with, for example, some nice geometric object like a CW complex  $X$ . And let's say  $M$  is an abelian group. Then one can look at the EXT groups in condensed abelian groups between the free condensed abelian group on  $X$  (or rather, on the corresponding condensed set), and  $M$  treated as a discrete abelian group.

Basically, if you treat  $X$  as a condensed set, then in any topos there is this internal notion of cohomology, which one way to express is as EXT groups in the topos. So this is the internal notion of cohomology of  $X$  with coefficients in  $M$  in the condensed world. And fortunately, it turns out that this precisely recovers the singular cohomology of  $X$  with coefficients in  $M$ . Okay.

So the first thing I probably want to do today is sketch a proof of this theorem. Maybe I should say that to a large part, in this lecture and maybe also the next one or two, I will still follow the first course that I gave on condensed mathematics some years ago. You can find lecture notes online, so for much of what I'm talking about today and last time, a reference is this "cond.pdf" that you can find on my webpage.

Okay, so proof sketch is that, well, as  $X$  is a CW complex, it's an increasing union of some  $X_i$ , where the  $X_i$  are compact. It's filtrable, built from finite dimensional cells. And then on both sides, you take colimits (direct limits). Yeah, you probably need a slightly better statement there to really compare the complexes, not just individual groups. But basically, you can reduce to the case that you just have a compact space  $X$ , a compact CW complex.

Then there's actually a more general statement that works for any compact Hausdorff space  $X$  that may not be a CW complex built from cells. That's the Čech theorem. Again, you still have these EXT groups between the free abelian group on  $X$  and  $M$ . It turns out that in this case, you can always compute this as what's known as Čech cohomology on  $X$  with coefficients in  $M$ .

So here, Čech cohomology is defined in terms of the category of sheaves of abelian groups on  $X$ . You consider a topological space  $X$  (or the corresponding site), and abelian sheaves on  $X$ . Then you have a global sections functor from abelian sheaves to abelian groups (I'll write this as  $H^0(X, -)$  to distinguish it from other notation). You apply this to the constant sheaf  $M$  on  $X$ , and take derived functors.

It is known that Čech cohomology, when restricted to CW complexes, satisfies the Eilenberg-Steenrod axioms and agrees with singular cohomology. So for  $X$  a CW complex, the two notions coincide. But not in general - in fact, singular cohomology is really only valid for CW complexes. In general you should use something like Čech cohomology (or really the same thing).

So here's a key example to keep in mind, of relevance to us: If  $X$  is totally disconnected, then you can actually show that the global sections functor  $H^0(X, -)$  is already exact. So in this case, Čech cohomology vanishes in positive degrees. Okay. So here is a proposed correction of the transcript:

Compact, I mean... Yeah, I'm still... Sorry, yeah, the prof fin set. These are the global sections and the global sections are just locally constant maps from  $X$  to  $\mathbb{N}$ , equal to zero, and there's no higher cohomology.

But if you would consider the singular cohomology, this is defined in terms of the singular chain complex, which is built from just mapping points and simplices into  $X$ . But from the perspective of all simplices, there's no way to tell apart  $X$  from just a discrete set of points, right? Because simply connected... So any connected space will just factor over one point.



But when you... So you can also compute the singular cohomology of  $\mathbf{N}$ , and again it's  $\mathbf{R}$  in degree zero, but in degree zero you get all functions.

Yes, continue. So this means that thinking about cohomology doesn't really see anything about the topology of  $X$  anymore. The sheaf cohomology does.

I think another possibility is to consider locally contractible spaces. I believe that when you cross  $X$  with an interval, this sheaf cohomology group should not change. I'm not completely sure what it looks like, and then one can do for locally contractible spaces a comparison with sheaf cohomology. And if in addition it is paracompact, then you get comparison to singular cohomology. But the usual results on...

Yes, yes, right. Yeah, I think the key statement you really need about  $X$  is that it's locally contractible in this funny sense. "Locally contractible" doesn't mean it's covered by opens, or that any point has a basis of open neighborhoods that are contractible. But rather that for any point and any neighborhood, you can find a smaller, possibly smaller neighborhood that can be contracted in the larger one. And under this funny assumption, that's the official definition of locally contractible.

You can actually show that yeah, I think everything that I said about CW complexes extends to this. But to compare sheaf and singular cohomology, the usual treatment requires paracompactness. I'm not sure if it is... So yeah, maybe paracompactness...

Yeah, I think these paracompactness assumptions, they would kind of only matter in intermediate... If you go all the way back to here, I think it probably disappears again.

Yes, the question... So the question was, often one considers topological spaces up to weak equivalence. And then obviously any space is weakly equivalent to a CW complex, and for example  $X$  is weakly equivalent to just a discrete set of points. But here I don't want to consider topological spaces up to weak equivalence, because otherwise I wouldn't be able to treat totally disconnected spaces at all. I'm not using here topological spaces as a way towards pure homotopy theory, I'm really interested in the actual topology.

Um, right. So okay, so here is what I want to prove. If I have any abelian group, then I can compare these sheaf cohomology groups. And this can actually be further upgraded, and I mean, I'm not doing this just for fun, actually...

So here's a first upgrade you can do. Sheaves were defined... So there are two sides. On the one hand, you can fix  $X$ , let's call it the topos of  $X$ . So this is what's used on the right-hand side. And on the other hand, you can consider the category of slices of the topos, defining light contravariant set-valued functors. You can consider light profile sets together with a map to  $X$ .

And once you pass to sheaves, or once you pass to the corresponding topos, it turns out that that's actually a geometric morphism here. So... These here are light sets with a map to  $X$ , and these are  $X$ , and there is a map here.

So what I'm telling you is that whenever you have a sheaf over  $X$ , you can pull it back to get a light functor on contravariant sets over  $X$ . And to define what the pullback is, you really only have to define it on those generating objects. So  $\mathcal{L}$  of  $U$ , when  $U$  is an open subset of  $X$ , is just  $U$ .

Okay, so here's the usual sheaves on  $X$ , and you can pull them back to get light sets over  $X$ . And correspondingly, you also have the topos of sheaves on

This means that  $X$  is a sheaf group, which are some kind of forms. In particular, for all  $\mathcal{F}$  which are abelian groups on  $X$ , the cohomology on the site of condensed sets over this coefficients in  $\mathcal{F}$  is the same thing as the cohomology. If you apply this to a constant sheaf, you recover the previous result. This is for  $X$  locally compact.

The kind of  $X$  I was having in mind is the case where  $X$  is a compact Hausdorff space, probably profinite would be good. So if we apply this to the constant sheaf on the abelian group, we get the previous result.

Maybe this all sounds a bit scary, but the point is just the sheaf is defined in terms of cohomology on one side, and the other is basically also just cohomology on the other side. The original claim was that for a certain very specific sheaf, namely a constant sheaf, they have the same cohomology. But actually something much more robust is true: that any sheaf  $\mathcal{F}$  which can be represented in terms of such a functor is fully faithful.

Okay, so let me sketch. We need that on any object, the pushforward of  $\mathcal{B}$ -cohomology sheaves on  $X$ , the adjunction map from  $\mathcal{A}$  to the pushforward of the pullback of  $\mathcal{A}$  is an isomorphism. This is a certain map in  $\mathcal{D}^+(X)$ , and then just a general fact that you can check whether such a thing is an isomorphism by checking

it on stalks. This is really the key point where I'm kind of using the topos-theoretic formalism in order to make a reduction to checking something on stalks.

We want that  $a_x$  maps isomorphically to—oh, right, we wonder what this maps isomorphically to. Now the key point is that you can actually have suitable base change properties that allow you to pull taking the stalk into this pushforward operation.

The key base change property, taking stalks at  $x$ , uses the following: this is where you actually use something about the nature of the stalks, namely you use a general cohomology result for taking the stalk of some kind of  $\mathcal{F}$  and taking cohomology of some kind of formula. There are certain statements when you can interchange them. Such base change results hold in the generality of what's called coherent topoi.

Coherent topoi is just basically the same thing as being quasi-compact and quasi-separated. So on the end, what you see is that the key thing you really need at this point is that this condensed set corresponding to  $X$ , in this internal language of topoi, means to be quasi-compact and quasi-separated. This was precisely the thing I mentioned last time: that if you want to do this, for example, for the interval, you need to know that there is a surjection from one of the generating objects, the contractible sets, towards the interval.

So implicitly here you're using that you can cover something like an interval by contractible sets. Sorry, yes, thank you. Why did you use the boundedness assumption? Because otherwise this statement doesn't hold true in general. If you don't have things that are bounded to the left, there's always some issue of how cohomology changes with cosic limits, and there are all sorts of questions.

Basically, there are at least three versions of which way you can do this. There's Lurie's version, which may be the best one, and then there is just the rough category of sheaves, and then there is the left completion. If you want the functor to be fully faithful here, you should always take the left completion.

All of this matters if  $X$  is not locally compact, or if it's five-dimensional, then everything's the same. But this kind of general statement, that it always holds for any Grothendieck topos, only works when you're bounded to the left. Otherwise there are some issues. I believe you want to regard the point as a limit of its closed neighborhoods rather than open neighborhoods, yes, exactly.

So if you actually want to execute what I gave here as a hint, then you actually have to use that. I mean, this is cofinal with all open neighborhoods of  $x$ , but when you're in a compact Sure, I'll help clarify this transcript. Here's the revised version with added punctuation, capitalization, and paragraph breaks:

You can take the closed neighborhoods here, which has the advantage that if you pull back a closed neighborhood, you actually get some profinite set. You want to stay within the realm of proper, compact, separated objects, for which you need to take a closed neighborhood. But you can just do that.

So, we can pull in—taking the stack  $X$ , this basically means that we've reduced to the same statement, but now the space  $X$  has to just become a point. Let's do this. Okay, but if it's a point, then all these are just abelian groups. Well, this is still something—abelian groups embed fully faithfully in there.

This basically means the integers, as a profinite group, are still projective. So the  $X$ —yeah, maybe you have to explain, but it's obvious that this  $\text{Hom}(S, \varprojlim M_i)$ , with this topology—you have to use that these are in between the open and closed neighborhoods. So if you pull back to open neighborhoods, but then the transition maps, if you have two open neighborhoods and their intersection, then the intersection on the closed neighborhoods sits in between. So it doesn't matter which one you use.

Let me just pause a second and let me try to understand what happened here more completely. So we're interested in computing  $X(S, G)$  for some  $X$ , and we're interested in computing, well, let's just say, in arbitrary degree. Then we would like—now I took a very fancy approach, but we could try to do it more down-to-earth. How would you actually try to compute  $X(S, G)$  in abelian groups? You would try to find a projective resolution, and if you cannot find one that's projective, at least you would try to find one that's a free resolution.

This actually breaks the proof in two steps. Step one is to show that if  $X$  is actually totally disconnected, then you don't actually have to do anything. So in this case,  $X(S, G)$ —in this case it's one of these generating objects in our site, it is in this case some profinite set  $S$ . The continuous maps from  $S$  to  $G$  are equal to  $G(S)$  in degree 0 and nothing in positive degree.

Okay, so for  $S$  being, for example, the profinite completion of the integers, this really just follows from the statement from last time that this is a projective object. But in general this is not true, and for the general case you still have to resolve this  $S$ . But for—so this comes down to the following:

If there was some, for example, some  $H^1(S, G)$ , you could always split this  $H^1$  after you pass to a cover of  $S$  by what it means that it's injective. Similarly, if there's any higher  $H^i(S, G)$ , you can always split it after passing to some kind of infinite resolution of this  $S$ . So concretely, what this amounts to is that for hypercovers—so this is a simplicial object in profinite sets—there's  $S_0, S_1$ , and so on.

Concretely, this means that  $S_0$  is a cover of  $S$ , and then you can recover  $S$  as a quotient by this fiber product  $S_0 \times_S S_0$ . But then the next one  $S_1$  is required to surject onto the fiber product, and so on. Here, it takes a little bit of effort in writing.

So whenever you have such a hypercover, there's actually some completely general thing that in any site, if you have an object with a hypercover, then you can build the corresponding chain complex. This is actually always exact, and you need to see that when you dualize and pretend that this should be the answer you would want, if I now pass to the corresponding complex of continuous functions, that this is still exact. This is not automatic, this is what you have to prove.

So we need to show for all hypercovers—the exactness is automatic, what we need to show is that the corresponding complex of continuous maps is also exact. There are several ways to prove this. One is to use the same argument I used, which is to argue that in order to prove this exactness, you can treat everything here as a sheaf on  $S$ , because instead of taking global

Stalls, and then again when you pass to stalks, you realize the same thing. Where now you're covering a point they have to cover a point, but then again, because it's a point, their cover—there's nothing.

So that's one way. The other way is to prove something, some abstract lemma, that whenever you have a hypercover of profinite sets by profinite sets, you can always write this as some cofiltered limit of hypercovers of finite sets by finite sets. And then this writes this thing as some filtered colimit of corresponding things where everything is a finite set. But then for finite sets also, hypercovers are split and the exactness is automatic.

I think that the latter argument is the one I actually used in the lecture. The first argument is like cohomological descent in SGA 4, and I think that they also check—I forgot in which reference, but in one of the proper base change theorems for separated, proper maps. And then the arguments in cohomological descent go through, and it will give this. The cohomological descent spectral sequence will give you this result. Right?

Yeah, I guess the previous thing on the right was, I guess, it's called Mor. Either that, or you have a colimit, and it actually takes a little bit of unwrapping that you can always do this, but it's okay, to reduce to the case of finite sets where it's obvious.

Okay, so that's the first step if you try to do this more concretely, but maybe also the less interesting step, because we're still just treating these totally disconnected spaces. And now, step two is to treat general spaces.

And so we would again like to find such an acyclic resolution, and now we actually found a lot of acyclics, because now we know that for these connected guys at least this is all right. So now it's enough to find a projective resolution, but one by such joins.

So we want to resolve  $\mathbf{Z}$  on  $X$ , but on  $SS^{-1}S$  is lol. And so how does one actually do that? But one uses precisely that—one uses that you can always find the surjection from the constant set, so from some profinite set on to  $X$ . And then you get some equivalence relation  $S_0 \times_{SS} \mathbf{Z}$ . And this actually, I mean, if  $S_0$  is already totally disconnected, then  $S_0 \times S_0$  is, and this is a closed subspace if you take the fiber product. So this is actually always totally disconnected. You don't even have to do any further resolution.

So you can take Čech's nerve, where this is  $S_0$ , this is  $S_{-1}$ , this is  $S_{-2}$ . In general, these  $S_i$  are just some  $i + 1$ -fold fiber product. We have some hypercover, in fact Čech cover, of  $X$ .

And so again, by the general principle that hypercovers give you resolutions, you now get such a resolution of  $\mathbf{Z}$  on  $X$  by these three guys. This is our resolution by things that are Čech for functions.

And so this tells us that, if you're interested in  $R\Gamma(X, \mathbf{Z})$ , this can be computed by taking these guys. And now we know that this guy here is computed by this complex, where you take here the continuous functions from  $S_0$  to  $\mathbf{Z}$ . Which is some really awkward formula for something like singular cohomology.

So you take your nice base  $X$ , maybe the interval, cover it by some constant set, and then take all the fiber products. And then everywhere you take just the locally constant functions to the integers, and build this cochain complex. And the claim is that this computes  $R\Gamma(X, \mathbf{Z})$ .

Now, it's not so clear a priori that it really computes the right thing. What the previous proof amounts to is to check that, to again treat all of these things here as sheaves on  $X$ , so as global sections of some sheaves on  $X$ . And then to check that this computes the right thing, you can again somehow compute on stalks, and then you're done. And then we check that it resolves the constant sheaf.

Sorry, sir So what are some examples of objects in here? Well, also the  $\mathbf{Z}$  are in there, and then I don't know, the real numbers  $\mathbf{R}$  are there, or something like the real numbers modulo the integers  $\mathbf{R}/\mathbf{Z}$ , or the  $p$ -adic numbers are in there. Also something nice like the adeles, which are the restricted product of the completions of the integers.

One nice thing about the adeles is that they sit in a short exact sequence where you have  $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}} \twoheadrightarrow \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ , and  $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$  is compact. So there is some kind of structure theorem for these guys.

Each object can be broken up into three pieces: one piece is the discrete piece, one piece is a finite-dimensional real vector space, and one piece is a compact piece. As you see, there are some kind of interesting exact sequences in there. So you definitely expect that there is some kind of  $\text{Ext}^1$  group, like  $\text{Ext}^1(\mathbf{R}/\mathbf{Z}, \mathbf{Z})$ , which classifies the extensions of real numbers by integers, something like this.

Two things that one knows about this category: it seems to be like an exact category, and computing Yoneda Ext from the category, one knows that all the  $\text{Ext}^{\geq 2}$  groups are equal to zero for these objects. I don't know about  $\text{Ext}^1$  groups.

So you can wonder whether something similar holds true now if you compute Ext groups inside the category of condensed abelian groups. Okay, so let's take any two locally compact abelian groups which are also metrizable. Then, again, this metrizable assumption can be ignored; it only comes from the restriction to metrizable condensed abelian groups. Then you can compute the Ext groups as condensed groups from the corresponding guys as topological groups.

I mean, just from the fully faithfulness, you already know that the Hom groups, they can't be changed; they must just be the usual topological homomorphisms. But a priori, there could be some weird  $\text{Ext}^1$  groups that you didn't know about. Here you're computing Ext groups within the whole category of condensed abelian groups, and there could be some really weird extensions between them. Actually, later, maybe hopefully today, I'll give an example where this actually happens.

But here it turns out that's not the case. All the Ext groups, all the extensions of a locally compact group by a locally compact group in condensed abelian groups, they all are themselves extensions of locally compact abelian groups. You can really identify the abstract Ext group with the usual thing, where we're looking at short exact sequences  $B \rightarrow X \rightarrow A \rightarrow 0$ , so exact sequences in locally compact abelian groups up to the appropriate notion of isomorphism.

Let me give some key examples. Actually, something slightly better: you could even consider this. So first of all, you can compute Ext groups of anything against the circle group  $S^1 = \mathbf{R}/\mathbf{Z}$ . This is actually  $\text{Hom}(-, S^1)$ , so this is Hom as condensed groups, and it's precisely the Pontryagin dual group as a compact group.

Or another example, one could also try to compute some Ext groups like  $\text{Ext}^{\bullet}(\mathbf{R}, \mathbf{Z})$ , where some of the intuition is that the real numbers are something connected, so they can never map to the integers which are totally disconnected. So you would expect all these Ext groups to be zero, and indeed, if I remember correctly, that's true.

I think actually these are more or less the key examples. To understand this computation, what you have to do is, you can do a devissage where  $A$  and  $B$  can be reduced to these basic cases. So for example, if you map from something discrete, then there's not really anything to show, because then the Ext groups are easy to compute. So you can assume basically you have a compact abelian group

Algebraic geometry to do some computations. But the statement we needed was never actually put in by Grauert. There is an unpublished letter of Thom to Grauert, where he proves the result, but it's some unpublished—but it's a very nice theorem. Let me use  $\mathbb{B}$  for this.

There is a resolution of the form, something functorial in abelian groups  $S$ . So what is a resolution? You're trying to resolve any abelian group  $M$ , and you're trying to find some kind of universal projective resolution of this. They're trying to resolve by free abelian groups, and there's of course a very easy way to at least find a projection onto them. You send some of the generators, given by some element of  $M$ ,  $M$  as an element in here, some the finite free abelian group generated by the elements of  $M$ , right? But of course, this is way, way bigger than this.

So the standard way actually to do this is to do some kind of monadic resolution, where you use a free abelian group monad. And then there is some kind of general thing that the next term—I mean, you might put here, and this would be (it's not what I want to do)—you could put as a free abelian group on the free abelian group on  $M$ , and then two maps here, and then the difference you use here. And then you can continue, but then these things get uncontrollably large. So this is not what you want to do.

And you realize that actually, you don't need something as big as this to generate the kernel. Because really, the only thing you really have to enforce is that when you add two elements of  $M$ , then they become the same as the sum, right? So basically, whenever you have a pair of elements of  $M$ , so here generators are certain pairs  $(a, b)$ , you can send that to  $a + b, a + b$ . And it's easy to check that this actually generates the kernel of this map, because once you prescribe those relations, then you can uniquely sum any such thing. And you realize you used to just ignore time.

And then you can continue. So each term will just be  $\mathbf{Z}$  joined with  $\mathbf{Z}^M$  to some power. And there are some transition maps that are given by some universal formulas like this, except nobody's able to write them down.

Do you have to take a finite direct sum or in each term of such powers? Or is it enough to have one power? Like when we originally wrote this up, we used to find some of these things. But you can actually, by some stupid argument, you can basically cover any such finite direct sum again by such a free guy. Okay, I see, I see. You will actually realize that there's a small issue with a zero, but you will figure it out.

You can just choose one term in each degree. And so all the differentials are given as some universal formulas, which actually have functoriality. All the differentials are universal. So it's a little bit of a mathematical result.

And surprisingly, the proof of the theorem uses a little bit of stable homotopy theory. So in some incarnation, it uses something like the finite stable homotopy groups. And that they appear in the proof is also the reason that you don't really know how to do this explicitly. Because at some point, you need to basically kill something like stable homotopy groups,  $\pi_n^s(S^k)$ , using surjections from finite free groups onto them. And you can do that, but you won't get anything.

But one very nice thing is that this is really functorial. And so this means that this immediately works in any topos. In any topos, you can write down the same complex, the same universal formulas. But whenever you have a sheaf of abelian groups, you can write down the same complex as sheaves of abelian groups on that side, and it will automatically be exact.

So functoriality, or really the universal formulas, also works for sheaves. And so we now get a resolution of our  $A$  that we're interested in, where now all the terms are some  $\mathbf{Z}[A]$  or  $A^{\square^n}$  or some such. And so in order to compute  $X$  from here, there's some reduction to computing Ext. Only did the case where the target was the  $\mathbb{S}^1$  group for the  $\mathbb{S}^1$ . I also need a case where the target are the real numbers.

Yeah, so if you look back at what I did in these notes on  $p$ -adics, then there are also proofs that for any compact Hausdorff space  $X$ , you can compute some of the  $X$  groups of  $\mathbf{Z}$  (join  $X$  into  $\mathbf{R}$ ) as also needed for all  $X$  to compute  $X$ , so from all the  $X$  for  $X$ , but now with real coefficients. And then the claim is that this actually doesn't have any higher cohomology, whatever  $X$  is. And  $\mathbf{Z}$  of course, it just gets a continuous  $\mathbf{R}$ .

This also works with  $\mathbf{R}$  replaced by any Banach space, but it doesn't work if it's just  $\mathbf{R}$ . But it really uses local convexity because there are some partition of unity arguments in the proof. You know, for the partition of unity to behave nicely, you need that the target Banach space is locally convex.

Sorry, that's important. So as a preview of something that will happen later: when we consider real vector spaces in  $\mathbf{R}$ , we will actually have to consider non-locally convex cases. And so we will actually really be interested in situations of such computations where comp, and then we do not have something on all comp spaces. And then this means that we will actually have to resolve. So when we want to resolve by a simplex, we really have to go to totally disconnected.

All right, so let me give an example of how such computation will come out. So when you're trying to compute the  $X$  groups of the reals against integers, then  $X$  is—no I think that's right, it's a shift from the free on  $X$ .

All right, if you want to  $\mathbf{R}$ , you just have to map into  $\mathbf{R}$ .

Yeah, thanks. So when you computed, when you compute that, then you're resolving this by the  $\mathbf{R}$  on  $\mathbf{R}$  on  $\mathbf{R}$ 's and finite vector spaces, and then we know—I mean, finite vector spaces, that's definitely a CW



complex. So we know that the  $X$  groups, you can see, um, sorry there's some  $2i$ 's here, that this is, well, it's the same thing as a singular cohomology  $\mathbf{R}$ . And so this is of course  $\mathbf{Z}$  in degree 0 and 0 otherwise.

And so this means that when you compute the  $X$  groups out of this, then each term will just give you one copy of  $\mathbf{Z}$ . You would like this to be 0 for all  $i$  greater than 0. You might be worried that there's some more lots of  $\mathbf{Z}$ 's remaining, and maybe you don't know what the differentials are. But one way to control this is to observe that you can do a stupid thing of also using this resolution for  $\mathbf{Z}/2\mathbf{Z}$ , which is also 0 by several properties of the integers.

And then you realize that when you compute  $X$  out of this sequence, it's the same thing as computing  $X$  out of here, because each term individually has the same  $X$  groups. And so then out of here is the same as out of here here, but this just results 0.

Okay, so this is how one can leverage this knowledge about the  $X$  groups from these free guys on reasonable things into such  $X$  groups from locally compact groups.

Let me actually mention the variant of this argument. So one might be worried that it's kind of weird trying to do very explicit computations by using an inexplicit resolution. But some of the inexplicit nature of resolution somehow never becomes an issue. Just the existence of such a resolution, its properties is enough. But actually there is a resolution that is explicit and that can also be used. This is something known as the Eilenberg-MacLane or  $Q$ -construction, which was rediscovered in the process of this formalization as well.

So this is an explicit complex. It starts just like we expect.

Do this on the other face too. So you take  $-a, c, -b, a + b$ . I hope you can check that if you compose the two differentials, you get zero. If not, then there is some easy variant of this that should work.

Now you can imagine how you do this one step up. Imagine  $\mathbf{N}^8$  is where the eight elements sit on the vertices of a cube. Then you take this side minus this side, and this side plus this side, minus the sum of the sides.

Then there is a theorem that this is kind of linear. More precisely,  $Q(M)$  is always quasi-isomorphic to  $Q(\mathbf{Z}) \otimes_{\mathbf{Z}} M$ . In particular, if you look at all the homology groups, everything is free. This just means that the homology of this cubic construction is just a linear functor. In general, there's some extra 2-torsion.

Yeah, this linearity in  $M$  turns out to be kind of sufficient. So  $X_i(A, B) = 0$  for all  $i$  if and only if all the  $EX$  groups from this explicit complex vanish. Basically, all we are trying to prove is that they can be put into the form that some  $EX$  group should vanish for all  $i \geq 0$ . Then we can use this explicit resolution of  $A$ . I mean, it's not quite a resolution of  $A$ , but for this purpose, it's good enough.

Another thing you can actually compute is what this thing is, using some stable homotopy theory. You can show that it is actually  $\bigoplus_{i \geq 0} (\mathbf{Z}[S^i])^{\oplus 2^i}$  shifted into degree  $-i$ . So this is where a little bit of stable homotopy theory comes in. Basically, whenever you write down something explicit, the explicit answer should have something to do with stable homotopy groups of spheres.

Okay, this is just to say that instead of an explicit resolution, you can also do all of these arguments with this explicit complex instead. It doesn't really change any of the arguments, except you have a little bit more comfort that you actually know which objects you are dealing with.

Just to clarify, it's not direct. I think it actually is a direct summand. You think no? Yes, I think the statement is that there are  $Q$  and  $Q'$ , so there is actually a modification that kills something extra, and then you just get this summand. But then you are removing the extra stuff.

All right, so finally I can say one corollary that's really important for the theory of condensed abelian groups. One thing you can now compute is an  $EX$  group of something that's not at all locally compact anymore, but actually quite big. You can take the whole product of a countable number of copies of the integers, so the profinite integers. Or if you want, you can take the product of  $p$ -adic integers.

It turns out that these  $EX$  groups are actually just a direct sum of countably many copies of  $\mathbf{Z}$ . This is very different from the classical answer, where if  $EX$  was the naive dual of a countable product of copies of  $\mathbb{F}_p$ , which is a profinite group, now we are looking at some continuous things, and we just get the direct sum. The really critical thing is that there's nothing in higher degrees.

So this guy will be the compact projective generator of solid  $\mathbf{Z}$ -modules, which is a full subcategory of condensed abelian groups. All the solid  $\mathbf{Z}$ -modules should be solid, and then if you want it to be projective, you definitely want all the higher  $EX$  groups to vanish.

Okay, so let's prove that. This actually uses a weird trick. Naively, you would now again try to resolve this by free guys on profinite sets, but the issue is that this is a really large thing. Here's a warning: if you

treat this here as a condensed set, then this is a union over all functions  $f : \mathbf{N} \rightarrow \mathbf{N}$  of the product over  $n \in \mathbf{N}$  of the interval from  $-f$

And so this approach would actually be extremely hard to execute, but there's a trick you can do. You can resolve now in the other direction, which seems weird. You can embed this into a product of copies of the real numbers. This is a similar thing.

At this point, we crucially use that products are exact in condensed mathematics. It might be easier to justify this specific case. Okay, so this is exact.

And now we're trying to compute  $X$  against—so this maps the  $X$  from here into the  $X$  from here, and the  $X$  from here. Let's do this one first. This is actually a compact Hausdorff space that is totally disconnected. It's still profinite, right? So it's a countable product of profinite groups.

And so we know what the  $X$  groups are. The  $X^0$  is a product of  $\mathbf{Z}$  copies. It turns out that this is, well, this is discrete and this is something connected, so there shouldn't be any maps. And then it turns out that in degree one, the computation will show that this is just a direct sum of copies of  $\mathbb{Q}/\mathbf{Z}$ .

Nothing else. This is what the result of the previous lemma refers to. Then if you write the long exact sequence, you realize that what you need is that the  $X^2$  groups from this product of copies of the real numbers is equal to zero for  $i \geq 2$ .

And now, still in the same kind of situation, we do some kind of huge diagram chase. And now there are two ways to finish the argument. One way is you can observe that if you do a similar operation now with this guy, then each of the terms here becomes a product of intervals. So each of the guys becomes an  $H^n$ -cube, and actually you can show that also for an  $H^n$ -cube it behaves like the highest degree—this comparison with singular cohomology is also true for the  $H^n$ -cube. There's no higher cohomology, so the argument we gave for the real numbers works also for this  $H^n$ -cube variant.

Oh, there's a slightly different way of arguing using a little bit of adjunctions. The source here is some condensed module, it has a module structure over the real numbers. And so this means that maps from the product into any module  $M$  will always be the same thing as the internal Hom in condensed modules over this condensed ring  $\mathbf{R}$ , from this guy into  $M$ , since internal Hom and external Hom agree here by this general adjunction.

But now this internal Hom is already zero by the result for  $\mathbf{R}$ . And so the whole thing vanishes. So we don't really need to know exactly what this is, it's enough to know that it's some module over the real numbers. And then because the real numbers don't map into any module was real analytic functions.

Okay, okay, so that's one of the key computations that we needed, which kind of gives me—so I have maybe five minutes left. Let me talk about a fun theorem with some set theoretic stuff.

So here's a theorem. All right, let me consider the following conditions, consider the following assertion (\*):

For all sequential limits  $M_0 \rightarrow M_1 \rightarrow \dots$  of countable condensed abelian groups, and all possibly non-abelian condensed sets  $N$ , the  $X^i$  groups from the sequential limit of  $\underline{\text{Hom}}(M_n, N)$  towards  $n$ , this is the colimit of  $X^i(\underline{\text{Hom}}(M_n, N))$ . Sorry, and of course this vanishes at least for  $i \geq 2$ , because these are just  $X^i$  groups and condensed abelian groups, and  $X^i$  groups in condensed abelian groups agree for  $i \leq 1$ .

So they vanish at least for  $i \geq 2$ . And then the colimit also, it's actually equivalent to the following statement: that if I take the  $X^i$  group of the product

So first of all, it's not just that this is excluded by  $\star$ . In fact,  $\star$  implies that the continuum must be really large;  $2^{\aleph_0}$  must be bigger than  $\aleph_\omega$ .

Basically what happens, I think, is that if  $2^{\aleph_0}$  is some  $\aleph_n$  for some finite  $n$ , then you will get some  $\text{Ext}^n$  problems. But if you make it larger than all  $n$ , then you have a chance.

In fact, it cannot actually be equal to  $\aleph_\omega$  by some GCH. What I proved is that actually the smallest bigger thing is consistent:  $\star$  holds and  $2^{\aleph_0} = \aleph_{\omega+1}$ . You can also—the first possibility after this can be realized.

In fact, it holds whenever you have any ground model. Then you can extend this ground model by doing a Cohen forcing; it holds in the forcing extension.

By joining  $\aleph_\omega$ -many reals, Cohen invented this notion of forcing that takes one model of set theory and builds another one, a bigger one, in order to show that the continuum hypothesis may be false. This is like the most basic forcing. I mean, now they have billions of different types of forcing, but this is still the most basic one, where you're just adjoining new real numbers, so to say, to your model. And you're adjoining quite a lot of them;  $\aleph_\omega$ -many would be the minimal thing that you can do in order to have a chance.

$\aleph_\omega$ -many, but once you join that many Cohen reals to your model, the simplest kind of forcing is done, which will ensure that this is true. It turns out, always.

Okay, and why do I mention this? Well, we don't actually ever in this course—I will never use this principle  $\star$ . But it's kind of neat to know that you can use it. Often when you try to compute certain things, it's easy to figure out what the answer would be if this was true, and then the things you really need, you can usually prove them without invoking this general principle.

But there are also some situations where you might want it. For example, in order to control Ext groups out of Banach spaces, where it really is the case that you get the expected answer under this principle  $\star$ .

But in general, these Ext groups are just some...

Question from audience: Why is it  $\aleph_\omega$  and not  $\aleph_1$ ? I mean, as I said in the first lecture, it comes from having the smallest possible .

Another audience question: I have a basic question. When you said that, um, when you computed the Ext groups for  $\mathbb{T}$  locally compact, and then you said that this is zero for  $i > 1$ , right? You said just because you ...

Lecturer: I don't know, I mean, I'm sorry, I didn't want to refer to—there's no simple reason. You actually have to do a computation. It comes out as zero, but in the end it's nice, it matches what ...

Audience: Thank you. I didn't catch the previous answer, but I want to ask again on this computation of Ext. So for example, if you have a compact abelian group and you take the  $\text{Ext}^i$  to the reals, via the condensed formalism, you have a complex that computes it where the terms will be continuous functions on various powers of this compact group to the reals, right? You have to compute that this is acyclic in higher degree. So I don't see exactly how you do it. For the group cohomology complex, you have this averaging by integrals, but I don't know exactly what is ...

Lecturer: I didn't actually look it up in preparation for this. How does it work? So instead of mapping to  $\mathbf{R}$ , it might also be that you want to ... How does it work? No, so it's different. You use that, um ... Where's—just go—I mean, I can't think right now, but you can find it in the notes.

I don't see any further questions, so let's—so in the set-theoretical setting, what was the colimit on the blackboard you're now looking at? What is the  $X_\theta$ ? This is the colimit of what?

So he writes this thing as this huge colimit of all functions, and then you can write, indexed by the same index category, where all the terms are now something you can write down. And then it's precisely these  $\delta$ -limits that they are studying.

Okay, I understand now, because we know that there is a cohomological dimension result for  $\text{Alf}^n$ . I see how it's related, because if I think there were all  $\dim_{\mathbb{Q}}$  functions—suppose that of eventual dominance—then under, for example, continuum hypothesis, this would be the same as  $\omega_1$ .

And then such  $X$ -groups indexed by  $\omega_1$  should be bad. But the specific order type of this poset of functions ordered by eventual dominance depends extremely much on the specific model.

## 5. SOLID ABELIAN GROUPS (SCHOLZE)

[https://www.youtube.com/watch?v=bdQ-\\_CZ5tl8&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=bdQ-_CZ5tl8&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

Right, so today I want to talk about solid abelian groups. The goal is to isolate a class of intuitively speaking "complete" objects. The point being that in the previous lecture, we've seen that when we work in the category  $\text{Cond}(\text{Ab})$  and start with reasonable examples, then the outcome is also reasonable. But when you instead do some free construction, for example you form some tensor product—I don't know, you take one power series algebra and tensor it with another power series algebra in abelian groups—then ideally speaking, there should be some kind of completeness, where this comes out as complete in both variables. But if you just naively form the standard condensed abelian groups, then the underlying object is still just the algebraic tensor product of these things, which is some nasty indescribable thing. Product-like condensed groups are not so nice.

So the idea is that you would want to define a subclass of complete objects, and then you would like to complete things, and hope to get a reasonable answer instead. Idea being, there should be some notion of completeness for condensed abelian groups. But we definitely also still want that complete objects form as nice a category as condensed abelian groups themselves, they should still be an abelian category.

You definitely want something like the integers to maybe be complete, and maybe the  $p$ -adic numbers should be complete. But then if you want this to be an abelian category, you also want some kind of extension of  $p$ -adic numbers by the integers to be complete. But this is a very non-Hausdorff thing. So you definitely can't phrase completeness as meaning convergent sequences have a unique limit. First of all, it's not really possible to say what a convergent sequence is, other than one that already has a specified limit point, in this condensed setting. Abstractly, "convergent sequence" isn't really a notion. But also, even if there was, you couldn't ask that they have unique limits, because you want these non-Hausdorff examples.

Okay, and so back in the day when we were first thinking about this stuff, this was really one of the key questions for us: how to define such a notion of completeness. Originally we wanted one fixed notion, where in particular the real numbers should be complete, the  $p$ -adic numbers should be complete, maybe all locally compact abelian groups should be complete. And still something where you can form cokernels and it's stable.

In the end, we couldn't directly make that work. But we realized that if we for the moment forget about the real numbers and just want something non-archimedean, then there is something that works quite nicely. So it turns out it's difficult to find a notion where the real numbers are complete, but there is a theory that works well for all non-archimedean fields.

Later we will recover the real numbers, but that's a different story. Today I want to talk about a theory that works well in the non-archimedean context. Originally we did things very differently, so today I actually want to give a presentation of the theory of solid abelian groups that's very different from the one you will find in the first lecture notes. It's a presentation that really only works in this nice way in the condensed setting.

Okay, but in the condensed setting, one can base everything on the following idea: being complete in some sense means any null sequence is summable. One basic nice fact in non-archimedean analysis is that sequences are summable if and only if they go to zero. You can just try to turn that into a definition. It turns out this is actually a definition that makes very good sense. So let me try to indicate how you formalize this idea.

Here's a formalization of the idea. Let me consider this projective object that we had:  $\underline{\mathbb{Z}}^{\mathbb{N}}$ , the constant sequence. We want to say something about convergent sequences, so this should play a role somewhere. Recall that this is an internally projective object. Let me recall what this means.

One way to say what this means is if you look at the internal  $\text{Hom}$  functor into any condensed abelian group, ...

And what does it do? It takes some life Beilinson group, sends to life Beilinson group. That is, the one that takes any life problem to the internal hom from  $P$ . So the underlying, so if we evaluate this at a point, and I just get hom from  $P$  to  $M$ , but then I've enriched this back into the solid Abelian groups. And then the thing is that this functor is actually exact, but also preserves all limits and colimits. Can you find all that? But actually these are compact objects, right? Okay.

Okay, so this will become important in a second. And now we want to phrase the condition that any null sequence is summable. And one way to do is the following. So let me write something down. I consider the endomorphism  $F$ , which is the identity minus the shift map, which is an endomorphism of  $P$ . So complete  $P$  has a basis generated by some  $\exp(n)$ , and we make  $\exp(n)$ . So this is the basic space of null sequences  $m_0, m_1$ , and so on. And what does  $F$  do? If you just translate the formula there, it just sends such a null sequence to a new null sequence, which is  $m_0 - m_1, m_1 - m_2$ , and so on.

And what would the inverse be? The inverse should send a null sequence here—well, you should be able to recover  $m_0$  as  $m_0 - m_1$  plus  $m_1 - m_2$  plus  $m_2 - m_3$  and so on, by telescoping, right? And so if there is an inverse, the inverse has to be  $S$ , which takes a null sequence and produces a new sequence where the first term is just the sum of all of them.

So this is one way to phrase the condition that any null sequence in  $M$  is summable. It's a very structured way of saying this, because maybe to say that, you would only have to say it on the actual hom, on the internal hom. But it turns out to be much better if you ask the condition on the internal hom.

Okay, and so the goal of today's lecture is to understand this  $C(\text{Solid Ab})$ . And in the original approach, proving that this is an Abelian category was actually the last thing one could really prove in this live setting. It's actually the first thing one can prove.

If you consider the subcategory of  $\text{Solid Ab}$ , it's actually a reflective subcategory, stable under internal colimits like tensor products, all limits and colimits, even internal homs, and internal exact sequences if you want. This also contains some objects—it contains the integers. And once it contains the integers, everything you can build by limits and colimits and kernels and whatnot will contain all this. It's not everything.

It contains the real algebraic numbers. It does not contain the reals, because of course the reals are just wrong. Okay, I understand. Because there is, yes. So it's definitely current.

And so Jaron, when you introduce later this proposition, it basically tells you that this is an analytic ring structure on the integers here. Okay.

And so previously, I mean, the way we set up the theory previously, it was a bit of an art to construct analytic ring structures, because there were a lot of things you have to check, and none of them were easy to guarantee. And so there were basically only two examples one could construct, which is the solid ring structure on the integers, and then there is this liquid ring structure on the reals and some related rings. Those proofs were pretty hard, and they were really very handcrafted things. But it turns out that because of this internal reproductivity of  $P$ , it's now trivial to construct analytic ring structures, because for any endomorphism of  $P$ , you can ask such a condition, and all this will come for free.

I mean, yeah, so let's just try to prove that for example, it's stable under kernels. Say you have a map  $M \rightarrow M'$  where this happens, and then you want to know that for the kernel it also happens. But the internal hom from  $P$  is an exact operation, so it just preserves the kernel. So of course the same condition is true for the kernel, the same for the cokernel. Or if you have stability under retracts and finite limits automatically in an abelian category. This limit is also clear, and this colimit also, because just the internal Hom has these properties. So it's all for free.

What about internal Hom from  $X$  to internal  $X$ ? This looks more difficult. It's also trivial, no? Because internal Hom you can also pull over, right? I mean, just by adjunction. Ah, but then you need only internal Hom on the second... I mean, you need only... Okay, let's try Hom, because maybe I didn't... I just wrote that down on the board and I didn't write in my notes.

Something guaranteed. Okay, so here is stability under... Let's first do the internal Hom, then see whether it works for the internal tensor product. Let's say  $M$  is solid and let's say actually any  $N$ . What's the internal Hom from  $N$  to  $M$ ?

Okay, I see it, because you can permute... Write it down. See, so what is this? For this, we have to prove something about this guy theater. But by the Hom-tensor adjunction, that's also the internal Hom from  $- \otimes N$  to  $M$ . But then, if  $M$  is solid, you can commute the two Homs and it becomes internal Hom from  $N$  into the internal Hom from  $-$  to  $M$ . But then, if you come as  $M$  here... So of course, another fun still.

And if you look at internal tensor product, then because internal Hom from  $-$  is exact, you also have that the internal Hom from  $-$  into the internal will also just come out as the internal tensor product from the completed tensor product. And then again, the internal tensor product from  $N$  internal Hom...

Okay, and so... Yeah, so I mean, this is something extremely... that when you just enforce such an "is solid" condition on the internal Hom from  $-$ , you get these very... Okay, so everything is for free, except



possibly that there is any object that satisfies it. But okay, so you can just compute that the internal Hom from  $\mathbf{Z}^{(\mathbf{N})}$  into the integers is just a direct sum of copies of the integers. Because if you have a null sequence in the integers, then it must be eventually zero because they are discrete.

Okay, it takes a little bit to see that it's really internally Hom, but that's true. Again by adjunction, if you want. And yeah, so if you just compute what identity on  $\mathbf{N}$  here... So you want to show that any null sequence is summable. But any null sequence is eventually zero, so it's easy to sum it.

Okay, before actually focusing more on this specific example of like completeness condition, let me draw some... It's good to know. So just by some abstract adjoint fun, this means that there... this inclusion here... will typically have a left adjoint, some kind of completion. So fun that takes any abelian group to a solid one. This is what we call solidification. And so, this is characterized by the property that for all solid  $N$ , what do we have... that for all solid ones, the Hom from  $N$  to  $M$  is the same thing as the Hom... The completion does make it the right adjoint, as opposed to the left? It's the left adjoint to the inclusion, because I'm using the inclusion. So I should write some kind of fun here, which is inclusion fun. Because this... No no no, my question is that... So but you see that both are stable under all colimits. It also has a right adjoint, which I've never considered. Yes, but there is some... also some small solid sub, I guess.

Okay, but over solid objects... requires need product... product... making the solidification. So concretely, this is just if you have two solid abelian groups and you want to form the tensor product, you first form the tensor product inside the abelian groups and then pass to solidification again. There's a little bit of unraveling to show that this is really some half-tor operation. But the key thing that you need for this is that the class of solid modules is stable under pullback. And...

Okay. Is this required to be symmetric? No.

Symmetric. Let me just... So the existence of solidification is just an instance of trying. Basically, if you want to have a left adjoint, it should commute with all limits. Then, under very mild assumptions, it also exists and they are satisfied here.

For the symmetric monoidal structure, well, we define  $I$  to just be the constant sheaf. Then we want the symmetric monoidal structure. So what we want is that for all  $M$  and  $N$ , which are just condensed abelian groups, first we individually solidify them, then tensor them back in the category of condensed abelian groups, and then resolidify.

To check that, you check that the solidification is a left adjoint. So to check that this is nice, you check that there are maps into all... For all  $Z$ , which is solid, we want  $\text{Hom}(M \otimes^{\mathbf{S}} N, Z)$  to agree with  $\text{Hom}(M \otimes N, Z)$ . On the right,  $Z$  is still solid, so it's enough. But then the two Hom's are the same, so it's also the Hom from  $M \otimes N$  to  $Z$ . This is not from the solidification, it's from the original one, but it's okay.

So we have a very nice category. It acquires its own enrichment, but now we would like to understand what it actually looks like. Actually, one thing we also need to understand is how to interact with the real numbers. We already said that the real numbers are certainly not solid, but something stronger is true. Namely, the real numbers do not admit any nonzero maps to anything solid. Or equivalently, the solidification of  $\mathbf{R}$  is equal to zero.

Okay, so for this, note that  $\mathbf{R}_{\text{solid}}$  is a ring, because the real numbers are a ring, and if you solidify it, it stays a ring. For a ring structure, to show that a ring is zero, it's enough to show that 1 equals 0. Now you want to make use of the fact that in an abelian group, you can uniquely form sums of  $\mathbf{N}$ -indexed sequences. There are lots of  $\mathbf{N}$ -indexed sequences in the real numbers that are not actually summable, so you would expect that using such divergent sequences, one can produce a contradiction.

It actually took Dustin and me considerable effort figuring that out, but just yesterday, Ian and then also Kobe found an argument. Here's their argument. We consider the  $\mathbf{N}$ -indexed sequence  $1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}$ , and so on. I hope you can guess how it continues. You take  $\frac{1}{2^n}$  and take each one  $2^n$  times.

This is a sequence  $\chi : \mathbf{N} \rightarrow \mathbf{R}$ . Then in the solidification, we have a map  $\chi : \mathbf{N}_{\text{proet}} \rightarrow \mathbf{R}_{\text{solid}}$ . When you compose further to the solidification, we have this map  $\mathbf{N} \rightarrow \mathbf{N}_{\text{proet}}$ , and then there exists a unique map  $\mathbf{m} : \mathbf{N}_{\text{solid}} \rightarrow \mathbf{R}_{\text{solid}}$  filling this diagram, because we are using that solidification forms a localization. So we can uniquely find such a "summable sequence" in the solidification, which intuitively speaking should be the thing starting the sequence with 1. In particular, if you restrict to the inclusion of 0 in  $\mathbf{N}_{\text{solid}}$ , you get a map from  $*$  to  $\mathbf{R}_{\text{solid}}$ , an element of  $\mathbf{R}$ .

Solidification is right exact if you have enough projectives. If  $M$  is an  $R$ -module, then you can resolve  $M$  by a solid projective resolution. Then, for example, if you solidify this resolution, you see that  $R$  solid

is zero. By a spectral sequence argument, the vanishing of Ext groups here reduces to the vanishing of Ext groups from the reals. But the Ext groups from the reals, they are  $R$ -modules. A different way to state this would be to observe that all the internal Ext's of  $M$  against something solid will be  $R$ -modules, but the solidification of  $R$  is actually zero, so any  $R$ -module works.

When you compute the internal Hom Ext's, since you don't have enough projectives, the classical approach is to use injectives. But then they are no longer solid. I don't know if that's solid, but of course if you have enough solid projectives... The internal Ext's I was referring to here, they are the ones in condensed abelian groups.

If you compute them by injective resolution of the right hand side, the solid argument... If you compute them by projective resolution of  $M$ , then by what we proved before, all the internal Hom's are solid. But if you are obliged to compute them by injective resolution... I think it's okay. This is definitely an  $R$ -module, right? But it's also solid. Why is it solid? Because internal Hom against something solid is always solid. That's something I previously said. But it doesn't follow immediately...

Okay, so everything on real numbers is completed by a limit argument. But on the  $p$ -adic numbers and so on, you get them by a limit of discrete things. So everything that's pro-discrete is definitely solid.

So, next goal is to compute the solidification of  $\mathbf{Z}$ . The idea is that when you solidify  $\mathbf{Z}$ , you need to adjoin lots of new elements, because  $\mathbf{Z}$  has no non-zero divisible subgroups. But then the divisible subgroups are summable. So the sum of the sequence  $1, 1/2, 1/3, 1/4, 1/5, \dots$  must now also be in  $\mathbf{Z}$  solid, and so on. We expect there to be lots of new elements.

One way to make this precise: Consider the subspace  $X \subset \prod_{n \in \mathbf{Z}} [-n, n]$  given by the union over all integers of the products  $\prod_{k=-n}^n [-k, k]$ . So it's a subspace of the whole product. Then there is a null sequence here, which is given by the sequences that are non-zero in only one term. Explicitly, let  $e_n \in X$  be the sequence that is everywhere zero except the  $n$ -th spot is 1.

Then actually on solidifications, that  $e_n$  sequence becomes zero. Actually, I didn't talk about the verification, so I would like to even say that not just on solidifications, but in derived solidifications, this  $e_n$  sequence becomes zero. A different way of stating this is to talk about Ext groups against solid objects.

That the solidifications agree means that the Hom's against any solid object agree. But in fact, all the higher Ext groups agree as well. I cannot read what you wrote, just after the  $\mathbb{P}$ . Is it the map from which  $s_n$  to... And then here you have a sum of those basis vectors  $e_n$  that are everywhere zero except in one spot where they are 1. Let's organize this into sentences and paragraphs:

These  $e_n$  are just the basis vectors, which is a null sequence in here because the sum has the topology of pointwise convergence. Can you send  $n$  to the sequence where only the  $n$ -th coordinate is 1 and the rest are 0?

Yes, I think so. We know that this makes  $\tilde{K}$  quite big, because the underlying abelian group of  $\tilde{K}$  is actually an uncountable abelian group. You sum only along finite sums of these countably many basis vectors, but  $\tilde{K}$  is a very large, uncountable thing.

So we've got quite a bit to prove. Let me draw some nice diagrams. Okay, maybe I won't do all the compatibility checks, but let me just draw the diagrams and then I'll leave a little bit of diagram chasing.

So we're trying to make good use of the fact that when you have solid abelian groups, the internal Hom's behave well. In other words, they commute with solidification.

Okay, so we have this diagram here. And then we have another map which is... Okay, so what are the maps here? Giving a map from  $\prod_{n=0}^{\infty} \mathbf{Z}$  or something to here corresponds to a map from  $\mathbf{Z}$  to the internal Hom, meaning it corresponds to a null sequence in the internal Hom. So to give this map, I have to give a null sequence of maps from the direct product  $\prod_{n=0}^{\infty} \mathbf{Z}$  to itself, which are the projections to the coordinates greater than or equal to  $n$ . In other words, in the first  $n$  coordinates it's the zero map, and on the others it's the identity.

And so then what does this composite do? This composite here also corresponds to... Let me first describe this composite here. This corresponds to the sequence of maps which are just projection to one fixed coordinate. Because if I think about the difference between two consecutive maps, only one fixed coordinate remains.

But the thing is, the projection to the  $n$ -th coordinate actually factors over the integers. So in each case, when I fix one  $n$  here, the corresponding map is just projection to the integers and then adding back in. But

this means that actually all of these maps that are parameterized here, all of them factor over the subspace  $\tilde{T}$ .

So this means that this composite factors over this subspace here. I actually need to use that I take a bounded product, because if the coordinates stayed unbounded, I wouldn't actually get this characterization.

Okay, so what's the idea here? You have the identity endomorphism here, and you write this as a sum of a null sequence of endomorphisms. And the null sequence of endomorphisms is the projections to the individual coordinates. So the identity endomorphism here is the sum of the projections to any coordinate.

But all of these endomorphisms that you sum, they all factor over the subspace  $\tilde{T}$ . And one other thing I should say is that because the first map that I project is really just the identity, if I project to coordinates greater than or equal to 0, I'm not doing anything. This actually means that if I come back, at the 0 end, this  $s$  here is the identity.

And so then I can solidify this diagram. I guess that's how to rewrite it.

Okay, so we have this diagram. Well, now this here is an isomorphism, because  $F$  solidifies to an isomorphism just by definition of what solid means. And solidification is symmetric, so both of these maps become isomorphisms.

But now this means that this map actually is a split surjection, because you can find an inverse by first going here, then going here, and then going here. So this is actually a split surjection, which is almost what we wanted to prove. It's actually an isomorphism.

To show that it's an isomorphism, you have to show that when you circle around this diagram, you get the identity. And this is another diagram chase that maybe I won't do, it's not difficult. Yeah, so all the maps are isomorphisms.

And so this means that this actually becomes an isomorphism too. I claim something slightly stronger, namely the agreement

So there's a derived solidification functor. And then the same argument is true on solid modules. This  $T$  that you solidified is  $p$ -completely solidified. A ring object now, it will be, but it's not obvious. Oh no, sorry, I think  $T$  itself is already a ring object. It doesn't have a unit, does it? Well, just zero, if you want. I mean, I'm not sure. Wouldn't the identity be the constant sequence 1? Which is not, I'm not sure how you get a ring structure component-wise.

I guess, in fact, once we're here, we can actually compute the solidification with another example. So that if we take  $S^1$  product, then the whole product, but I mean, this is already solid. We don't need to solidify. And it's also a condensed ring, okay.

And actually, this is a case where I do need a little bit of this  $\varprojlim^1$  business. Because it's actually, so we have this sequence here,  $\varprojlim S^1$  is pretty large, and then we have this funny  $\varprojlim^1 S^1$ , which is some weird non-separated guy, but whatever. And so what we have to see is that  $\varprojlim^1$  for all  $i$  equal to zero, and then this guy doesn't have any  $\varprojlim^1$ .

Now, of course, I've carefully planned my lecture. So now we know already one class of examples where the solidification exists, which is anything that admits a module structure with a real number. So the claim is that this guy is a condensed ring, which seems surprising at first.

But so why? Actually, there's a different way to write this. It's also the same thing as a bounded product of copies of the real numbers modulo unbounded products, where I define the unbounded product the same way, as an increasing union. And why is that? Because the difference between these two notions, mapping  $A$  to  $B$  to  $C$  to  $D$ , is the same as first taking ratios here and ratios here.

So what is the ratio here, or  $\varprojlim^1$ , whatever. Here the ratio between these two is, of course, just the product of  $z_i$ . But the  $\varprojlim^1$  here is also the same thing. Because if you want to surject onto a product of circles, then of course you can keep the projection, can keep them  $z_i$  and 1. So this definitely surjects onto here, but then the kernel is obviously just  $B$  sequences of  $z_i$ . So if you want, you can draw some kind of three-diagram of short sequences justifying this. Like a short sequence here, short sequence here, and then they give you a snake lemma.

Right, and so this one is visibly a condensed ring. Apart from this discussion, the upshot is that the completion of  $\mathbf{Z}$  is just the whole product, and the  $\varprojlim^1$  is zero. So then, now where here on the right I'm taking the ext groups, all the ext groups I'm taking are currently still taking in condensed abelian groups.

Noting that, I mean, of course this is zero for  $i$  greater than zero. Because, so in particular, we recover that  $\text{Ext}^i$  in condensed abelian groups from this product of copies  $\mathbf{Z}$  to say,  $\mathbf{R}/\mathbf{Z}$ , which is a discrete abelian group, is just a direct sum of copies of  $\mathbf{Z}$  and  $\mathbf{Z}$  for  $i$  positive and zero. So this is something that I also proved at the very end of the last lecture. And actually, I imagined it would be an input into today's lecture, but now actually one can present the argument so that it's a corollary here.

Did you use the solid analytic feature to show that  $\text{Ext}^1$  vanishes?" Yes, I did. Well, I mean, they're also in condensed abelian groups, but yes, I definitely used a different argument. Also, when you actually want to justify that this map here is surjective, I mean, it's actually, if you think in terms of

Why? I mean, this is just given by... Think that, but  $p_T$  of  $P$ , you can take this to  $P$ . And basically, there's a free guy on, like, an  $n \times n$  grid of elements that all jointly converge to zero. But then, well, you can just enumerate... Same, it's just, here is some kind of...

So, in particular, like, phrased in terms of... One way to think about power series algebras, in particular, two power series algebras together, and you get the power series algebra. And maybe up there I should have noted, in some cases, the product of  $n$  is some compact projective object. No, I'm sorry.

Alright, so we understand quite a bit. We don't yet necessarily understand all the objects and groups, because there are other generators in life groups:  $\mathbf{Z}$  join assets fin. And so we could wonder what their solidification is, but this can also be understood.

And then the... exist from the free... toward using solidification and on  $X$  solidification. So you see that also these solidifications of these guys will be a product of... And finally, any assumption on  $S$ ? Like,  $S$  should be non-empty or infinite, or you should... Yes, yes. Yes, infinite.

Canonically, one way to write a morphism is that if you write  $S$  as a limit of compact sets  $S_n$  with surjective transition maps, this will actually just be the limit of these... These are all kind of free being in groups, the... the transition maps. So it's easy to see that any such limit would be isomorphic to a product of copies of  $\mathbf{Z}$ .

So, in our previous way of setting up solid series, we were actually taking this formula as the starting point for defining what a solid guy is. So we were just, on all the generators of our category, all these  $\mathbf{Z}$  join  $S$ 's for profinite  $S$ , we were by hand declaring what the solidification should be. We were defining it to this limit, and then we were checking by hand that this gives a valid pretheory. But this argument is actually much harder than the way I've set it up now. In some sense, the definition of what a solid group is, is much clearer, and then this really becomes a computation.

Okay, so let me quickly give a sketch. So we have our  $S$  and it's written as this limit of finite sets. I assume that all the transition maps are surjective and I inductively choose... Well, first the section  $i\mathbf{Z}$  from  $S$  second  $\mathbf{Z}$  back into  $S$ . Then, on all elements of  $S_1$  that are not in the image of  $i\mathbf{Z}$ , you pick the first playing on on  $S_0$ . And then if you project back down to  $S_1$ , in particular, you've already some elements of  $S_1$  which you've already lifted to  $S$ , but then there's new elements of  $S_1$  that I didn't yet have. And so on, then I pick  $p_i$  new  $S$  sometimes on  $S_1$  minus the image, and so then jointly these two things together now define for me a section from  $S_1$ , which on some elements is given by projecting to  $S_0$  and then taking the lift you already have there. And the other elements, you make a new choice. And then you continue.

Okay, so this way, what in particular you will get is a countable sequence of elements in  $S$ , right? Because the union of all these maps is just a countable subset of this, but it will certainly be dense.

And so, let's also enumerate the elements. So, enumerate  $n$  as you start enumerating at zero, then start enumerating the elements of  $S_1$  that weren't in  $S_0$ , one and so on. And we get a map  $d$  from  $\mathbf{N}$ , which is some of the free guy on this copy of  $\mathbf{N}$ , towards  $S$ . Right, on  $S\mathbf{Z}$  this is given by  $i\mathbf{Z}$ , but on some higher  $S_{n+1}$  it's given about the difference, the next in

Canonically, it should be this thing. In order to prod such an isomorphism, I have to produce such an isomorphism. If you carefully think about how you would actually go about doing that, you have to choose a new base of elements. Then it is precisely what you want to end up to.

The argument is actually very similar to the argument we already did. We have our carefully here, and then sectors, where again, giving such a map means I give a pro-sequence of maps from  $S$  to  $S$ . The pro-sequence I consider here is a sequence of  $S \rightarrow S_{\leq n}$ . So you start with the identity, which we have to do because we want this splitting here, and then take  $I$  project to the  $\leq n+1$  and take the splitting  $\pi_n$ .

Okay, so when I write these maps, I need the maps induced on quotients. It turns out that because some of these maps approximate the identity here, the same as  $S_{\leq n}$ , and so then this guy here, this corresponds

to the sequence of the differences  $\pi_{n+1} - \pi_n$ . If you think about what these differences are actually doing, you realize that they are only changing something on a small part. I would mess it up if I try to say it orally, but you can check that you've exactly crafted things so that this difference will factor over the image of this  $S_{\leq n}$ .

The idea here is again that you take the identity here and try to write it as an infinite sum of maps, all of which factor over the submodule. For this, you use the sequence of functions  $\mathbf{Z}_S$  which are factors of something of finite range  $\mathbf{Z}_S$ , and then you can do it. The argument is just the same as before.

What becomes critical here is that you really have a pro-finitely presented  $I$ . If I were to set up the theory of solid abelian groups back in alter groups, then the condition I gave that just talk about pro-sequences wouldn't be enough, because you will never see this  $I$  as a quotient of the integers. You're really using that you can still understand this via its finitely presented quotients.

Okay, so now we have that  $\text{Solid}(\text{Ab})_{\text{ST}} = \text{Coh}(\mathcal{O}_S^{\text{a.cg}})$ . In particular, this includes that it has a compact projective generator, a single compact object which is a full subcategory of products  $\mathbf{Z}_S$ , and it's actually internally projective. In fact, if you tensor it with itself, it becomes isomorphic to itself.

Also, in the solid case, being solid is equivalent to only having the Hom-finite condition hold, which is what we took as our original definition. It had only  $\text{Hom}(\mathbf{Z}_S, -)$  preserve filtered colimits, and all maps  $\phi : \mathbf{Z}_S \rightarrow M$  have a unique extension to  $\mathbf{Z}_S^{\text{a.cg}}$  from the free object, which is the colimit  $\varinjlim_n \mathbf{Z}_S$ .

And this, and then also because  $\phi$  is actually zero. And maybe I didn't say, but you do it, you can do it by taking  $M$  to be anything, and then you go from the universal property for  $\mathbf{Z}[S]$  to the same one for any  $M$ , because you represent the solidification of  $M$  as  $\mathbf{Z}^S$ . So the fact that any map from  $M$ , from other  $M$  (I mean  $M'$ ) to  $\mathbf{Z}[S]$ , factors through the solidification, and then by then is clear that this is the same as  $\mathbf{Z}^S$ . I think this also represents  $M$ . Part of it is clear, a direct fact.

So, time, so let me just give one kind of philosophical way of thinking about this solid condition. The very end condition was that all null sequences are summable. And something that we get out of this is that one can always integrate against certain kinds of measures. So one can also think of  $\mathbf{Z}[S]$ , it's actually the same thing as the continuous functions from  $S$  to  $\mathbf{Z}$ , because the continuous functions, they have a colimit of the functions on  $S$  to finite sets. This gives a description.

And so from this perspective, these are some kind of  $\mathbf{Z}$ -valued measures on  $S$ . And so this means that whenever you have a map from  $S$  to  $M$ , where  $M$  is solid, and whenever you have a measure on  $S$ , then this induces by what I said here, and this is the measure  $M$  to something here. So we can and this to the integral of  $f$  against  $\mu$ . In other words, whenever you have a map from some profinite set into  $M$  and some  $\mathbf{Z}$ -valued measure on  $S$ , then you can form the integral.

One thing that I should stress here is that this characterization of solid at the end, I only talk about homs, not about internal homs. The first characterization I gave where I talked about differences, I had to talk about the internal homs. Here it's a problem enough.

All right, I'm out of time. Let me check why the composition is zero. Because these are 0, solidification is also 0. I'm slightly confused about my diagram, why it's still a presentation of  $M$ . That's maybe... First, you definitely get a right exact  $M$  with a solidification. Solidification is right exact. But then, and you definitely always have this map here, but then the observation that there exists this map back, which is zero here, means that this actually retracts. So  $M$  is a retract of its solidification by this argument. But retracts of solid guys are solid. So there is a single compact projective generator.

Right, so I mean we'll discuss this more next time. So one can give some purely synthetic algebraic descriptions of what the category of solid abelian groups is without talking about anything. And so you can say what they are, right? Products of sums of  $\mathbf{Z}$ , they are just infinite matrices which in every row are eventually zero.

Are you going to give a similar new way to define, a new definition for liquid modules? Or is this new formalism available only for the solid setting? The question is whether one can also characterize the liquid vector spaces in a similar way by mapping. I think it should be possible, and we're currently figuring out the details of what works. Let's see, we still have a few weeks.



## 6. COMPLEMENTS ON SOLID MODULES (SCHOLZE)

[https://www.youtube.com/watch?v=KKzt6C9ggWA&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=KKzt6C9ggWA&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

So today I want to uh before Dustin takes over on Wednesday again uh today I want to give some some compliments on solid modules sorry being so just briefly recall the definition so with  $P$  has always denoted the subjective object which is as a free  $G$ -module on the normal sequence and uh then we consider this anamorphism of  $P$  which is the identity minus the shift  $n$  which takes the generator  $n$  to  $n+1$ . One  $n$  and then the new definition of a solid module was that to the old definition but on  $P$  so solid is  $\text{Ed}$  by  $F$  on internal homeomorphism where intuitively speaking this is a map  $I$  mean  $M$  from  $P$  to  $n$  some of the space of  $n$  sequences in  $N$  so you take a  $n$  sequence in  $n$  and to the sequence of differential and intuitively speaking that this is an isomorphism means that conversely if you have any more sequence than  $M$  then you can sum it because if you think about it then this  $m \in Z$  must be the sum of all so so this looks like some definition of derived complete modules like periodically derived complete so I suppose there is some mathematical relation that derived complete let me talk a little bit about uh little bit of yeah uh I mean I think on the face of it it's not related but okay so there will be some dvc complete this today um and so then the theorem we proved last time was that uh this  $PO$  guys and try all good be stable under all four extensions let me just say limits or coits um tend product product making the left joint so this inclusion  $L$  join um oh solidification um making this left joint symmetric mon meaning Comm product and uh it is generated by a single compact projective object um this is a product and on this generator uh the Sol tender product is just  $u$  and maybe the other thing I should say is that one can compute on the solidification of all these three modules on a Prof finite set  $s$  they are just the limit of the three modules on the finite set and okay so I think this basically summarizes starting from last time and uh maybe okay what's the first thing maybe the first thing okay then then let me first uh discuss the derived enhancement of the situation um so let's let's assume that instead of uh just one  $Con$  in group you really have not  $Tak$  then actually the definition still makes sense and we take that as a definition so we say that a complex of like which is solid if again internal alism but again because the internal  $H$  from  $p$  is actually exact this actually equivalent to asking uh that for all  $I$  in the the internal home from theology so there's equivalent to asking that uh all the homologies of  $a$  also and again this is also true uh in our previous discussion of  $s$  being groups and is in fact a general property for we call an analytic  $R$  structure that you can check completeness of the level of groups classically this is a little subtle to prove but here again it's an immediate consequence of the internal projectivity of key okay so uh then some of cor area of this  $C$  said uh um the class of sub solid  $a$  is itself a triangulated subcategory or if you would work in the infinity setting it would be a stable inity uh and stable under all direct and so all liit and liit uh and and also one can again show it's also stable under our home uh and again from same proof as last time and then if you just repeat this discussion of tender products now at the level of deriv categories uh you actually see that um oh actually it's also true that uh so I did give a notation to the subcategory uh and I didn't want to give itation because in the end it just turns out that if you take the drive category of solid guys um then drive  $C$  are functorial so you get a map this is actually  $F$  faceful and the essential image ex a solid base this is actually something that's not completely automatic this uses that um these project generators here that they stay they some have exp banishing also in light condens being groups Orly that's the solidification actually here is okay um uh so is it the case that if you have any complex possibly unbounded whose terms are direct sums of those generators these generators and the the taking solidification turn by ter is is a good arve that that gives you the yes yes that's right okay I yeah solidification preserves like on the level of for more maybe forget about uh the thing so then you have this whole sub category in the dra of see solid objects and again by thej function you will actually incl will have left joint some kind of dvc solidification fun uh which again will commute with all  $Co$  elements back into because this is still under alls and then this and on the generator we kind of determined what it does last time that it comes this thing even in the  $d$  s uh so yeah um okay so uh also also on the of  $D$  categories we have a full inclusion really um so it's as nice as it could be and uh and again by some my joint fun serum uh this has a left joint hey nothing to let me write this a deriv solidification  $L$  box so in the joint Factor theorem usually there is some set theoretical condition because you want thatly I'm thinking of everything as Infinity categories and then everything here is this presentable Infinity category and then  $L$  has this general joint fun serum there I think there are for tri categories there are Ser of nean and others which would also

to justify this I believe uh but I mean they often assume actual compact generation which is not quite true in this Cas so I'm not sure but L version is definitely sufficient okay so so for B for would need the abstract Z because we can just prove the existence by hand in this case because on on a class of generators we can we have identified that the exists these guys already know that they exist but the abstract theorem is both for Left Eye joint and right eye joints assuming that there are enough limits and Co limits right I mean so there one of presentable Infinity category which is one which has all Co limits and which is generated by the set of objects um and in this situation every functor that preserves all col is R joint and every functor that preserves all limits and all sufficiently filtered full limits also all sufficiently filtered coits so it's accessible one also um uh and uh a unique uh symmetric product prodct um making uh a goes to R and actually something even better to that this is literally the derived left derived fun of the thing you have being categories uh and similarly you can also I mean the drive tender produ is also obtained by driving the thing you already havean level but here how do you know that the solid category is presentable in your before you you prove the existence um yeah I think there are General theor that if you just ask locality for such a map there are some automatic presented work okay so uh so once we pass to the dri t i can St things from last time that really the deriv Sol application of z s is still limit as Z end so it is uh the other solation there are no higher homies and uh so also the Der solidification of p is this perect and the Der C product of s to copy is the under okay so you might think that nothing arried it really ever appears in practice but not so um here's the very F proposition uh let's say again that uh for simp the xcw complex so recall that in this situation we previously had this thing that the X groups against the they are the singular Gres right um but now we can also interpret the homology of X and it turns out that a singular homology of x is exactly the iology of the the solidification more precisely um I mean there's some of to complex Computing singular modin and this is really isomorphic to the rication in particular the RO ification is actually comes from the complex of the street being groups so again this is some kind of version of the if you take this three group on X and this models so as example like if you take the three intervals is some large thing right and the underlying thing is finite sums of points but if you solidify them the interval just goes away it just becomes Z and but if you take the circle and solidify that that becomes a copy of Z degree zero and a copy of zg one and in general if you take uh this is Union of circles of all dimensions and you get something which is concentrated degree zero but solidification goes arbitrarily in part to the left um right and so uh again there's the formal reduction to the case of complex so justs assuming I keep theor which um and so for financy w complex uh how do we actually comput this so so assume comp and also it's to fin complex we know this homology is finite generate eache okay so I I may miss the point but you said that the direct solidification with GX under bar is a singular uh complex right so it is a Qing it is one so in above this statement you said that say yeah sorry it's just a qu isomorphism ah qu isomorphism sorry I see sorry not the yeah sorry it's only find yeah yeah yeah yeah that do makes sense sorry sorry sorry I see I see sorry um okay so how do we compute the the appication so uh so we do the use those things so we take some subjection from some life profile set onto X so like the cont said and then uh we get all fber products and this usess resolution of this condenser being here by the guy on S and then uh the guy on the F product which is again some light profess and then and all the fiber products okay so that's the resolution of of this guy and so to form the d uh solidification we can somly the D solidification to all the terms here but for these terms we know that it's concentrated degree zero uh so we know that the derived solidification is computed by uh this let me not underline okay now this see might seem a bit hard to compute uh but we know that each term here is actually just internal home from the continuous functions from s into z z right because uh that's something I discussed at the end of the last lecture I mean this formula that this limit of the SN can be uh translate into this formula so these are some kind of measures on X and similarly for the other so this actually means that all the terms all the terms in this complex are isomorphic to their double dual right so the continuous functions from s to Z is besides the Duel of sorry the Dual of of this one and then I dualize again uhoh that's solidified is actually just home and I I could also put put the r here it doesn't change high in those cases um and so this actually means that uh the same formula will be true for this guy if you think about it so because resoltion so this means that also this gra solidification here one way to compute it is that it un isomorphically to the r form R form X but now we're in business because we know that this guy here is just some isomorphic to quic to singular Co chain complex and uh because we know that the singular chology for fin CW complex is fin in each degree uh it's some taking the Dual of singular corology against singular homology okay and so you get the qu the Der solidification of X and singular and so this means that to some extent this passage

from light condens being groups or solid being groups uh is like passing to homotopy types A little bit like at least for CW complexes it's some contracts the interval and then some identifies two homotopy equivalent W complexes but on like totally disconnected things it's much finer information or it's yeah you could still invert homotopy equivalence but not invert V homotopy equivalence so is it the case for locally contractible spaces in the sense that you explained yes yes so whatever I said about CW complex also works for a locally contractible session actually uhan pointed out to me that uh the par price assumptions actually not required in comparing singular cohomology and sheaf cohomology no no they are not required there are some recent papers Su that effect um but I don't care some sense right now so right uh so I know it's fun it's probably not really relevant right now but F to know um okay so now let me go back and try to describe the the category of solid a little better um and um so yeah so uh we want understand the structure Al Sol being better so so there is a comp projective generator so there's definitely a notion of fin generated objects and these are precisely the Cs of a dias and then there's a notion of fin presented object uh these are uh the coreel of map and it turns out that so the modules behave like modules in the following sense uh that's a presented objects form itself in a category and like the only critical thing is stability under kernels but once you has that it's automatically stay under kernels Co kernel and really really only the kernel are something to mention um and the whole category is just the end category of those also any finally presented object actually AIC to cor of injective Meth this actually something slightly better than what DUS announced in the first lecture uh he claimed that the science presented objects has a resolution of length two and uh but actually length one is good enough any and there's a resolution just stop okay so there some the finite objects in your C which are exactly the of injective maps and then everything is a f um I the key spe for this is the following that uh so trying to understand all the five presentive ones then okay you have a kernel you can always find the subjection onto M and then there's some kernel and the kernel you still know is at least a finally generated subm module of this and so then it would be good to know that all the fin generated sub modules of this are actually such a product and this is what I claim now zero no sorry sorry let's say product of copies of Z generically it be product course could be zero fin um so this definitely implies this statement here but it also implies like okay so object are always St on the C and extensions um things that they're stable under kernels uh but then I mean this also always for reduces to identifying the finally generated sub modules of like you're generating objects and if those are all finally presented then then you're in good shape and this some so the Ser is really easy to deduce from okay so let's proof Sy so we know that uh it's finally generated so we know that there is some subjective map from product of copies of Z onto M which injects back into of by the way uh this is again the result it's previously a lot of what I was talking about about the general properties of the car Sol being groups they basically extend to the full condensed setting not just the line one this Serum is again one which only holds in the L setting um and be because we're again using some countability in just a second um okay so so we have some here let's go g um and we know that the uh the MS from a product back to Z uh they are given by direct sum so we actually know that this is dual to a m uh in the other direction from a direct sum of these and basically our task is to show that whenever we have such a such a anamorphism of direct sum of these and when we dualize then this image here is itself a prod right so for the proof I will use the following fact which I'm not sure if it's known uh that well known uh uh let N accountable grou um that into AAL I mean it's just an abstract yeah discret no condens that EDS into uh Z ISE okay maybe um uh okay um let me leave this as an exercise um I no is definitely fult if you drop the accountability assumption because if you take an actra prodct of copies of an AB being group it's definitely not a free being group um but one thing this implies is that if I look at the image of AG uh then this is definitely countable and it embeds in the direct sum do also into D implies that the image of H three for this you don't need this it's just image okay I will use a spe again in a second so I want mention it way to justify this to use f but okay may it's easier uh so the image is free um but then you have a subjection here from this direct sum onto another free group so this splits in particular like there's there's a kernel here but this must actually split back so the of H uh split as a summon I mean maybe actually this completely irrelevant maybe I should focus more on the potion but let me just try to understand a little bit about the structure of such such snaps and what it implies on the Dual side um uh so we kind of split but this means that uh you can basically replace this by the other direct factor which is also freeb and replace H by the corresponding thing because then this corresponding this prodct will split into two two copies and this n will just embed into one of them uh so we can definitely assume that H is indexes um Okay so so now we have an injection I'm going direct some of these and now we have some quation here and now the next thing is to understand

structure of the portion you um so uh there is a certain quotient of  $Q$  which is the image when you embed it into double um sorry let me just write as a product over all over these over all possible maps from two into  $V$  so you can look at all possible maps that  $Q$  maps to a free group or just to  $Z$  and takes the corresponding maps to  $Z$  and then there will be some quotient here uh and then by the fact uh this group is actually free okay and so this means free or particular projector splits at a direct summand and so uh and also if you look at the tensor of this map then it's easy to see from this that this  $Q$  Prime will have no more maps to  $Z$  right because if this had a map to  $Z$  then because this is a direct sum there's another map from  $q$  to  $Z$  which would make  $Q$  okay so this means that uh this has this portion  $Q$  Bar which is free and so this splits back here but then also back here if you want and so then you can also get rid of that sum uh okay so without loss of generality so you can replace some  $Q$  by  $Q$  Prime and then there are no more maps to  $Z$  those generalities um  $Q$  has no more maps to  $Z$  a different way to think about what this I'm doing here is to observe that um the category of countably generated free groups has core kernels and this  $Q$  Bar would be the core kernel okay okay so where are we in the proof so we're trying to show that whenever we have a direct sum of  $Z$  then once we dualize the image of this map is itself a product of copies of  $Z$  and now we split it off the kernel we split it off part of the core kernel and now we're down to a situation where we have this map and  $Q$  has more maps to  $Z$  and we're trying to understand what happens if we dualize okay so um so we have here product of copies of  $Z$  copy of  $z$  uh this map here is our old map  $d$  and uh all right uh sorry the map this is precisely uh the internal hom from  $Q$  to  $Z$  and then there would be actually  $X^1$  um and no but so we ensured that the hom from  $Q$  to  $Z$  is zero but actually it turns out that the internal hom from  $Q$  to  $Z$  is zero as the points are hom from  $q$  to  $z$  join  $s$  into  $Z$  which by using the adjunction this different order is the hom from  $Q$  into the continuous dual which is a discrete group and where this group is  $S^1$  uh and so uh but  $Q$  has no maps to  $Z$  and just no map any being group and okay and so so what does this mean so this means that the image of  $G$  is actually just this product of  $G$ 's so some of the reductions we made in the beginning precisely ensured that actually our map became injected just and okay I guess strictly speaking I did some of these reductions it could have happened that an infinite direct sum became a finite direct sum but just take the direct sum with an in thing whatever um uh okay so that's good um um right so actually call a proof is that the map any  $M$  which is a presented solid group um here's a direct sum or a product of copies of  $Z$  and the group of the form  $X^1$  of free against  $C$  for some countable being group  $q$  without maps to  $Z$  and so if  $Q$  is actually a torsion group uh that these are precisely is a profinite group there are some kind of weird non separated groups okay uh uh ah and here's another fun um uh so this this call is again something that's only true in the uh in the live setting namely that the product of copies of  $V$  is actually flat so next Che it's not always true that projective objects are um we could check that once we pass the  $P$  numbers um something like this holds true in like without the Lf assumption but we were kind of confused about the case of solid being groups and then but then Sim observed that actually it's false uh in all solid being groups the flatness but once you restrict to the light connect becomes true again which is also kind of the reason that we never saw any obstruction to this in actual mathematical practice um and why uh so we need for all and solid The Rock Solid and the product the  $S^1$   $T$   $R$   $T$   $C$  limits so generality to prevent uh so we actually have a resolution of length one uh but now I T this with a product of copies of  $Z$  then this map stays injected this  $S^1$  I mean it's just I mean actually this actually what you really see is that if you take the  $d$  uh it's called  $T$  this the product of this you actually just get a product of  $c$  of for because so you can just write this as a flat fin presented groups and all right uh so uh and one thing that's really nice about this whole tensor product is that uh really gives a lot of I mean many many computations come out right um in sometimes non obvious ways just like maybe previously a lot of EX computations came out right now obvious ways and so maybe uh let me do some tensor computations um so often uh you maybe care about things like you start with some in being group and a speed being group and then you form the per completion of them and and then in this for in this such situations uh one often uses with complete the  $T$  of product uh which is like completion of the usual if you do some kind of formal geometry or something um and here's a theorem that is actually uh what the  $S^1$   $T$  product does uh so actually like in full generality for any being group is actually better to replace this by so-called derived completion which is the derived  $P$  operation um very where this derived thing just means a complex represented by multiplication by where this sits in degree zero is this sits in homological degree minus one or homological degree one uh almost always this agrees with usual completion it's degree zero but in complete generality that's a better operation um and so generally uh and then it makes sense to talk about derived I mean these things are the derived complete  $S$  um uh and uh so here's here's a proposition uh and

because we're doing the is slightly better to from the start uh work in the dra category and let me assume I have things which are conc non negative degrees so go to the left so see I have two of them and they are complete so there isomorphic to D of the D production from P to the n n obviously it's actually also a condition you can check on grps uh then also there Sol products um let me prove this in a second let me just Note One cor which is for example that if you take a I don't know infin direct some of copies of Z and take a period completion of that and then take product of that with such a period completion of you just get the similar so this means that in such kind of situations of more or less formal geometry uh d i complete things the S product does what you would expect it to do and there's really nothing special about the number P here I me could work with any base ring and solid modules structure with that ring and then any element of that ring would so this PRD completion of the free free ailan group you consider it as a as a solid as a light condensability group or as a is it is it solid I found the completion in condenser being groups so it's it's a all the z p to the end all the but takes a limit to requires the non condens strategy and it is solid the it is solid because solid I mean the class of Sol is stable under all limits liit and so on so start with C which is solid take a direct Sol take to end it's solid take a limited um uh remk uh there's nothing special about P for any R um any elements or and and and then which are solid a for now I mean this at this moment it really just means a modules inside solid B groups uh which are the T complet then the dve 10 the product of M is I should mention is actually a COR with the proceeding so we know there's definitely a map from here to here because the standard product represents B maps and so there is obious B map to here um uh but both sides are deriv promete the right hand side by definition the left hand side by the proposition um so to check whether we can actually check after reduction P because for dve complet things is can be checked modular P but modular P both sides just become theet and you're just taking the usual TCT of product of FPS and product a direct sum of FPS and a direct sum of FPS and just um okay so uh let me give a pro sketch uh so first of all maybe in this situation you can actually work with a zp instead because uh the D centerct of zp was CP uh is just zp um that's because you can take C power series T C power and then you fren by T minus p and uus p um then tus P you get here uus P get here but also when you take this power Series in two variables and but tal to P I mean then you get zp power but thenal to zp um so this actually means that uh the drive ceg of solid zp modules is automatically full sub of solid Z module so being a ZT module is not a structure and a solid being it's just a condition and if everything's complete then uh uh everything becomes a zp module so so m&n is actually here and everything is taking place in the subcategory okay and let's actually let me do the case where which is maybe the most destructive case uh where m is really just uh take a direct sum of copies of zp uh but you complete a direct sum and then small then in order to compute what the Sol T product does we have to write this in terms of our generating objects and this is actually a slightly non trial exercise because there is actually rather large to liit of compx detective objects um so what What's Happening Here is that this actually is a CO limit overall functions from n to n uh which go to Infinity so they only find the many values where below some constant um all the FR of copies Over N of to F of so what's happening here so first of all why is there a map so whenever you have a map which goes to Infinity you can take the product of these copies of P to the F and DP and this actually match to the completed direct sum because modu each power of P almost all of the terms in this product will go to zero that's because this is going to Infinity um but uh uh and quite obviously also an injector um but then you still have to show that it's a surge action so you have to show that whenever a live Prof set map is completed direct sum it actually effected over one of these terms but if you have a map into this uh completion then yeah Fe Vector think any light from and the map from s this um I'm just thinking whether in the most General case I'm actually um uh so pick any sest and the map from to copies of D which is by the permission liit and so so we have I don't know G here and so we have a collection of M GN here and then G each GN vors over some direct sum over uh integers at most some uh a of um right because uh this is a like I mean a compact object and Maps lot so it must what is this and uh so okay so let me go right so so there are just a few bunch of elements where you you get something mod P one and then there is maybe a larger Bunch where you get something non zero uh mod P P squ and then there's something even larger where you get something non zero mod cubed but then you can just find some slowly very slowly increasing function f uh uh sub effect over this product right so what's what's my diagram here uh these are my M these are my M and okay this is AF one this is AF okay so so this means that this tend of M and M um each of those so it's a of f of of to the F of n p and then I take Sol product with a similar expression on the other side and let me call now this function for the other Factor T and let me call a variable on the other side n zp and the S



TCT canes Coit so I can pull the Coit out so it's a CO liit over two functions from  $n$  to  $n$  which goes both to infinity and uh and then the individual hand up these I mean both of these are just products of zps uh and yeah I mean I probably didn't say it so far but product of CPS they just become the B INF product again just for like for product of c um and so now it's the product of  $N$  and  $m$  in  $M$  of  $P$  to the  $F$  of  $n$  plus  $P$  of  $m$  c and now this has an obvious map so  $f$  on it over all functions let me call them  $H$  which are now functions on both coordinates which just go to Infinity on yeah is a function of two variables uh and then you put the function of  $H$  and  $N$  here uh where this this is uh completed and now at first you might think that this will surely not work out because here you're allowed to quantify over all arbitrary slowly increasing functions of two variables whereas here you're just getting those that are sums of functions of one variable individually but then there's just maybe at first slightly surprising but not Al how to prove combinatorial L that whenever you have such a slowly increasing function  $H$  you can always find one that's even slower increasing uh which is a sum of two functions of individual coordinates leave it to as an exercise to figure that out but that's what I mean that the Sol T Pro like for non obvious reasons giv you the correct awesome um and so yeah General argument some saying so you can redu the case at  $n$  and  $M$  are just some uh yeah completed direct sums of copies of the generators and then all as the same argument you actually have to be slightly careful because upor you can also have the case that  $M$  and  $M$  are completions of uh a direct sum which is over an uncountable index set and then you still want that uh but then there's actually an argument that the uncountable case just by more formal aru to cont Cas because this der completions they always commute with I mean you can always ruce uncountable things to contable things because just anys um right so this is a nice computation um there are some even there are some further computation that come out right uh which some really appear when you do someone solid functional solid functional now so let me work over QP for Simplicity uh but most one of what I'm saying uh works over any me so then again uh solid PP displaces or even the D category it's just the full subcategory well P modules and those where  $p$  is invertible and it's the full subeg of be group and if you form 10 prodets and just some St was in the sub um and uh this category itself it now has a compact projective generator well here will be a product of zps and here you have to invert  $p$  take of  $zp$  this is a slightly curious kind of topological Vector space or like condens vector space but actually comes from a topological Vector space so it's it looks like one where some the unit which would have a norm where the unit ball is this thing where the unit ball is compact but actually there's is General thing you cannot have bner spaces or Norm Vector spaces of infinite Dimension ver unit ball is compact so it actually turns out that if you were trying to indulge this with a norm in the kind of obvious way where this would becomes the unit ball then the nor map would not be a continuous map and so this yeah it's not really a norm Vector space is what it is uh so uh these things have appeared a little bit also in the classical function literature uh May firstan called um uh I mean the the more usual objects the consider thetic function analysis which space you wrote Smith theic Smith space and she was uh studying the analog of those things over the real numbers they can also consider some type of topological Vector spaces where which as well ining Union uh gotten by scaling out some compact convex set the more usual objects we consider B spaces and at least separable examples of those exactly the on that we considered previously you take it accountable copies of  $P$  complete and then and there certainly also in this category and we just saw how we would present in terms of the generation just in and these things are actually in reality uh Mi places SP so either you put separable and light on those sides or you so separable on the bio side means you can take accountable D somewh and light some on the other take accountable product and this is just the obvious where Take A Smith space St but you can also take B of space like so yeah so the generator that we use here the sp there and sometimes du of spaces okay we're basing everything on them uh uh here's again a curious remark um and maybe people ask me this so it seems relevant uh so what happens if you try to do the sage well this is the smth this projective and even turn projective so there actually deriv so um uh um so if it takes the  $r$  home here it is just the internal Zer that's true and kind of obvious um but if he go the other way you can all wonder if you take a  $b$  space and takes the drive to of that one is it just the usual dual and the answer is uh depends on your model of the set the in the same way as this came up before so again it fails under the contines but under this principle star that I mentioned that is consistent uh it's because a BOS space is a special type of Pro Group limit of direct sums over end of QP mode  $p$  toz  $mzp$  and so this AR de like between Pro things and this precisely the thing that's controlled this is princi Yeah the more I think about the more I'm actually temped to just uses work model is true uh because you have the choice of either these higher  $X$  being some junk that really has no meaning at all all them being zero um okay um

but other like what I DUS and I have been trying to do this this question was never really relevant so that's sometimes you run into this thing I mean it's definitely like if you do some kind of computations now with in this kind of solid function analysis you're doing lots of home lots lots of tenses lots of homes and so on at some point surely you will at some point take some our home of b space against D somewhere and then you can decide whether you work in a model where it's just some joint or you work in this model where it's what it's supposed to be so um but what I really wanted to say is uh so you know you also have to fre spaces um which are uh countable limits of FAL spaces along them these are really uh this really pop up a lot um and there is a notion of uh there's a standard notion of computer trans for them uh in particular it's the one that's compatible with limits so on B bases it's a usual BF tender product and when you have f space inv such a accountable limit and it's the other one also in some corresponding limit of complete T of products of B spes and so like a general thing that usually when you define such T prods and function analysis is that you're trying to make some compatible with limits here right for example in this case uh which is not readed because our Sol T produ was defined to be compatible with Co limits and usually there are very few funs that are compatible with limits and to limits and certainly thect is not compatible with all limits uh but fortunately it so turns out that in this situation it does give the correct answer so here's a proposition if CMW this let say considered as topologic spes for the moment uh then you can pass the condensed World from line and then take the Sol tender product or even the dve Sol tender product and it turns out that this is really just uh the completed T PR spaces and then pass in particular it's deg Z for example if you take a product of copies of p and take another prod of P it becomes the by infinite which might seem like it's a standard thing on compact projective generators but it's absolutely not because these things are very very far from being compact projective only these kind of unit balls there there as compact projector cars but this is a huge space okay and so let me again just give an proof on an example in zamp I me the general proof is in some sense combining uh the proof I did there with those poit with the one I will do here uh and we kind of spray by was doing a minimalistic approach to combining this but Sasha e actually has a very fancy way to combine them that he needed in some of categorical stuff um okay so so so okay so how do we write this product of tpce so it's again such a huge po limit and this time you're taking such a unit balls but uh where you makes the denominators larger and larger so you take of P the minus FN \* DC um where again f is some function from n to n but now we're taking forward over ever faster increasing functions so it's certainly going to Infinity but but very fast say um right because again if you have any complex project I mean any profile that mapping into here in each factor it will Factor over some uh over some pie of F mCP and then you just well take this function and I mean generically it's probably a function that I don't know just for any n you give some value but for each new number you can make it much much larger okay so uh and so if you take this tend product it's a Col by same Principle as before now we have two functions which go to Infinity very fast um of a product n n of tus FN minus zp and again this next to a similar col or functions which are functions of both uh variables going to Infinity very fast um and again you might naively think that surely if I have two variables I can build functions that are so fast increasing that I can never dominate them by something that's uh some of functions in both variables but turns out again by basically the same argument as there is you can always dominate any such function by function all right that's fun um all right uh I should maybe say I mean uh this is some f poet here but the precise structure of this Index postet this is actually where all the set appears so here we kind of don't actually have to understand what this looks like because this combinator can just prove but for these X computations you really have to be some starting with such a description in the case of b space and then do that and so then this this this huge call liit becomes a huge derived limit and then you really have to Grapple with a structure of this kind of postet of functions that grow arbitr fast and uh this is something that very much depends on your model of set here particular on the cardinality of SP set just a all right um yeah I think that's basically all I really wanted to say but if there have questions we have a few minutes so for functional analysis in some very large B of spaces does it make a difference if you work in light condenser or all condensed sets it doesn't make a difference because all B spaces are lied all for spaces are lied no matter whether they're separable or not it's just the pred to Smith bases only they uh they see okay it's clear Factor yeah because it's actually have some kind of uncom dimensional V Space it's still the filtered just naive filtered col of okay okay any other questions is the sorry it's a very V question but is the relation between the homology of the C complex and ification kind of the accident or should should I take it some deep I mean can I can I can I expect that we will have another example in I don't know in a condensed Spectra or something like that for other well instead of

uh like taking the dra C being groups you could everywhere work a category of Spectra and then you could also yeah I mean maybe a good point to point out the following um there's Spectrum call little K like to think of as some you take the algebraic Cas of the complex numbers but you take into account the topology that the glnc has right um uh and some makes this homotopy say uh and classically this a little bit hard to phrase what this is supposed to mean I believe orbe there way to do anyway so one way you can however do it in our formalism is you can take the cas of the complex numbers as a condensed spectrum because I mean the complex numbers this is a condensed ring and so this actually means that F Cas series is kind of has some automatic structure as condensed Spectrum right because to give a condensed Spectrum I have to give a function for each maybe live profile set I have to produce a spectrum but uh I can just take the case theory of the continuous function from  $s$  to  $C$  um so this this is sorry so far this is not at all true um because here the  $\pi_1$  for example will be  $C$  Star as a as a condensed being group but uh now you can take KFC and solidify like just like we had solid draft of solid being groups and draft car of condensed being groups you also have solid Spectra inside of all condensed Spectra and this sonification does exactly what you wanted to do it somehow contracts all the deal NC that this was build out of Toopy Ty essentially by the proposition I mentioned in lecture today and then you see that this is actually exactly the and I mean you could do the same thing really not just for the complex numbers but for any algebra over the complex numbers or even a category over the complex numbers and then take the case as a condensed spectrum and solidify and this actually gives a construction of what's known as a semitopological case mhm so it is solidification of what semi topological uh so if  $a$  is any kind of  $C$  algebra was it SC topology or uh with a topology I don't care um then you can similarly take the cas of  $a$  and solidify and this recover something that's known as a sar topological case the I some some some hope in the area that when you have a proper smooth DG category of the complex numbers there's some way to define its topological case Theory and I think this was one of the reasons people started uh defining the same toy Cas Theory so the actual one should be a module over periodic KU but you can just take this one and just inverts the B element then you have a candidate for something that might be the correct thing but nobody knows how to prove much about it I think can you get the similar result for KO or KR sure uh just write R yeah I mean uh I mean all the obvious VAR yeah so in particular KO is take Cas of the real numbers and there's something I know PSP or whatever it is and then maybe related to the so here the K is the periodic or the the full spectr I mean in those notation I the algebra Cas Spectrum uh it turns out to be connective for the inputs that I'm using uh maybe not quite obviously so but it's true um and so there is a connective KU uh it is much more subtle to recover the periodic you I mean you can of course just invert what El at the end but uh and sometimes some this antic case the that do an I develop which is a different way to approach this kind of stuff is aimed at defining some kind of version of the cas the of the complex numbers but now analytic ring where the answer well certainly has a St negative degrees Ian some FS is it somehow related to the description of the small KU or small K or as some configuration space like description given by gam seal um I don't know about the ah sorry there's some uh do like description for uh small K and for using a like vector given by SE uh I I think this statement is much more naive I mean this is really just this statement is really just one way of encoding the intuition that or maybe see but I you you you some in some sens this is false some something that's true before group completion namely that if you take uh the infinity monoid classifying space of  $c$  and then pass to homotopic types here um then then I mean KU is a good completion of that and basically solidification uh factors over passing to homotopy types and so this is seconds um but they have I don't think there are any figuration spaces any other questions to what extent does the Sol tens product combine two topologies so say we have  $m$  and  $n$  and  $m \times$  complete and directly and there was a thing that I mentioned that if you take power series  $T$  and then  $U$  then it becomes the UT edic one so there it does do this thing um and this more b space setting it doesn't do it I so when you TOR over  $a$  and you have this exotically complete statement now when you can take the completion of the Ring  $a$  but this one is no longer the derived completion does it make a difference for the proof uh yeah so sorry I actually realized that probably most General version for St might not be expected true um I was getting confused about this um well I mean if you want to compute some direct product of a solid or whatever um one it's kind of resolved by the the absolute one and then you're also taking a proct  $a$  and so to show that this here is the completed it's enough to know that all the other ones are and this way you can kind of reduce the case that the ring  $a$  is actually just  $z \times x$  here so then the DED X completeness of this one reduces to the one here here here but a itself is derived X complete I mean if it's basically X complete it's also derived X complete um so uh you can kind of reduce this statement to

the case that the ring is really just the join  $x$  what did you mean by passing to  $\text{otop}$  type in  $\text{BGN } C$  so  $\text{glnc}$  is viewed as a topological space and therefore as a as a condensed set or or notari I mean yes so when I when I Define  $\text{KFC}$  one way to do that is to take this classifying space of  $\text{DG}$   $\text{onc}$  and then I have some uh infinity group in condensed groupoids  $\text{um}$  and then I group completed but then uh yeah so some physically I'm here using the category of condens  $\text{Alima}$  which contains condensed sex or condensed groups and so on as one for sub category but contains  $\text{Ana}$  or  $\text{typ}$  whatever there another for sub which are some of the discret objects  $\text{um}$  and then on a certain class of objects here there's a left joint which is passing to the homotopic type uh and this is not always defined but on some object it's defined and for example on  $\text{CW}$  complexes it is defined during testifying space of  $\text{Jo}$  and so and so what I meant there was that I kind of appli this left joint all right let's stop here

## 7. THE SOLID AFFINE LINE (CLAUSEN)

[https://www.youtube.com/watch?v=fUjn2rGw9SA&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=fUjn2rGw9SA&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

Today we're going to be talking about a little bit of geometry maybe the solid affine line. So let me start with a recap of what we've seen so far. So we had this category of solid abelian groups which was a full subcategory of the light condensed abelian groups and this was some kind of analog of the complete non-archimedean topological abelian groups but it's kind of from a formal perspective easier to work with. It's an abelian category in fact it's yeah so it's abelian category closed under the limits colimits extensions and many other things besides and also we had this left adjoint called solidification and there's a symmetric monoidal structure here which you think of as a completed tensor product and such that this completion functor is symmetric monoidal and the definition was in terms of this object  $p$  which was kind of the free module on a null sequence so the definition was that  $m$  is solid if the map from null sequences to null sequences given by identity minus shift is an isomorphism. Okay so that's what we had last time. Now I want to start doing a little bit of geometry so we're going to be modest and look at the affine line which is actually the most important case. So about this  $P$  so it turns out that this  $p$  even before solidification so  $p$  is a ring so there are two ways one can look at because you can shift one to the left and one to the right. Yeah or both are equivalent. I want the one no I think only one is good I mean I want the one that's injective. So  $p$  is a ring and maybe I'll be clarified in a sec maybe well  $p$  is a ring and in fact there's a ring map from the polynomial ring in one variable to  $p$  and the shift is multiplication by  $T$ . So that's that's what you have before solidification but after solidification something extremely nice happens so when we solidify well we actually already explained what the free module on  $P$  is so the the the uh there's a compact projective generator here which was a countable product of copies of  $\mathbb{Z}$  that's kind of the basic object from which everything is built and you can get that object just by solidifying this basic sequence space  $p$  and if you take into account the ring structure what happens when you solidify is you just get the power series ring in one variable so in the solid context the universal carrier of a null sequence is just this power series ring but moreover there's a very nice property of this situation so so LMA how do you multiply things in  $P$  that looks like a bit by something which is not the null sequences is like ring out unit usually yeah so let me explain how you get the ring structure just very briefly so it's uh it's better to think of this construction as being attached to  $n$  rather than  $n$  Union Infinity so more generally if you have a set say or a countable any countable set you can define measures on it as the free module on any compactification module the free module on the boundary and it's independent of the compactification and it's functorial for proper maps so now and it's symmetric monoidal you can see that using independence of the compactification if you take the product of two countable sets then measures on one tens or measures on the other is measures on the product so using that and the fact that addition on the natural numbers is a proper map has finite fibers you can see that addition on natural numbers induces this ring structure on  $P$  okay all right. It's the same and you know addition on natural numbers also in some sense induces this so that's why uh yeah all right. So the Lemma is that uh so uh so if you do the tensoring in the solid world of this power series ring with itself over the polynomial ring uh you just get the power series ring again maybe multiplication map is an isomorphism you could see and even the even the derived tensor product and well this is quite elementary uh because we know how to well we know how to do tensor products in solid  $\mathbb{Z}$ . Peter gave lots of examples of calculations in this category and the tensor product of two of these basic guys uh is just another one of them and so if you do so proof. If you do  $\mathbb{Z}$  power series  $T$  let's say  $T_1$  uh tensor over solid  $\mathbb{Z}$  uh  $\mathbb{Z}$  power series  $T_2$  that's just uh  $\mathbb{Z}$  power series  $T_1 T_2$  by the basic calculation of the solid tensor product and then if you want to get to this uh you just do modding out by you just identify the two different variables  $T_1$  minus  $T_2$  and uh power series ring modulo  $T_1 - T_2$  is the power series ring in one variable. There's maybe another perspective Peter also mentioned this fact that if you take the solid tensor product of two derived complete things then the result is still derived complete so well maybe you have to worry a little bit about this but morally if you want to check this is an isomorphism both sides are derived  $T$  complete you can check at modulo  $T$  and modulo  $T$  it's really a triviality. So kind of okay. The the derived solid uh Tor product is it defined as the derived  $T$  product then you solidify or is it how how is Def exactly yeah it's the you can either say it's the derived



the left derived functor of the solid tensor product or you can say that you take the derived tensor product and then derived solidify it it's it's all the same and do need to know there are somehow enough flat things to construct the the you don't need to know that but you but we but it is actually true that all of these guys are flat I don't know if Peter did Peter mention that I forget uh yeah um okay uh yeah so a priori you need to resolve both variables but since you have flat objects you don't you would only need to resolve one um okay so what is the interpretation of this so this uh this corresponds to sort of the apine line you could say and then since this uh satisfies this item potent property uh sort of like a a when you invert an element in a ring you kind of get the same thing you can think of this as corresponding to some Subspace of the apine line so a Subspace as opposed to just a random space with them with a map to the Aline line and that's the interpretation of this item potency so then you could ask okay how how should one think of this Subspace what Subspace is this um so the naive thing to say would be that it's the formal completion of the Aline line at the origin um well because it's a power Series ring but that naive interpretation is not the correct one and so so does not correspond to a formal uh neighborhood of zero in  $A_1$  and the reason is that that interpretation is not stable under base change so so look at base change by which I mean this is all implicitly over  $Z$  but we could tensor it to any ring or any solid ring and see what pops up um and let me do a little calculation so so let's uh let's base change to a non- archimedean field say I'll just take the simplest example  $Q_p$  and we'll see what pops up uh so we take the Aline line over  $Q_p$  um uh oh sorry well maybe I'll just say yeah we we take we take  $Q_p$  and then we do a we we do a solid tensor product with uh this thing here and we want to compare this to  $qpt$  um so this so  $Q_p$  is  $zp$  with  $P$  inverted and the solid tensor product commutes with co- limits in both variables so this is the same thing as  $zp$  solid tensor product uh  $Z$  power series  $T$  and then you invert  $P$  but this  $zp$  uh has a resolution by two copies of so so  $Z_u$   $U$  maybe mod  $U$  minus  $P$  it has a resolution by two copies of this power Series ring uh so we know how to do these tensor products and it just does the naive thing of you can pull in the limits uh so the result of that is that you get  $zp$  power series  $t$  uh and then invert  $p$  on the outside which is not the same thing as the formal completion at the origin namely  $Q_p$  power series  $T$  so it's contained in there and it's the subset where the coefficients uh are bounded in piic Norm so so what is the interpretation then of of this ring uh this ring is the ring of functions on the open unit disc uh in in  $A_1$  over  $Q_p$  so if you you could ask uh if you take say maybe after an arbitrary non- archimedean field extension of  $Q_p$  you could ask when you plug in a value in that field uh when will this when will such a series converge and the answer is it will converge if and only if the number you plug in has absolute value strictly less than one the Ring of bounded functions actually what's that it's the Ring of bounded holomorphic function of open Unity ah yes uh what not the full because sometimes yeah there's another one Ah that's true yeah that's the bound thank you yeah yeah that's right sometimes people use the the not necessarily like the robing and stuff which is not about I don't know that all kinds of things yeah I don't know what well okay certainly all these functions Converge on the open unit disc um but they don't converge past that um so uh well so it yeah so then the interpretation to that this suggests is that uh that uh this uh  $ZT$  uh  $Z$  this should be thought of as corresponding to some sort of open unit disc and it's just that over a discrete ring like  $Z$  you can't tell the difference between the open unit disc and the formal completion at the origin but the difference pops out when you make a base change to something that has more more freedom to play with okay um but in uh rigid geometry uh uh it's not really the open unit disc that's fundamental uh usually it's the closed unit disc um so what about the closed unit disc well one way to understand the closed unit disc if you understand what an open unit disc is you can imagine putting the open unit disc at Infinity instead so then you're looking at then you get the complement of the closed unit disc and then that's one way of talking about the sorry the complement of the you yeah you get the complement of the closed unit disc and the complement of a thing is is just as good as describing the thing so let's take this guy and put it at Infinity instead so the complement well this is all just at the level of a uh loose thinking so we can take this power Series ring in one variable uh and we can tensor it over well but let me call the variable  $T$  inverse instead because I want to be thinking of putting it at infinity and then I tensor over  $ZT$  inverse with uh  $ZT$  inverse so then I'm puncturing at Infinity so that I actually live in the apine line so I have a homomorphism from  $ZT$  um uh and what is this ring well I'm just taking this power Series ring and I'm inverting the the variable there so another way of describing this is as  $Z$  power Series in  $t$  inverse so that's the open unit disc centered at Infinity um uh so if we want to understand the closed unit disc we should in some sense be uh kind of localizing away from this complement or another way of saying that is that we should be killing this object so so to get we need to kill uh kill

this thing here so um so well I'm just going to make just a preliminary so this  $Z$  power series  $T$  inverse is a module over  $ZT$  as I just exhibited and it also has a very simple resolution so uh note um so uh well  $ZT$  inverse we can think of that as let's call let's disambiguate and call the variable  $U$  um we can think of it as  $Z$  power series  $U$  and then invert  $U$  um but one way to invert  $U$  is to adjoin the thing that you want to be uh the generator uh I mean want to be the inverse and then enforce the equation that that says that they're inverse to each other um and when you look at it like this uh so so I I'll rewrite that so we get  $ZT$  inverse uh has a two-term resolution where you have  $Z$  power series  $U$  bracket  $T$  and  $Z$  power series  $U$  bracket  $t$  uh and here you have uh  $U$  minus one so two term resolution and what are these objects well these are just the base changes of our fundamental uh  $p$  uh not to solid  $Z$  but to solid  $z$  uh and then adjoin a variable  $T$  so so these are resolutions by the compact projective uh generator uh of the category of of  $z$  braet  $t$  modules in solid ailion groups so killing this thing should be the same thing as requesting this map to become an isomorphism and now this suggests the following based on an analogy with the definition of solid up there so so definition so let's say let's let's say we have a one of these guys let's say that is  $ZT$  solid um if and only if when you take uh internal  $H$  from uh  $P$  to  $M$  and then internal  $H$  from  $P$  to  $M$  uh and then you take the map which is given by so now  $U$  corresponds to the shift and  $T$  is the extra thing we have acting on  $M$  because  $m$  is a  $z$  bracket  $T$  module so we have shift time  $T$  minus one uh we want this map to be an isomorphism and then this note up here is saying that that's the same thing as requesting that if now I should maybe pass to Arham uh it's the same thing as requesting that there are no  $R$  homs from this object we're trying to kill into  $M$  okay so now uh theorem is basically that um this definition the name is well chosen so that this  $ZT$  solid Theory over  $ZT$  is very similar to the solid  $Z$  Theory over  $z$  uh so so let's say and we can even I mean everything is also inside condensed little light this uh is an aelon category closed under limits and colimits extensions um so if uh if  $m$  is an arbitrary say condensed  $ZT$  module uh then internal  $x$  uh and  $N$  is in solid  $Z$   $ZT$  then internal  $X$  over  $ZT$  from uh  $M$  to  $n$  uh is also in solid  $ZT$  uh there exists  $d$  uh there exists a left ad joint to the inclusion uh call it uh let's say well I'm going to at least temporarily denote it by upper  $T$  solid we're forcing the variable  $T$  to be solid you could say so that's uh  $z$   $z$  and  $Z$  is the same thing one you wrote in the first yes I wrote I use different notation yeah sorry yeah this is this is supposed to be the same as as this yeah yeah um there exists symmetric monoidal structure on this solid this drive solidification over  $ZT$  um and then uh the driven analog uh holds as well so the derived analogs of all these claims hold and moreover the drive theory is naive in the sense that an object in the derived category is derived solid if and only if each homology group is solid in the sense of the ailion category and it is the derived category of the aelan category um okay so that's all the kind of formal stuff but but that in the solid  $Z$  Theory it was also important to to get a to understand what was going on it was important to understand the the basic compact projective generator which is always gotten by just well solidifying the sequence space um so let me make a claim about that so the if you take the  $Z$  power series well maybe I'll just I'll say let's not think of it as a ring we're thinking of it as a module now so if you take a product of copies of  $z$  uh you base change it to  $Z$  bracket  $T$  and then you solidify you get well a product of copies of  $Z$  bracket  $t$  um so maybe a little remark about an interpretation of it so I said solid  $Z$  was kind of analogous to complete non archimedean topological ailan groups non archimedean means Say by definition that there's a basis of neighborhoods of the identity consisting of a billion subgroups um so what would be the analogous interpretation of solid  $ZT$  it would be that you have a as you have a complete non- archimedean thing where moreover there's a basis of neighborhoods consisting of  $ZT$  submodules and you you can think of I mean the and for a basic example of a  $ZT$  a  $ZT$  module which is non archimedean but does not satisfy that property you can think of this ring um you can not find a basis of neighborhoods of zero which are stable under multiplication by  $T$  because they're stable under multiplication by  $T$  inverse instead and so but in some sense the the theorem is that if you kill just that one guy then you've explained the difference between the two Notions um okay so so in the previous Theory without light it was a little bit different it was not defined using this  $P$  but well okay it basically was done done this way uh I mean maybe we didn't make this explicit and we maybe more talked about uh just this but we definitely talked about this um yeah but it does make it more clear to to to think of it that that way with the  $p$  okay so the proof yes confused about the  $ZT$  itself is  $ZT$  solid ah so it is that's part of the that's part of the theorem because I'm claiming in particular that this is  $ZT$  solid and and  $ZT$  is a retract of this so it's something that we need to prove and we will prove it but it's not solid what  $ZT$  is  $UN$  solid  $ZT$  oh no it is it is so as an ailan group you mean or yes no no every discrete ailan group is solid because it's generated under co- limits by  $Z$  no it's an important point Thank you for for bringing it up

yeah other questions okay and  $Z$   $ZT$  power series is also solid  $ZT$  power series is also solid yes yes because well because it's a limit of things I mean you can build it from limits and Co limits from  $ZT$  so it's yeah all right uh so for the proof uh well the all of these properties so all except the last those are that's exactly the same those the arguments for all of those things were completely formal just based on the fact that you have this internally projective object  $p$  and you're asking that some endomorphism of internal  $HS$  from  $P$  to  $M$  become an isomorphism that was all that Peter used when he was proving the analog of these claims for uh solid  $Z$  so the fact that we have a good formal the theory is already contained in there and then in Peter's lecture the hard part was identifying the free modules that when you solidify this sequence space  $p$  that you actually just get a product of copies of  $Z$  fills up the whole thing thankfully uh so that that part is actually going to be easier here because we already have solid  $Z$  and  $um$  and basically thanks to this interpretation of killing some object so so the claim uh the key claim is that  $um$  the that you can give an explicit formula for this derived solidification uh so I apologize for the janky notation now this looks really bad uh  $D$  derived derived solidification with respect to  $T$  so uh that this is just an internal  $ROM$  over  $z$  braet  $t$  uh from the quotient of this ring by this ring to  $M$  and note that this this is this does have the required map from  $M$  uh because  $m$  is  $ROM$  over oh sorry oh sorry minus one uh shift by minus one uh soor I'm just trying to read what that last thing says that that's and  $m$   $t^2$  yeah I so derived solidification yeah the deriv functor of this thing here yeah which I I kind of prefer actually to think of as just the derived analog of it  $um$  okay and the  $AR$  underline is either in all condense or only in solid uh it's the same so basically because of this I mean they yeah the derived the the so when you compute of course you have  $Gren$  the categories so you have a well defined even  $Rec$  bounded complexes but how do you check exactly I think Peter must have briefly said something I don't remember but how do you check that the the  $AR$  or underline are the same for some Junction  $arum$  I don't remember exactly I mean you basically I don't know I like I said I I sort of tend to think on the derived level from the beginning but the basic point is that everything has a resolution by these uh internally projective guys and then on those they're the same and then it's some derived limit of that and it's just I di internally projective in  $h$  in uh in all condensed in light condensed I mean so but it's really not that's that's really not necessary either I mean this also works in the in all condensed so I mean I I claim it's formal and I also claim I don't want to get into the detail right now so uh yeah so let's maybe discuss after if you still have questions  $um$  okay uh right so the proof of claim so this follows formally from uh the fact that this ring  $Z$  power series  $T$  inverse uh is item potent over uh over  $ZT$  which follows from the very first description I gave of it as the base change of the we check that the power Series ring is item potent and we put that at Infinity  $um$  and the item potency is preserved so for example using so that the item potency of that so what is this thing here it's just the the homotopy fiber of the inclusion of this into this and this is the the base ring and you can easily check that item this item potency is equivalently to equivalent to the derived item potency of this object and that means that if you take this expression and you apply it again you get the same thing back so that's kind of an item potent operation and again the same item potent will prove to you that if you take this operation and you  $Arham$  from this guy uh you get zero so this thing is  $ZT$  solid and that's not quite everything you need to check but it's basically everything you need to check it's an item potent operation and it uh  $m$  is solid if and only if this map is an isomorphism  $um$  so it's a it's just completely formal and I won't write out all of the all of the details  $um$  so maybe I'll just say note so this yeah okay so that's a a formula for the derived solidification of a general  $Z$  bracket  $T$  module but uh something nice happens if your  $Z$  bracket  $T$  so we still haven't proved the claim  $um$  but something nice happens if your  $Z$  bracket  $T$  module is base changed from a a solid  $Z$  module so and next claim uh is that the functor from the sort of pullback functor from solid  $Z$  to solid  $ZT$  so sending a a module  $M$  to uh you take  $M$  tensor  $Z$   $ZT$  and then you solidify  $um$  this functor uh is  $T$  exact uh preserves limits and colimits and it sends uh  $z$  uh to  $Z$  to uh  $ZT$  so if we if we prove this then we're in particular we in particular get this claim because this is of that this is that functor applied to product of copies of  $Z$  I claim the funter commutes with products so then we could just try to it's enough to understand what happens with  $Z$  but I already also claimed that  $Z$  goes to  $ZT$  so uh we'd be done so the proof  $um$  so we just take this formula for the solidification  $um$  and we plug in the case where  $m$  is uh also induced so we get that so  $M$  tensor  $ZT$  uh  $LT$  solid this is the same thing as I'll just write it  $m$  brackets  $T$  by that I mean  $m$   $t$  for  $ZZ$  brackets  $T$   $um$  and now we're Computing an  $rhom$  over  $ZT$  so the way to do that is to disambiguate the two occurrences of  $T$  and then equalize them at the end so that's uh calculated so  $um$  let's say  $Z$  power series  $U$  well this is so  $u$   $z$  power series  $U$   $um$  uh uh so two term complex I'll write it vertically so to speak  $um$  so then you equalize  $U$  and  $T$  or no sorry  $U$  and  $T$

inverse no uh yeah um uh or so let me let me say we no I should I should say rather that we equalize  $T$  and the and the and the shift operator on here which is induced by multiplication uh by  $T$  here so let's say let's denote that by shift here and then I'm switching variables uh from  $T$  to  $U$  um okay uh but uh note that now we're taking  $R$  homs in solid  $Z$  but this is one of our compact projective guys so humming our humming out of that commutes with  $\text{Co}$  limits and this  $M_t$  is a  $\text{CO}$  limit uh it's just uh so we can put that out here like this and then we're taking uh this thing bracket  $T$  and we're modding out by  $T$  minus something so that means that or we're not modding out by it that modding out would be the  $\text{co-}$  fiber we're doing the fiber instead but we also had a shift and those two actually cancel and in the end what you get is that this is  $\text{ROM}$  over  $z$  uh from  $u$   $z$  power series  $U$  uh to  $M$  and that's the key formula oops and uh we've kind of along the way we sort of lost the  $ZT$  module structure on this thing uh but you can recover it it's induced by uh so the multiplication by  $T$  is here given by sending  $U$  to zero uh  $u^2$  to  $u$   $u^3$  to  $u^2$  Etc so if you do that there then that induces an endomorphism here and that's the multiplication by  $T$  on this resulting thing sorry I I miss what you say so what what is the isomorphic to this green uh this this this bit a somewhat silly notation uh is homotopy fiber of this map or some kind of shift of a mapping cone and that's the same as this it's also the same as this okay and now we're done basic well now this functor again this is internally projective in solid  $Z$  so this fun is  $T$  exact and it preserves limits and  $\text{Co}$  limits in solid  $Z$  but limits and  $\text{Co}$  limits in solid  $ZT$  or are calculated on the underlying level because it's just part of a module category um and then the last claim is that  $Z$  goes to uh just the usual polynomial ring in one variable but what happens when you plug in  $Z$  here you're taking  $R$  homs from this product of copies of  $Z$  to  $Z$  it turns into a direct sum of copies of  $z$  um and you can check that uh what you get is just the usual  $Z$  bracket  $T$  even with the  $ZT$  module structure or really there's a natural comparison map which you see to be an isomorphism okay so there was this some someone prove it's not related directly to the theory but someone proved just obr and gr at home from the product infinite product  $\text{cop}$  of  $Z$  to  $Z$  is a is a direct but this is not here you are doing it in this is easier this is easier yeah yeah yeah okay so uh we can also do other examples of such a thing so let's do another example so that finishes the sorry that finishes the proof of the theorem so we now have a grip on this solid  $ZT$  Theory and I want to advance the interpretation that the solid  $ZT$  theory is is like uh like working over the the  $ZT$  without the solidification is like the Aline line and  $ZT$  with the solidification is like the closed unit disc that was kind of the interpretation that I started with um but let's do an example again in non- archimedean let's base change to a non- archimedean field and see what happens uh so let's take another example in above so let's look at  $QP$   $T$  so that's functions on the Aline line and now we want to restrict to the closed unit disc in the sense that we've just described so we take the  $T$  solidification of this um so we this is an instance of this this funter here um and I just said this functor has all the properties in the world so it commutes with  $\text{co-}$  limits so I can again take the  $T$  to the outside oh sorry the one over  $P$  to the outside and it also um commutes with limits so I can take the the  $\text{ptic}$  so I can take limit Over  $N$  of  $Zod$   $P$  to the  $NZ$  bracket  $t$  uh  $T$  solidified and then one over  $P$  at the end um and now we're applying this funter there to  $Zod$   $P$  to the  $N$  but that's just two copies of  $Z$  so um you get the same answer as for  $Z$  it's just discreet so this is just inverse limit Over  $N$  of  $Z \bmod P$  to the  $NZ$  bracket  $t$  uh one over  $P$  or in other words it's  $ZT$   $P$  completed  $P$  completed one over  $p$  and these are the functions uh on the closed unit disc kind of as as desired okay so maybe it's I'll take a five minute break uh maybe I forgot to give the interpretation of the coefficients here so again this is some power series with coefficients in  $QP$  but here the coefficients uh tend to zero periodically um so it's a smaller ring of functions so it converges on a bigger region and you're allowed to be now on the bound well boundary so to speak I mean yeah it's yeah yeah um okay now I want to let's play a fun game uh so I've been talking about the open unit disc and the closed unit disc or maybe I'll say yeah or the closed unit disc and the complement of the closed unit disc but in it's kind of there's always this fun question of what exactly is open and what exactly is closed because in rigid geometry the closed unit disc is considered as open um and the open unit disc well I suppose it's also open in a sense but it's not quasi Compact and um so let's play a game also you can use a below Viewpoint which changes somehow is another way another then it is closed not way yeah so I claim that our formalism is going to give a definite answer to this question what's open and what's closed yeah okay well all right um where was I oh yeah what's closed and what's open so let's let's take a look at the the the the categories you could say we've attached to the various geometric objects so for the apine line we said we're just thinking of it as the polynomial ring in one variable but then the linear algebra category is just  $ZT$  modules and solid ailan groups and let me actually go to the derived perspective so uh um but then inside there we had this derived solid so we had  $D$  of a solid

uh  $ZT$  um but actually it's better to think of that as a quotient really um because the symmetric monoidal functor is the derived solidification functor uh which goes like this so uh but then on the other hand we had the complement uh and the complement was uh the open unit disc or rather the open unit just move to Infinity so that's this thing and that corresponds to  $d$  uh mod uh luron Series in  $t$  inverse um solid  $Z$  I think someone asked before but I for about that you had the solidification which is one ad there should be another on the other side wait wait wait wait uh so then and then this is just the natural inclusion well it's it's a forgetful functor a priori you're forgetting the extra module structure but it's in fact a fully faithful inclusion because of the item potency uh the item potency of this thing means that there's at most one  $ZT$  module structure on any  $ZT$   $Z$  lauron series  $T$  inverse module structure on an  $e$   $z$  module um so this is actually like a localization sequence or what have you so this is the symmetric monal quotient of this by this thing which is kind of an ideal in there um and what kind of adjoints do we have and which of them are well behaved and so on so well we know that this one has a right adjoint uh which is given by the inclusion um but it also has a left adjoint so so um and the left adjoint is given by sending  $M$  uh to uh I guess the fiber so you take  $M$  and then you base change it to Infinity uh and you take that uh cone diagram here so it's the fiber of uh  $M$  mapping to  $m$   $t$  over  $ZT$  with the  $Z$  lant series  $T$  inverse so uh and this left adjoint is in a sense better behaved than the right adjoint the naive inclusion and by the the measurement that it's better behaved well they both commute with co-limits but but this one satisfies a projection formula with respect to this fundamental functor which the symmetric monoidal functor so this left adjoint is kind of linear over these two symmetric monoidal categories so this is better so uh and I'm going to call this one  $J$  upper star and this one  $J$  lower shriek and this one  $J$  lower star this inclusion here um and on the other hand for here uh we have this funter here which I'll call  $I$  lower star the inclusion there uh so it has a left adjoint also  $I$  upper star which is just the base change functor uh which is symmetric monoidal so that's kind of the more fundamental one from this perspective but it also has a joint  $I$  upper shriek which is given by some arom um but it's less well behaved again uh because here this one satisfies a projection formula with respect to this one so this is a linear this functor is linear over this symmetric monoidal category just like this funter is linear over this symmetric monoidal category so star is an adjoint the inclusion has an adjoint on it does but that one we don't talk about yeah now this was the question I yeah yeah um because already this one's not as nice as I mean the good adjoint is actually up here so um so the interpretation that this suggests so this is exactly like uh uh if you have a if you have a topological Space  $X$  and then you have  $z$  a closed subset and then  $U$  uh the complement open then you get if you have on DED categories of sheaves uh you have exactly the same thing where uh you have the open over here you have this thing satisfies a projection formula you happen to have this other guy but it's not as well behaved um and then you have  $I$  lower star uh from  $d$  shees  $z$  uh  $z$  um  $I$  upper star and then  $I$  upper shriek so it satisfies formally speaking it behaves exactly the same way and actually they're both special cases of the same thing which is you have a symmetric monoidal category um and you have an item potent algebra in it and uh it generates the whole situation in this case you have this category and this item potent algebra here you have this category and then the push forward of the structure sheath from the closed sub or push forward of the constant sheath from the closed subset and it completely analogous formalisms um so that tells us that from our perspective uh the closed unit disc should definitely be thought of as open and the open unit disc should definitely be thought of as closed so as far as I'm concerned the question is settled um but also uh this this another funny thing is that if you think about this perspective it's also telling you that zarisky opens and usual algebraic geometry should be thought of as closed and I think that's also true um um I can explain why people want to know uh all right but not for usual you don't like no you start you start from the usual theory of like patching of sh like is used in many context like in I mean it was used but then the for usual is used for the usual topology like oh the risk of a but then you somehow develop it further and decide that now that the risk opens are well close this I think I'd like to try to convince you but but not right now uh all right open sorry which import open I don't understand the question no I heard the question I don't understand it what because you said the closed things correspond to an ENT object in symmetric monal category so you said that there risky open is like closed yeah yeah so for example a distinguished open is just the ring with the function inverted that's item potent yeah yeah so I mean it's so so maybe I should say quasi compact opens or closed it's probably yeah the deriv direct image of the structure sheaf is important yeah yeah um okay all right so now uh um yeah so this by the way this uh yeah so this this we're kind of going to be guided by this sort of thing in in setting up the definition of analytic stack and so on like the idea of so one of the things we



discovered is that when you move to this condensed solid what whatever context that you you actually get six funter formalisms in large generality on derived categories of so to speak quasi coherent sheaves and um and they have I mean they really have nice interpretations so for example if you think of this thing as being what sits at Infinity then it makes sense that this is extension by zero because you're taking your sections and then you're killing the ones that live near the boundary and it really all just plays quite nicely and we showed how to give proofs of things like sidity and so on using these formalisms and um so we take this perspective seriously that the derived categories and the funs between them are going to dictate to us what what the geometry looks like um there was oh there's another thing I remembered in the break that I forgot to mention is that so I said that this is  $T$  exact and preserves limits in  $Co$  limits but I want to caution you that this this uh  $T$  solidification from from solid  $ZT$  is not  $t$  exact uh but it's only uh so it's only off by one so so um I said it was given by  $RH$  humming from this object which has a two-term compact resolution so it's only off by one from being  $T$  exact so it sends everything connective goes to something connective everything anti- $c$  connective goes to something that's at most a one shift of something anti- $c$  connective um so it's not deriv  $t$ - solidification is not exactly exact but it's it's very controlled but the composed map is yes exactly so each the first map is the exact and the composed map and the second map is not except that's right yes it's a bit funny all right um okay so now I'm going to say something like I want to motivate what I'm going to do in the rest of so I'll say Vista so where what is one place we're probably going to go is uh so we want to look at solid rings so IE commutative algebra objects in this tensor category solid  $z$  um uh and then we want to see uh so sort of generalizing our discussion when  $R$  is the  $a$   $r$  is  $z$  bracket  $T$  we want to see some subsets like closed unit discs open unit discs things like this but of course when you have a big algebra you'll get many more such subsets and we want to organize the organize uh what you see in a in a nice way and it's not necessarily the most General thing you can do but what we're going to do basically is just take the things you things you see over  $Z$  bracket  $T$  and kind of Base change them along all possible maps from  $ZT$  to  $r$  that is uh all possible functions in  $R$  um and what we're going to end up with is some is the statement this this derived category of uh  $\text{mod } R$  solid  $Z$  even this tensor category uh localizes uh along the valuative Spectrum the spectrum of all valuations uh of well the underlying usual commutative ring of this solid commutative ring localizes so you take the valuative spectrum of  $r$  ah just of  $R$  viewed as a as a discrete ring yeah as a discrete ring and the valuative spectrum is okay it's like a residue field and valuation without the valuation doesn't have to be integral on  $R$  just any evaluation yes that's right it's a big which is and it has the topology which is okay I know the topology but it's kind of the it was yeah is topology which induces the topology and things like  $\text{spa}$  and so that yeah but it's kind of because sometimes one gets  $M_s$  which are not  $SP$  in this okay but okay it's possible to consider so it's yeah yeah it's possible to consider it indeed and it's you know a priori more General okay and so so I I won't remind you what evaluation is right now but I will but I will remind you that this space is similar to usual spectrum of a commutative ring in that it has a basin basis of uh quasi compact opens I mean it's a spectral space and it even has a particularly nice basis which is in some sense analogous to the distinguished apine opens in algebraic geometry and while a distinguished Aline open in algebraic geometry is parameterized by a single element of the ring uh here you kind of have to take a little bit more data so so this uh so the basic opens are the so-called rational opens uh so  $X$  you have to take finitely many functions uh and then you form this thing here where the interpretation is that uh you invert  $G$  so we're inside the zisy distinguished open for  $G$  but then we shrink further by requiring that the the sort of  $F1$  be the the absolute value of  $F1$  be less than or equal to the absolute value of  $G$  the absolute value of  $f$  to all of the absolute values of these guys should be less than or equal to the absolute value of  $G$  um  $G$  yes that's what I said yeah  $g$  yeah so so uh right and so I'm saying that this localizes on this meaning you have actually a sheath of categories a sheath of symmetric monal categories um and what you're going to attach to this thing uh is a version of the solid theory of the solid Theory so you look at this those  $m$  in  $D \text{ mod}$  are solid  $Z$  such that well first you want to say that multiplication by  $G$  on  $M$  is an isomorphism and second you want to say that if you take this uh internal homs from  $P$  to  $M$  uh and internal HS from  $P$  to  $M$  uh and you take  $f_i * \text{shift minus one}$  this should be an isomorphism for all  $i$  so you can or in other words uh oh  $f_i$  over  $G$  sorry so in other words you want so you think of these all these guys as Maps from say  $\text{spec spec } R$  so to speak to the apine line then you want that  $g$  lands in the uh standard you know zisy nonzero Locus and you want that the  $FI$  over  $G$  uh land inside the closed unit disc so so so okay so I'm not going to go into details about that um but uh where where should I go now I don't know maybe here this as  $PB$  just refers to the local corespond to uh

I mean yeah the only I mean I don't uh you can think of it as a local but it's also a topological space um yeah and the points have a nice description and so on uh okay so what do I want to do today well or part part maybe partly do today so in particular so I'm claiming this category localizes along the space and in particular you get a structure sheath uh uh on the space and um what I want to do in the next bit so goal for rest of lecture is make this this structure of Chief explicit and uh compare well maybe probably start to compare to Huber's Theory so we're we're eventually going to produce this data by very easy formal means but it requires some language and setup so we can't do it yet but I want to make at least this part of it explicit uh already at the beginning okay so let's see so um let me make a so ah let's let's let's start with this generality so we have a solid ring um um and let's take an element  $F$  in  $r$  or really I should say in the underlying discrete ring of  $R$  in case there's ambiguity um and that in particular well that that gives you a map from  $ZT$  to  $R$  which sends  $T$  to  $f$  um and then we can sort of see how these  $\text{Loi}$  that we've identified uh can be or correspond to properties of  $R$  so let me make a definition um so uh  $f$  is topologically nilpotent uh if uh this map factors through this homomorphism I should say factors through the power series ring and uh  $f$  is power bounded uh if uh well I want to say that if if this map well the map geometrically factors through the closed unit disc um but the way to say that is uh if so let's say if  $R$  is actually so  $R$  is a algebra over  $ZT$  in particular it's a module over  $ZT$  and we can ask that it be solid in the first definition the factorization is unique yes it's Unique if it exists because of this item potency so that's an interesting fact actually this is also the free module on a null sequence and so um so even though there are sort of house solid aing groups that have like non-h house dwarf Behavior Uh still uh kind of this limit is unique if it exists um okay so basically in the definition of solid you're imposing that certain limits exist uniquely even even though you have non-house dorf Behavior Uh okay so that's the same thing as saying that uh if you take  $H$  from  $P$  to  $R$   $h$   $p$  to  $r$   $f^*$  shift minus identity that this is isomorphism that's just oh someone's talking hello yes yes the first condition lit saying that  $St$  nor sequence which is the  $PO$  of and zero so it's literally yeah yeah it's I mean yeah I was going to I mean I was going to explain the relation with classical definitions but it's it's yeah it's quite immediate for this one that it's the same as the classical definition so maybe I'll just repeat what Peter said uh so this is the same thing as uh this  $P$  thing solidified so yeah this map sends  $T$  to  $F$  but it's a ring homomorphism so it sends  $t$  to the  $N$  to  $F$  to the  $N$  um and then saying that that factors through  $p$  is the same thing as saying that this sequence uh so  $1$   $f$   $f^2$  so on extends to a null sequence which is  $B$  basically exactly the same thing as being topologically nilpotent um yeah the new phenomenon in the solid case is that it's Unique if it exists such a limit that's kind of fun uh okay it factors through in the sense of Rings or in the sense of the uniqueness factorization okay you can ask factorization as a condens group or in the sense of rings in the sense of rings yeah so you could have another factorization just in the sense of condens yes that's true that's true yeah yeah yeah yeah I was maybe being too imprecise earlier thank you yeah or in the sense of  $ZT$  modules is enough or I'm not sure I don't think so so yeah thank you for the thank you for the comment yeah just as Rings just as Rings okay um so all right so here's a a Lemma giving basic properties ah so no sorry let me say so so we write uh  $r_0$  so this is again kind of standard notation in Huber's Theory so set of power bounded elements and our  $z_0$  uh set of topologically nilp putting elements which is a Subs well yeah I'm going to prove that uh yeah so Lemma is that  $R$  zero uh inside this ring here uh is an integrally closed sub ring and uh  $R$   $z_0$  first of all is contained in  $r_0$  and it is a radical ideal so I'm still confused about this  $n$  sequence you something is  $n$  sequence in the just the sense that the map extends to a another sequence then you don't know that it factors as a ring that this is a ring map that's that seems correct to me yes okay so it is doesn't mean topologically important in your definition in your sense but on the other hand maybe uh yeah no I mean yeah you're right uh yeah well it's the same it's okay that's a good point but if you have something that comes from a uh if you have something that comes from a house doorf topological ring uh for example what if if the target is quasi separated then independently of asking about the algebra structures then the limit is unique if it exists so yeah oh quasi separated was this uh analog of how  $DF$  in the in the condensed setting so so still if you start with yeah I then it is automatically a ring map if you yes because of density basically okay so we are not sure whether topologically OS in the null sequence sense is always the same as this well okay I mean house door I mean yeah all right um so hello um okay so proof so why is this uh a subring um well I guess maybe the first thing to check is that it has a unit that's part of what I mean by subring um but if you look at it that's the definition of solid that's one way of saying it so put  $f$  equals  $1$  here okay um okay now I want to show that if  $f$  and  $g$  are in there uh I'll prove that it's a subring by showing that any polynomial so if  $f$  is a uh and if you apply any polynomial to  $f$  and  $g$  then you're

still uh in there uh so how can we do this so kind of maybe there's different ways of doing it but I think the cleverest one is maybe so we so we can look at the map from the polynomial ring in two generators to  $R$  which sends  $X$  to  $f$  and  $Y$  to  $g$  and our hypothesis is that  $R$  is a solid  $ZX$  module and it's a solid  $ZY$  module and what we want to conclude is that for any map here uh from the polynomial ring in one generator  $T$  that  $R$  is a solid  $ZT$  module okay so so we can so first resolve uh  $R$  by it's it's a solid it's definitely solid as an aylan group so we can resolve resolve resolve  $R$  by these guys we can make a res resolution of  $r$  by direct sums of our compact projective generators um but we know that  $R$  is a  $Z$  bracket  $X$  solid uh so but uh then so we so if we solidify uh if we if we solidify with respect to  $X$  then what does this turn into it turns into direct sum um ah sorry  $R$  is also a ah I'm sorry I'm sorry we should I should put I should give myself my variables uh  $X$  and  $Y$  cuz  $R$  was also a module over  $Z$  bracket  $XY$  um and now I solidify with respect to  $X$  um and I get direct sum of product of copies of  $Z$  bracket  $X$  bracket  $Y$ 's by the properties of derived solidification that I proved earlier but then and then it doesn't change  $R$  so that's still a resolution of  $R$  and then we solidify uh with respect to  $Y$  and we get a direct sum of produ of copies of  $Z$  bracket  $X$   $Y$  Again by the same uh reasoning so uh in total we see that  $R$  can be resolved by these guys but each of these is clearly uh but each of these is solid over  $ZT$  because it's a limit it's a co-limit of limits of things which are solid over  $ZT$  this is solid over  $ZT$  because it's discreet um and then this is a product and that's a direct sum so in total it's it's solid over  $ZT$  okay uh so I didn't directly use the definition of solid instead I used the description of these generators so um right now how about showing that our uh the topologically nil potent elements are all power bounded um you can use a very similar argument so what we have is a map from uh power series  $X$  to  $R$  which sends  $X$  to  $f$  um which means that  $R$  becomes a  $Z$  power series  $X$  module uh so now if we resolve  $R$  by direct sums of products of copies of  $Z$  uh uh we're allowed to tensor this with uh  $Z$  power series  $T$  but we know how to compute this tensor product they're both just products of copies of  $Z$  so this is direct sum of uh product copies of  $Z$  power series  $T$  and this is also solid over  $ZT$  all um so we conclude that  $R$  is solid over  $ZX$  oh t i I switched to  $T$  somehow switch to  $X$  somehow so so we conclude that this composition uh  $R$  is actually solid as as a  $ZT$  module oh I didn't prove oh I forgot to prove it's integrally closed oh I'm sorry uh uh oh let's let's do that uh so so it's again a very similar argument so let's say that we have a an equation of the form uh um so where all  $C_i$  are power bounded um so again we just make the universal things we have  $Z$  bracket  $x_0$  up to  $x_n$  minus one and over that we have the ring where you adjoin another variable uh let's call it  $t$  and then you set the equation uh uh yeah and we have our solid ring  $R$  and by hypothesis we have a map here such that when we compose to here uh  $R$  becomes a solid module over each of the variables and what we want to show is that when you compose here uh it becomes a solid module over this variable so uh I'll just say it quickly in words you use the same trick so you resolve  $R$  first as just a  $Zx_0$  up to  $x_n$  minus one module in solid Aon groups and you solidify with respect to each of the variables you find yourself built out of product of copies of this ring but you're also a module over this so you can tensor up to this but that's a finite free module over that ring so then that will just go inside the products and you find that you're resolved by a direct sum of products of copies of these guys and then because this individually is solid and solid is closed under limits and colimits you deduce that that's solid as well so the key here is just that this is finite as a module over this so that tensoring with it you can bring inside the product finite free yeah although well that's not really necessary the Rings no Theory and you can resolve yeah finite would be enough in fact yeah you mean finite with all with representation by finite well it's an ethereum ring I mean yeah yeah okay um so uh oh and the rest so yeah I mean I'll leave the rest to you it's completely analogous arguments like why uh why it's an ideal and why it's a why it's a radical ideal even um so fun exercises in that style of argument so um so now we can describe this structure sheath uh so now suppose we have a again a solid ring and then we have  $G$  and  $F_1$   $G$  comma  $fub_1$  comma  $FN$  in  $R$  solid um then the claim is that well so now yeah so there exists a universal or an initial solid ring uh so and I'll write I want to just I don't want to use exactly the same notation as who so I'll decorate it with a solid in the super stri script um with a map from  $R$  uh such that uh so first of all  $G$  becomes invertible uh in there and the second thing is that uh  $f_i$  over  $G$  is power bounded uh in  $R$  uh for all  $R$  so that kind of uh encodes the idea I was talking about with you want  $f_i$  over  $G$  to go to the closed unit dis you want  $G_i$  to be invertible um so the proof is uh you can just construct the guy so um you can start with so well you could first invert  $G$  but I'd rather maybe I'll do invert  $g$  at the end so what you can do is you can take  $R$  and adjoin just polynomial variables  $X_1$  to  $x_n$  and then solidify with respect to all of them which recall um does something when  $R$  is not discreet remember we had the example of one variable and  $R$  was  $QP$  then this gave us the

Tate algebra um uh and then we can say that these variables are supposed to be fi over G so maybe uh GX IUS F so G gx1 minus fub1 uh gx2 - F2 gxn minus FN and then it doesn't matter at which point you invert G but let's do it at the end and um this kind of obviously satisfies the correct Universal Property so we first we freely adjoin solid variables uh then we impose the relations which guarantee that kind of thing there's one one small thing to check which is that after you take this in which these are definitely solid and then you do these operations you need to see it's still solid but that's because it's all Co limits um okay uh so um so this is the kind of thing you're looking at um and now I want to make a caution here uh yeah know exactly so just a sec so this is this is not exactly the value of the structure chief on uh  $X_{F1}^{FN}$  over G it is  $\pi_0$  of the value of the structure so you have AED yes so I said that this I when I remember when I was discussing this six funter formalism and this localizing the category I was always careful to say it was the derived category and in fact these things really only work uh these six funter formalisms and so on really only work at the derived level and what that means is in what you're producing a prior is a derived chath not an ordinary chath um so so that's uh okay but but now I'll say but in in most practical cases almost all practical cases uh all  $\pi_i$  equals  $z$  for  $i$  bigger than zero so in practice it doesn't seem to cause trouble but it's an important thing to keep in mind and the second warning the second warning is a so even if  $R$  is an is very nice some kind of Huber ring uh then uh this quotient uh may not be uh so this uh this may not be uh quasi separated so um okay so it's a kind of a nonous of quo of of restricted form of Series in gender yes exactly so this this will always be some this will always be the usual Tate algebra and many generators that follows basically from the arguments we gave for any sort of Huber ring  $R$  but then when you take this quotient the ideal generated by these elements might not be closed so in principle you could be having a non-house dwarf quotient here um so you start when you say Huber ring you mean complete hu yes I mean complete Huber ring thank you yes yes um but again but this theory is defined for complete ubering as well or no this theory is only defined for complete Huber Rings the theory I'm discussing because you cannot Define the condensed of any topological thing you can Define it but it won't be solid unless unless the thing is complete in general ah okay you wanton Sol it so you won't complete and then the other guys when you do this it is still solid but doesn't come from it's kind of nonous of yes yes yeah but again uh in almost all practical cases or maybe all practical cases uh it is quasi separated I.E house dwarf um so and then so so what I was trying to aim for is the relation to hu's Theory so so I but I I so I Pro I guess I probably won't get there uh so if so I think we'll probably we'll definitely discuss this more detail later but just as a preview so if  $R$  is a a Huber ring and don't worry if you don't know what that is it's just certain nice kind of topological ring that people use in non archimedean analysis ah and I let say complete h no just hubber I'm just doing the Rings now I'm not not doing the pairs um and if uh so and if the ideal generated by  $F1$  up to  $FN$  uh inside  $R$  is open which is the condition that describes rational opens in the space of continuous valuations as opposed to the space of AR arbitrary valuations then in this case hubber defines uh uh the Ring of functions here and you could ask what the relationship is with this thing that comes from the solid Theory um and this is the uh quasi separ ification uh of this more generally defined thing that we have here so the solid thing or maybe for emphasis I should maybe put the  $\pi_Z$  as well so you can get the Huber thing functorially from this more uh General thing in particular from our structure Chief but they're not necessarily equal in general but in all practical cases sort of they are equal that's the the general outline of the story so General okay is not and so so in general if you always use the structure shft do always satisfies the yes so it's always shy yes yeah so and yeah so maybe that's an important point to mention there was this little flying the ointment in Huber's theory that for in the general setting uh he had he defined a structure sheath except it wasn't a sheath it was only a prief and in all all practical cases it was a sheath but still it was a little bit the general theory was kind of missing something something for that reason um that's fixed by this so if you don't if you no longer care about things being quasy separated and non-derived then you get a a good a good plain old structure sheath and you get even more you get derived category of quasy gent she which localizes and also you you get the possibility of um uh of of defining all of these things even without this condition being present um okay I think probably I'll stop there thank you for your attention yes do we have any use of don't continuous varations any use yes uh uh I know Coles likes them so he he came he has this has this notion of AD o spaces and and those correspond to certain yeah but you don't really use them um I think he used them but you oh me well look I I just build the theory I don't he calls him ad hoc spaces I don't think he developed the theory he just kind of did it in an example um so he wanted to he wanted to exactly include things like this open unit disc kind of uh so this yeah things like the  $z_p$  I think in his

paper where he does this  $GL_2 \bmod P$  Langlands uh is that right yeah I don't I don't know so so he wants to this this thing is a prototypical example of a topological ring that's not a Huber ring so um so in the usual theories of rigid analytic geometry the open unit disc you can't think of it as an adic space because the ring that it corresponds to is not one of your allowed rings so they always so you always have to think of the open unit disc as a union of the closed unit discs contained in it it's this non- quasi compact thing to fit it inside Huber's Theory but Coates Advocates that sometimes it's better to just work with this thing as if it were adic and that's something that our Theory easily accommodates yeah there are people who who did I forgot the name of this and construct some notion of rigid I forgot the name some draw some paper on this on like variant of three I about the name of the there was some paper some time ago but not sure okay any other questions yes yeses every Sol  $z$  algebra uh yes so you're yeah so this this notion of analytic ring we haven't defined it yet it kind of organizes a lot of discussion but indeed yeah if you have a if you have a solid if if you have algebra in solid  $Z$  modules you get what we call an induced analytic ring structure so it's just all you take all I mean I said the module category you have and that that sits inside condensed our modules and that is an analytic ring yeah but in case of  $Z$  we had something more natural more natural it's arguable it's more complete uh there're two things they exist they do I mean they you want them both I think and can be extended can be extended to any discrete ring right yes that's right not in general not not General  $Z$  solid  $Z$  algebra well for a general solid  $Z$  Algebra I don't know of any necessarily know of any like completely canonical well like maybe I mean one thing you could do is you could take all of the Power bounded elements and force all of them to be solid that would be kind of the maximally complete thing that you can get via the stuff that we've developed Ved but there could be there could be further completions of that as far as I know I mean I uh you're welcome yes with the definition of solid anal Rings we used in the very first lecture if we use this instead of a regular ring here do we get something similar to like Huber's theory for Huber pairs instead of just regular could you could you repeat the question I'm not not sure I understood in the very first lecture we find like solid analytic ring solid analytic Rings yes yes yes with the ring and the direct category uh if we try applying something similar here to analytic Rings what do you mean what do you mean ah you mean like this discussion of this this thing here oh yeah yeah yeah you can do that yeah but do we recover like oh yeah you you recover the theory of with with the  $r_+$  in there as well yeah so elements  $R$  plus you require solid solidity exactly exactly not for all of the guys exactly exactly so you can you can you can yeah and we will discuss this you get you can add that extra flexibility into the picture yeah and the fact that  $R_{\geq 0}$  is this is automatic that those are the to they're automatically solid exactly yes okay yeah so it actually it actually fits remarkably well with Huber's Theory a Pretti like you take tensor product of analytic  $R$  you have to first like solid one another have to take yeah yeah usually yeah so here it still the same here you don't have well these all these solidifications all all commute with each other so there you don't need to yeah they're all given by  $R$  homing out of out of something and any two  $R$  homing out of commute with each other yeah other questions okay well see you on Friday or next Wednesday or whenever



## 8. HUBER PAIRS AND ANALYTIC RINGS (SCHOLZE)

[https://www.youtube.com/watch?v=dIwBTJNN7a0&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=dIwBTJNN7a0&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

Right so last time Dustin started talking about um like the relative solid Theory and uh the relation to tic spaces and so uh I want to kind of continue with that um okay so I guess I want to talk about the relation between these basic objects that appear in Hub Ser that are called hu pairs uh and the kind of basic objects that appear in a analytic ring uh right um so motivation uh but now we've seen we've seen several examples of a pair of um likeing a a present notion of complete now then same so uh the first one we discussed was the integers and solid or should stress that it's some one over  $z$  uh solid beinged inside of all condenser beinged but then last time Dustin discussed the example where you take the polynomial algebra over the join  $T$  and then within all condens zint modules um well you can somehow just take the ones that are complete in the sense of this one so you can to taking modules over this uh algebra inside of solid groups uh but then Dustin argued that it's actually quite natural to uh also isolate a stronger condition giving the solid  $C$  joint  $T$  modules um which where geometrically speaking um well this corresponds maybe to some kind of line and this corresponds to those unit and all I mean in these cases the condens thing was actually just the scre but uh this motion of completeness is still interesting you could also I don't know take I know VP or QP or whatever and then takes follow GP modules or Tak QP follow QP module um and me in those case these are actually just the ones that are with the underlying condens group is solid there's no me meanful way strs on this um okay so uh so notion of analytic ring exor this situation um and uh on the other hand uh um yeah on the other if you learn the sertic St um then you uh run to this definition I mean the basic objects there are these hoirs so let me recall what these are um these are ho although ho of course use different names um a mover ring a topological ring um let me also just pause a now um uh a that itms an open sub a zero then um such that there exist some Finly generated idea and because now it's not clear what IDE means because there are two ring in place is an ideal for I zero um uh subset a  $z$  has the icy um Let me give examples in just a second let me just uh finish the definitions um and then who this notion of a ring of integral elements this is an open in a Hu ring a is an open and integrally closed sub range  $A+$  uh contain actually in the power bom elements um second um and third uh who by par the pair well a plus uh ho range a anding okay so so what are the examples to keep in mind um let me first give some stupid example uh any discreet ring is wor so in this case you can take a  $z$  to be  $a$  and  $t$  to be  $z$  uh um when ring that itself has has the is anic ring for fin generated DUI ring with the I or some generated IDE so in this case you take a  $z$  to  $B A$  and I I um and maybe like actually interesting is the when have something like B algebra um so I don't know so example QP is super um or any non field SCU and in this case you take a  $z$  to be VP and I generated by which really is only an ideal in  $z_p$  and not in  $C$  so basically the idea is that Uber rings are basically certain kinds of localizations or something exist of of such AR rings with some itic topology um and so also whenever you have any kind of boff algebra over QP or some other nonan local Feld there's also always Uber Rank and as a Zer you can some take the unit ball in your B algebra and as the idea you can take the one that comes from a uniformer and you um uh remark is that um the completion of any hu Rings again Uber and we will generally only consider examples completion now in the classical sense uh like allow all converging to sequences like sequen mod no sequen um is again and in fact if you if you start um with uh such called R of definition so one which has the I topology for some fin generated IDE ey uh then it will be the case that the completion of a  $z$  will be an open suffering of the completion of  $a$  and this is actually just the tion of and uh at least when when one uses hu R rings and hu pairs to do addic spaces then the TIC spaces associated with a Hing and this completion are definitionally the same um so in this sense it's non-complete h rings have most a technical role in the whole the so uh and we will only consider complete once so from the following whenever I say h ring PA and so on I always assume that's the underlying H ring is conf all right okay and so uh here last time Dustin discussed some Notions of uh like toing new elements power Bon elements and so on uh this can also be defined uh in here so so if you have a uing um then you can Define element and power element in a um this is the set of all  $F$  and a subet FN goes to zero and goes to Infinity which once I assumed it's complete is really just a condition right now um and these are all the elements such that the setup its powers uh it's bounded um it's actually problem to saying that it's contain some such a zero some such thing definition actually a different way to think about this said uh um first of all I me you have these canonically defined these are kind of canonical objects this

a St and a double St um but on the other hand we all these things uh appeared in the definition that there is some uh so ring of definition uh and some are here of definition of same and these will always consist of elements and the ideal consist of to elements all a um we have this so for example example of pp uh there zp is actually the  $A_3$  and the p is actually the so in this case these kind of inclusions are uh equalities um of course this can't in general be true because you could also as ideal of definition here take it's the ideal generated by p squ and but then you don't have all elements um but in some sense this as better gets in fact the collection of all such a z and the collection of all such eyes they form filtered Collections and a is actually uh actually CL Co Li of all possible such a Zer and uh also I sorry and similarly uh these paic NE elements Al the F exp of all of defition Ring of definition of a all right this was part uh one of the definition together with a little bit of discussion part two was that oh no sorry there there is no I wanted to Define I wanted to say that if I also have a ring of an elements and I can define an  $A_+$  but it's kind of wake is because sorry okay so uh when learn Ser exp um I at first I think it's extremely hard to appreciate the significance of this ring of integral elements  $A_+$  it is somewhat necessary for H to set up a theory but it's kind of hard to feel why it's necessary um and uh and but it actually turns out that uh uh the series that we develop using the solid modules gives you a very good understanding of what it actually does Nam precisely uh ma right okay here's an example um I actually several possible of I just this in this case is just integr for example you could take just instead of own or you could take four thing and in our series sum uses and Associate to this analytic ring with of modules it's some only complete with respect to Z and to this you will associate um the relative and so the role that this  $A_+$  will play is precisely the one of sing out which modules to be considered as some f all right so so maybe I should give this definition of Analytics great uh oh by the way um sorry maybe I can make from notation um Hoover uses the single letter a uh to the note the ru pair consisting of some topological ring and a ring of integral elements and uh we will follow him follow the uh so also when I discuss analytic rings I want to use single letters to denote my analytic rings but then they will have an an underlying well condens ring that St and we needed some symbol to denote that underlying condens ring and we didn't come up with anything good so we chose to just follow Hoover's lead okay so uh let's say a triangle now some lightens and then I want to say what is an analytic ring structure on this thing so what is an allowed class complete modules um okay here's the definition it's equivalent to the one that Dustin gave in the first lecture but presented from a slightly more Elementary perspective okay so it's a full sub category um that the modely ni um so this is the condensed being group and together with a M from T product um the data was just said and now I will make a lot of conditions on this uh but are those conditions uh that we had already seen before uh twice like twice we stated that like solid groups and the solt modules that they have a lot of nice properties and it was a long long list um and sometimes because we don't want to save this list all this time we we make this definition um uh so first of all to be stable under kernels and Co kernels uh but it's also stable in fact under all limits and Co um all extensions so you have an extension of two complete C should also be complete uh uh um then there is a Cy condition uh that you want to check it's also stable under all X I from some  $n \times i$  over triangle um where any anything any um looking something um and contains we want the module talk with something I don't think so so uh can I speak yes okay does it imply maybe you can one can prove from this in some way so for example does it imply that it is a grow in the category where the condition that I allude to is the existence of a set of generators is it automatic under these conditions yes yes yes did you hear his answer over what he said yes he said yes yes ah this is oh and does it imply that the X groups in the subcategories the same as the X in the full or the oh no uh yes I don't I don't know what he said but the answer is yes yeah I know the answer is yes but you didn't hear the I mean in this presentation it's the drive completion might not say in De zero so if you really phrase itly the bilon level you have to be slightly careful when you say that right because it not be the case that like the second I will Define the derived C of a modules and it's not the derived C of being it's not no does he hear what I say yes I hear what you say so you said that the the category yes so the second I will Define a certain triangulated category or stable in whatever um as a full sub category of condensed the of those things yes which is a correct one but in general it will not be the same thing as the drive category of being cat don't say ah so you in this case you you have a you you in this case when you have this full subcategory you say that you have a derived category which is not the dve category of this subcategory yes this is the heart of a t structure ah this is the those whose chromologic groups are in this so it doesn't imply that the X are the same so it doesn't imply does not and sometimes it does sometimes it is true but like in most practical cases it will end up being true fullest generality no okay okay so I can press that yeah press um uh now G put me

uh made me confused so uh so I want to claim that there is automatically left the join to inclusion uh Dustin do I really need need something for this no I think that's fine so what's the argument again for well I mean it's again this General principle uh I think it's not quite trivial but I think it's a general principle that if you have a presentable category and a subcategory closed under limits and sufficiently filtered co- limits in this case all filtered co- limits then um then the inclusion has left the joint so there's a very related statement that involves this SE thetic nonsense as for p principle but I think there you don't have the limits right yeah or no well it's something or maybe you don't have the sufficiently filtered Co limits for something I I believe the the statement that I just mentioned which I believe you also mentioned at some point earlier is just a a fact in zfc right if not then just assume that there is a left to join okay what is the fact that there is it's like if you have a a presentable category and then you have a full subcategory closed under all limits and under all sufficiently filtered co- limits uh then the inclusion has a left ad joint and the category is presentable yes the big Cate no the big Cate only the big category has to be presentable the small one will automatically beat yeah but it is not clear from the stability that it tell up the set of generators no that's right that's it's it's a non-trivial claim yeah it's a non-trivial claim which is true in in and what was this I me X implies something here I think it maybe implies you can leave out sufficiently filtered Co limits or something I don't quote me don't quote me um right um okay end of definition um uh right so okay so so so first I want to explain some general structures of this um so uh first on the idea level so the claim is that uh the excess left includ that I will write as uh sending module  $m$  to the base change from a triangle to a  $um$  uh so so mod  $a$  right now is just purely notationally but I will think of it as the modules over this analytic rinking that I call a  $um$  and so uh it's such a basic  $s$  function uh the kernel of this fun the  $T$  ideal and mod  $a$  requires a unique symmetric structure making that's a base thing um all right uh let's sketch the proof um so we already discussed the question of the assistance of the left joint you think it's formal nonsense if it's not then just make it part of the definition okay um now the question is about that this is a  $t$  IDE so what does it mean to be  $T$  you so so  $s$  joint  $F$  it's definitely this all coits um uh so to show the  $T$  you have to show that if I have something in the kernel and the  $t$  is anything then it's still in the kernel so assume a module a triangle module  $suang$  meaning it has no maps to any a module any um then we want to show um I take the  $s$  zero uh this means set for all mod  $a$  um so home from  $aner$  standard zero because this what this fun by definition left joint um but now using home in Junction this here also the  $H$  uh from from  $M$  into the internal uh from  $n$  to  $l$  but we assume that this guy is still complete by assumption this was one of this assumption that was stable under all all internal likes in particular internal homes uh but now this is mod  $a$  then again because I mean this precisely the condition right also right only mod  $a$  only base stange but this guy uh uh structure so uh uh so the cender product on  $a$  has to be given by uh taking the send of product and uh of see this as cond in modules and then completing again uh the question is whether this uh this makes a fun of symmetrical um and uh again so to check uh for all  $m$  and  $n$  I can first tendler and then uh or I can first keep both of them um WR this a  $s$  product where this was by definition okay now it looks really ugly um but okay only only at the very beginning when you first set up the series to disambiguate all these things um but the point is just that uh um this fun here uh uh this localization about the kernel of this is a  $t$  IDE and so um right I think you don't use the kernel but you just repeat the same time of the type of argument homing this to to Something in the in the subcategory and and using intern I did this in the solid case and you just same argument same let me not try to do this again Yes actually uh I realized that on the level the better thing to say would have been that if I have a map which is turned into an isomorphism under a under under this localization then also if I tender this map with anything else it also becomes an isomorphism and then the which follows from the same argument as as as this one really um and then the point is that for example  $f$  is from  $M$  to its completion which becomes an item of localization because an important operation and so if I  $T$  this within some the same stage it so so this is some structure you automatically have antic range  $C$  of modules some kind of localization of condenses the underlying ring and requires in terms of product um and now we pass  $d$  let not just say that a structure on an underlying lightens string a triangle um then undine the of a modules uh the full sub subset for all  $Z$  uh of all let me still just call them  $n$  so compx of modules um so group let me think homologically um uh all the homo groups line Define for Jesus and okay so here here here's already the warning so there is a natural comparison function from the of mod  $a$  but it's not always and in essentially all I mean basically yeah all Cas I'm aware it will come out to be an equivalence uh but it's just not a general effect um uh but yeah so the good thing we are to focus on is the thing that we simply call  $D$  and so the previous proposition has an analog on dou um so  $e$  of a triangle triang so I mean

probably in one or two lectures we will probably switch to the infinity categorical language um where we would say stable Infinity category instead uh for now it's not really required so let me just TR more classical terms um uh stable under all so again in stable Infinity case we could say stable under all limits and Co limits but General limits and Co limits are not well behaved work Str categories but you can say something equal and stable under all  $s$  and for product um which are well behaved to what they're supposed to do um uh I'm trying to say um right um the inclusion again has left on um that I will call the  $dve$   $S$  the product and again this has a property that if you have something it becomes an isomorphism here then if you  $T$  it with anything else it's still the same and because this is now a triang  $c$  you can actually phrase this equ in terms of the  $co$  so if you have something in the kernel of this uh then you tend up with something  $sa$  the and then again this the  $T$  here so if one wants to do the previous type of argument for this fact then uh one lends into the question of whether the of course there is internal or in the full derived category by unbounded and so on but is the question is whether if you have internal oh from anything to Something in  $D$  of a then it lies in  $D$  of a right and this is not because of unboundedness I don't of course you have a bounded complex you have a spectral sequence I mean you still have to work with that here you have unbounding in both directions I can see in One Direction you have IND light conding is still repetitive so the derived category is left complete and so I think you can control the question okay let me do this in a second when I come to the pro uh  $s$  the product making face Chang all right um uh yeah so so first of all triangulator um so see you have I don't know  $M$  Prime to  $m$   $m$  Prime  $M$  Prime One Prime and there are modules and let's assume two of them and by shifting it doesn't matter which to uh  $R\&D$  of a then we want to show that  $m$  is also to show but for this we just look at long sequence so we have  $hm$  Prime have  $H$  and these are  $a$  so if I uh have some quotient here uh and have some  $Cur$  kernel right and this is a kernel of the thing which a mod is a  $qu$  of a modules so this of both a modules and then this one is an extension right here we use stability on the kernels and cions and then we use stability on extends no I think standing here actually realize okay so uh and the directed to the  $H$  uh and not standing realize that possibility the product uh countable ones they definitely reduce to  $H$  uh and then the uncountable ones that was okay when back when we were working the full condensed setting uh and preparing the lecture I overlooked that there might be an argument to do here Dustin should I just assume that there is a leage on the level of the categories or I'm sorry I was busy with the chat what what's going on why is it stable under all products why is what stable under all products the sub of complete ones oh oh all products oh instead of just countable products right oh ooh oh all prod all products exist but is it exact in your category yeah this is problem problem like lightens all products are not exact yeah this is a problem okay we'll have to think about this uh chip chip chip not an actual issue in some sense but I screwed up to definition so we should ask for the existence of left the  $Z$  I mean Dustin did in the first lecture and I just threw it up when I prepared to a lecture uh uh for existence of left joint inclusion uh the definition of I definitely once he admitted left the  $Jo$  call um sorry uh all right um so now I made this next thing actually part of definition that exist uh so  $Cur$  so let's say  $m$  uh is uh it's complete and then is any con um no sorry uh what I want to show  $n$  isn't the and then there anything um then you have to show that for all the on are complete um Ander and uh and first of all because uh light set what's known as a repeat this means it's countable limits of  $s$  or subjections uh notion that was introduced in my paper the SW but on the pro side um and one thing we saw there is that this implies that uh any such any say sorry for all I don't know I didn't use the letter  $K$  right so for any  $K$  which is for example condensed a triangle module um  $k$  isomorphically to the derived limit of its trations in degrees at North End some kind of of liit usual us being groups I mean it's somewhat true right I mean when you truncate up to some degree and then just well take a limit of these things you're some stabilizing to the correct answer in general that's so an issue because you're taking accountable limit here in general accountable limits NY thing but under this assumption you can control them um so this means that I can certainly assume that  $L$  is bounded here right what I need so first of all um and to show this I can again use tender  $H$  Junction and uh I assume that  $M$  has Ser completion so it's enough suffice is to show that the internal and condense a triangle modules from and sorry it's not CLE because then you can rewrite this as a  $h$  from  $L$  into this guy but I assume that  $M$  has trivial completion so it doesn't have to anything  $D$  okay so this I want to reduce to the level where I kind of had the statement that I something in mod and some anything then all the internal on mod a the issue though as already pointed out is that here we need to ask this condition for all possibly unbounded complexes uh that's why I mentioned this fact so this at least allows us to assume that  $L$  uh here is um so so can assume  $L$  and  $D$  minus or I people put this goes to the right um because also Al trations they are still in

D OFA but because the condition was just on the um on the other hand  $n$  can be written as a CO limit uh of the transations to the left I mean this is always true that there a CO limit ofation and less equal to then this much easier because Co limits for Ed to direct sues are always good um and some some similar you can pull the co out into a limit and because we know that this is stable at least under accountable products all of them I can also see that  $n$  is minus but I say now I'm in business because once this is goes to the left and and this goes to the right everything we usces kind of to uh X um okay sequence L just being uh Z and also  $n$  the condition it's the internal X okay I think that's fine and once you have that uh the existence of the tender product is just the same formal diagram chase that I didn't execute just previously but did solid all right so another thing I should have really mentioned as part of the general thing but didn't and so let me now is that uh Mr Ro tender product ah sorry uh yeah so uh the D of a has a natural key structure so making T EX and uh the the left is not General exct as we've already seen the solid base where solidification could turn something to the left but if uh but still uh this the face change it preserve connector part structure uh T structure uh is that you have a notion of like things which uh have a notion of truncation of complexes kind of so you have a notion of like ration complexes which certain non- negative degrees complex which certain nonpositive degrees they satisfy all the usual properties um we definitionally made this triang sub which is stable under all different right so um so this inclusion is T exact and then it's just a completely general fact that if you have left joint to T exact function at least it preserves the connected part um let me do it check whether this map to anything which is on concentrated on the right but this is a left joint so you can compute the uh Mor pH in in the larger category but then this is still in this category um right and so uh so in particular you can talk about the heart and the Heart All is also definitionally is just a and uh so if you take this and to to the heart this is uh this thing and if you take the temper product and pass to the heart so in this sense the Der and the V level are compatible and then there's the other question whether if you start here and just animate all these construction to R category um whether we recover those but this is just not true in general yeah so in general you don't even recover DFA but even if you do there are separate questions where you Rec recover the correct function and again not General Al uh I think if you do recover the correct categories Ed on also the correct one just by function reality of but the t produ is a sub but again in practice it is true yeah the C of a is just the C and all these funs are all right uh with that out of the way I'm almost done with my lecture unfortunately um so okay the back uh to the comparison um so when we had hoing we had these topolog new elements power and so on um and Dustin already gave a variant of this uh first of all H Rings himself or hubber rings and car um the m to condens rings of course all this is actually fully faceful yeah all all my ones are complete um uh but actually uh this L and Sol I uh on where the underline solid um right maybe I should call this not uh right and so then DUS already last time gave a definition that uh a trial a solid ring we can Define subsets  $A_3$  and  $A_3$  of the underlying R where let me recognition  $A_3$  was set up both elements uh in the underlying ring such that the corresponding map  $t$  uh to Triangle factor I um there was some discussion about how much structure do you need to check here um and so the condition was that it factors as condensed Rings it's actually enough to check that it factors as condensed modules over this this one uh and then saying that there is a sequence no sequence one f f s and so on up to zero uh but together with small condition that F time the sequence of shift of the sequence make it okay and so uh if you apply this to the case where this Sol ring AR arose from a Hu ring then this is precisely the set of to and um elements those def to elements such when you regard AER as a z display the same way that actually becomes and again you can show that this is precisely the same thing as h conion of being power um and so then uh Dustin show last time that this is always an integral St and this here is always an what's the definition uh yeah sorry I mean given f i can again let me write again why this me I can as a and then the condition is I'm already speaking some of modules over see modules over the Clos unit disc some motivated last time um some this means that should be at most one so should be also check okay but now uh I can also make the point so assume  $a$  is an analytic ring structure on a solid  $r$  a triangle then I can also Define an  $A_+$  I realized I didn't didn't find any so let me do this in just a second um such set the map from  $d$  t a same as always steps um um this map uh in use of a map of analytic  $r$  uh from V solid corresponding solid modules um yeah 6 precis so something that I should have said previously uh but forgot what which is pair L strings um and of modules to some V which similarly is aens of modules is a m of condens rings which has a property that if you use this to restrict the condensed B modu to condens a module uh then module say complete restriction of s here and here you have the a module uh and in this case you can pass the left the joints on level the same stat on level of the right cator because they can check it on



the level of mod and once you pass left joints left joint to restrictions of scalars left joint here is what I term the base change functor uh and so you get it also left joint here which is and if you want you can compute it by first base changing as condensed modules and then complete it and you also get a deriv t okay so the claim is that uh first of all once you have such anru can get data as in Uber um and so I will immediately check that this actually automatically satisfies this list of conditions that Uber puts on this re integ elements um uh conversely I not sure if I have time I hope I can say it is whenever you have a ring of integral elements uh in h Theory you can actually produce an analytic ring which some of the initial one in some sense um okay so uh right so first of all I can also rewrite this the Smalling conditions it's all those things such that  $1 - s^*$  which is an endomorphism uh from the d f change from so P was this so P call is always a spech based on the nor signal and we characterized being solid over  $z_t$  by well being solid but this we already asked and that oneus  $t^*$  morphism on this projective generator so uh what's actually to ask is that if I look at this thing here uh an object in G of a then this is an why so uh if if I admit such a map then this already happens here like already here oneus D time shift becomes uh autm of this object like definitionally but on the other hand because this precisely characterize the solid modules here uh you can also show the Congress like if I want to show that I have such I need to show that all complete mod here restrict the complete modules here but being complete precisely meant that if i h oneus t time shift into there becomes an nice moris and so just transl so basically we we like whenever you have any element of your underlying ring uh you can ask this condition that one F time shift become morphism of p and this will defin for you analytic ring structure by General yeah basically whenever you have an endomorphism of your comple project object p uh declaring that this should be an as from all exptic structure so you can take any any subset up of this ring and ask this type of conditions uh so I want to ask a question I just asked DUS but it's not completely clear so since you define the notion of M of analytic just in terms of moduls so if you do this you instead of just the derived tensor L you send to so this P tensor l means that you take the tensor product and you apply the left agent in the derived sense but you can just take the tensor product and apply the left agent on modules H and that so this is what do this means P1 Za without the L I suppose but then so to check this condition you get what I said so if you you try to to check the the induc analytic ring you can translit to this kind of statement but with this Factor on modules but then probably it's equivalent to what you have written if you think about it but in the derived way but I just want to confirm that the two versions are equivalent the one yes right because you can actually detect analytic R structures on the level of modules it's enough to check it on the underlying uh on the indan level that is also true yeah okay so it's equivalent it's equivalent it seems um I'm smaller uh yeah so with with a drive on I would be more confident but I think the argument just sketched this all through so here um uh right so the point is that uh this satisfies basically all the this this subset Al automatically uh satisfies the conditions of so it's always uh contains the toping new Els which are always an open subset Uber Series in particular this open when where it comes from hu um it's always containing the power bonded elements and it's always an integral to St ring it's um so why is that so well f is an A3 then we actually get a power series T about um that's almost yeah if I have a module that's actually Z power series T modu in Solid V that's automatically Peter actually I mean this proof is just exactly the same as yes exactly the same as last time me just because all I used was arguments about modules being solid over one implying they're solid over another and so on yes let me just St again uh okay um so okay so these are actually all there but this means that whenever I have a a module here over over a so in particular becomes a modu here which is everything was solid so it must be an underlying solid braing um uh and so it must be Sol with uh I should um if a plus then I get a map from here joint solid to a um but in particular this means that a the under condensing which we always assume is a complete module uh by restri becuy so f is an a um right so this this group seeing inclusions up there o and uh there for suffing that's the same argument so in fact yeah so the argument that Dustin gave there was already talking in something not just about a triangle but about any module uh and so if you just run his argument to see if this what he actually um okay so right th for Hing a triangle I have the solid anal stru solid instruct one which lives over solid Z and Uber uh consider this motion of the Rings of integral and I just gave you a recipe here that was taking some such analytic bring structure here and produced a ring of inte elements A+ in here and it's actually functorial so you can actually show that one rror is contained of the yeah I mean if you haveing and you get an inclusion of the and uh this actually has a left joint I assume um so whenever I have a ring of so I can produce and musicing stru solid s yes so solid admits and necessary Heat uh from the solid solid is I mean it's Unique because the M of rings is unique and then it's just a condition it's just

a condition that all the  $A$  modules are actually  $Sol$  um and the different way to phrase this is to ask that  $1$  minus fifth as an anamorphism of key  $10$  a on doesn't matter um uh all right so I I wanted to say that it has left design uh so if I have a move up  $PA$  then I can send this toing a asso where mod  $a$  by definition all those condens modules the underlying ring sub for  $f$  and  $A+$  when I take internal home from  $P$  into  $F1$  minus  $F$  shift acting on that and is and because some of this such a classes are all is integrally closed it's actually enough to ask for some kind of generating set whenever you have a subset of  $A+$  that generates it as an open sub ring into elements then it's enough to checked for the subset so usually there's just one or two elements something where really have to check Al the same thing as those modules and only for those two elements uh uh so yeah so to connect spe to the beginning so for example like  $Z$  Comm  $Z$  pair this just follow  $Z$  modules um if I take  $Z$  uh that's the pair so if I only put  $Z$  here then I'm only asking that some solid over  $Z$  so I take all  $T$  modules and solid  $Z$  modules but then when I take then this becomes  $Ro$  for and then of time so let me just State one last proposition so when you start with  $h$  um and then go to uh then if I go back then this actually match back to  $A+$  um so if I right so I have an analytic ring and then I can take its so I can take this this thing here and take its plus ring so this is actually a plus equivalently the left fun flyable um from who thank so all brings but they all and yeah I mean really still quite struck by how how closely uh the series that solid antic Rings really matches hub's classical theories if you restrict to antic Rings where you only ask allow yourself to put conditions that one minus some element shift operator on  $T$  becomes an isomorphism then you're precisely getting those analytic ring structures that are induced by rings of integral El in  $h$  sense so which is kind of very strong a poster motivation for for for this definition of all right I should stop

## 9. LOCALIZATION OF SOLID ANALYTIC RINGS (CLAUSEN)

[https://www.youtube.com/watch?v=1JTLj8gYAtg&list=PLx5f8Ie1FRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=1JTLj8gYAtg&list=PLx5f8Ie1FRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

So welcome back everyone. So today we're going to be doing a little bit of localization in a certain framework in the setting of solid analytic rings. But first I want to start with a recap of last time. So last time we defined the notion of analytic ring and that's a pair  $(R, \mathfrak{t})$  with  $\mathfrak{t}$  a topology on  $R$  and then  $\mathfrak{t}$  is the whole pair being denoted just plain old  $R$  and it consists of an  $R$  with a topology  $\mathfrak{t}$  and what we would want to call the category of  $R$  modules where  $\mathfrak{t}$  is a topology on  $R$  is a condensed string commutative  $\mathfrak{t}$  and this  $\text{mod } R$  is a full subcategory of  $\text{mod } R$  triangle by which I mean condensed  $R$  triangle modules satisfying some axioms some closure axioms and the first one is that  $R$  is closed under all limits colimits and extensions in particular it should be an aelon subcategory but even more because I'm allowing infinite colimits and limits here and the second is a bit funny so if  $m$  is in  $\text{mod } R$  triangle and  $N$  is in  $\text{mod } r$  then we want to require that the internal  $X_S$  in this category  $\mathfrak{t}$  from  $M$  to  $n$  should also lie in the smaller category  $\mathfrak{t}$  for all  $I$  and this condition was necessary to get a good tensor product  $\mathfrak{t}$  on this category and on the derived analog as well. I'll make a couple more remarks about that in a second and then the third condition is that our triangle itself the unit in this category  $\text{mod } R$  triangle should lie in  $\text{mod } R$  what about the existence of our joints which said one so there were a couple of technical points that came up last time that were not fully addressed that I'll take the time to address now. So there's a and one of them was indeed about the whether the existence of a left adjoint is automatic or not and there was a claim that it was and it's justified by this theorem of uh uh ademic rosi they call called reflection principle so if  $\mathcal{C}$  is a presentable category  $\mathfrak{t}$  which used to be called locally presentable but and if  $\mathcal{D}$  is a full subcategory so it's a version of an adjoint functor theorem  $\mathfrak{t}$  closed under all limits now that's the condition under which you might reasonably expect a left adjoint to the inclusion but there are set theoretic technicalities that get in the way in general but if you also  $\mathfrak{t}$  Plus  $\mathfrak{t}$  there exists a regular Cardinal  $\kappa$  such that  $\mathcal{D}$  is closed under all  $\kappa$  filtered  $\text{Co}$  limits  $\mathfrak{t}$  so if  $\kappa$  is the first infinite Cardinal so  $\aleph_1$  if not then this is the just the just  $\kappa$  filtered  $\text{Co}$  limits is just filtered  $\text{Co}$  limits but if you make  $\kappa$  bigger it's a weaker condition that it be closed under  $\kappa$  filtered  $\text{Co}$  limits  $\mathfrak{t}$  so but  $\mathfrak{t}$  then  $\mathcal{D}$  is presentable  $\mathfrak{t}$  and there exists a left adjoint okay so this is nice inclusion okay the was I'm not about the Der the categ and the infinity category version  $\mathfrak{t}$  is proved by by ronov schlunk right  $\mathfrak{t}$  let me make sure I got the name completely correct  $\mathfrak{t}$  yeah ragimov I'm so sorry ragimov I'm so sorry ragimov schlank yeah but you have to there was some circular thing I see in one of the discussions I don't remember exactly that if you want the reservation under limit that is let say infinite let me make another remark concerning a technical point that came up in the last talk so recall that we were discussing the derived analog of this and in this context of this definition we made the definition  $\mathfrak{t}$  that we Define the derived category of  $r$  as a full subcategory of the derived category of you know  $\text{mod } R$  triangle let's see let's just write  $\text{der } R$  consists of those  $m$  in here such that  $\mathfrak{t}$  on homology you lie in this ailon category  $\text{mod } R$  for all  $I$  and we wanted to show that this derived category satisfies the analogues of all of these conditions in particular these closure properties under limits and colimits and for proving limits the the thing to prove is to prove closure under products and there was a sticking point that came up that we needed  $\mathfrak{t}$  that if  $\mathfrak{t}$  say  $m_\alpha$  and  $a$  is a collection of elements of objects in  $\text{mod } R$  we needed that  $\mathfrak{t}$  if you take if you view them now in the derived category concentrated in degree  $Z$  and you take the product in the derived category then it should satisfy this condition and the subtlety is that infinite products countable infinite products are exact in our setting of light condensed ailon groups but arbitrary infinite products are not so the product functor has right derived functors and you need that this lies in  $\text{mod } R$  for all  $I$   $\mathfrak{t}$  okay but but this can be proved as follows  $\mathfrak{t}$  from the axioms but consider but note  $\mathfrak{t}$  so but Set  $\mathfrak{t}$   $M$  to be the direct sum over  $\alpha$  and  $a$  of  $m_\alpha$  then this guy is a retract of  $r$   $i$  product  $\alpha$  in a  $m$  because termwise it's obviously you know this this system is a retract of of this system here  $\mathfrak{t}$  but this is just the same thing as  $X$  internal  $X$   $I$  from direct sum of copies of our triangle to  $M$  ah okay okay and what about product of infinite complexes unbounded below complexes yeah the argument given in the last lecture I mean this this was treated in the last lecture ah because there are I mean kind of it's POS liit so that you can reduce to countable do this one exactly okay so that's that's and I think also that I don't know maybe you know the reference so if you have in general  $\text{Gren}$  the category

and then the subcategories that closed under colimits, filtered colimits and direct limits but then in the derived category when you consider object cohomology, your lies in the subcategories and this is also present this is this I think can be shown I don't I don't know the reference for this I don't know yeah okay but this is just an elementary thing and you can do it the of complexes and I think this can be used to to give a less I mean once you know this which actually one can formulate it in terms of every complex kind of a limit of small ones with some bound then then this can slide this well yeah I don't think yeah to to this without using the I don't think it's necessary to use this I mean actually you can kind of explicitly construct the left adjoint on the derived level just by taking left derived functors of the um the left adjoint you have on the a priori level well maybe it doesn't quite work like that but you can well yeah I I I'm willing to I'm very much willing to believe that the infinity categorical version can be um can be um avoided anyway let's you can first the actually presentable because you're just that all the  $\text{hi}$  right right right right right yeah category yeah that's that's a good argument yeah because this is not necessarily the DED category of of mod are but what Peter said was that um there's a general principle about like presentable categories being closed under limits and the category of categories as long as you have functors that commute with co-limits and the homology functors commute with co-limits so then you can see that this thing has to be presentable and then the infinity categorical adjoint function theorem the more naive version proved by Lurie would give you the left adjoint okay anyway enough of that right uh let's move on to some real math um okay so and when you did before without light you did everything so is there a close relation between the notions in the light and the general set that is so a condensed ring in the light gives one in the general and subcategory subcategory and so on maybe or it is more Twisted well the the first statement is is completely accurate so light condensed Rings embed fully faithfully into all condensed rings but when it comes to the analytic ring structure it's a little bit more it's a little bit more subtle um yeah because you a priori if you have a light analytic ring in the light sense you've a priori only decided what the free module should be on light condensed sets um okay so uh we also in the previous lecture we also um discussed some examples of analytic ring structures in the solid context so let's say we have a solid uh solid ring so I use this notation uh so it's a condensed ring whose underlying um condensed abelian group happens to be solid um and then we had this uh subset of power bounded elements um and then for any subset  $r_+$  contained in the power bounded elements and maybe I'll say that one lies in the subset um we get an analytic ring structure um on  $R$  uh denoted uh like this our triangle  $R$  plus solid and um well as the as the notation indicates the underlying condensed ring is just our ring our triangle and then the category of modules uh um uh such that if you look at the this space of null sequences in  $M$  um then you have an endomorphism from this to itself given by  $1$  minus shift time  $F$  this should be an isomorphism for all uh  $F$  in  $R$  uh Plus and let me make a remark uh so so the the condition that one is an  $r_+$  ensures that uh every every  $m$  in  $\text{mod } R$  is actually solid is in solid  $Z$  that was actually so if you plug in FAL 1 that was our definition of solid  $z$  um and then uh this condition that  $r_+$  is contained in  $r_0$  that's what gives that uh the ring itself is complete so to speak so I recall that the interpretation of this uh second coordinate in the analytic ring structure was that it's specifying some notion of complete modules adapted to whatever situation and you kind of want the ring itself to be complete otherwise you would complete it um and that's a so that's the condition on power bounded Elements by definition just translates to saying that the the underlying ring itself should be complete which was one of our required axioms um and another remark is that there's no loss of generality uh that uh  $r_+$  uh is an integrally closed subring uh with uh containing all the topologically nilpotent elements um because we saw that um well that if you have a solid analytic ring structure on a solid ring so meaning a analytic ring structure such that every module over it is solid uh as an underlying abelian group then the collection of  $f$  for which this is an isomorphism for all  $M$  uh is actually an integrally closed subring containing all the topologically nilpotent elements so you could always just throw in this these guys take the subring generated by them and take the integral closure and that will not change the theory um okay and then then there was an example uh so if uh if this is a Huber pair so recall that means this is a Huber ring uh or let sorry excuse me sorry I'm going to rephrase um if so so it's a Huber ring um when those things can always be considered as solid condensed Rings uh just by taking the associated condensed ring um then uh so then the integrally closed sub Rings uh are  $r_+$  uh like this are the same thing as as open uh integrally closed sub Rings uh like this and these are exactly uh the the  $r_+$  pluses in in Huber's Theory so the general setup we have for an arbitrary solid ring of the possible choices of  $r_+$  when you specialize to the case of a Huber ring it recovers exactly the the choices of  $r_+$  you have in Huber's Theory complete Huber ring yes thank you yes I let me say that from now

on whenever I say Huber ring I'll mean complete Huber ring unless otherwise specified so so there is also sometimes people use SoCal derived comp complete things and if you have a derived complete thing it also seems to give a condensed thing and then it's also possible to say that this is also a solid yes yes Yes actually I think I might I was I was considering talking about that there's a fun story with that that maybe I'll talk about it later in the lecture yeah yeah but yes the answer is yes um uh right uh and moreover in this case the  $r_+$  is actually recovered from recovered by the analytic ring is it well it's basically equivalent to the you know yeah so the so hubber is  $r_+$  or I should say  $R_{\triangle} r_+$  is exactly the same thing as this solid analytic ring structure so in the general case I didn't quite claim that uh if you start with an  $I$  didn't claim and it's probably probably not true that if you start with an integrally closed subg satisfying these conditions and you form that theory there might for some other reasons also be other things that  $F$  that satisfy this property for all your  $M$  possibly but in the Huber case you can show that that no there aren't not we did not do it or we did do well maybe we didn't do it last lecture but it was done in a previous lecture so the the argument is in a is in the the analytic PDF okay yeah maybe the second to last lecture I think yeah okay questions so now we're going to discuss uh localization so um so let me make a remark so localization so I'll start with a remark which might be a bit uh shocking at first glance but it's actually trivial um so if  $R$  is a  $r$  is a solid ring so and then we have this  $r_+$  satisfying these conditions and again you can feel free to assume it's an integrally closed subring containing the topologically no potent elements um note that this condition defining the analytic ring structure is just a condition that you're imposing for all  $F$  in  $r_+$  and  $R$  plus was by definition a subset of the underlying discret ring  $R$  so so um all the the data that you're using to define the analytic ring structure actually already appears at the discret level so uh then get another pair uh just with the same  $r_+$  and the power bounded elements in the discret case are just uh are just all the elements so certainly it's still going to be power bounded inside there um and this argument shows that uh or this observation shows that if you you're interested in solid modules over your original  $RR$  plus um uh sorry  $RR$  plus solid well you can take the ones over where you have a discret ring uh and then that already has all of the uh the information about the analytic ring structure and All That Remains is to observe that  $R$  will be a commutative algebra object in here and you just kind of abstractly take our modules uh in this ailon category so it's important to note that since we're doing condensed modules even when you have a discrete ring you have a huge amount of new modules besides the discret modules you can consider and in particular over this discrete condensed ring you have  $R$  the honest condensed ring and the theory for an arbitrary  $R$  is actually base changed from the discret case in this completely naive way okay um so we're going to discuss localization so how kind of uh how these categories glue but it's actually going to be sufficient to treat the discret case because if you understand how this category glues then you could just put the  $r$  module structure on top of that and you'll understand how this category glues um and I want to stress from the beginning that I'm talking just about one kind of example of gluing I'm not claiming this is the most General um but it is a it is nonetheless quite General um but it's just a certain framework for gluing you can call it um so now uh let me make an analogy uh so well so we're going to be in the world of discret rings now but uh so if  $R$  is a commutative ring uh then we have its usual derived category of  $R$  modules So This Is Us of and I'll for EMP emphasis I'll put discrete  $R$  modules um this localizes on over uh the  $Z$  risky spectrum of  $R$  and I'll I'll say more precisely what that means in a second but just uh and um uh and what is this  $\text{Spec } R$  well it's the yeah so  $\text{Spec } R$  are the set of prime ideals um and the there's a basis of Quasi compact opens closed under finite intersection these are the uh the so-called distinguished opens something called principal principal uh principal opens well it depends on I don't know I've seen the principal in some references but it is not I'm not sure what it what do you prefer I don't know because uh actually myself I didn't remember I remember sometimes I said basic open because I did not know but then I saw in some text books it was principle but I don't know how standard it is are you okay with distinguished open okay okay uh distinguished opens so let's say  $UF$  those are well set of prime ideals  $P$  such that  $f$  is not in  $P$  so  $F$  maps to something nonzero in the residue field um and um there's a structure Chief so and it's its value on this distinguished open uh is so this ring  $R_1$  over  $F$  localize at  $F$  and it's kind of important to note that the neither  $UF$  nor  $R_1$  over  $F$  kind of determine  $F$  right so but they they do determine each other um uh and and in fact uh this  $UF$  actually gets identified with spec uh of  $R_1$  over  $F$  and this matches up uh up distinguished opens um and then um so if if this is an inclusion of distinguished opens uh Sor well sorry sorry sorry sorry um so if  $U$  distinguished open uh to this we can assign uh the derived category of the value of the structure sheaf on  $U$  um and if  $U$  is if  $V$  is contained in  $U$  then you get a base change



functor and then the theorem uh completely classical I guess uh is that um this sheaf or this pre-sheaf uh is a sheaf no that's true yeah so it's a sheaf of infinity categories um so well in in this setting you also have a sheaf of a billion categories I could have told the same story with the mod the usual yeah yeah yeah yeah I mean yeah okay this is but also there is a certainty of a sheaf of hypercover versus check true in both senses but yes that's right so this is proved in in Lurie somewhere or where is it proved I assume it's proved in Lurie somewhere yeah uh but but people sometimes don't know EXACTLY okay it's kind of in this language I'm sure it's uh it's lousy yeah so um I'll explain the argument in a second but um right so let me so in yeah in this case we also have a sheaf on the level of a billion categories and even more as Gabber says a hypersheaf but uh never mind that um I just want to warn you that in the setting I'm about to discuss just to discuss here we will only get a sheaf on the level of derived categories we won't get a sheaf on the level of a billion categories and the basic problem uh is one that uh came up in my my previous lecture so uh in this situation the localization maps are flat so  $O_U$  going to  $O_V$  uh is a flat map well it's just a localization and so if if you have a derived statement then uh using the flatness of localization it's not too difficult to deduce an aelian statement but in the setting I'm about to discuss um these localization maps will not in general be flat and I kind of uh so one example of such a localization map is going to be this localization from the affine line to the closed unit disc which was this t solidification or ZT solidification and I already mentioned that it's not exactly T exact there's a discrepancy by one and that AB actually obstructs uh the um uh the chief condition on the aelian level this closely related statements so one statement is that the derived category of a ring is the same as the Quasi-coherent derived category of spec of the ring and I think in without boundedness is proved in prob in the tax project okay and then there is another statement about the shift property which I I also don't know the reference but I think that people that it is this line of Lurie see that you have if you have any ring topos then you can associate to every object in the site or the top most the category of uh of this uh  $O_U$  and this should also be a hypersheaf but is it what do is it so this this is in Lurie also or so first of all what I said should be correct I think is it I'm sorry I'm I'm my brain is a little not working very well right now I actually I actually zoned out while you were talking my apologies any toos and you have any object  $V$  the right category of the topos restricted to you  $V$  in the sheaf yes then this is a hypersheaf hypersheaf no Chief yeah hyp and also hypersheaf well why why would it automatically be a hypersheaf no no I think I was able to to do it in some classical more classical formulation but it's it's a is it so what do you know about this statement uh oh yeah the usual derived category I maybe it is maybe it is a hypersheaf yeah okay I don't know I don't know yeah you don't know the ref I I mean I don't know the statement it's I guess now that I think about it it sounds plausible but I mean the hypersheaf but there's a certainly lousy proof it's a sheaf yeah and hypersheaf maybe I I don't think proves it in one of his books yeah I'm sure yeah there is also something that I saw that you men I mean in some text that they found on the internet instead of looking at the derived category you can look at functors infinity functors or shifts this is the case in the anabelian group on the yeah and you claim that this is the same as the D category of the lousy hypersheaves hypersheaves yeah if you do hypersheaves with values in  $D$  of a billion groups that's the same thing as the derived category of the category of sheaves of a billion groups yeah and this is also proved in in I assume it's somewhere in Lurie yeah now uh I'd like to move on so maybe we have this discussion at another time um okay so that's uh one part of the analogy kind of uh and the second part is uh so now we have uh  $R$   $r_+$  a discrete Huber pair so that just means  $R$  is an ordinary commutative ring and  $r_+$  is an integrally closed subring um well then we've assigned to this uh this  $D(R, r_+)$  solid um and the claim is that this localizes on something else on the valuative spectrum of this pair  $R, r_+$  um okay so what is this so so that was the set of prime ideals and the the kind of purpose of a prime ideal in this setting is to let you know where functions vanish or don't vanish so kind of you could think of it that way so kind of a binary condition of whether you're zero or nonzero and in the valuative spec you are allowed some more refined information not just information about whether a given function vanishes or doesn't but given two functions you can ask whether one is bigger than the other uh um and the the way you can measure that is by means of evaluation so so it's a a function from  $R$  to  $\Gamma(U, \mathcal{O}_U)$  so this is an aelian group written multiplicatively the actively um and then there are axioms so multiplicativity  $vfg$  equals  $VF \cdot VG$  and there's the obvious rule about  $0 \cdot$  anything equals  $0$  um there is the non-archimedean condition  $VF$  plus  $G$  uh is less than or equal to maximum of  $VF$   $VG$  ordered thank you thank you ordered but the order is reversed when you pass multiplicative yes yes yes yes yes but thank thank you yes um uh  $V$  of  $0$  equals  $0$  and  $V$  of  $1$  equals  $1$  and then we involve the subring  $r_+$  so we ask that all of these uh kind of be integral with respect to the valuation um so may  $VF$  for all  $f$

in  $r+$  and then there's an equivalence relation on valuations because you could always for example enlarge the order to be in group in some arbitrary way without really changing what's going on and one way to describe this equivalence relation is uh that so uh so  $V$  is equivalent to  $W$  if and only if for all  $f$  and  $g$  and  $R$  we have  $V$  of  $f$  less than or equal to  $V$  of  $G$  if and only if  $W$  of  $f$  less than or equal to  $W$  of  $G$  so um so in other words that what's really important about evaluation is this binary relation on functions which is how you testing whether one is bigger than the other is it the same in this case is the same as Spa in the sense oh it's also Spa yeah yeah yeah um yeah because the continuity condition is vacuous when the the ring is discreet yeah um okay that's a bit of a mouthful if you've never seen it before there's another perspective on these things that maybe explains exactly in what sense it's bigger than Spec  $R$  is um one Val this is also the same thing as a uh just a pair of a prime ideal and then um for uh evaluation sub ring so evaluation domain in the residue field so you have the information which records whether your function vanishes or not and then you have this extra information saying when uh a function or a fraction of functions uh where the denominator doesn't vanish uh should be less than or equal to one basically and that's the same thing as uh as this yeah thank you yes yes yes yes yeah yeah uh oh you yeah um containing  $A+$  thank you yes containing  $r+$  the IM plus yeah thank you yes plus uh right okay um so the the point here is that we now we have a much bigger category and there's more flexibility for how to localize and um it connects with this classical discussion of valuations so if you've never seen this before then uh you know you can look for example at uh I don't know like the rational numbers or something then maybe it's maybe you know the classification of valuations there's the trivial valuation um which I guess corresponds to equality here for every Prime ideal you have the trivial valuation where it's zero if your element is zero and one uh zero if your element lies in the prime ideal and and one otherwise but then also for every prime prime  $P$  you have a petic valuation so you have generic point of spec  $Z$  you have the special points of spec  $Z$  but then you have you have these things in between which kind of are nearby  $P$  but not equal to  $P$  these ptic valuations um uh oh sorry I was talking about  $Z$  not  $Q$  um and then but then the fact that you can classify those is kind of um is a little bit misleading because once you add an extra variable then all of a sudden uh things explode and there's many different kinds of valuations basically because you in a surface you can have lots of different kinds of Curves passing through a given point and uh you have valuations of so-called higher rank which introdu additional complications into the theory um so I'm not going to go too much into into this but yeah so I'll I'll stick to mostly formal aspects for now um okay so let's continue the table of analogies so um so there we had Spec  $R$  and we had this particularly nice basis for the topology quasi compact closed under finite intersections and each of them was also of the same form as the global guy just for a different input datum so  $R_1$  over  $F$  and we have the same thing here so we have a basis of Quasi compact opens uh closed under finite intersection and these are called the rational opens in this case and they're uh they dep depend on the choice of some elements in your ring so you choose arbitr  $F_1$   $F_N$  and  $G$  inside your ring then you can form this thing and what is it it's the set of those valuations  $V$  uh satisfying all of these conditions such that moreover so  $V$  of  $G$  is non zero and  $V$  of  $f_i$  is less than or equal to  $V$  of  $G$  for all  $i$  so in some sense it lives inside the zisy the distinguished open the zisy open given by just deciding  $G$  should be non zero and then we use this extra flexibility of we can also impose inequalities so we're shrinking this risky open a little bit using some inequalities and we still get an open subset um okay continuing um so uh so there's a struct there there actually there's a structure sheath but actually there are structure sheaves so uh on this  $F_1$   $F_N$  over  $G$  uh you have one thing which just takes the algebra risky localization um but then you also get a choice of integral elements um and that you get by it's going to be has it's going to be a sub ring of here and you get it by taking uh the integral elements you had before or rather their image in there and then adjoint and then looking also at these elements  $F_1$  over  $G$   $F_N$  over  $G$  and then that might not be integrally closed so you take the integral closure so basically you just uh just look at all of the elements all of the elements which the valuations in your open subset think should be less than or equal to one so you've kind of already have it for this by Fiat and you forced it for these and then the collection of those things is an integrally closed sub ring so um you have it for all of those guys um and then again you have this nice recursive property that uh  $U$  of  $fub_1$   $F_N$  over  $G$  is just the same thing as the valuative spectrum of  $o_u$  o plus  $u$  and this matches up rational opens and here is another place you can see the kind of necessity of including the data of this  $r+$  in the general Theory because if you you I mean you could have said okay well I I want a bigger space I'll leave out this condition why should I ask for it but then you define these rational open subsets and they will no longer be of the form  $SPV R$  for some ring  $R$  because you've forced certain elements these elements  $f_i$  over

$G$  to be less than or equal to one and the abstract ring  $R_1$  over  $G$  doesn't remember that so if you want this condition that the  $r$  opens themselves should be of the same form as the global object then it's absolutely necessary to include this extra data of  $r_+$  from the start so of course this okay implicitly so this imp that when Russ Russ including the other you say that this  $M_s$  this but what what the statement matches up Russi what do you mean there I mean that the so there's a a continuous map from uh this to this and uh it induces a bje the pullback say induces a bje between rational opens in here which are contained in this open subset and rational opens in here okay no this open  $R_1$  open rational open okay remember okay yeah okay okay um and then uh the theorem uh ah well no I should say again okay so u a rational open to that we can attach uh  $D$  of  $o_u$   $o$  plus  $u$  solid uh let me I'll put this um and then if  $U$  is contained in  $V$  inclusion uh then we get the pullback map well there's a in fact there's a it comes from a map of analytic Rings uh from uh  $o_{u,u} o_u$  plus  $o$  plus  $u$  uh solid to  $o_v$   $o$  plus  $v$  solid in the sense of the previous lecture so we have a map of condensed Rings which is just in this case a map of discrete Rings such that if you have a complete module here then when you restrict scalars it's also complete here so that's the kind of forgetful functor um and then that always has a left adjoint which is this base change functor and explicitly you get it by taking your module here abstractly tensoring up from this ring to this ring and then recompact completing in this Theory here that's the base change functor um and then the theorem oh I've kind of run out of space but maybe I'll put it so uh this pre- that one over there uh is a sheaf of infinity categories and I'll put the warning that this is not true on a bilan level in contrast to classical case these these pullback functors our functors are not t exact in general because the pullback involves a solidification a t solidification which as I said is not a not a flat operation does it bound theological Dimension uh yes it does yeah so it's  $B$  it'll be bounded by uh  $n$  the solidification is bounded by I mean the homology is zero up yeah yeah okay so I think we'll take a 5 minute break um before I get to the proofs proof is it I complete or not um probably not in general but there's an abstract result that if so that if you're space has finite CRA Dimension then hyper completeness is automatic this is sometimes useful so if I start with uh a Sol ring I can take it underline set which in is discret to and we mention it's the same as modu versus thing and is there Cas is it's actually poent of this to algebra and um there uh no I don't think so so the things these tend I mean so for example like I don't know  $Z$  so we we showed the  $Z$  power series  $T$  is item potent over  $z$  polinomial  $t$  but I mean this this Des this discret ring is is going to be way too big I think yeah so I mean like this is not going to be item potent there's no extra reason why this should be I mean I didn't think about it carefully but I would assume the answer is no what's that right right right so let's um so so I've I've stated the theorem um and now I want to uh explain the proof um but uh to motivate it I'll give a a certain proof of this classical theorem here and there so there's many actually in the this classical case there's many different possible arguments um for this um especially because these localizations are flat there's lots of flexibility in how you set things up um but I want to describe a particular argument for this claim here which will uh kind of translate over without too much difficulty to to this case here so but maybe I I erase some boards um I don't know there was maybe one remark that one can make in both settings that I forgot to make uh so I said so I defined a um I Define this prief of infinity categories on this rational opens okay not every open subset is rational they're just a basis for the topology but there's this General result and when you have a a basis for the topology closed under finite intersections that condition being actually necessary in the infinity context um then a she on that basis uniquely extends to a sheaf on the whole space in kind of the naive manner of have an arbitrary open you take limits um yeah so we're only describing this sheaf of categories on the rational opens but after the fact you get also a category attached to an arbitrary open whether or not it's rational okay um so proof of of classical theorem so meaning uh descent zisy descent you could call it uh for  $D$  of  $R$  um so you can start with the so we're we're interested in this uh you could say this this site of uh distinguished opens inside  $\text{Spec } R$  so we have this all with the open cover topology and there's um and so we we kind of have an understanding in of what it means for to have a cover um you know the distinguished opens cover a ring if and cover a spec of a ring if and only if the you know if you're you're inverting some elements and those elements should generate the unit ideal um but you can do a series of reductions actually which will show you that you only need to check the sheaf condition for very specific a very specific example of such a situation so the Dilemma is that this Gro de topology is generated by covers of the following form so you take u specr a distinguished open you take an element of the structure sheaf on you and you form the cover uh which is uh  $U$  of  $F$  and  $U$  of  $1 + f$  covering  $U$  so this is a very simple example of two elements which generate the unit ideal uh inside this ring and the claim is if you want to check something as a sheaf you

only need to check the sheath condition in this one specific situation yeah this was originally in quance proof of the okay it doesn't anyway it's easy but it was there was something of qu and he proved the cell conjecture okay reduces to I mean you want to prove that if you have some Vector bundle on a fine space over a ring which is driven over local ring then it is extended from the ring and it did it by reducing to this and it was a bit trickier yeah kin's a clever guy so let's give the proof um so well I said you know we know you can describe algebraically the covers so if you if you in general the covers would be described like this you take  $F_1$  up in  $F_N$  in in  $O$  of generating the unit ideal so such that there exist  $X_1, \dots, X_N$  in  $O$  of  $F_1$  plus dot dot dot plus  $X_N F_N = 1$  and the general the general cover is the  $U$  of  $f_i$  oh darn it I can't believe I didn't think of that will not be okay because you cannot generate the empty the empty set from non empty things damn it I I can't believe I forgot to check those things yeah uh thank you I should know better by now but um uh plus empty cover of empty set okay checking the sheath condition there just means you check that the value of your sheath on the empty set is the terminal object in the category that is the target of your sheath okay so that usually can be done without much difficulty um okay any is there a question or comment from Bon no okay um but note that uh this cover here is uh but this is but this cover is refined by another cover uh where you take  $f_i * X_i$  so this is a smaller distinguished open um and those still generate the unit ideal because because of the same expression uh so so then we can assume just that  $F_1$  plus dot dot dot plus  $F_N$  is equal to one um and then you can do an induction on  $N$  so you can then induct on  $N$  so let's say for example you had uh  $F_1 + F_2 + F_3 = 1$  and you want to check the chief condition uh then because you're assuming the sheath condition for covers of this form you can localize uh to uh you know  $U$  of  $f_1$  plus  $F_2$  and  $U$  of  $F_3$  and it's enough to check the sheath condition when you do these localizations but when you localize to here then  $f_1$  plus  $F_2$  is equal to a unit and by dividing by the or multiplying by the inverse of the unit you reduce to that case there um and then when  $F_3$  is a unit um uh well there's something similar right uh first there is a statement about generation of the topology ah sorry wait topology containing this and then there is another statement which is probably in L that under some conditions that it's enough to check the shift condition on generators of on coverage to generate topology and the Bas changes yes yes yes course assuming it seems like the intersections exist yes exactly but the yeah I chose these collection to be compatible under base change so that um yeah so I this this collection here is closed under base change so that that is an important technical point when you get into making this argument completely precise but I set it up so that it's true oh yeah and sorry when you restrict your cover to this the cover is split I mean the because this was one of your covering elements so the the chief condition is automatic here and the chief condition here follows by by induction so um that's that's basically the the argument fairly simple um okay so but what about this here um okay so what are we trying to show so in now so checking the sheath condition uh for well I can it's a distinguished open and an element  $F$  but there's no now now there's no loss of generality in assuming  $U$  equals  $\text{Spec } R$  so uh uh um so well so what is the sheath condition in this case so in this case we have uh well we have just two elements and then we have their intersection so the sheath condition says that if you look at the derived category of  $R$  and then the derived category of  $R_1$  over  $F$  the derived category of  $r$  one over  $1$  minus  $f$  um and then the deriv category on their intersection which is you invert both uh this should be a pullback of infinity categories um and now let me make a pause because I didn't spend much time discussing what this notion means like sheath of infinity categories limit of infinity categories and so on but I can make it completely so to speak Elementary in this case of these pullbacks so you get a feel for what what the claim is so yeah so claiming that this is a pullback of infinity categories what does it mean concretely well you have a it means the functor from  $D$  of  $R$  to the pullback category uh  $1$  over  $1$  minus  $F$  uh should be an equivalence of of categories so now how how to think of this Infinity category well so you give yourself an object in the derived category here and an object in the DED category here and then you give yourself extra data of an isomorphism between them here um and that's uh but it's not an it's not it's not an isomorphism in the usual derived category it's an isomorphism in some infinity version so you can imagine for example if this is represented by a complex of projective objects objects this is represented by a complex of projective objects then you'd actually want to give a chain homotopia equivalence between their images there let's I let me say they're bounded above just for Simplicity and then you make an infinity category out of that so you def find some notion of chain homotopy there and so on um uh right so then what does uh exential surjectivity mean it means you can GL glue in the derived category if you have a chain complex here a chain complex here and an explicit identification between them so maybe you choose some quasi isomorphic models and make a chain homotopia equivalence between them then that collection

of data uniquely comes from an element here um up to you know up to quasi isomorphism so the point being that you actually have to specify the data of the chain homotopy equivalence here in order to get the well-defined object there that's the essential surjectivity the fully faithfulness says something else it says that if you have two objects here and you want to know the the homs between them so you can think of calculating  $X$  groups for example so the  $r$  homs between two objects here you can get it by you base change here and you take  $R$  homs you base change here and you take  $R$  homs here and you take  $R$  homs and then you do a homotopy pullback of those complexes for aroms you have there uh which is the same thing as like a shift of a mapping cone of some direct some of these two mapping to that but is it equivalent to like the  $D$  of the diagram that you can say in the a billion level a module over diagram is just giving module over every ring and marks and then not necessarily without then you take the Dory with something posing some conditional mology maybe uh it sounds sounds reasonable but I'm not entirely sure yeah yeah that's certainly not how I think about it but okay um okay so that's kind of how to think about this result it's it lets you glue objects that are defined locally in a driven sense sense but it also lets you do Global if you can do Global  $X$  calculations by localizing um okay uh but how do you now how do you formally prove such a statement so note that so the proof so note each base change fun has a a right a joint uh which is just the forgetful functor so from dve category of  $R_1$  over  $F$  to derived category of  $R$  and then it so that's for each of the individual maps in this diagram but then it actually follows formally that this functor uh also has a right adjoint um uh doeses two and you can explicitly describe what this right adjoint is so it sends if you have a pair so  $M \otimes N$  Alpha so Alpha is an isomorphism so  $m$  is a module here  $N$  is a module here and Alpha is an isomorphism between their common base changes uh then you just take you apply the right adjoints to each of these objects and then you take a limit so you just take  $M$  cross over  $n$  with  $M_1$  over  $F$  which is the same thing as  $n_1 \otimes F$  okay didn't exactly Define those things so it's not yeah you have to open up lurry and find the precise version and but but it works yeah um okay um so the trick to get used to conver by two op this is just to have a EAS EAS diam yes yes yes yes yes in principle you could you could also do this argument without doing the reduction that's correct but it's certainly easier to talk about okay because it's fin many intersection just inter like alter important fin maybe yeah I think uh I think in the um once you get to this the the statement we're trying to prove with the valuative Spectrum then you really probably don't want to um well I don't know maybe you could probably organize it cleverly but I think doing the reductions makes it much easier so um okay right oh so so and this is good news I mean this is the great thing about proving something as a Shea of categories it's like uh you have a you have an automatic candidate for the inverse it's some right adjoint so you have a functor you want to prove his uh an equivalence you have a right adjoint that means you have a unit and a co-unit you need to check our isomorphisms so then you need to check so one of them will be a map in this category and one of them will be a map in this category um so so so for example for the unit uh you need that if  $m$  is in  $D$  of  $R$  uh you need that  $M \otimes M_1$  over  $F$  uh  $m_1$  over  $1 - F$   $M_1 \otimes 1 - F$  this is a pullback so that would be the claim that the unit of the adjunction is a an isomorphism um and um so this so This follows from uh the statement that if uh if you have any element let me call it  $n$  in  $D$  of  $R$  uh such that  $N_1 / f$  equal  $n_1$  over  $1 - F$   $0$  uh then  $n$  equal  $0$  so you want to test something as a homotopy pullback you can measure the difference between this and the homotopy pullback by a mapping cone and that object you can check with this um will satisfy this condition and then you want to conclude that the object itself is zero um and then well this is well this is for for example now you could reduce to the aelion level because an object is zero if and only if its homology is zero and these are localizations that are flat so it's reduces to the same claim on the aelan level which is kind of very easy to check um and then the that was the that was for the unit and the co-unit uh is the it actually reduces to the exact same claim again um but what so um so that's kind of the proof but I want to point out what's uh what's formally used okay so we use it localization by  $F$  somehow and this right adint commute that is I'm going to say it I'm going to say it over um so each uh base change is a localization so the right adjoint is fully faithful um the localizations commute with each other just a sec I'll say what it mean I'll say what it means it means that if you take something which is local with respect to say this operation so something on which one minus  $FX$  inverti and then you invert  $F$  on that it'll oneus  $F$  will still act inverti um these are things which are all just obvious in this context but I want to point out exactly what's being used in this argument um and the third condition was that yeah if  $M$  uh maps to zero on each each element of cover then  $m$  equals  $z$  so if you have a pullback diagram satisfying these uh three properties then and then I mean yeah and then um then you're going to you're going to automatically get this uh I mean a limit I mean a square like this satisfying these



three properties then you're automatically going to get a pullback okay H okay so then the proof of the solid analog you use the  $D_r$  when you invert  $F_1$  minus  $f$  is some of the intersection of the two that is those which are yes yes yes yeah you want to know that when you yeah exactly you want to know that when you pass to right adjoints this thing is just the intersection of those two things as well yeah maybe I should have added that to the list um yeah thanks yeah well well we have to understand it in a subtle way I guess you're right and of course we have to know the language of  $L$  to make it precise yes so it's not doesn't work maybe to well I mean it's a the the wreath joints are fully faithful so it really is just kind of an object-wise condition you could say just yeah yeah um Okay so so what's the analog so so again we have uh the site of rational opens Uh u in this Valu of spectrum um and the gr topology open covers and then we have a Lemma that topology is generated by uh so the m cover of the empty set and for all rational opens and all  $F$  in  $o_r$  uh now we have to take care of two different kinds of covers so uh we have  $U$  uh  $F_1$  uh and  $u_1 f$  cover cover u a geometry that you have those two yeah right so the way you read this is well you read this  $F$  less than or equal to one so  $F$  should be integral and the way you read this is that  $f$  is non zero but even more it's  $f$  is bigger than or equal to one so it's it's well away from zero it's complement of the open unit disc that um and uh you so then uh yeah  $1$  over  $f$  u and  $u$   $1$  over  $1 - F$  so this is again so again this is where  $f$  is non zero and you require  $F$  to be bigger than or equal to one and similarly here so this is actually a refinement of the zariski cover we had previously um where you where you just inverted  $F$  and  $1$  plus  $F$  and that was a cover this is a smaller a refinement of that which still covers and the last thing I think one it's enough to do it when  $f$  is in  $R$  plus okay I don't think that will be helpful but uh it's that's nice to know yeah it's like in read geometry in Tate's original work where he proved a Simplicity theorem yeah he reduces to he didn't have rational domains but he has this uh two two types of covers for which you can prove a cyclicity and it turns out that it generalizes to the a Cas yeah I mean I don't know if um maybe in the Tate setting that uh one can arrange that  $f$  is no and there  $f$  is in you need because you have the you call them  $v$  and  $L$   $v$  is no not not him maybe some other people read it anyway then the  $V$  is just  $F$  in the  $a_z$  i mean in that case a  $z$  is  $A^+$  you know about CL but I think that in Uber that okay then I also wrote some letters to other people some to explain okay um so I will not give the argument for this it's it uses well it's it's just it's a bit more complicated because well the value of spectrum is more complicated than spec but the idea is basically well the idea is somewhat similar you could say but it's actually a somewhat complicated argument so but it's um Uber there is maybe a statement that it's enough to have a rational that is you have a  $r$   $f_1$  generating the unit ideal and then you it's enough to check for those yes yes yes you can do some little bit of work to reduce to exactly exactly exactly yes yes so so Huber so so maybe I'll yeah Sketch So based on what gabber said so Huber shows uh every cover refined is refined by one of the following form so take  $f_1$   $F_N$  generating the unit ideal so and then you look at  $U$   $f_1$  up to  $F_N$  but then you leave out  $f_1$  and you put it on the bottom instead so and the collection of these  $I$  and  $I$  so again it's a refinement of the usual zis cover you get when  $F_1$  through  $F_N$  generate the unit ideal but you can check just on valuations that it's still it still covers is the the valuative Spectrum um and then you do some kind of you do something similar to what we did previously you have with this this here you can using these covers which refine all those risky covers you can assume one of the elements is equal to one anyway you keep playing and playing and playing and eventually you get you get the desired thing um okay uh right so now what are we reduced to analogous to there so so if  $R$   $r^+$  discrete hu pair and we take  $F$  uh in  $R$  um then we need uh that um uh oh all right so there's two there was two different kinds of covers there was the  $F$  over one and  $1$  over  $f$  um and then um yeah um uh so I'll do that one first um sorry  $R$  and then um okay so uh solid solid solid we need that and we need the other one so um uh do we need that as well okay um so we're going to basically just verify that all of these conditions hold so so what is this so there's only there's only really two types of covers here if you look at it there's the cover  $UF$  over one and then there's the cover  $U_1$  over  $F$  this is of the form  $U_1$  over  $F$  where  $f$  is just one minus  $f$  um so so if  $F$  in  $R$  so what is this  $D_r$   $r^+$  uh goes to  $D$   $R_1$  over uh sorry let's say  $r$   $R$  plus a joint  $F$  integrally closed um this is just uh solidification uh  $t$  solidification uh for  $ZT$  goes to  $r$   $t$  goes to  $f$  um so by definition uh so these are both these are both analytic ring structures on the same ring right and uh the only difference is here we've enforced extra conditions of uh it's just by definition so uh if you have so here we have for all everything in  $r^+$  um you're solid with your  $T$  solid with respect to that variable um and here uh well and if you enforce this then you also have that condition not just for  $r^+$  but for  $F$  but then we said you automatically then get it for the integrally closed sub ring generated by those so then you exactly get this condition here so the the category that you're getting here is exactly this condition here and

the left adjoint is just the the  $T$  solidification  $u_m$  and recall that this was given by some  $\text{arom}$  uh  $u_m$  okay that was the first type at least need to invert  $F$  first sorry do at least to invert  $F$  and you no I'm not doing this sorry I'm doing this one here yeah so sorry thank you so this is  $U$  of  $f$  over one and then there's the one where  $U$  of  $1$  over  $f$   $u_m$  so that gives  $d_r r+$  uh goes to  $D R_1 / f$  and then  $R$  Plus plus  $1 F$  integral closed  $u_m$  so what is this this is so first invert  $t$  or first invert  $F$  uh then uh  $T$  solidify uh for uh  $ZT$  uh  $T$  goes to  $1$  over  $F R_1$  over  $F$  so the modules here are full subcategory of  $R_1$  over  $F$  modules which is a full subcategory of  $R$  modules and the condition is  $FX$  in  $\text{vertily}$  and you have this extra thing that's supposed to be solid  $u_m$  but now let me make a remark about that second situation so I claim that that whole process inverting  $F$  and then solidifying with respect to  $1 / f$   $u_m$  so that uh base change there is also described by just an  $\text{rhom}$  so now now I'm going to take  $ZT$  mapping to  $r$  with  $t$  going to  $f$  not one over  $f$   $u_m$  and I take  $R$  home over  $ZT$   $u_m$  so  $Z$  power series  $T$  modulo  $ZT$  shifted by minus one there so  $IE$  so and based on the formalism from two lectures ago this is the localization which kills uh the item potent algebra item potent object uh  $Z$  power series  $t$  uh in mod in  $DZ$  uh  $Z$  solid so we had  $ZT$  modules and solid  $Z$  modules we said that this was item potent and I also said that when you have an item potent algebra then when when you you can kill it by just taking the mapping cone of or  $\text{hom}$  homotopy fiber of the the unit map hitting that thing and doing this  $\text{Aram}$  formula  $d_z z$  this is what you call  $d$  uh this was yeah this would be the same thing as  $D$  of modules over a  $z$  bracket  $T$  in solid  $z$   $u_m$  that's kind of this is kind of the new notation fitting it in the general framework because before for for  $\text{Uber}$  pair you wrote the a  $Dr R$  plus you did not write solid no no you had a integral Clause ah okay you you wrot  $D$  so let me explain what the point is here so this localization was supposed to be given by first inverting  $F$  and then doing the solidification but again this solidification yeah but the first claim is that this functor already inverts  $F$  so if you have something  $F$  tors and it's going to be killed by this and the reason is you're killing this whole guy and therefore in particular you're killing any module over this guy but  $u_m$  everything  $T$  torsion is a module over over  $Z$  power series  $T$  So this automatically uh inverts  $F$  because anything  $T$  torsion is a  $Z$  power series  $T$  module so if you have a solid aylan Group which is a filtered colimit of things killed by powers of  $T$  then it is a  $ZT$  module that's just a condition so you can to check it you can it's a condition closed under limits and colimits so you can reduce to checking for something which is uniformly killed by some power of  $T$  but then it's obviously a  $Z$  power series  $T$  module because it's a module over the the truncated power Series ring  $u_m$  so then it would be the same thing to write this formula where you invert  $T$  but then you if you that's just after a change of variables it's exactly the same thing as  $t$  solidification as described by the or  $t$  inverse solidification as described by the previous formula so inverting  $F$  and then solidifying one over  $f$  is just the same thing as as doing this here okay  $u_m$  so basically all you need to check now if you look at those conditions most of them we already know so it's a localization kind of by construction the localizations commute that's because they're both given by  $R$  homing out of some object and any two functors  $R$  homing out of an object commute with each other just because the tensor product by a junction and the tensor product being commutative  $u_m$  so and then what does this translate to in terms of these uh item potent algebras which determine these localization functors so three translates to well it's a different condition in so so for uh so  $u_m$  it's just a a simp translates to a simple condition on these item potent algebras here so there's that one and then there's that one it's that if you you take  $Z$  power series  $T$  and tensor it in solid  $Z$  modules over  $ZT$  with  $Z$  laurant series  $T$  inverse uh you get zero so if you have something that dies on  $\text{ROM}$  out of this and dies on our home out of that then by messing around you will using this condition conclude that it just has to be zero  $u_m$  and this is a so it's kind of uh yeah so you can use the geometric series to to see that this is zero uh yeah so what's the interpretation here by the way remember like uh so this so this you can think of this as localize away from the the open unit dis and this was localized uh to  $u_m$  localized to the closed unit disc or away from the open unit disc centered at infinity and the reason those two cover intuitively is because if you take the open unit disc and the closed unit disc then uh or sorry if you take the sorry if you take the closed unit disc centered at infinity and the closed unit disc centered at zero then those Union is the whole space but in terms of the complements that's saying if you take the open unit dis and the open unit dis at Infinity then they don't intersect and that's exactly a you know this is the algebraic translation of that fact  $u_m$  and then similarly so for the second kind of cover uh you need that  $Z$  power series  $t$  uh tensor over  $ZT$  uh  $Z$  power series  $1$  minus  $t$  you need this to be zero  $u_m$  uh but again this is just  $Z$  power series  $Tu$  and then  $1$  minus  $U$  plus  $T$  and that's also zero for the same reason and again the interpretation is that the open unit dis centered at zero intersect the open unit disc centered at one they don't intersect and that sounds strange

until you remember we're doing non-archimedean geometry and then sounds reasonable again um yeah okay  
 uh all right so I want to finish with just a couple of remarks so so remark um so there's the the Corollary  
 so that if  $R$  is any solid ring and then  $r_+$  containing one I don't know sure contain in um so the uh so now  
 you have to be a little bit careful so you you I could say that  $\mathrm{Dr} \, r_+$  solid localizes on the active spectrum of  
 this discrete ring um but be careful namely so there is something that is formal which is what I said that so  
 on the global sections uh that this thing is just our modules in uh the category we assigned to this discrete  
 Huber pair but if you then want to get a sheaf uh so then you send you a rational open you have to send  
 that to modules our modules in this uh  $\mathrm{D} \, \mathcal{O} \, \mathrm{U}$  uh discrete uh I don't maybe I want to say uh uh so o for  
 the discrete ring okay um so this recover the topology in the in the  $g$  mode nistr okay yeah so um so the and  
 in particular so what is the unit object in this category so uh you take you take  $R$  um and then you invert  $G$   
 and then you derive solidify uh with respect to uh fi all the FI over  $G$ 's so um necessarily when you do this  
 object for a completely General solid ring um you're going to end up with some derived phenomena here um  
 and we'll I I was hoping to get to it today but we'll probably discuss exactly how that happens later um and  
 I want to make but I want to also make another remark uh which is that if uh so in fact uh this this this  
 this sheaf so  $\mathrm{Dr} \, r_+$  actually lives over so it localizes on this big topological space for the discrete ring but it  
 actually lies over a much smaller subset uh so the closed subset uh so let's say  $\mathrm{SP} \, R \, \mathrm{star} \, R \, \mathrm{plus} \, r$  you if you  
 remember the set of topologically nil potent elements then here you add the condition uh that if  $F$  here is  
 topologically nil potent uh then you want the valuation to be strictly less than one uh so for an individual  $F$   
 that's a closed subset and then it's a big intersection of such things that's a closed subset of this topological  
 space and my claim is just that if you take this sheaf of categories and you restrict it to the open comp  
 ment you just get zero so it's really living over this closed subset here um uh on the other hand so Huber uh  
 studies Huber considers uh  $\mathrm{Spa} \, R \, r_+$  um which is the continuous valuations um which are less than or equal  
 to one on  $r_+$  and that's the same thing as uh so sub the subset of of this it's a potentially smaller subset  
 a generally smaller subset uh such that satisfying a stronger condition says that if you're topologically nil  
 potent uh then the uh for all  $\gamma$  in  $\gamma$  there exists an  $N$  in  $N$  such that the valuation of  $f$  to the  
 $N$  is less than  $\gamma$  okay so there's some subtlety here uh that uh the space that hubber localizes over is  
 actually smaller than the space that we localize over um but yeah so the the the distinction only Cur occurs  
 for higher rank valuations but I shouldn't say only because in in higher Dimensions higher rank valuations  
 are just everywhere so there's actually a huge difference um but uh but Huber shows uh and and and I and  
 I should say that this um yeah continuing the remark so this this  $\mathrm{drr} \, \mathrm{plus}$  does not lie over uh but does not  
 live over the subset uh  $\mathrm{Spa} \, r \, r \, \mathrm{plus} \, r \, r_+$  in the same I mean I I claimed here that the sheaf here becomes  
 zero when you uh restrict to the complement of this subset it's not that's not what's happening here but  
 there does exist a map a retraction huh what's up Peter a what oh thank you thank you well yeah does not  
 live over the subset but uh there is a retraction like this um which is actually a quotient oh sorry  $R \, \mathrm{cus}$  yeah  
 which is a realizing so by definition it was a Subspace but you can actually realize it as a quotient and then  
 you do get a sheaf of categories on  $\mathrm{spa}$  and that's the correct way to so to speak get a a sheaf of categories  
 on Huber's topological space and it's the retraction that's kind of the good map in the sense that this is the  
 Quasi compact map so um yeah we'll probably so in in in general you get more flexibility for localization  
 using this picture than with Huber's picture and the kinds of extra things you get are something that we  
 already for example the things we already discussed something like this so-called functions on the closed unit  
 disc will arise from the structure sheaf in this general setting but doesn't arise from the structure sheaf in  
 hubber setting and um you can uh you can analyze these things but I think I've now said enough thank you  
 for your attention so as farther remember in this retraction context yeah so actually  $B$  of spectral spaces the  
 inclusion is not spectral in general the retraction is spectral exactly over it is such that when you have a shift  
 on the bigger thing then the direct image by the retraction is the same as the Restriction to the Subspace  
 for an arbitrary Chief no no I I as far as I remember uh the ah yes yeah so let me make the so so it is a nice  
 situation where the in in particular the higher direct images so the retraction are zero because of this yeah  
 and now you're dealing with shifts in this Infinity so probably the same thing we Lo but I'm not yeah so um  
 yeah so there's a the basis of rational opens here you can actually parameterize it by similar data but with an  
 extra condition that these things generate an open ideal and then if you pull those rational opens back here  
 you get exactly the corresponding rational opens as expected but if you take a general rational open here  
 not satisfying that condition and then restrict it no no if it does satisfy that condition and you restrict it you  
 get the correct thing but if it doesn't satisfy that condition and you restrict it you get something new which

is not necessarily even quasi compact not a rational open so you have You' have to write it as a union of rational opens and this is kind of like yeah taking the open unit disc and writing it as a union of closed unit discs is a typical example of that phenomenon so you yes so you said something about getting a structural shift last time yeah can you comment on this or or will it come later uh yeah I I was hoping to get to it again today but I didn't so it is the the structure Chief would just be you take  $R$  which is living globally and you apply the localization functor to to get something living in in here instead and that's that's this uh it gives you this object here and one can analyze it and so on and so forth yeah and it's also some Center Cate some category is there a way to think of it as a center of Cate Center I don't know yeah oh no but these are your symmetric monoidal categories so it's just it's just the unit I mean uh the unit of the symmetric monoidal derived category of in this sense yeah okay so this ah so you claim that when you take mode  $R$  of this this is actually a good thing which so it is associated to the in good cases it is associated to the rubber associated to the rational do except that sometimes you have to do yes so this is I mean this so this will also correspond to an analytic ring but you know in the to say in derived sense so you have to the notion of analytic ring that we've discussed so far you had an ordinary condensed ring in a full subcategory here you need to not just remember that ordinary derived you need to remember some derived enhancement of it as well but then it is enough to just remember the ordinary aelan category of modules over the ordinary thing yeah and and besides the derived stuff there's also a quasy separated issue where the value of the structure sheath might be different from Hubers even if it lives in degree zero it might you know the quotient might not be by a closed idea and so it might still differ from Hubers but again in Practical cases that doesn't show up um and yeah even I guess even inverting  $G$  can introduce non- quasy separated behavior in general yeah we'll I think we'll discuss this in coming lectures all of these so in the so considering those two spaces in  $U$  Theory where you have a retraction which I think he maybe use a slight different anotations but anyway this SPV AI so you have this Subspace living in slightly bigger thing is a retraction which is spectral so you have shifts you can consider shifts on both things what I said I think is correct that the direct that the Restriction to the Subspace is like the direct image okay then you have a shift like in this shift of categories and some sense on the full  $s$  and then you take the direct image yes to the southpace which is like restriction something I probably but the question can also be asked about so you have particular shields on the full scene which are direct images by the inclusion of shields on the Subspace so the the so the question is like in this context so you have your let us say you have a a rational open in the figure  $C$  and you consider its intersection with the smaller  $s$  you said that you can write is the union of  $C$  so you can evaluate your shift by In This Way by inverse limit of those is it equivalent to the shift to the value on the original thing in the big in SPV so like whether the sh of categories is the direct image of its restriction to the Subspace which so you have on SPV  $r$   $r$  plus  $r$   $z_0$  you have let us say some stock let us say of categories you can you can take his direct image to which I think is the same is a restriction to that okay yes and then it is AAL Junction map for to our lower star to the let's say is the inclusion it yeah it's not the same because for example on global sections you can see something like a ah well no it's not the same because um for the category you have a a rational subset corresponding to the closed unit disc uh giving you this and in this in the category of modules over this guys say the unit is compact but when you when you go through that procedure again analytic function is not this  $R$  the global analytic function is just certain where you don't invert a fixed power you just have convergence on each yes so you get another ring of function this is something else this is another the so it is not so the answer is not no there's really more data in the in the in the in the structure chief on this guy of what one can do in usual exactly exactly which is okay okay so uh yeah maybe if there are more questions I can handle them on a personal basis let's release people from the room

## 10. ANALYTIC ADIC SPACES (SCHOLZE)

[https://www.youtube.com/watch?v=YLQt\\_tV4tHo&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=YLQt_tV4tHo&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

e good morning um so today I want to talk about yeah what's called analytic adic spaces um right uh so but I want to immediately point out that there is some conflict of terminology here or conflict of terminology um so basically using what we've said so far all adic spaces give rise to analytic spaces in our sense of spaces whatever they will be in our sense uh but within the world of adic spaces there was already a qualifier called analytic um but within the Serre of any space what is this conflict of terminology oh um uh and I didn't come up with a good name for them and but there's one name that's already to some extent used in this world of adic spaces also uh so I will call them  $T$  in this lecture um Peter would help us um if you could write a bit bigger because uh it's it's a bit just a bit unclear there is a space or something okay uh yeah uh so uh yeah just that I don't want to overuse the word and take in this um okay so here's the definition where this already used this name uh so defining a space it's called  $f$  so this is terminology that's already in use um if it has a topologically unit space saying it's also called  $f$  um uh right so so the example to keep in mind is for example  $T_p$  where  $p$  is a topologic unit or in fact any non-archimedean local field complete of course we always assume complete or any kind of  $f$  over or  $f$  over any  $u$  or  $f$  over because when you have a topology you can put unit in sum then for anything above you also and let me call space state so that's usually called analytic if it is covered so we didn't really discuss the full definition of adic spaces but we discussed the algebra that the  $R$  for them and then there's some way of doing them which will not be all that important for us right now um and basically we told you what the open subs are then you kind of do the obvious thing um it's covered by such uh  $A^+$  uh it's actually equivalent to a point condition that uh here I speak in the the green I speak green space so just to clarify the first you did not define analytic spaces in our sense because you you not yeah okay and also adic spaces there is there was classically there is a subtlety about finiteness so there is this condition in  $T$  that and then it seems it if I say at space right now I probably mean shy okay yeah but then there is also the question of whether in this theory of localization and Dustin talked about last time when you do this you get some condensed thing possibly you get some condensed thing for an adic for a rational domain and it is over you say you come to this I will come to it okay okay okay so you need some in all cases of interest I mean usually it gives the right thing but it's I don't know if for any space think it get I'm not sure offer I will come to um um so if you have any space you can actually each point you can define local completed residue field and this can either be just the residue rings or can be complete on median fields and here some asking that there are no points where this is a discrete not um okay and so then yeah basically the series set up so that you always the residue fields are always of either of these forms but then when you have such a complete non-archimedean field and there is some unit in there and you can lift this to a small neighborhood and that's why there um and so uh the intuition is that there are some there's a world of schemes this sits on some world of formal schemes maybe with some adjectives that I don't want to discuss right now because it's not really relevant um these all set with the Serre of adic spaces uh but then these are some of things that where you start with the scheme and there is some still some kind of large space where it seem like or least like underlying reduced but then these adic space areas are some of the ones more speaking where start with a formal scheme but then remove all the scheme like points so spaces which are at the other end okay and so I mean here you might have something like which you can also consider some kind of  $SP$   $TVP$  that um but then um the generic fiber such sense okay um let me say few words about structure of such things so let's say uh so we always have this range of definitions so okay so let's say  $P$  in a such a topologic unit so um then you can take out the ring of definition and because and you can always assume that they contain any given power bounded element and I mean some particular power bound it so you can result of assume that your um definition contains  $P$  you could also do it such way to take any and then some power of  $P$  will anyway  $SP$  in there any power of  $P$  still unit and then actually it's automatically the case that a zero of  $P$  that's an actual exercise and looking at the norms of the topology on  $u$  so up you might think that it could have some rather more subtle uh topology for generator  $I$  but actually it's always  $u$  and then you can actually uh into a with Beau all a  $z$  for example you could say that the norm of a function  $f$  is  $2$  to the  $n$  is a maximum all integers  $m$  such that  $f$  is  $u$  because it could also be in  $u$  uh and some particular this means that the norm of five if I said it as right  $p$  is minus the norm of  $P$  a half the normal by inverse and of course there's some kind of arbitrary choice that I made here both first of  $s$  uniformize  $p$  and in like the number a half here



um the trivial counter example with the zero ring so the norm of  $\Pi$  would be zero I think there um is a zero I don't think it has a top unit right or I mean maybe that's a counter examp to make think that anything that is an algebra over no but zero be  $\mathbb{Z}$  probably okay so whatever you figure it out what um uh and so um because actually Le an equivalence between facing and alge B of rings um admitting and may should first of all much the objects are the objects are B Rings admitting some B rings admitting some  $p \vee$  such that the norm of  $\Pi$  is basally between  $\mathbb{Z}$  and one and nor nor inverse equal to one but as a moment you you actually only care about the topology used by the by the mon so and again before somebody complains everything here is completed um B ring and continuous so you mean ultrametric ban Rings otherwise you have a and of course the zero is still an counter example to what happens to zero uh um um right um okay so so these are uh like who basic algebras that we consider here um so you could also talk about this in the language of B algebras but you shouldn't really fix the norm but you should just ask def find by some Norm this this property because we have um all right so then let me already uh come to right so so now I want to State a Ser that K and proved um when some 10 years ago or so and the serum that I found uh extremely striking and surprising at the time so as as was discussed there are all these issues that for General Huber ring or Huber pair uh what Huber defines is not always uh actually a she so um so but let's just say that  $A^+$  is GC um if uh what hu defines U defines a PR she of hu Rings if this is a sheet yeah some kind of stupid definition um and I did not I mean I did certainly expect that in those cases where you can prove that something is chy you can probably also prove other nice properties because then that's probably reason that things are well behave what I did not at all uh expect is that you could prove any nonf serum whose only hypotheses is something is chy and then other good things happen because some only asking for the like there the first thing that could break that this is not a she but they expect that if you're in a situation where I mean there might be situations that just for stupid is a she and then why why should other okay uh but here Pro the following St z uh so seee a is f and a so uh then I mean any final protective module over a um you can take it and tender over sorry M uh tender it was over a she and I mean a is just a direct module so certainly this is still a sheath that's good so it's a sheath of finite projective modules um actually it's one that's locally free so this is a locally free f bra and so whenever you have uh uh ring space you can always talk about these modules which are locally free or fin rank so these are I guess called us called vect B with respect to and okay so it's easy to see that this a only faceful fun but the highly nontrivial surprising fact is that there actually an equivalence so when you can glue Vector bundle can do find protection mod and uh somewhat later um they also proved that there is some version of this that works for some kind of coherent coherent modules but I don't want to State precise the because of precise is actually Vector extremely let me just say there is a burden then at least if the ring is the recovers the expected result that you can even the Rings you have to be care because loation not be yes all right so this is a very nice result it's uh like basically me that their aim at the time was just to prove that on perfector Spaces you can R Vector bundles uh but the argument some worked actually whenever just chy um and I mean I could kind of follow their argument line by line but it's some kind of rather tricky argument where you really have to I mean when you have a desent item for Vector bundle some some kind of invertible Matrix over some completion of your ring and then you have to do some changes of bases to make things better and it's actually B it's an actual analysis um uh and so today I want to explain the different proofs that we can give using our general theory uh of solid modes I mean using this way I should maybe say that next some 10 years ago um when this result I was also talking to K and he was some already telling me that actually like if you work in the deriv category and you look at some kind of INB up modules then you can just glue and I was like okay but what does it even mean like we have all these problems I think it's not a structure shap and it's like seems highly nony through Vector mounts and so on so what does it extremely General result actually tell you about such complete questions and uh I don't know whether he ever like figured that out but some go lecture is to show that this is not just some fancy I mean the S result for this SP mod was kind of I don't know abstract and so maybe didn't see much happening and so so I want to show how you can actually get down to something more concrete today okay so they are basically uh right okay so the outlines um so first of all and this is some addressing what G already wants to know is a relation between um the thing that def Andy from last time the thing that naturally comes out of our Ser um in particular we will actually uh it's actually true that if it so happens that this thing is is Achieve then it actually happens to agree always with our she uh which is why suddenly the good things happen because then it actually has a correct series over and finds the correct series um and uh the second thing is that well in general uh sending any any rational substate U um to uh let's say the to

coherent modules in non-negative homological degrees over four where this is now the thing that's in general right um the subcategory of um this is a sheaf of we cannot just glue within this  $\mathcal{H}$  category but also like more reasonable objects they also GL to reasonable objects and then once you like in the most General case that's the best thing you can hope for um IT addresses like the subcategory and the precise formulation of this result for two modules uh but once you restrict the vector bundle you can then prove this very clean result maybe I should also say that the reference basically reference with this uh done in more detail um is this paper of okay so okay so the first thing some the relations so let me call the other just oh um um right so first of all here's a that Som just computes those sides um let's say you have such a um subset uh so I say from do I need it here in my mind I'm making the blanket assumption that a state from throughout this lecture it may or may not be necessary for everything I'm saying um okay so this be the Ral Subs um here I'm really using this uh edic spe of continuous valuations um which concretely means that when I form such a rational subset I should assume that uh the ideal generated by  $\alpha_1$  is an open ideal but actually once you have top open unit must be equal to  $\alpha$  um then first of all uh  $\mathcal{H}$  uh is given by taking the completion power series algebra variables  $G_1$  to  $G_N$  and then modding out by the ideal generated by  $d_1 - f_1$  and taking your closure um I this is basically yeah so why is that so on on this open subset but first of all you didn't invert  $G$  because now the ideal generated by  $G$  contains  $f_1$  up to  $f_N$  but they generate unit ideal uh so the ideal generated by  $G$  must be the  $\mathcal{O}_U$  ideal so  $G$  has become invertible even without taking the closure um and then on this also so you have now  $T_1$  which is  $f_1$  over  $G$  and all it's uh completion Power Series so some of you kind of this is what could happen on this subset where this less equal to this uh but I mean Huber he was always working with complete topological ring so when you take completion by something you also take completion by the closure and then well for example but I mean I mean this is a  $b$  algebra and the  $q$  by any close that still a  $b$  algebra still okay so um this is maybe not so hard to show and uh all of you uh I recall the endomorphism part yes uh the image of  $\alpha$  well I it's a thing that's just defined last time yeah um um you take you define a category ring structure by asking for this completion for all these elements and uh then you somehow look at what the completion of the unit actually is this is what the  $S$  use and um this has a very similar formula okay want but then take the Der um where here or and elements I want to  $R$  uh this derived quo signifies um the Der base strange from only algebra but they just modify all the variables  $X_i$  so basically you're setting  $X_i$  also to zero but not just in the stupid way but in the derived way um and completely this is computed by possible complex and in the middle there some eyes there a costal differentials and this just by taking the standard resolution of  $Z$  over here it will be an animated condens and that's why I will consider this um I don't really want to talk about animated things today uh but yeah so uh just yeah so it's certainly some kind of algebra let me just give a sketch it um and for the second part so you can write  $A_1$  over  $d$  as  $A_1$  mod and now it's not really necessary whether I take the drive or not but it's also turn the drive  $S$  because what I will write down is a regular sequest um and then you solidify uh solidify solidify this so some exact operations category so this thing is just computed by possible complex which is  $\text{Con}$  a fin complex fins of cop of The Green for some to understand what this some of this  $C$  thing just partial solidification so you just have to understand what happens when you solidify uh these things but this is exactly what DUS already computed that something uh he had to say that if you uh take a ring or take a fun takes a module and joins a new variable  $T$  and then solidifies for variables all limits and  $\text{Co}$  limits and so some of then you can write a first of all a zer bring of definition  $\Pi$  but then a  $z$  is the limit of a  $z$  not  $\Pi$  to the end so this way you can like operation of joining this variables and then solidifying  $\text{Jo}$  commute this description of  $\alpha$  as a limit  $\text{Co}$  limit of limits and so okay and so car are your things actually that saying it's the same thing as stating all which is a context but zero just a usual Str um and then you take the  $C$  separated a small inter um the inclusion uh from light or not it doesn't really matter proba separated cond set inside of all cond set it hasn't left the joint I'll take any  $x$  to the  $q$  separate portion and this is making the operation of taking the maximum  $h$  of portion and I can practice that what it do um one very nice feature that this left joint has for non obvious reasons is actually the commut with finite products it does not in general finite limits but it does those finite products uh this has a good effect that uh if  $x$  has some kind of alic structure like being a group or ring and so on and the qual function all that isct which is a preservation of final product is available both for the light setting and all condens yes I mean also if you if start with the live condenser and then from the maximum  $q$  SE also live and so on so thank you um right uh uh so completely if you WR  $X$  as a of some by rware it's I'm telling you to computers left the joint and you can always write any uh any condensed set as a portion of a  $q$  separate just take different of profile subjecting on to it uh then

there exist the minimal um for compact injection our bar and  $R$  that is also an equivalent relation and uh containing um  $R$  so it's in some sense taking the closure of  $R$  intuitively speaking uh but sometimes this adds new things which where solution is not spreading you add that as well and then you do some kind of trans induction or you just intersect all possible contain intersection station and so then  $X$  this is  $x$ tion this is  $X$  Mod because uh I mean this  $x$  was separate start only if this inclusion here relation was called the comp and so um so for example yeah so if if  $x$  happens to be a group then you can always also find a subjection from separated condense to be in group and then the relation is just the qu by subgroup in that case you're really just taking the qution by the closure of the subgroup the clure in the sense of small SP injection okay so here's one other thing I should recall uh which is what do qu comp injections have to do with slau as I was pretending they have um let's say  $x$  is any sequential space to recall it's topological space for those uh for which the me to to cond set is fully faceful uh in the close substance of  $x$  tively to the uh to the qu compact injections underl taking any uh taking it easy here so for example why if you start with the close subset here why is it actually forther compact injection while to check that the definition is that whenever you pull back to some profinite set then the pre-image uh this is a  $f$  compact injection but this precisely means that the the other  $g_u$  is also and so the preimage in this case is uh just the close stop of this profile set byose close subset here um and yeah close subset of Prof set is a profite set and compa using using that  $X$  has some of potion topology from profile set you can kind of also reverse it okay um and so uh okay so back to the SC what is 5 uh well this is just the thing and then you take the ex okay um and then if you take a qu separated potion well this guy is already qu separated and then by above and because this here is actually sequential comes from a sequential topological space um this precisely means that I should just replace this one here by Disclosure which is uh what um right and so uh this are gives us some relation um but we interested in a somewhat more precise relation um which is a following serum that's potentially do tralia uh except that he didn't uh use this language to phrase it but I mean the proof is really his um that uh a plus CP if only who stru or if you just pH in terms of all series all you rational or VI fits in degree zero and okay so let me give the proof um uh this is clear because uh by the general desend results from the last lecture we have the she of categories but in particular like you get a sheath of rings and we just take those unit and uh this is over and uh so sheep and then well she um the non-trivial and I think somewhat surprising result is that if it so happens to be Shey then actually it is the right thing let me get SK the other direction um uh and before I start uh note that uh to check this property that all you6 degre  $Z$  and prob separated it suffices to show that this true on a further refinement of you  $U$  further cover of you because you can always recover it by all the smaller values as a limit um so and then as St didn't use it last time there are all these arguments that General Ral coverage and so on they can always refined by this uh simple wrong covers uh and actually the second type of cover that you use it's actually not necessary uh in this the state setting um so uh right so so so so to get take the good property of all view after be further uh Sur and I don't want to go through all those Comm does refinement and just uh notes that some following uh ke also and always refin but we can always any space with Locus at most one and at least one uh so the other type of cover is not necessary when you to new units um and so this reduces to the key to the following uh s uh so sa inte and  $F$  the  $L$  and assume o this a sequence of EX um actually what does chiness mean chiness I mean so this is of  $x$  all ubber s so let me to zero here um this a Lo and this just BL achieve condition says right so if you want to give a thing here you have to give one here and this open cover and they to agree on intersection it's actually always true that the last  $m$  is objective just because any element in here can be written as of sum of  $f$  to the ends and one over  $F$  to the ends and the  $F$  To The Ends come from here one  $F$  to the end come from here so so activity on the right is always true um never mind that uh so assume this which is always Pro of  $H$  because it's just one specific instance of the sheet ex um then actually uh this is the deriv Cent by  $F$  and the other thing is also the I don't so in other words it agrees with with uh the thing that we Define or I also what does this  $D$  I mean it means that modification by minus  $f$  is actually injective here and it has closen um right so uh let's Pro this so what you have to see what I just said so that if you look at on and then multiply by say  $F$  or by go that this is injective with close sure um but actually because of star um like for example if you want to show injectivity uh let's assume you have something in the kernel that also still lies in the kernel when you replace a by one of those two rings but then over those two rings uh it's I will argue that it's easy to see that this net actually is injective um uh but then this map here we assume to be injective and so uh we get injectivity and similarly this map actually has closed image because the image is precis the crel of the next map um and uh and so using this you can also uh um check that if if on over

these Rings it has closed image and also here it must have closed image so it's enough to take it when once we replace  $a$  by those two rings in other words we can assume uh they see that either the abute value of  $s$  letal to one one of the um uh but uh when for example  $f$  is less equal to one then  $T$  minus  $F$  you can actually easily show that it's a close IDE deal because it can just successively like heill off the highest coefficient uh and something similar here so uh yeah those two Cas actually something you you can really just I mean if you have an element here you can really just by looking at coefficients kind of see what happens and also like I once it it's a base change actually one of them becomes a toy anyways because like not doing any further localization it's only the other one you have to check uh basically say  $f$  one then the second thing is the one where you ask that  $F$  has also absolute greater equal to one but this basically just means that you invert  $F$  the Ring of integral elements and so already I'm the level of a  $z$  if become the localization and then things are good okay so this is this funny argument that K found that if you just assume the  $T$  thing is a CH then you can yeah reduce this problem and it's some criticals that you can reduce to the simple wrong covers where you just take a potion by one element because then the question what do  $R$  potion is is really just the question whe this one  $M$  here is injective is close image um yeah this can be red okay so this uh um finish is my discussion about the she condition and the relation between  $H$  struct um and so now uh to the second part so here here definitions what does the me so let  $R$  Qui say no cond structure for now um uh then some something like um this definition  $S_6$  I believe uh if it can be represented by complex goes up to some cus  $n$  but  $Z$  go all the um yeah so there is some stuff here here it goes only to the left um and all these terms must be finite you could also say try project this with negative let actually fin projected because I want to say something else in a second and then important um so if your ring was an aan then this will just be the or more generally conent this would just be the condition that each each I mean it's bounded to the right and each group is a coherent module the case just finally generated module but can go arit part of the um if you're notan range then there's some cerent modul St form in a be category usually and so uh because you didn't ask that the relations between the relations are fin generated and so some of the infinite complexes they capture the idea that you have modules which I finally generate with  $s$  many relations  $s$  many further relations so like I guess point is that over in the Ser it's a good class of like  $f$  generated modules behaving nicely you don't have that over General ring but the ring still the right nice of complex um and uh let is perfect if there is such a representation which just wanted to found it just a fire compx project mod and then the serum is that um let's say a  $A^+$  is any  $H$  um um the a state then sending any you start uh to the following uh things so on the one hand you can take perer guys um all the ones or you can take two clearent ones that are say in degrees greater equal to zero or greater equal to  $n$  for any  $n$  um we could also take all perfect complexes but you could also take those perfect complexes which have such a representation in some interval so per ad means those that can be represented using a complex sitting The Source degrees uh so these are all funs that take a  $R$ al subet to Infinity category and they all threee cly and claim that they all sh of on the right I'm always just thinking of a as a satisfy that I really just Lang so these are modules being reason Peter you didn't have a dependence on you on the right hand side I did not thank you uh um and I guess inally I'm uh now extending all these Notions to animated Rings um and so okay so either you know what all these things mean when this guy is just in animated ring then it's true as stated or you secretly that you maybe in the she case where is just con zero and then I just told you what they are either way these are she of in particular the case uh Ser 0 if I exed the vector bundles this cover yeah Peter is there a reason you didn't State the like pseudo coherent AB uh uh uh no okay you have to be be careful that the good way to make uh uh UD restrictions on CHS and a perfect complex are different yeah for perfect complex you assume that there is a representation as an actual complex that's in some range of degrees for carean Chiefs you can just Mak more a naive thing like  $B$  to degrees in which it's although no there is actually a reason uh if I no there is a reason because look I don't think localizations are flat in any generality here so you could have something that's zero then you localize go further to the left won't won't they be flat I mean on the level the discreet Rings maybe not always I still I think an open question whether for perfector space is a localization set ref okay everybody expect wrong but I don't think anybody okay so I guess I only have half an hour remaining so let me uh uh just give an overline of the overview of the argument so I mean I said this is in this paper of Gregory andf um and we also gave the we proved the analog of this result in CA on complex geometry and uh the same argument as there works here uh but okay let meall theum uh so we we know that this fun takes any you to like um and okay this will have a meaning some point when this is animated uh we didn't discussed yet so uh if you feel more comfortable just assume this sits zero and this cas

um then this is a sheet of any cator and so this certainly contains fully faceful uh just the modules over the underline the SC ring that's always true um for any antic ring whatsoever um and this certainly contains all these other subcategories but yeah basically some kind of yeah this is some kind of fin solu here CL um and so by virtue of this being fully faceful uh the it means that the only thing actually have to prove in all these settings is that if you have an object here uh which locally happens to line all of the subies and actually so globally um and maybe at the expense of pretending that I can replace all of you by a again um that this is this is such that uh over over cover like the base change from a plus col one of the sub categories then so does so the conditions that we put they they GL okay um and so maybe the first idea you might have is that well let's first check that this condition of being like module discrete module the sense and then you the sess conditions but this actually does not work warning this is a warning that Dustin already made but I want to reiterate it because it's important um uh the conditioning noise sorry uh this does not just say Global it's do locally that will not be globally um let me actually quickly sketch the one example um I'm not sure it's the easiest example but somewhat destructive um let's consider take the curve so it takes the um TM Spains over let's say PP um and then you can take the qu by let's say other unit so this what's called table different with the analog of taking C Star Complex space c star and po by some topolog element and then this becomes atic curve complex numbers do similar SLE or fi this some how rigid geometry started um uh and let's let's let's assume that our spa a plus is some large open large open subset here so basically remove a small disc around the war one Genesis um but large some see that there is non with global monog um but should spit back should really be sub only ofon and not this yeah and then there is C the projection that yeah ring um and then you can um find what I will call a lower streak of sub here um what do what is this well locally locally this map is just split and so locally on the base like the P are just disc Union of copies of of the base in Tak by the integers and then the she is just a direct sum of copies of this of this same and and so this this how defined locally and well it's obviously a sheath and so it gles to some sheath on on the base so this this defines for you object daa plus solid because locally it does um and it is locally discreet because locally as I said this coverage of splits and this is just a bunch of copies of of your Bas instead men um but you can check that globally it is the speed and exercise okay so um we can't hope to um I this category instead we will use to to other Notions so we said you can first also Define this kind of PUD in oan um uh where now I mean yeah it's again the complexes which are represented um by um something that's bounded to the right so goes to the left and oci are in this case I just say they are equals to all right like projective mod uh I guess in when I say it like this I assume that that P is also still degree zero um in general you should do the appropriate thing where St there's a different way to characterize this as X Out of the Sky Computing with f you meant mod AA plus solid yes thank um the right e um so you define this and show that property applying their globalized and you actually just use I first of all being bounded to the right is a proper that globalizes because you're locally bounded to the right and then globally you're just some kind of finite limit of suction so there still bound to right um and so then you need to check this x condition and and to check that the only thing you need is that these localizations have found the to amplitude uh otherwise there would be issue with some issue but during that localizations b to the left this far um for okay so this fire condition on the big category uh this can you to S it and then actually once you have the fire this condition then the thisness is also something that can be uh just globalized but this actually not you need to how prove it and some equivalent what you want to prove um so the issue there up there that it'sin um and so to do the actually is second subass of n objects and this a plus and these are soal nuclear objects object so maybe we will discuss them much more generously uh in much more detail later let me just give a um quick definition here so all the speed ones will be modular but things are somewhat more General than that um so in particular they contain all B modes but not the Smith ones um so let's say C so the name is inspired by this uh class of nuclear Vector space gr defined in function analysis um it's not a completely precise translation but like in spirit it is one it's the following condition satisfied so it takes the internal home P so p as usual is my I should have said before I mean this is my usual mod so you have that the other one way to produce Maps is to start with a dual of B and then tensor uh and these are in some kind of of Trace CL so this in some way incling the idea that all M from uh so a couple of comments about the definition um so in general the nuclear actually in this case uh the nuclear modules yeah generated the colon shifts um by uh just B up module Z function from if you want it also a Jo T but just as a mod um so in general um you can generate them by full limits um the um in general referring to more General analytic Rings um by where these are all Trace meaning they come from they come from element and Qi dual s whenever you



have an element in the stand Pro you can produce a  $m$  from  $Q$  one these are all Trace class and then if you take a sequential form of Trace class Maps uh you can check using that internal home comms uh limits that people always have and conversely when you try to start presenting nuclear object in terms of How It's generated by compact projectors and some take a five comple compact projectors mapping into  $C$  but then that map will actually fact will be Trace class and you can Factor it over like Trace class map right okay and so yeah in particular like over  $QP$  this is stuff generated by  $B$  uh so this is still a rather large class of guys it will actually I mean there actually Ser about cies and so on one very nice property this nuclear module that form a category Universal lra approximation in some sense to the right hand side um and using this you can also Define the cas the of tic spaces uh which is and see this but taking us the field uh um so back to here uh so you have the sub of ble guys and uh so the finest Class end then you show that this top Global and so so you want to check that if you have some  $C$  and it satisfies this condition locally then you want to check it globally uh and for this ideal situation is that just that those sides of the isph can just localize well the right hand side is just some tensoring so the right hand side definitely like localizes um the question is for the left hand side also does and it actually does and so the key thing is that uh here said theal from  $P$  from loal but this is the same argument that just use last time to show the different localization with each other because they actually show that all the localizations are given by internal home from some object uh that's a different internal homes can meet with each other okay so I mean Peter I mean strictly speaking uh that's not um I mean for like an algebraic uh localization inverting an element um you know it's not literally internal hum from some object spaces and they all rational subsets like inverting an element is not a rational subset so but a general I should say what a suitable the large bations but yeah yeah I mean there's a and yeah in any case you can always refine further so that it is yeah I just wanted to just in case there was some confusion about that um well actually also right Ian also with the other types of because it's also Comm with element because it limits but okay um and so so now we isolate with two classes these things that are the sense of spe category uh and these Nu so excuse me so the the consider this question of commu localization it seems that the sketchy argument you gave doesn't use that you you put their  $P$  you can put anything not necessarily compact instead of  $p$  is it right and this community localization which is contrary to what we have for schemes where you have to take underlying home from something bit surprising so actually in this case also localizations um uh they Comm with all products because they have a left to joint which is something we'll discuss uh and which is extremely important for this whole theory of compax SP coherent mod that actually in these situations the pullback funter there's a kind of coherent lower Street functor which is yeah is the left joint so the pullback all limits but it's also uh satisfies projection formula lower stre and this is precisely uh equivalent to this internal home so it is true for anything instead of  $P$  it's true for anything instead of  $p$  uh this okay um for these variant it do mentioned it's however again important that this col but for the precise localization we're interested in here to Russ open subsets it's actually always true um right and I should wrap up and so then okay so we made two steps in our proof we isolate two different classes of uh modules that uh can be glue uh and then the last step is to like isolate then also intersection and then the last step is show that to Cent guys over Soo it's just an inclusion uh so they are certainly in  $s$  of spe categor and they also this is actually called and so then you finally shows that you can GL this two ones then there's still a little bit to show that all the other super and subscripts I put also glue but this is actually easy and so right so far I didn't do anything that seem remotely like analysis um but some of doing a vctor bundles there is actually some new verus power series that you have to undertake somewhere um so somewhere there has to be some like actual word right and so it's some on this step and let me actually uh show how it's done uh intersection and uh so we can assume we have some compx and by shifting you can assume it starts  $z$  um and and I also said you can assume that these are all just okay you have such a complex how we know that it is nuclear right but now but now I certainly have a  $m$  from  $P$  to here which is just the identity here and like my complex is just zero sare right complexes um this is a map from from  $P$  into  $C$  right now being nuclear it means that it's uh Trace class and uh so if you actually have a section in here then again you can Factor this over some approximation to  $C$  and so uh when you think a little bit about what this means uh then you realize that can actually um there exist a diagram like this uh where  $G$  say where where Gus PR but then this is actually subjective so this  $F$  can actually be split back and da da da da and so you get actually some morphism here and actually what you uh see with this is that the map from  $P$  to  $C$  tectors over um the me phone uh oneus  $g$  to  $p$  for some and  $G$  of  $P$  is just some playing around with I know the fact that  $P$  is projected and so on and the definition of nuclearity redu pH  $F$  things

and so so right so in the end you see that there is actually a way to put oneus here and so then it's actually enough to show that this guy here uh it's actually lies it's actually perfect cont and certainly this is this representations have written it doesn't show that but they claim far as to compx that these are just three modules here and this is basically classical fre home Theory so if you have a trace operator the identity map is fre operator and then your this by TR C in particular compact operator and then any perturbation of a fre Operator by compact operator still fre fre operator some have properties that are kernel and cor kernal dimensional and this is argument that you just have to execute slightly more carefully in the situation so this is some pH operator on this like bace something like this and then uh uh you can show that yeah some the cone on F operator is a perfect complex which is better way of thinking that's the K and generate and so this is the only place where you actually have to play R with some I mean you have to Matrix representing G and then you play lower it's easy but there this is a step where you actually feel like you're doing somewhat a little bit of analysis okay that is questions so anyways to understand this so before you you erased actually the Blackboard is fuzzy you can't see it you can't read the the right Blackboard from here uh I don't know what what's but before you erase what you have written before you had a factorization where P okay now it's okay so P2P you said that you factorize it through p and one map is Trace class right but it it seems to me that it is you have to use hom I mean the data will not give you that the map from P2P is is you you need some homotopy to do it I mean uh I think the data is probably that where it's executed look at the complex geometry not where we do precis the same argument like this liquid Vector space and so it is UN for thing that I'm probably screwing up if I try to do it right here but but in any case the data probably means that the map from P to the complex is homotopic to a map that factorizes through a finitely generated freea module up to homotopic yeah and then you can use this finally generated frea module to modify the so this way I can modify the complex that now degree zero it becomes zero and then I can just keep going and yeah yeah okay well maybe it's the same approximately yeah thanks thanks Peter did you figure out the question by the way no sorry see

## 11. SOLID MISCELLANY (SCHOLZE)

[https://www.youtube.com/watch?v=87-wuqGA8GE&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=87-wuqGA8GE&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

Okay so I'd like to start um welcome back everyone uh so I think um we're about ready to move on to a new topic but I just want to um finish off our discussion of the solid Theory with some miscellaneous miscellaneous uh facts so um so recall that we had this uh procedure that if uh we had a pair say a  $A_+$  for this is some solid ring and this is  $A_+$  uh a sub ring of power bounded elements uh oh sorry just one um then we associated to this well a certain solid analytic ring uh which was specified by this mod  $A_+$  um and it was um obtained by uh forcing all of the elements in  $A_+$  to be sol to become solidified variables um but recall that we also had the thing on the level of the derived category and this is so what would if you if  $A_+$  if  $A_+$  is not a subring of a  $z$  ah then then when you enforce this condition you might lose the uh a might not might no longer live in that category so you'd have to apply the localization and get a new a a sort of sort of the completion in some sense okay yeah um but we also had this uh so and recall that this was defined by uh this was defined by as it was defined as a full subcategory of here consisting of those objects whose homology lies in here for all  $I$  and we showed that uh you have a nice left adjoint with the exact same properties as the left adjoint here symmetric monoidal and so on and so forth but there was this subtlety that this is not necessarily um well in the full generality of an analytic ring this is not necessarily equal to the derived category of this and the basic reason is that if you take some basic object here and apply derived solidification it might not live in degree zero uh that's kind of the obstruction um so in fact uh the generator here so so there's a compact projective generator which is equal to you take the compact projective generator we know in  $\mathcal{D}(Z)$  Theory and then you simply tensor it in the sense of the solid  $Z$  Theory with this solid ring  $A_+$  um and that we know that it doesn't matter whether you take the derived tensor product or the ordinary tensor product there because as Peter showed in one of his lectures this is flat in solid  $z$  um so that's what means that if you did made that the if you did the derived analog of this definition it would literally just be the derived category of this ailing category um but then what's the generator here compact generator generator of this derived category uh sorry uh of this thing so so what do you do you take this object here and you localize it with respect to this derived localization functor uh so you derived a plus solidify and uh maybe I'll say uh so theorem so first the I want to explain is that this this lives in degree zero and that means it's a compact projective generator for the ailing category uh mod  $A_+$  plus solid and uh and the derived category of this ailing category uh is the is the derived category we defined there so as long as your underlying solid ring is not derived in any way then in fact the whole theory is you know or that doesn't have any the whole theory is kind of flat so to speak uh determined on the ailing level um does it so is it a change from what was said before or or because I remember that it was not clear in the previous talk I mean it was not clear that this was well that's that's why I'm explaining it now yeah yeah um okay so so how what is the how do you access this derived as solidification so to so being being uh being  $A_+$  solid means that you're solid for any variable mapping in so um what I want to claim is that uh this uh so claim is that the this derived  $A_+$  solidification um uh you can actually write this as a filtered  $\mathrm{Co}$  limit so it's all happening element wise in  $A_+$  and you can write  $A_+$  as a union of rings which are finitely generated over the integers so let's say  $R \subset A_+$   $R$  finite type over  $z$  um and then you just forget to write the square in  $A_+$  plus  $d$  oh yes thank you yes I did yeah uhuh yeah because irritating asking the difference between when there is a square yes yes uh sorry that's yeah um so you so you can take  $M$  which is a priori an  $A_+$  module but you can  $A_+$  maps to  $A_+$  and therefore  $R$  maps to  $A_+$  you can treat it just as an  $R$  module um and then you can do the derived uh  $r$  solidification um so this is a filtered  $\mathrm{Co}$  limit so the point is well there's two things to check first of all you need to check that well there's a map you need to check that this is derived  $A_+$  solid um and you also need to check that um in this functor if you have something which is uh well yeah and then you need to check that if you map out to anything derived uh  $A_+$  solid then it doesn't see the difference between this expression and this expression um so why is it derived  $A_+$  solid um because derived  $A_+$  plus solid uh is the same thing as uh so is the same thing as derived uh our solid uh for all our subset  $A_+$  plus finite type just because it's simply an element wise condition on the algebra um and uh so so certainly the  $r$  term in this thing is derived  $R$  solid but also any further term in the uh filtered  $\mathrm{Co}$  limit after the  $r$  term is also going to be derived  $R$  solid because it's even has an even stronger property of being derived like  $\mathrm{RP}$  Prime solid where  $\mathrm{RP}$  Prime is bigger than  $R$  um so there's a co-final system of things in

this filtered cimit which are derived  $R$  solid for any fixed  $R$  and we know that a filtered  $\text{Co}$  limit of  $u_h$  of  $R$  solid things is  $R$  solid so it's going to be  $R$  solid for all  $R$  and therefore it's going to be  $A^+$  solid  $u_m$  and  $u_m$  so and then mapping if you map out to something a plus solid then in particular it's  $R$  solid for every  $R$  and by a similar argument you see that there's no difference on maps from  $M$  to this or this filtered cimit to this okay so so it suffices to analyze to show  $u_h$  that if you take this product  $z$   $u_h$  tensor  $Z$  solid a  $u_h$  and then you derived  $R$  solid  $u_h$   $R$  solidify  $u_h$  this lives in degree zero  $u_m$  but  $u_h$  this  $u_h$  we can do it in two steps so we can take we can see this as you take you first base change to the solid  $R$  Theory  $u_h$  so you take this thing and then you derived  $R$  solidify  $u_h$   $u_h$  and then you tensor in the derive tensor in the solid  $R$  Theory  $u_m$  with a so it suffices to see  $u_h$  that  $u_m$   $u_h$  it suffices to see that well first of all that product  $z$   $u_h$  tensor solid are derived solidify  $u_h$  well this lives in degree zero and is flat  $u_h$  with respect to our solid tensor product okay so by the way let me not to before I continue with the so basically we have to analyze  $R$  solid when  $R$  is finite type over the integers and prove analogues of what Peter already explained for solid so  $u_h$  we have to calculate this basic generating object make sure it lives in degree zero and we also have to check that it's flat  $u_m$  but before I go on and do that let me note corollary which is kind of a corollary of this filtered cimit type of argument so  $u_h$  if  $u_m$  so we have a compact projective generator of this aelon category and I basically just analyzed for you that it's gotten by some filtered cimit of  $u_h$  the thing you see over a finite type ring  $u_m$  so let me say that  $u_h$  so so if  $R$  is arbitrary commutative ring  $u_h$  then well one thing you can say is that  $u_h$  solid  $R$   $u_m$  by which I mean solid  $R$  comma  $R$   $u_h$  so you solidify all of the all of the elements in  $R$   $u_h$  this  $u_h$  is equal to in of the finitely presented  $u_h$  elements in in solid are  $u_m$  and the finally presented things are the things which are just co- kernels of  $u_h$  of maps between finite direct sums well you don't need that but of these comp of this single compact projective generator that you have here  $u_m$  and so also  $u_h$  if you want to know about this finitely presented category it formally reduces to the finite type case  $u_h$   $u_m$  so in some sense all of the completion is happening at the level of finite type sub subrings  $u_m$  and then the rest is just some algebraic filtered cimit  $u_m$  all right  $u_h$  so right so now we need to do this analysis here  $u_m$  so I'll make a a claim  $u_h$  which is basically that it it works just like in the case of solid  $Z$  so if you take product  $Z$  tensor solid  $z$   $r$  and then you derive solidify  $u_h$  then this is just the same thing as a product of copies of  $R$  and if I forget to say an index set it's always going to be countable  $u_m$  okay sorry finite yes thank you yeah our finite type over  $Z$  thank you very much so by this discussion if  $R$  is not a finite type what you would instead be seeing is so if  $R$  not finite type then  $u_h$  it's equal to the union over all finite type subrings of the product of copies of  $RP$  Prime which is smaller than the product of copies of  $r$  okay  $u_h$  so proof so so so first consider  $u_h$  the polinomial ring on finitely many  $u_h$  variables  $u_m$  then we've  $u_h$  essentially already seen this  $u_m$  so we saw that the derived  $u_h$   $T$  solidification of of a free module  $u_m$  commutes with all limits and colimits and  $u_h$  and sends  $Z$  to just the to to  $Z$  bracket  $T$  so  $u_m$  well so if you do  $X_1$  solidification so  $u_h$   $u_h$  then that just gives you product  $u_m$   $u_h$   $u_h$   $X_2$   $x_n$  and then you just keep going you do  $X_2$  solidification  $X_3$  solidification and so on and it always just throws the variables inside the product instead  $u_m$  and in the end you you do end up with product of  $zx_1$  up to  $x_n$  so here we're so here we recall  $u_h$  that the if you do if you want to if you do  $ZT$  solidification of a a module tensor over  $Z$  with  $Z$  bracket  $T$   $u_m$  this commutes with limits  $u_h$  in  $m$  and it sends  $Z$  to  $Z$  bracket  $T$  and we're also using that  $u_h$  the different solidifications commute so that you can actually do the solidification with respect to all of them by just doing them one by one and when you do the  $X_1$  solidification and you  $X_2$  solidified it'll still remain  $X_1$  solid you also use that if you solid if it's  $X$  and  $Y$  solid then it is Sol for everything yes in the subing generated yeah yeah yeah exactly so that you know so that to solidify for  $R$  it's enough to solidify for all the variables in  $R$  yeah don't well right but we we're using it oh well this is yeah this I guess it also follows yeah that's true the argument for that was basically this anyway but okay  $u_m$  it's probably good to mention it in any case  $u_m$  okay  $u_m$  now in the general case so let's say we have a a quotient so  $X_1$   $x_n$  rejecting on to  $R$   $u_h$  then  $u_h$  sorry  $u_h$  product  $z$   $u_h$  tensor  $Z$  solid  $R$  derived  $R$  solidify  $u_m$  you can get it in two steps again you can  $u_h$  solid  $u_h$   $u_h$  and then  $u_h$  you've already solidified with respect to a generating set for  $R$  so you're actually already are solid and the only thing you need to do is do an algebraic base change from this ring to the ring  $R$  so derived everything should be derived  $u_m$  but we already figured out what this is that this was the product of  $zt_1$  up to  $T_N$  and then we  $u_h$  so we're left with just analyzing this  $u_m$  and proving that it's equal to a product of copies of  $R$  but  $u_h$  so  $zt_1$  up to  $T_N$  is mean that means  $R$  can be resolved by potentially infinite resolution but a resolution by finite free modules and for each of those finite free modules it's clear you just bring it into the product  $u_m$  and then it follows that the same holds for  $r$

on the level of this derived tensor product  $\otimes$  so then so you can look at the class of  $\mathbb{Z}x_1$  up to  $\mathbb{Z}x_n$  modules for which this claim is true with  $R$  being replaced by that module and that class contains the Ring Of course and therefore it contains anything that can be resolved by uh direct sums of copies of the ring as a also the product the infinite product of copies of  $\mathbb{Z}$  TN is I think flat over  $\mathbb{Z}$  TN again it's a property of  $\mathbb{Z}$  or coherent yeah at least in the as a the usual product but here it's condensed Mass prod but still I think for every condensed set it's the value of this on it probably is yeah yeah that's another that's another argument that's another possible argument but I mean for me this is a little more a little more more straightforward yeah yeah because the flat I I believe I don't remember the flat thing just uses coherence ahh okay so the flat the product of flat is flat is coherent dring but now you use ideal is is a finite  $Ty$  to actually see that when you quotient out you get a product of copies of  $R$  otherwise it would you would get a limit of product of of  $\text{Co liit}$  of product of finite fin type so and and again it is condens set by condens set it's the same it's it's verified by Elementary way yeah I mean for me this is pretty Elementary but okay I I don't know no because in theory to compute this tensel  $L$  you have to resolve on the left I mean  $t$  is different meaning I mean when you  $B$  change from of ring to another ring you are supposed to resolve the original not okay you can also compute it by resolving the other I don't need to I mean I I really this argument just works in the  $D$  category I don't need to I don't need to worry about resolving this it just I the the argument is I I claim there are no gaps in this argument no because when you reserve a  $\mathbb{Z}N$  you are living the categories of  $R$  and you have a  $d$  category of some of no no no no no then but the map is Def find already so the yeah the to cheog you can pass to the  $D$  categor if no I have a resolution in this algebraic category and then I just I have this tensor funter taking me to this  $DED$  category and that tensor funter preserves co- limits and I get a resolution there it's it's really um it's really fine um okay so yeah this is this is quite technical I'm afraid um but uh so we've proved now this uh this claim that this drive solidification lives in degrees zero and that was the first thing we needed to show but the second thing we needed to show is that it's flat uh with respect to the tensor product  $\otimes$  so for this we want uh we're going to analyze this category of solid  $R$  modules in a little more detail so this structure of solid  $R$  um so again so let me say maybe theorem so again  $R$  finite type over  $\mathbb{Z}$  uh then well something we already know um from this being a compact projective generator is again that solid  $R$  uh is just the in category of its compact objects which are the finitely presented ones um so these are the things that are co- kernels of maps between the compact projective generators um but this CLA second claim is uh so you could say coherence uh this category this subcategory uh is an ailan category is is closed under uh under kernels co- kernels and extensions this is same as this Corine uh which Coraline yeah the first part is the first part is obvious or sorry well the first part follows from the fact that we have a compact projective generator of this category then uh the compact objects will all be built from finite co-limits from that and they will be generators and it follows formally that it's the end category of of that um so what we really need to check is that we have this this is an a bilon subcategory now let me remind you a bit about some classical commutative algebra so a commutative classical commutative ring is called coherent if you have the analogous property in the setting of discrete modules so that the finitely presented  $R$  modules form in a billion subcategory but you can check by some by playing around with short exact sequences that you only need to there's only really one thing to check uh which is that every uh finitely generated ideal uh is actually finitely presented and those same arguments in this setting uh show that it suffices to check so every finitely generated subobject of product of copies of  $R$  is finally presented so so CF coherent Rings now I'm going to use a bit this notion quasi separated so note that uh if you have any subobject of product of copies of  $R$  well this is quasy separated I.E house dwarf and any subobject of anything quasy separated is also quasy separated so this is also quasy separated um so it suffices to show for this it suffices to show that every finitely generated quasi separated uh solid  $R$  module is actually finitely presented so let me make a Lemma so so if  $m$  in solid  $R$  is quasi separated uh then the following are equivalent um one is that  $m$  is finitely presented two is that  $m$  is finitely generated and three is that well you can say exactly what  $M$  looks like uh so  $M$  uh is an inverse limit of  $m_n$ 's or each  $M_N$  is a finitely generated  $R$  module discrete  $R$  module um and the transition Maps forus projective and the and the category of these uh is just the countable Pro category of the finite of the category of finitely generated  $R$  modules so the maps between such inverse limits are just the the maps of pro objects Pro objects with subject of transition yes thank you thank you yes yes yes yeah yeah okay so so the non-trivial implications are two implies 3 and 3 implies 1 um so for two implies 3 uh so well we finally generated by definition means we have Sur some surjection from product of copies of  $R$  and then there's some kernel um but since  $m$  is quasi separated uh it follows that this inclusion is a



quasi compact map of of of condensed sets which means so uh this object is quasi separated so it's just some filtered Union of compact house doorf spaces and this means that when you restrict this inclusion to any of those compact house door spaces you get a closed subset but um this is one of those sequential spaces metrizable spaces in fact um and so being a closed subset on any compact subset is the same thing as just being a closed subset so this is the same thing as just a closed subset in the topological sense uh with the product topology CL Subs which is also in our module this is the same associate condensed module is the one that we are talking about right uh so let me so then um but now we can look at for any  $n$  we can look at the projection so we have  $K$  uh subset product  $R$  we can look at the projection onto like the first  $n$  coordinates uh and then we can let  $k_n$  be the image in here uh then it follows that uh  $K$  is just the uh the inverse limit of  $K/N$  uh and these have subjective transition Maps I.E so  $K$  is as in three no you know that  $m$  is as in three but this also uh right I wanted to know that  $m$  is as in three but this also follows yes um uh and that yeah then yeah and  $M$  analogously be uh inverse limit Over  $N$  of  $M/N$  uh where  $M/N$  is the product uh okay and note that this argument shows that um if you so in particular if you start with an object as in three uh yeah okay so now let yeah so now um we also see uh to finish I.E to show that 3 implies 1 it actually suffices to show uh 3 implies 2 because if we know that this guy is fin generated then it fits into something like this and we just prove that the kernel is of the same form and then it will follow that it's fin presented um um so but then this is actually Elementary so you have like  $M_1$  subjecting onto  $M_2$  subjecting onto  $M_3$  you can make some uh finite free module rejecting onto  $M_1$  um and then you have some here uh kernel of this map here uh and then you can make some  $R$  direct some  $D_2$  rejecting onto that uh and then uh you can make a system  $R$  direct some  $D_1$   $R$  direct some  $D_1$  plus  $D_2$  uh uh um and in the inverse limit you get a product of copies of  $R$  rejecting on to  $m$  so in fact uh if you're wondering about surjection on the level of condensed objects in fact you can even get a topological splitting for the map of topological spaces so by kind of section here make a compatible section there and so on if you're not worried about our linearity which you're not then uh you can get this quite easily um okay so that's the proof of the Lemma so the quasy separated objects are just these kind of classical these classical  $R$  modules um and that also uh proves the um it proves the uh the coherence here in extensions is easy extensions is formal yeah yeah it doesn't require anything uh because it's projective projective generator so it's yeah you can lift exactly I mean actually uh because you have this compact projective generator you're equivalent to the category of modules over a ring and so it's really it's you can actually just quote the usual theorem from commutative algebra um okay yes all right like so here um so if  $M$  was finally generated in particular the kernel also had this really nice property of being of having property three as well uh if  $M$  was quasi separated and finitely generated yeah and so that in particular implies that if  $m$  is finally you can find a resolution of by really nice like yeah that's right that's right you said you said something about Comm in fact the endomorphisms of the projective object is an associative yes oh yeah sorry yeah not not commutative algebra sorry thank you thank you it's not commutative algebra it's it's module Theory over non-commutative ring yeah thank you yes wait we're still proving the theorem uh we're done with that theorem where did we use the fact that was finite type over  $Z$  sorry ah well it it was used in order to say that this was the compact projective generator so when we were so a prior you take product  $Z$  tensor up to  $R$  and then solidify with respect to  $R$  and only in the finite type over  $Z$  case when you had this rejection from The polom Ring yeah but for any narian ring you can take the product of copies of yeah you if if you just say that you build a category that has this as compact projective generator with the endomorphisms what they have to be then the rest of the arguments here work in the for an arbitrary etherion ring yeah okay um right and we're not quite done with the uh the first theorem because we still need to show that uh product of copies of  $R$  is flat so now but so now we make a claim that if uh  $m$  is finitely presented uh in solid  $R$  uh then if you take  $M$  derive tensor product  $R$  solid with product of copies of  $R$  you just get a product of copies of  $M$  and in particular this lives in degree zero no um and that implies uh that this theorem or this claim implies that this guy is flat uh because the derived tensor product with any finally presented object lives in degree  $Z$  but every object is a filtered cimit of finitely presented objects so the drive tensor product with any object will live in degree  $Z$  which is the same thing as saying you have flatness so incidentally to define the the right pens of maybe we didn't spend time on this but in the usual approach you need to know there are enough objects to relative to  $T$  to define classically  $t$  so here what is the approach what's the meaning assigned  $d$   $t$  do you prove that there enough  $L$  it's not really necessary so first on the level of the derived category of light condensed ailan groups um I mean you can take this perspective of sheaves with values in the OR hypers sheaves with values in the

derived Infinity category and then you can just sheify the sectionwise derived tensor product of aian group  
 derived sheify the sectionwise tensor product of derived aian groups and that's a fine definition of the derve  
 tensor product on that basic thing and then it just gets ported to all of the other contexts uh you know you  
 have a ring object in there you have uh then you just do a relative tensor product and then uh you have  
 some localization but we already argued how to get the symmetrical structure on the localization of given  
 given by an analytic ring structure and so on and so forth so it's not actually to make the definition it's not  
 actually necessary to know anything about flat objects so for for condensed a billion ah okay so let us say  
 without Sol you just have the T of okay there we know we know initially there enough FL in indeed in fact in  
 in light condensability groups we do know there are enough flat objects so that's I didn't need that Infinity  
 stuff really to you're right yeah yeah but I mean yeah I have to say I was kind of off the top of my head  
 just saying how I think about it yeah these free object free object on a light condensed set will itself be flat  
 and um yeah okay uh okay so where was I ah so we have we have to Pro to finish the uh discussion of so far  
 we just need to prove this claim um so proof um so we are finally presented so we can make uh a surjection  
 here where the kernel is finitely generated but the kernel is also uh quasy separated um so you know it'll  
 also be finitely presented well we can then it'll we can basically just continue to make an infinite resolution  
 so uh and that reduces us to the case uh where M itself is equal to this product of copies of R um and then  
 uh and then we can remember that this was product of copies of Z tensor R and then R solidified and then  
 that uh will reduce us to the analogous fact for products of copies of Z that it distributes over uh infinite  
 products so for prod infinite product of Z for Z ah then you reduce to to Li condens abil groups is infinite  
 produ of Z and yeah there are even the thing so then you have this nice object that solidifies to a product of  
 copies of Z this uh this thing we were calling P basically n Union infinity and that light condensed set has  
 the property that if you take its product with itself it's isomorphic to itself in the in the expected way  
 um so or like that that object for n times itself is the same as that object for n cross n and yeah you just  
 well anyway it was explained um okay um any other questions because this is projective uh this is the prod  
 copies of the projec generate okay but if you did not define t l is a classical Left Right F you cannot use the  
 fact that for projective object I mean this requires a classical treatment yes but right but so the actually  
 theun of the T right that that I could have added to the um I could have added that to uh yeah that's right  
 that's another important claim in addition to the fact that the derived category of the aian category is the  
 derived category that we defined you also have that the derived solidifi or derived solidification is the left  
 derived functor of solidification you have that the deriv tensor product is the left dve functor of the tensor  
 product and you can resolve an either variable this all follows once you know all of this this flatness and  
 living in degree zero stuff so again to so the foundational way you define tensor L for your deriv category of  
 solid is what I mean without using flat objects yeah so the way you do it is that you of course it lives in some  
 dve category without without solid so you can just take tensor there and solidify yes exactly of course you  
 have to know that that somehow then part of this story is that solidification commut I mean if two things  
 have the same after I mean you have to know but then to define derived without solidification then then it  
 is just derived over this is just modules over a ring supp you already know over Z of the find yeah then over  
 a ring you use flat object or what do you use you you never have to use flat objects to make the definitions  
 okay so how do you define the the derived tenso product for without the solid yeah yeah you have a you you  
 have a commutative ring in solid Z and you want to do dve tensor product I mean it explicitly it's you just  
 you know it's some geometric realization of a bar construction where all the tensor products over solid Z are  
 derived I mean so Lori sets it up if you have a commutative algebra object in a symmetric monoidal Infinity  
 category then you get a tensor structure on our modules in there um okay so you can have in this General  
 setup you have some way to yeah I mean it's the it's basically like you take M tensor Ln M tensor LR tensor  
 Ln You Know M tensor LR tensor squar okay and then you have some simplicial object and you take the co  
 limit all in this Infinity sense okay so you have this General way of viewing the tens of products and then it  
 always in all cases there enough flat objects and you can use flat or okay in in the in in what we've discussed  
 so far the solid theory yes that's what we've just proved although as you see it's not a formal argument  
 by any means you really have to analyze uh what's going on other questions okay um so oops so I want to  
 spend just a little more time on the homological algebra of solid R um prove because Peter also explained  
 something else in the case of solid Z basically that every finitely presented module it doesn't just have an  
 infinite resolution by products of copies of Z but it actually has a two-term resolution by products of copies  
 of Z so that kind of at least from the perspective of finitely presented objects you have sort of homological

Dimension One for solid  $Z$  just like you have for usual  $Z$  so um I want to present a generalization of that  
 um so so uh theorem so if  $R$  is finite type over  $Z$  and is regular a regular ring of Dimension  $d$  uh then if  
 you have a  $m$  in solid are finitely presented and you have  $n$  and solid are arbitrary say uh then  $X$  I even  
 internal  $X$  why not uh uh the  $X$ s vanish uh in degrees bigger than the dimension and the to also right so a  
 corollary of this will be that if you have a finitely presented uh module then you actually get a projective  
 resolution of length uh in most well depending on how you define length  $D$  or  $D+$  one yeah maybe yeah um  
 and then we already saw that the projective objects are flat so this also implies a bound on the tours so so a  
 corollary is so if  $m$  is and that and that passes from fin presented to arbitrary uh let me say hi yeah okay  
 um let me caution uh uh this this this this statement does not extend to non-finitely presented  $M$  so unlike  
 unlike in the case of discrete modules over a nean ring regular nean ring of finite Dimension but there you  
 actually have the  $X$  Vanishing uh for all modules um there's some extra argument for going from the finitely  
 presented case to the general case that argument does not work in this context and in fact um I think you  
 can have arbitrarily High  $AXS$  even for solid  $Z$  so let me give a here's a fun a fun exercise you can try to  
 do so you take two distinct prime numbers uh this is just just cute so here's something that's not finitely  
 presented uh because you're quotienting by something which is  $z$   $Z_1$  over  $L$  um then if you do there's an  
 $X_2$  which is nonzero and it's given by switching the roles of  $P$  and  $L$  uh so that's just if you want to have  
 a little fun with some calculations and the  $n \times$  to and you claim also there arbitrary high  $x$  I believe that  
 so certainly there so certainly uh there exists a model of  $zfc$  in which there are arbitrarily high  $x$  so there's  
 logicians analyzed a particular uh  $X$  Group which I think Peter also mentioned in one of his lectures but  
 I think even in  $zfc$  it should be possible to produce a arbitrarily High non-vanishing  $X$  I mean even under  
 in any model of  $zfc$  I mean you run to a priori if you want to go from the finitely presented case to the  
 general case you have to analyze derived inverse limits along some arbitrary filtered system but do you  
 have a boundary cardinality because if you are in a model with the Contin hypothesis for  $FS$  yeah and if  
 the you look at or maybe generalize I don't know if you look at all possible finite types of modules yeah so  
 they  $M$  it's related to maths from product of  $Z$  to product of  $Z$  yeah then cality of this space is like one of  
 those some finite aln then we have chological Dimension results for this yes yes yes yes yes so it suggests  
 that that there the would be bound for sub for quotients of  $s$  by by of a project something but then there  
 is a classical argument in ring which probably extends to this case that that if you know the  $X$  Vanishing  
 for this kind ring mod IDE and you have it for arbitrary yeah but so but you need to but that the analog  
 of that argument shows you need to analyze something like an arbitrary subm module of product of copies  
 of  $Z$  not necessarily finally generated and yeah and yeah and then yeah but I guess yeah there's a bound  
 on the cardinality there yeah so yeah you're right that the cardinality of what's that Peter it could be that  
 something have maybe yeah it's possible yeah um but if the cardinality of the Continuum is larger than alfn  
 for any  $n$  then then you should probably get um okay uh arbitrary higher  $X$  anyway let's let's uh let's not go  
 there um right uh where are we ah so let's let's try to prove this so um so the first claim so first we're going  
 to so first case uh we're going to take  $n$  so ah so we well so so by the Sorry by the same argument um sorry  
 just a sec let maybe I'll do that at the end so the first case let's say that  $n$  is actually a know a classically  
 finitely generated module over so the uh nean ring  $R$  um and then let's take  $M$  to be quasy separated a  
 finally presented solid  $R$  so then that's an inverse limit of  $mn$ 's uh over all  $n$  uh with surjective transition  
 maps and in this case the claim is um that  $X$  I uh from  $M$  to  $n$  uh or I guess it doesn't even need to be finely  
 generate let's just say discreet uh is the same thing as the filtered Co limit of  $x_{ti}$  over  $R$  so this is  $R$  solid  
 um  $x_{ti}$  over  $R$  from  $MN$  to  $n$  didn't you Pro that par present impli qu separation uh no no certainly not uh  
 I mean in solid the solid  $Z$  case something like  $ptic$  integers mod integers will be fin presented but not quasi  
 separated so what what did be Pro proved that if you're quasi separated and finally generated then you're  
 finally presented uh and if you're fin presented are you ah you are not necessarily qu SE exactly exactly  
 yeah so for example  $zp$  modulo  $z$  uh is finally presented so if a map from product of copies the product of  
 copies of  $R$  is the coal is not cly separate not necessarily that's right okay I misunderstood this uhuh yeah  
 that's important um yeah um so we need to show that so we need that the arom uh you can pull out the  
 inverse limit uh basically uh by the way the claim for internal follows from the claim with underlying it's  
 just replacing  $n$  by like continuous functions with values in  $N$  for some uh for some profinite set  $s$  um but  
 so uh we have the mitag Leer so we have the resolution uh some kind of usual identity minus shift sort of  
 thing or maybe it's um or shift times  $F$  or some some kind of shift map um so this distinguished triangle  
 here so that tells you that the question of pulling rhom the inverse limit out of the ROM is the same as the

question of uh pulling out a product and turning it into a direct sum um and then we can resolve product  $MN$  by product  $R$  to reduce to  $ROM$  product  $r$   $n$  equals direct sum  $R$   $RN$  uh which follows from this being uh projective and yeah and the basic calculation on home okay so from this it follows that we get the desired bound uh on  $X$  if  $n$  is uh a classical discret  $R$  module and  $M$  is finally presented and quasy separated now you can always resol so now  $M$  arbitrary finally presented not necessarily quasi separated then you can always resolve oops  $M$  by product of copies of  $R$  and then the kernel will be both both uh finitely presented and quasy separated and uh then you analyzing the long exact sequence in  $X$  mapping out to  $n$  you would get the desired result but with uh you'd lose one degree of chological Dimension you'd get that  $x_i$  vanishes for  $I$  greater than  $D+1$  um so the  $x_i$  vanishing here for  $I$  greater than  $D$  when you Chase through the long exact sequence would only give the vanishing in degrees greater than  $D+1$  there so uh so this naive argument only gives uh  $x_i = 0$  for  $I$  greater than  $D+1$  so we have to do a little bit of extra work to improve this for that conclusion there we didn't use regularity uh well we did because I should maybe recall the classical fact so weall so if  $R$  is regular mean Dimension  $D$  then the classical  $X$  groups you know between discrete modules so this so that the the question now for quasy separated things in the solid context reduces to the classical question in discret commutative algebra and that's where we use regularity yeah um so we're going to do a less naive argument so we're going to continue the resolution one more step so so instead so we have a product of copies of  $R$  we can Sur again from a product of copies of  $R$  and now at this point we take a kernel  $um$  and we get a resolution like this  $um$  this is still quasy separated and then it suffices to show uh to so to prove uh  $x_i = 0$  for  $I$  greater than  $D+1$  I'm so sorry um ah so let me let me say  $x_i$  uh for  $I$  greater than  $D$  it suffices to show uh for  $I$  greater than  $D-2$  so we bought ourselves one extra uh uh drop um because we continued the resolution One Step more um say  $Z$  and one have to be head that's true yeah so this will I'll be assuming so assuming  $D$  greater than or equal to two the case  $D$  equals 1 well you can use similar arguments in low Dimension yeah thanks Peter one the same as for integers I'm not entirely sure but a modification of this argument for  $D$  greater than or equal to two will will also work for  $D$  equals 1 um right so it suffices to prove this  $x$  Vanishing so now we want to um uh now we want to find a nice expression of  $n$  as an inverse limit of finitely generated  $R$  modules so there's actually two different two ways of writing uh  $n$  as an inverse limit of a finitely generated  $R$  modules so the first way is just the what we have the argument we already had where this is  $um$  this is the kernel of a map between quasi separated things so it's a closed subm module here um so we can always just look at the  $um$  well maybe I'll make that the second method meod sorry the first method would be uh we analyze this map here  $um$  this is a map from product of copies of  $R$  to product of copies of  $R$  again that's the same thing as a map on the Pro system so if you uh if you restrict to any initial chunk of this then there's some corresponding initial chunk here where the the map projecting onto that factors through projecting onto this chunk and then just a map of discrete modules there so if you reindex and allow like uh finite free modules here instead so uh then you can you can assume that uh this map here comes from an inverse limit of compatible maps from  $um$  you know the initial chunk the  $n$ th initial chunk here to the  $n$ th initial chunk here just by just by reindexing um and then if you take the kernels of those Maps so you take uh so well let me say I don't really need it to be a product anymore so so we take an inverse limit of this type and we say that this is given termwise  $um$  uh so then we can take so we can let uh so let uh so then we have  $n$  equals inverse limit Over  $N$   $NN$  but the other thing we can do is we can let uh  $n$  Prime  $n$  which will be contained in  $NN$  uh contained in  $FN$  be the image of  $n$  mapping to inverse limit of  $FN$  mapping to  $FN$  or the particular  $FN$  inverse limit  $FM$  mapping to the particular  $FN$   $um$  so uh number one has uh so number so each of them each of these presentations of  $n$  has something good and something bad about it so uh good news for one is that you have this uh  $XD$  minus 2 Vanishing  $x_i = 0$  for  $i \geq D-2$   $NN$   $x$  equals zero for all  $i$  bigger than exactly what you want  $D-2$  um but the bad news no guarantee that this system  $NN$  is MOG leer uh which is needed for uh  $X$  to  $I$  uh  $NX$  equals filtered Co limit Over  $N$  of  $x_i$  and  $NX$  so that's uh that's unfortunate so in the AR the argument I presented here in the case where the transition I said the transition Maps were subjective but the argument really only used that it was a MOG Lefler system  $um$  to get this resolution here  $um$  and then in two prime the situ in two the situation is opposite uh MOG leer in fact the transition maps are rejective by construction but no guarantee that  $X$   $um$   $D-1$  uh  $n$  Prime  $n$   $x$  equals  $z$  so  $n$  Prime  $n$  is this condensed module of no no it's just discreet so any any subobject of a discret object is itself discreet all right so we know that this is a but we know that  $n$  is the limit of both in the condens it's limiting the condenses of both of these systems yes okay but and certainly of you can produce an example of map from  $limn$  to  $limn$  Prime for which the the the kernels

would not be Mead left yes exactly yes you can yeah URS in nature yeah yes um right so what we have to do is find something in between that has the good properties of both so I apologize this argument is a bit technical by the way um the reason we have this nice property here uh is that it's a kernel of a map between uh projective objects uh so it's it's like the it's the you know so if you took the co- kernel then it would be two steps away from that Co kernel so the vanishing in degrees greater than  $D$  for the co- kernel will imply The Vanishing in degrees greater than  $D$  minus 2 for that long kernel over there that's why you have this and um so and but for this NN Prime you you don't know that it's a kernel of a map between projective things you only know that it's sitting inside a projective thing so You' get Vanishing of  $XD$  one better than for a general module but you wouldn't get Vanishing a priori in degree  $D$  minus one so that's what we need to fix um so what is the difference so find so we're going to find uh  $n$  Prime  $n$  sitting in between them uh and such that we get  $X$  Vanishing uh uh uh and uh NP Prime  $n$  is mler so if we get this then we're done uh because the inverse limit uh we would get the desired  $X$  bounds for the inverse limit of the NN primes but that's sandwiched in between these two things which have the same inverse limit which is the thing we're interested in so the vanishing of  $X$  for this inverse limit would imply it for uh our our desired Vanishing for  $n$  okay so now we use a bit of commutative algebra so so what is the obstruction we have a module where we know the vanishing in degree  $D$  but we don't know the vanishing in degree  $D$  minus one uh the obstruction uh having uh uh this is uh that so that the obstruction is in some sense concentrated at the closed point so it's a question of depth so the Outlander bbam formula tells you that uh at a for a regular nean local ring the projective dimension of a module plus the depth of the mod module is equal to the dimension of the Ring um and because we have the uh we have uh the one better estimate on the projective Dimension um the you know the only rings that are going to give us obstruction are the local rings at at Prime ideals which are actually maximal ideals so that the local ring has maximal Dimension and the only obstruction is going to be moving from situation of depth one at that maximal point to depth two so so is the fact that at a maximal ideal uh so  $n$  and Prime  $n$  only known to have depth one yeah yeah yeah yeah yeah many closed points and you know that it lies inside the other one because the other one is as this yes is reflexive yes and so uh and then you have to establish but only close point the classical L argument work that's that's correct yes okay all right maybe I'll just repeat what oer said so so what you can do if you let's let's say for Simplicity you only have a problem at one maximal ideal in general you'd fix a problem at finitely many and then if you increase the number of them eventually the situation would stabilize by an ethernet this so um so let's say there's a problem only at one maximal ideal then you can look at the inclusion of so Spec  $R$  minus that closed point  $x$  uh closed Point into Spec  $R$  and then you replace  $n$  Prime  $n$  by uh  $J$  lower star  $J$  upper star of  $n$  Prime  $n$  and uh that receives a map from  $n$  Prime  $n$  but it's in fact an inclusion due to the depth one assumption uh on our module um but then on the other hand by this procedure of this extension and restriction so depth is you can characterize depth in terms of local chology at the maximal ideal and uh that's exactly what comes up when analyzing the difference between this and say the drive version of this and using that you can see that this uh improves the depth so this is depth two uh at  $X$  um on the other hand if you were to perform the same construction for NN since we already know this has depth to you wouldn't have been changing it so you really are producing something uh sitting in between which fixes the problem at a given closed point then there's the neeness which tells you well it's only finitely many Clos points that are going to be involved so you can by the same procedure you can fix the problem all the problems for  $n$  Prime  $n$  um and in a compatible manner in the tower okay so that's then you've got it to be depth to at all the closed points which gives you the desired  $X$  Vanishing for these uh  $N$  double Prime ends uh and then what about the mid leer so so uh mtag ller so this Tower  $N$  double Prime  $n$  sits in a short exact sequence with  $n$  Prime  $n$  uh and the quotient  $n$  Prime  $n$  mod  $n$  Prime  $n$  so to show that this is Midler you need to know that this is mogler or it suffices to show that this is MOG Lefler and that is MOG Lefler this one we know because the transition maps are subjective but this one this one here is supported at but it's supported at finitely many closed points and it's a finitely generated module over the ring and that implies that it's finite as an ailan group because remember our ring was finite type over  $Z$  the maximal ideal is all of course have you know the resid field is a finite field and a compactness argument shows that any uh inverse system of finite aan groups satisfies the mogler property so compactness gives M left okay it's a bit of a yeah yeah um okay so that gives the  $X$  Vanishing so so the to sum up summary of what we've done so far we've seen that if  $m$  is fin presented solid  $R$  and if  $x$  I've switched to  $x$  uh is a discrete module you know over the ring then we get the desired  $X$  Vanishing um now uh so now suppose  $x$  if



$x$  is quasi separated and finally presented in solid  $R$  you write  $X$  as an inverse limit of  $x_n$  uh with surjective transition Maps  $u_n$  and then the  $R$  homs into that will just be the uh inverse limit of the derived inverse limit of the  $R$  homs into each of those terms now in principle you might get one worse again because there could be a limb one in the last inverse system but because that last uh uh in the that's that's talking about the XDS into this but since these transition maps are surjective and  $D$  is the largest degree at which you have nonzero  $X$  Vanishing you actually see that on XDS the you get surjective uh Maps so that there's no limb one potentially giving you something in degree  $D+1$  um so the you get the claim for  $D$  um and then uh  $X$  arbitrary finitely presented then you resolve it by uh and the  $X$  with values in  $X$  can only be better uh then the  $X$  with values in these two so you get that you get that situation as well and then finally for an arbitrary so for  $X$  arbitrary uh so if you write  $X$  as a filtered Co limit of  $x_i$  highs or these are fin presented then uh  $X$  to  $I$  uh from any finit presented  $M$  to  $X$  is the filtered Co limit of  $x_i$  oh no I shouldn't use  $I$   $X$   $J$   $M$  to  $X$  by pseudo coherence of  $M$  so you can resolve  $M$  by product of copies of  $R$  and then this follows from product of copies of  $R$  being compact projective what about  $X$  line it's the so yeah I should from this point on so uh this claim for an arbitrary discrete module implies the same claim for underline because it's the same as uh  $s$  valued points of this thing is the same thing as  $X$   $I$  from  $M$  to continuous functions from  $s$  to this discrete thing which is just another example of a discrete thing so then you get this there and then from that point on all of the arguments actually work at the uh at the internal a level okay including the Lim one argument including the limb one argument yes because you reduce it to to an  $X$  you reduce it to showing that a limb one vanishes in condensed to billion groups and the terms in the system are discrete uh and the system is MOG Lefler and that those properties are preserved by and if you take continuous functions with values in that it's still a system of discrete things and the transition maps are still mler or surjective even yeah and and in the condensed a bilan group again you have up to only up to  $L$  one yes you don't have more because you can compute it termwise yes because products are countable products are exact that's the reason ah okay the countable ah okay okay and you can compute the Lim one term by no maybe this is not you can compute Lim one on each object not necessarily project okay you don't have projective Compu one it's not like in any site where it's it's delicate you cannot compute but here you can compute the Lim one on any test condensed Set uh no no for some things yeah so certainly for this  $P$  object you can because that's projective um and then that's enough for solid because we know that the solidification of  $P$  generates um so that's that's one argument you could give I'm sure there are other arguments as well but yeah I mean but you know the limb one is always just going to be the sheaf of the uh you know the condensed limb one will always be the sheaf of the naive sectionwise limb one and so if prove the vanishing of the of the limb one section wise that's enough to prove yes I missed where coherence comes from ah right so that comes from the the the claim that the finitely presented objects form an ailan category which has as a corollary that for any finitely presented objects you can build an infinite resolution where all of the terms are products of copies of  $R$  you're welcome and to have non qu separated presented object the ring has to be of Dimension at least two or non quasy separated finitely present has to have Dimension at least one at least one yeah so if you have a if you're over a finite field then the every every finitely presented object is quasi separated but once you move to say the integers then then as I said  $\mathbb{Z}$  mod  $\mathbb{Z}$  is an example of something that's not quasi separated but is finitely presented also summary the step of reducing to uh reduce from qu separated fin presented to fin to discrete one uh is that EAS to it's not  $L$  one here like which Step sorry this step here yeah you see that there's no limb one because the the only the only limb one that can contribute to degree  $D+1$  is the limb one of  $X$   $D$ 's and because because on the level of these guys we know that there's nothing there's no  $X$  for any discrete module in degree  $D+1$  one you see that if you have a surjective map uh with then uh XDS into it will also be surjective because the obstruction is an  $XD$  plus one of the kernel which vanishes so then the XDS is also a mular system yeah okay that was uh that took longer than I anticipated um I had a couple of other topics I wanted to discuss but at this point I probably have to choose for so maybe I will um  $U$  maybe I just want to make a point um and illustrate it um so so recall so if  $R$  is now a discrete commutative ring so we we showed that that solid uh or  $D$  let's say  $R_{\mathbb{Z}}$  uh solid localizes over uh this valuative spectrum of of the Ring um this localization had the property and this is maybe why it's um convenient to anal one reason why it's convenient to analyze that if you on the on this discrete level if you take the unit object here which is  $r$  and then you restrict to any any of these basic quasi compact opens then you still just get another discrete ring in fact if it was the rational open like given by  $F_1$  up to  $F_N$  over  $G$  then the ring was just  $R_1$  over  $G$  so you with this localization you're kind of staying in

the world of of your ambient ring being discreet but this is by no means uh there are other are also other ways to localize the same category this is by no means the most General possible localization it's the one we discussed because it's the one that's most closely related to Huber's Theory and our goal in this class is to explain well the basic definitions in our Theory and their relations to more classical theories of analytic geometry but I just want to point out very briefly that um there's also so a whole different avenue you can go uh for um localizing this which even more radically I guess departs from Huber's formalism so we already saw a small departure in that we were allowed to do slightly more localizations of a Huber ring than before because of this difference between valuations that are less than one on topologically Neil potent elements and ones which are valuations which are continuous in Huber sense um but kind of even more drastic things are possible so um so in fact so if uh let's say  $C \subset \text{Spec } R$  is any constructible subset so that that means an intersection of a quasi compact open with the with a complement of a quasi compact open so it's some locally closed subset of  $\text{Spec } R$  which is sort of finitely presented and of those uh finite Union of those yes yes yes yes so the ba the basic let me say the basic objects which um sort of form a basis for the constructible topology would be something like you take  $D$  of  $G$  uh uh intersect um the complement of yeah the common zero set of finitely many functions uh uh uh then you can define an item potent algebra in uh in  $d, r, z$  solid uh namely uh to this on the level of this basic object um  $D$  of the complement of the zero locus of  $G$  and then intersect the zero locus of  $F_1$  to  $F_N$  uh to this you can assign the thing you get when you take  $R$  and then you invert  $G$  and then you uh derived complete along  $F_1$  up to  $F_N$  which uh this final object only depends on the the constructible subset um and you know by the way if  $R$  is an etheron then this derived completion is just the usual completion but I want to emphasize that I'm taking this derived completion this is a discreet object but I'm taking this dve completion uh in uh solid ailan groups or deriv category of solid ailan groups so I'm so to speak putting the inverse limit topology on this uh derived inverse limit here um no such things then you you say that you ah you view your constructive subset as a union of locally closed you also look at the intersection so on so for each of them each fin intersection you do this and then you take the what okay you take the limit of this type thing and it won't be concentrated in hom they'll have some chological yeah so is it the case the intersection of two basic thing what you get is a TENS of product yes okay so it's it is okay uh intersection oh yes yes that's right that's right sorry uh right so maybe I shouldn't actually say that you can assign item poent object to any constructible subset maybe I should just say that you can uh or is it okay or no well let me just say that you can assign a item potent object to any basic constructible subset so you want to take the kind of the derived inverse limit of that important object Associated to all basic thing inside the given thing but then you have a problem to prove properties of this I don't know if it follows that it is an algebra orru looking those subset May what yeah constructible locally closed is if it's construc locally closed Som take  $R$  GMA of this so it will could have some chological degree depending on the on number of Alpha and  $C$  to cover yes yes yes yes so that's but thank you Peter that's that's uh that's better yeah so then the only derived Behavior comes from uh kind of uh Union of principal opens thanks yeah that's what I should be saying um uh okay so now why are these item potent so so recall that um uh recall from one of Peter Peter's lecture that um the solid tensor product theyif me uh item potent means that yeah thanks so let's say that this let's say that this is a item potent means that a tensor over  $R$  uh solid derived a is the same thing as a um of derived complete  $RZ$  solid yeah um is derived complete so to check the item potency you're asking that that that implies that uh so you're asking whether a map between derived complete objects by this fact is an equivalence and that you can check by reducing modulo um uh modulo the uh regular or the sequence  $F_1$  through  $F_N$  that you're um that you're looking at and then it just becomes kind of an obvious fact about a a ring tensor itself being itself uh over ring tensor over itself being itself rather any localization of a ring tensor over that ring with the localization is itself so you did something like this before yes in one of Peter's lectures he proved the well at least a special case and and claimed it worked in general that the if you take a solid tensor product of two derived complete things um exactly in a setting like this so you have a ring and you have finitely many elements um or any number of elements I suppose and the drived solid tensor product of connective objects so in homological degrees uh uh will still be derived complete why because we proved it or yeah we Pro well Peter gave a argument for the special case of like  $P$  um and then but the the general case is quite similar so he he gave the the heart of the argument of the general case yeah you have to write yeah you have to you have to do some work you have to do do some analysis but it's not um yeah it's something we've we've discussed in previous lectures um okay so now uh and then so now the claim that I would like to make is that if  $C_1$  up to  $C_N$  are locally

closed uh constructible and if their Union say set theoretically is all of  $\text{Spec } R$  than uh these guys cover let's say these the corresponding  $A_1$  these item potent algebras uh cover uh  $\text{DRZ}$  solid uh in what sense so in the sense that uh so if you have  $m$  in deriv category of  $\text{RZ}$  solid uh such that  $M \otimes a_i$  equals  $\text{Zer}$  for all  $i$  then  $m$  equal zero but  $C$  yes thank you thank you very much yeah yeah so uh and this implies that you get uh  $\text{DRZ}$  solid is some limit over so in the first term you have product of product over  $I$  of  $A_i$  modules in  $\text{DRZ}$  solid and then you have product over  $I$   $J$  uh less than  $J$  it doesn't matter modules over the tensor product  $A_i \otimes a_j$  in the same thing and then so on you'll have some finite check uh thing what sorry cover if they love yeah I like that they're they're they're giving it a big hug that cover um uh yeah so if they love  $\text{DRZ}$  solid uh okay anyway um and the basic idea uh idea of the proof is that it suffices to show that  $R$  is generated by uh  $A_i$  modules for varying  $i$  um but um so so for if I just take the example of uh  $\text{Spec } R$  is  $\text{DF Union } Z$  of  $f$  uh then you use then  $r_1$  over  $f$  is okay because it comes from here and then the difference between  $R$  and  $r_1 f$  uh is  $u_m$  so generated in a finite manner finitary manner um but this is the union of like  $R \bmod f$  to the  $N$  with some shifts um this is actually uh the fiber  $\text{SAR}$  here uh and this is actually an  $\text{RF}$  complete module so it uh lives in in  $\text{ZF}$  um so if you take all modules over  $\text{RF}$  inverse and all modules over  $\text{RF}$  complete uh and then generate things in just by triangulated category nonsense eventually you're going to hit  $R$  and then tensoring with anything you'll see that you hit anything and then using that you can check that you get this kind of covering situation so uh now let me just um say what this looks like in examples so uh so all of these yeah so we have the sort of the category attached to each element in our cover the category attached to the pair wise intersections is just modules over this tensor product which will also be item potent so let's take a so example uh let's say  $r$  equal  $zxy$  um so let's take  $C_1$  equal  $Z[X]$  Plus or  $-1$   $y$  uh so we've got the whole Locus where uh  $X$  is non zero and then we can look at the locus where  $X$  is zero but  $Y$  is also non zero um and then we can look at the locus where both  $X$  and  $Y$  are zero that's the power series uh oh sorry I should add  $X$  and then complete at long  $X$  so that's  $z$  braet  $y$  plus or minus one power series  $X$  um so then what happens when you start taking intersections of these guys the most interesting thing is what pops out when you take the intersection so the algebra attached to  $C_1$  intersect  $C_2$  intersect  $C_3$  um you can just calculate these things using the fact using this basic fact that um uh uh solid tensor product of derived complete things is derived complete um so what do you get so when you do  $C_1$  intersect  $C_2$  you're tensoring this with this that's just inverting  $X$  so then you get uh  $z$   $y$  plus or minus one laurant series  $X$  and then you should tensor that over  $zxy$  uh in  $Z$  solid with this third guy  $XY$  um now both  $X$  and  $Y$  are inverted um uh but if you imagine not inverting  $X$  then you'd have a power series here and you could use this derived  $X$  complete trick to see that you can move this power series onto the inside here and you get that this is  $Z$  laurant series  $y$  power series  $X$  with  $X$  inverted which is just laurant series  $X$  so you get this kind of object here where you're in you you're iterating op so you iterate in general when you intersect a lot of these things together you iterate completing and inverting and it famously it matters in which order you do this um but it all comes from this sort of commutative situation nonetheless in this way of setting it up here and I want to make the point that the so these things are like kind of like higher local fields that people have studied and they tried to study them from their perspective of topological Rings topological fields and there were just terrible problems Matthew's not here right now but he even wrote a paper explaining that everything is horrible um but uh if you put them in the condensed world then they're just perfectly well- behaved objects that you can work with um it arises from this natural procedure and it's item potent even in the  $\text{DED}$  sense over this ring and um it's just uh it just functions uh functions quite well um so I invite you to try to draw pictures of these covers um to get an understanding for what's going on with these kinds of constructions but um okay I think I'll stop here yes for finite Type  $\text{zra } R$  explain  $R$  solidification of the countable product of  $C$  and  $R$  is just a count countable product of yes and I thought it also true for something like the ring of power  $Z$  that's true which is not finite time that's right but it's also not discret so so yeah it's detered by yeah so you could also take a finally generated ring uh and an arbitrary ideal and look at the completion and then you get the same statement so the okay uh the basic the compact projective generator is just the product of copies of the itic completion of  $r_i$  see and that's a same time if ring is not fin at type you also said it's Union of five is it easy to see for this no but  $Z$  double bracket so sorry that was a statement about discret rings so in general if you maybe if you have some uh ring that's complete with respect to a finitely generated ideal you'll get the good answer if your ring modulo that ideal is finally generated maybe this is a good way to if is the ring module the ideal is finite type over  $Z$  then then you'll get the the naive answer I think yeah sounds reasonable at least yeah solidification solid

is determined by underr yes so the same formula doesn't work for  $Z$  bra what so  $Z$  double bracket  $T$  is a discrete ring yeah as a ring you're looking at  $Z$  double bracket  $t$  as a discrete ring yeah well then it's just no then no then the doesn't no it doesn't hold no but if you if you look at  $Z$  what to say if you look at  $Z$  double bracket  $t$  uh non-discrete modules in  $Z$  double bracket  $T$  discret uh solid Theory that's the same thing as same as in in uh in  $ZT$  solid Theory so um once you've decided to yeah yeah once you're complete then it's enough to solidify uh stuff modulo that ideally you're complete along okay yeah so does that address your your concern yeah okay so what again you discrete ring and takech the associal condensed ring and you consider solid relative to to ah relative to the elements of the Ring yeah okay just each element of the ring and then you so if you consider this condensed ring with the product topology you could either solidify every element in the underlying discret ring or you could just solidify  $T$  and it's the same in fact you don't even need to solidify  $T$  I mean you don't need actually it's just it's just the same as in zolid theory so if you look at a  $Z$  power series  $T$  module in zolid then it'll automatically be solid with respect to everything so in fact this follows because I mean we showed that the topologically nil potent element ments will always be solid um and uh yeah um and then modular topical noo elements all you have is  $z$  which is is there paper on picture I mean paper of  $P$  about this about what this here he gave he talked about it in this class yeah are you wondering if it's written down somewhere yes it is uh Guido Bosco wrote down at least the case of  $P$  like complete and I think also I think also isn't it right Peter uh uh I don't remember the title of guo's paper but I think also Lucas man wrote yeah Lucas man wrote The General case in his thesis and the in the I think so I forget but I forget the title of the paper in which it's contain but he has some appendix on ptic functional analysis from the solid perspective the title of the paper you say I said the in this in this here it's has some appendix to one of his papers whose title I can't remember and the appendix is called something like solid uh functional analysis or solid ptic functional analysis something like this which kind of writes down the the argument that Peter went through it was before but okay yeah just yeah yeah very quick um maybe this was a computation done earlier in the course I'm just forgetting but like regarding the control of the  $X$  groups like The Vanishing of of them like so we had the proposition for finite type so if I just like for example like  $z_p$   $z_p$  the ptic integers yeah was there computation oh so you're wondering about these homological Dimension results for  $z_p$  yeah so there you get uh yeah now I have to remember I think um yeah there you should get also it should just only be  $X$  ones uh yeah that that's actually even that's easier to show than in the case of of solid  $Z$  because  $z_p$  is compact so you can there you can directly see that all of the finitely presented objects they're also they're they're it's quasy separated and it's the they're also all those objects are quasy separated and they're just like yeah inverse limits of finite  $z_p$  modules uh countable inverse limits of finite  $z_p$  modules and that makes the whole analysis much easier so to summarize so for let us say  $R$  is a field yes then the fin presented are qu separate in fact they are all given quite simply by you just have the either infinite or final product of the cas so that's okay for our domain except the fact that you can have a the parar group is maybe not trivial but still it's a product of uh you you have a resolution you have product copies of  $R$  and then the Kel is something like product of maybe ah actually you can use the the AR Swindle to get rid of this anyway so you can it's always this this okay so it is a resolution of length to like like before because we don't have to worry okay and so we get and then in dimension at least two you do your the pro the uh that we you discussed yes okay that's the yeah and then there's if you wanted to move to non-discrete rings then you'd have to do more analysis and so on and yeah yep okay thanks everyone

## 12. WHAT HAPPENED SO FAR? (SCHOLZE)

[https://www.youtube.com/watch?v=VMgZSP9sRdo&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=VMgZSP9sRdo&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

e I guess it's time to start uh good morning um for the people oneone you can fill out the survey there uh now or anytime this week or next week um all right so so where are we in the course uh so let's try to uh take a little bit of a broader view so um so what happened so far um so we started with this idea like I know condens set condens the being cured and so on so cond mathematics and we try to make the point that this is a good framework uh for for combining homological algebra uh and functional analysis and the goal is to use this framework to develop a general good notion of geometry um but then at some point we Su made the wanted a notion of completeness so uh and um and at that point like we we went into a certain direction namely with know uh were concentrating on the solid Ser so we had the solid modules um first over the integers but then over our let's say in finite type the algebra D discussed last time particular um or in any and uh well this this is closely related to this it's good for nonan geometry um and a specific uh framework we discussed is really extremely closely related to Hoover theory of edic spaces uh and I mean there were certainly uh some conceptual idea that emerged from uh from this namely that uh this notion of completeness shouldn't be something that you define once and for all uh but rather is is an extra datum that someh part of the datm of like what a commutative ring is in the setup um so completeness is a more relative not some sense so it's not AB no concept for all but it's part of dat of antic which modules are con as part of the data of what we then defin as an analytic and so uh so what we are we trying to first through in the course uh um so we have the St analytic G and maybe um for reason that we already saw a little bit that when you localize some of the structur might become not 7 de zero anymore um we might change the preise definition here a little bit to some of our deriv rings from the start uh but this is something I don't want to go into today um but so we want to start from this C of analytic rings and G say all basic building blocks to then form like as usual algebraic geometry you start with Community rings and then you build schemes by gluing the Spectra of commun rings and similar there should be some procedure where you start with this C of analytic rings and then do some kind of gluing to produce some kind of notic spaces or maybe we will directly actually go to some C of specs um some kind of procedure um so there will be uh some very general such class that we will introduce at some point um and but then within within this world we want to find all sorts of uh to find this and find within the where is CL no okay um but certainly want our Theory to not just be a nonen we definitely want uh to accommodate also geometry over the anal geometry over the real or complex numbers and so uh so but first uh we would like to see more examples um so some key key questions maybe are um um real numbers uh have an natur you know that's Som suitable for doing complex or realic geometry and another important question is uh if you can put some natural search on the real numbers um and so can do some kind ofi geometry but the other also have non geometry using the solid modules um maybe and then there's a question what a meaningful way to combine uh combine the two settings so is there a good way um some combine um and there might be several different ways of how one might try to start to thinking about this um I want to develop one way uh uh um with the following example really uh uh in mind so maybe the most prime example really of of one one should be able to do in some world of geometry so here yeah key example that we want to accommodate um is the famous table look the um so let me spend a little bit of time just talking about this uh what is this which um right so uh so T elliptic curve say is some elliptic curve e e or EQ um that can be defined for example over the ring D wrong series and uh here's one way to define it that's basically uh using uh the framework of Ed it's actually uh if you look at Uber's paper I think it's the one called the generalization of formal schemes and something like that um there really is also one of the first applications of a Ser he's again discussing like the ti of De curveent that's a generalization to high dimensional BM varieties um so you have the edic spectrum of like WR series Al series Al and then over there you can look at some of the gem as an space over so inally speaking that's all the like there's some coordinate T here and T should have the proper the absolute value of T um is bound with between some uh some BO so Q is some on top element in there and then you look at the part uh of multiplicative group where uh the absolute value of T is abounded by by power um and then acting on here you have multiplication by Che and uh this is actually a totally discontinuous operation because it multiplies the absolute value of T by the absolute value of Q which is like something between zero and one um so it's actually uh free and totally discontinuous action uh and so in the world of



X basis you can really pass to the quotient and the taking this quo is nicest kind of quo the other way uh just get where thetion is like locally split and locally locally the space just looks like this and so you can def find EQ to be uh this analytic DM and then you potion by Q to Z and and I mean how it look like I me you started with with I don't knowm and then modification by Q it's moving everything towards the origin but then when you take a qution this someh the same thing as taking some Ang of radius uh one one of of Q and then identifying boundary anul and so using this you can actually see that this is some uh yeah it's actually proper so take far the Compact and all the separated um it's also smooth because it's locally just a gem it's proper smooth connected curve analytic curve over and actually also I mean there a group structure on gmed by sub groups it actually has a group structure second and then and basically in any world of analytic geometry that should be a theum that something purple smooth one dimensional is always algebraic because you can always find an ample devisor by just taking any close point and then taking the corresponding inverse ideal sheet um this gives an EMP line bundle and then I if you have any kind of version of RAR nor your theory which should always say show that they not function to Pro algebraicity of the thing and so it's definitely true here so there actually so here there implicitly there is some Gaga for over a nice narian H basic Uber pairs but also the statement about Relative Dimension One it works over a non archimedian field that the proper curve prop is a projective but if you have a more General basis like Spa of something like here then of course locally for the analytic topology you can use formal model some device to get formal model with relative dimensional it is easy to construct bundle but to globalize it here you have a section actually which gives an ample device so the identity section but in general I'm not sure if it is clear that a proper Relative Dimension one over a nice base of course locally analytically it is projected by using some formal models but I'm not sure about what happens globally over Spa of nice pairs so but it's not I have a section okay which I right so but anyway so in this case lot hard to this automatic algeb right and from the end it just defines an algebraic tic curve some defines antic curve over just in the end it's just Cur of the abstract policy I mean in between we Som use use some kind of topological condensed structure on this series Q but in the end we just get an litic P of the abstract um and okay so it's a litic curve so you can write down in R press equation first um and uh uh so if you write down the wi press equation I mean for I guess if you remove the zero section as usual um then you can write in the following form um I copy this from Wikipedia so I hope it's correct uh okay so to write our integral and usually can omit this term but usually have to Inver two to do this so if you don't you should put it Mo form um and then these are some power Series in Q and what are they uh my notes tell me I should write the phone okay so I know I mean this definition looks rather simple to me these formulas look rather complicated to me but when I first saw those formulas I remember that I was struck that if you actually look at the coefficients okay so you don't actually see the coefficients of Q to the n in here but I mean okay so it's easy to invert oneus Q to the N right it's 1 plus Q Plus One Q to 2 N S and so if you imagine doing that then I mean here's the coefficients they are just polinomial and N right and I mean when you do this this geometric series here I mean the new coefficient still stay poal on end uh so some kind of funny observation is that uh uh uh the for coefficient the coefficients of Q to the end uh po gross um this implies in particular um like after we just to find something form series room but then when you actually write down this equation you realize that this is something that you can specialize uh to any uh also not any Arian value uh between zero and one and well that would have also been clear from this kind of analytic description I mean you can certainly take like C Star as a uh as a complex analytic space as a complex manifold and for any complex number between zero and one I mean now this picture makes literal sense I mean you can literally take that potion and you get an actual Taurus right and so so you can do that but also uh the analytic description SDM keep V uh makes perfect sense then so so we would really like to have a uh yeah to be able to do geometry so that we can Som perform this kind of construction of like an6 GM Q2 Z not just over the full wrong series algebra but really over some sub algebra but we put some growth condition on the coefficient so that we can also later on specialize to median part um note however that the precise grow condition that you get here is quite a bit stricter than just the observation that you can plug that it converges when the have to VAR CU less than one uh uh so convergence um this is just a subexponential growth of the coefficients um but I mean I even you think about maybe if you think about yeah so it's pretty clear that there there should be a geometric reason that geometrically should be pretty clear that this cor series should converge when Q is less than one uh so there there's some kind of direct geometric way of seeing that the whatever coefficients are getting here that their coefficients much have at most uh sub exponential growth uh but it's

not so clear how you would see that they actually have most poimi goes but there's a geometric reason for that um all right um so uh let me actually talk before um really getting again to this Mark median Theory let me actually talk a little bit about uh the geometry of the space space so this The Continuous valuations on on on  $Z$  WR series  $Q$  so so I don't know so if you have some they call variations but I still want to denote some as absolute value so it's this to  $G$  zero and so in particular the absolute value of  $Q$  will be a certain element in here will actually not be zero so it's certainly actually  $G$  part um and it turns out that like you have this to new poent unit and this is actually an extremely convenient structure to have because it allows you to compare ABS values of all other functions against the absolute value of of  $Q$  just like we did here when we had some other function like like the corrent function of  $GM$  we can somehow gauge how large It Is by comparing it to the absolute value of  $Q$  and this is something you can always do once you have such a top LOD new poent unit um and so this in particular this means that there's actually unique  $M$  here to the real Great equal to zero um which sends the absolute value of  $Q$  to some pre-specified element between zero and one and let's just choose a half um and for most of the valuations that we care about here um this will actually be injective but there's some rank two valuations where there's a little bit of extra information in the GMA that's not remembered by this qution but but in first approximation you can really um think uh right so this gives a map from this Ed Spectrum um towards what's known as the burage Spectrum maybe there are several any 11 call it nber um um here for f recall that um uh uh the is an example of such a t ho ring um and said last time that this can really be endowed with a bout Norm where this this thing becomes a the unit ball and then again if you spe uh specify what the absolute value of  $Q$  should be you can decide what the absolute value of anything is looking at the power of  $Q$  that you need to make an integral and then taking the corresponding power of a half so for Bing R um could you write slightly larger yes I think sorry maybe just use next St perpose here a set of all now again called absolute values but now they really take  $Y$  is and RoR equal to zero and now you ask the following things uh where first of all I don't know there are some stupid things like normal  $Z$  is  $z$  normal one is one um it's multiplicative nor  $X$  nor  $X$  nor  $y$   $B$  which I mean here valuations which in particular satisfies this strong Triangle inity  $C$  the is not restrictive to non setting so it considers the usual triangle inequality but you also ask that they all of them should be bounded by the norm um you have specified on on  $r$  um exactly how he sets it up say less equal to a constant time but I think in this case it doesn't and so this naturally match to the product of copies of  $R$  greater Expos to zero enumerated by all elements  $X$  and  $R$  and uh he end this with a Subspace topology maybe so actually uh because of this Foundation you can actually replace  $R$  greater equal to  $Z$  here always by the interval from  $Z$  to the absolute value of  $x$  and so then these are all compact intervals and then an arbitrary comp product of compact  $h$  space is still compact  $H$  so there still a compact space and all the other conditions that you see here they are all closed subspaces this is actually a closed Subspace and so particular the verage space is actually a nice compact  $h$  of space yes and then actually effective of this situation that actually the only difference between these two spaces here is the possibility that in the TIC Spectrum you can have higher rank things but these higher rank things they give rise just to some uh very inmal changes in the space actually uh this potion that's or this NE here from the Spectrum the  $C$  Spectrum this actually maximum House of potion so the two are extremely similar note here that implicitly I'm endowing the integers here with a norm where zero the nor of Zer is zero and the nor of all elements is one because they all containers unit for the ring I'm considering okay so he has this tic space for this and so I still want to understand a little bit about this geometry it's more or less the same thing as anyways as the  $b$  space um but that actually Alat down to the Bur space of the integers and I mean usually if you have any kind of spectrum of the integers it doesn't have a rich geometry maybe I mean they have a prime and maybe one generic point or something I guess but the bir space of the integers is much more interesting um here's here the proposition uh maybe I should stress here yes Norm on  $Z$  implicit  $B$  Norm on  $Z$  it mean yeah it's the one where the normal zero is zero and but the norm of any  $n$  which is not zero is one because all of them are contained in um but there is a sometimes more natural uh Norm you can put on  $Z$  just the absolute value right so let me draw a for the norm which is  $z$  or one um which to noted um um this is the following thing um so actually this very Norm it is satisfies all the properties the nor of  $Z$  is Zer nor of one is one uh so actually defin point it's a more less a generic point but of course you know many other natural absolute values on the integers namely for any prime number  $P$  you have the pic absolute value and it also satisfies all those properties there but whenever you take the pic absolute value you actually have a choice what is the absolute value of  $p$  and so for each prime number  $P$  you actually have a full line uh of possible like full interval like going from zero to one I'm

not running here I guess um see and then there's also a close point at the end of this Ray uh where you go to FB um so you can also get an absolute value if you first project from Z to FP and then do this kind of thing that zero goes to zero and everything that's non zero goes to one okay you can do this corresponds to two this corresponds to three so on corresponds to five and then for each prime number uh you have such array and then some kind of inverse limit of joining one Ray at a time uh right so F points so the points and all correspond to maps to complete B fields um and so there's one that corresponds to Q with the value so first go from Z to q and then do this thing which we just recover this absolute Val here here then you can go to QP for any p and the absolute value with the norm of P being any specified element in the interval from 0 to one and then FP again with a tri um all right uh let me also already discuss now um the side variation of this so you can also take the b space of the integers but the usual absolute value I mean so Norm of n is plusus n um positive version of it um so this the same pictures uh the difference between these two things I means the same data satisfying these three same conditions the only condition that's different is this boundedness condition that you have here so previously someone asked that it's uniformly bounded by one so in particular the absolute value of two can never be two like it would be for the real numbers uh but now I kind of fix that so now the absolute value of two can be two uh and so this is actually some of the same picture so there's again this thing like for each FR number part before but now there something extra then there is some kind of Heth interval um um so this corresponds to the map from Z to R and then you take the usual absolute value of r to any power P where p uh go from zero up to one and now it looks like this was also an interval from zero to one but actually uh it's better to think of this if you take the usual absolute value Peter yes can I vote that you use Alpha instead of P when talking about raising an absolute value to the P power thank you yeah um just in time uh to the alpha so if I if I parameterize this in terms of some fixed absolute P absolute Val say the one where absolute p is one over p uh then this line here there corresponds to this to the alpha where now Alpha can be anything from zero to Infinity uh but for the real you fix the usual absolute value then you can raise that you can try to raise that to any real power and it will definitely satisfy this condition and this condition but actually if you want the triangle inequality to be satisfied you realize that this happens only if Alpha is at most one uh so in this sense this this line for the real numbers is actually some stops in the middle compared to the others okay uh right so where are we uh right so we are trying to understand uh a little bit about geometry of what this add spectrum of Z long series Q actually looks like and it's basically the same as the ver spectrum and this is fire over the B spectrum of the integers and now let's actually try to understand uh what are all the fibers right of this map and so for example if if you take uh the fiber at a prime P of like just FP um well then is just one point then as usual some of the formation of dark space commut with some kind of fiber products and so Rings FP series q and I mean this is just a point um because there already non field and you fix the value of Q to be the half like okay so in characteristic P the thing has just uh one fiber which is this kind of the wrong series r maybe I should have said I mean this proposition is basically just zero right because find the absolute values um um but then uh you can also okay you have this dark space of the integers and then for each P you have this off line here line of FS so if you how Bas change this whole half line of QPS um someone looking at the you're in some sense taking now this is some kind of punctured open unit dis and now you're this bace change to QP will actually be some kind of punctured open unit QP so this will actually be a punctured and uh this map here to this to this uh line zero Infinity depending on how you parametrize at 01 or whatever um uh this should be an incarnation of the radius matter uh but actually in a slightly funny way it say one over the L of the radius or something like this I won't get it straight um so there there is a whole function of unit here of which has Like An Origin um and a boundary so to say there not point in there and whenever you fix a specific point on here then this fiber will be some specific anual in here of where the absolute value is fixed um and now you can wonder what happens as you move towards is invisible anyways but uh I'm always tempted to try um see the image is this circle here and then when you move towards uh towards this characteristic P point right uh so at the end of this r at the end here that would be q and at the end here that would be FP um and as you move towards FP actually this anular here it goes towards the outside of funed open unities and some sense the F wrong series t sits or nor Q it sits near the boundary of this of this Ang um whereas when you move upwards on this Ray go towards Q uh then you will get other Ani and they will get closer to the origin okay so what what does this thing actually look like um so there is one lary point which is qong series q and then uh there are special points which are wrong series and on the way there you have the punct open un the region in the middle the punct open and the slightly mindbending

thing is that yeah how the different parts are GL to each other I mean the whole thing is a compact h space so it makes sense like to ask when you go in this direction where you end up um and so yeah so if you move towards the puncture of this open unit disc you move towards this point and you move towards the boundary we end up towards this point so this whole Space is space that has peric regions for each P each peric region is a function open QP and then they glue to each other in this funny way yeah where for each one if you go towards the center you will end up at the common point which is is a kind of generic point and for each one when you go towards the boundary you end up the characteristic B so you mentioned this fact that this map is ma is this special for this r no no this for like T U ring t Okay than yeah so for these three things actually the two series are extremely different right because we saw that like if you take just the integers here then the addic spectrum of that would just be like non very nonous thing like the spectrum of ring is like very non um uh but once you introduce this extra top variable allows you to fix absolute values and then it's actually the two series you're closing your leg oh all right but uh but I need already suggest to look at a varant uh of Z WR series Q as b r where I we would like to now like we've already seen that for the B of SP of the integers there's a natural way to enlarge this picture by instead of doing this kind of Noni Norm we can put the usual Norm then there is some aian part that's naturally contained in there um and we could of course try to do the similar thing for zero wrong series Q um where we put the usual Norm on Z and well like today from the start I kind of made the choice that the abute Q was a half and so far it didn't really matter now actually suddenly starts to m a little bit and it's a bit weird okay but you simply declare the abute value of Q to be a h uh so concretely and like Z wrong corals was an nor the normal sum and Z almost all zero to the N is the sum of the ne Norms of the ANS times one two to the and then we can just complete this range like operations unar doing their kind of Tor to match literally B which is Original Series where which is about B frames when we actually do this in our the we will do something something slight different operation but it's extremely closely related um complete as thing to get some thing call the a um which are those half 2 uh so these are concretely um uh nasty but um these are I mean this convert for values Q in the complex numbers that are sorry yeah yeah yes they do converge and um they Define a holomorphic function of Q in the region where Q is less than a half and this hph function send to continuous function on the whole uh and now right so uh I apologize for doing this people taking notes um but now what happens if I look instead of the verage space now of this ring I put this extra aredian convergence condition so now uh there the reason maps to the nonent part anymore so it's now just snaps to uh this SP space was a fual absolute value which has this extra real part and now we can try to understand uh instead what happens when you pull back uh this half interval of wheels and that PB will be precisely uh uh this opuni disc uh actually mod compex conation because two complex conjugate things they give the same Val and again uh right uh now it's liter like this basically this projection here is some kind of radius of function where the points of radius of that's to a half to what this boundary Point here and when you go towards zero then you go down this line and so then again you end up end up here so so what does the space look like so now it has some kind of function open uni this for each place of the rational numbers so pic place and the acum place um and everywhere when you go towards the center uh you always end up with the center point of picture the sly awkward feature of this picture is that I had to now specify the ABS value of Q in advance and then as the acum part of this picture this this punctured op just stopped that the BR new although like from the perspective of the ttic curve there was no reason for stopping at qal to half the I right and so then there uh right so then one but like this is precisely the kind of picture in which we would like to combine non median geometry I mean so this this space his has Parts which are literally complex analytic or real ptic analytic things but they sit together in one joh same um and so one kind of PR version of the question uh about existence of antic R structures um is now uh like K1 and this guy or more any albra for theator um was a natural and uh this has been a question that was very much in our minds uh back when we first found out about the solid Theory and then tried to really go further and um and which we eventually answered with a serum uh yes need liquid construction I mean it actually turns out that to Define this one it's slightly better uh to work with not just precise the health convergence condition but functions which Converge on some rate some slightly larger dis so there is a union over all radi which are bigger than a half of the things that converge that's more techn point and some well generally speaking like for any kind of B offing which has a top Lo unit in the usual way uh you can produce this kind of liquid analytic and the resulting the will be extremely close uh to B which Theory and this is something that we want to discuss at some point um today however uh I want to also talk about some something else something

that we only found out a few weeks ago I maybe so once you you have this with liquid and structure over  $\mathbb{Z}$  long series  $Q$  maybe greater than a half you can literally repeat the construction of the ttic curve that I did in the beginning over this ring and now they all frame here I guess you could also just do it in  $B$  here um and then someone show that the ttic curve is definitely defined over this ring and then okay in the end you could also make a half larger and larger and would get that it's defined over the whole open unit disc but you would not get this way this strange poomi draws bound on the coefficients so uh yeah let me make a kind of somewhat philosophical or personal digression um so the said back when we did this so back in 2019 or back very recently um was that if you want to Define aning structure what you should do and what we've somehow done in all the examples when we produce them uh uh is to Define is to describe the pre complete mod and okay maybe actually have enough time to actually specify a few examples so when you take so in all the examples I will take  $D$  to be an image limit of Finance ACC end or let me actually use I think so let's say live uh and so something we've seen is that if you want to describes the three solid modules on  $S$  and they have a very very simple description um this is actually just the limit of all the three modules on and this also has a meaning in terms of uh measures in the sense so it's also like integer values measures from this profile access so to define a measure you just have to assign an integer to each glal subset in which way a way which is Finly additive um uh actually let me discuss a different example I think I already said it at one point but let me Str it again you can actually also describe the abstractly free modules without any completion they actually turn out to uh sit inside there um and it's some the  $L$  zero bounded card uh in which which sense uh so in this case actually more natural from um it's a union over all integers of the imers limit  $I$  of the part of the free module on  $SI$  so this is just a finite free group with a specified basis and so then you can define an  $L$  zero nor where you just uh yeah okay I mean maybe well I guess in this case same as one but that's still like to call  $L$  zero um so all to Summit most and Sumit most and what the difference so on each of these finite three guys you have the basis vectors and then you're low to take some final bounded Summit by most  $n$  of them so this gives you in each of these things it just gives you a finite subset and it takes a limit of all these guys gives you some Prof finite set it takes a unit of all these gives you a certain small subset um so there is actually a description uh of um of the liquid structure of the three modules um right um so yes let until um  $S$  as  $I$  times to the end I uh to be the sum of Val of the  $n$  as usual then in this  $R$  thing we take have of  $Q$  to be  $R$  and uh right and this  $s_i$  um then the three liquid modules um on  $S$  over this ring were kind of defined to be and then the hard the is to proes that this actually antic ring comes from antic ring structure def find to be the following thing bit of space here so I mean just like the  $r$  is this Union of all bigger than a half also the fre modules are this Union over reg bigger than a half and then but then also as in like the free discreet guy um we have this Union overall size vals and this time I maybe want to index them by some real numbers  $C$  greater than zero but then once you fix the radius and size you're just taking an inverse limit of such free modules of the part where this Norm that I've just defined there is at most  $C$  and so similar situation where you can show that if you bounce uh the norm here and you give this a certain kind of topologies and this actually becomes the compact  $H$  thing the limit is still compact take describing it as condensed set and I mean this is really the thing naturally suggests itself when you try to Define some notion of complete modules over the screen because I mean what the thing in place is like the  $B$  Norm on this ring that you have which is defined just this way for one element and then of course if you have a a five free module you're just summing uh summing the absolute values and so this proposal for what to free complete modules in some sense wres itself when you're trying to find antic instruction on this one  $y$  um let me actually just the course example um that like like in the beginning of my lecture today I was asking two questions first is there natural stru on the reals second is there one which allows you to like combine the things now I'm starting to answer them in the opposite order so I first I said that there's something that combines them uh let me now also give the answer to the first question what are the antic ring structures of the reals so uh you can specialize from  $\mathbb{Z}$  Ser  $CU$  greater than a half to  $R$  by sending  $Q$  to some number  $T$  here less than two a half and well this defines the point of the  $SP$  space now which maps to the  $b$  space of  $\mathbb{Z}$  and so must actually to some power of the usual absolute value of the reals so it actually maps to like inside there you have this half interval for the wheels um where Alpha and 01 corresponds to the absolute value on  $R$  to the alpha and so so in here in here you also have to interval of  $T$  which me from from zero to to a half and now this map is actually realizing some isomorphism between these two things we writing the value of  $T$  to some number  $R$  and this can also be made explicit namely not screwing up  $T$  is a  $h$  to the off no teach the off I think that's right okay so so the value of  $T$  determines



some Alpha and I could have just told you the formula let Alpha be the D such  $T$  to Alpha as a half but this would seem slightly curious what does it mean I mean the meaning is that you have a point in  $S_p$  space maps to  $b$  space of the integers and you get some point there this is the AL okay so we have structure here so we get one here but the complete modules are just those that are complete when you just go to SC what are the three modules so this means that for zero less than Alpha at no one gets a so-called alpha liquid ring structure on  $r$  um okay let me use next point to describe Al liquidations uh so as before here it was a  $c$  Union of a radi bigger than a half here will be a certain Union over let call betas maybe which are less than Alpha and again Union over constants that um and then you do same thing so here you have the three modules on SI and then like recall that some of the alpha corresponded to the the absolute value on the  $re$  which was the absolute value to the alpha now I'm using this Union here so I take some of it the Ala nor uh is at most so here's um of  $x$  i s i the Ala nor is a sum of of  $X$  beta um note here that zero is less than beta is less than one so this is not one of the usual kinds of norms that you would usually consider in real functional analysis this is not locally convex usually when you put  $l_p$  Norms the  $P$  Li between one and infinity here we us going to the left and use the non l o l e s so you mean it's not Ultra it's not doesn't satisfy the triangle inequality this this this is not a norm in the standard sense there is no triangle inequality no it's better than the triangle or is it better it's not real it's not over is it don't take one of so this this nor satisfies the trying inequality but it doesn't satisfy usual scaling XM so you it to the beta power of the norm but usually would ask it satisfies the scaling with respect to the usual real numbers but this this doesn't do but I mean most conely like if you if you just I know for two dimensional Vector space you look at what the unit balls so to say look like then they they have this very SECU shape that they look something like that and uh yeah so maybe when you see this you're tempted to think that this is a stupid uh way to put a ring  $R$  structure on the real and you should rather uh do something else but this do work so when I E guess um that the threee complete modules on  $R$  uh should be the Rong measures on  $r$  measures one way to define them is we just the Dual for continuous functions um does not work does not find instent uh so the way to describe the  $S$  by the way would be to do the similar construction here but just  $U$  one more um and this would be really closely related to the usual the of complete loc compx Vector spaces in about a clean as way as the usual theory of solid I mean the Ser of solid modules Rel to the usual series of like linearly complete modules um so we were when we developed the series we were really hoping that this one here is an PR structure on the reals and was really uh yeah somewhat impressing and you realized it isn't but it's Rel to some fun stuff uh uh In classical function analysis um so one thing we do always want um in structure is that the classic complete modules stable under extensions but this is simply false for Loc convex Vector spaces so you can have natural extension of locally convex Vector spaces even just of an  $L_1$  Space by the real numbers where the extension is not locally convex this was realized around 1918 by Reba and somebody else and it comes from the entropy function so it's it is not some completely crazy thing it has some kind of natural meaning but it destroys the option but then once you allow some local compx Vector space into the picture the Ser works again but you're really forced to uh use stly Conex and I guess I should also discuss one other thing very quickly um uh so in for we specialized uh this this uh the uh to the real numbers uh we could also specialize it to the pic numbers um you would expect that like in terms of verage space we were really just trying to extend over this missing part at the real numbers so you would expect that maybe at the non part of this picture we didn't really change anything and like also the convert extra convergence condition we put on our on our on our function was just an extra convergence Condition it's a real numbersss uh so naturally you would expect that when you specialize liquid structure uh on on this convergent Artic drone Series ring that back to a piging then uh then you would just get the solid structure but that's not true uh actually you get new ones I can specialize as TP um and uh if you do some kind of similar analysis and now okay there's a slightly tricky issue that I can dis send  $Q$  to any fractional power of  $P$  here uh this can be rectified by long the half to VAR instead um and what you see from this is that for every Alpha that's in the non meting case allowed to be anything between zero and infinity so in other words again for any point the bage Spectrum really of the integers um we get an alpha liquid restructure on QP um again the pr modules can be described and the formula is just the same  $Q$  the op liqu one and it's again Union over less Al in the unit of all constants foring a norm of the three modules on the si uh which you now endow with the exactly same as in the uh and this should be compared to uh the solidification this is just described similar way as limit Union of all constant of limit of joint s i but now on this guys we put the infin Infinity the Supreme so these are all sub space in here so suddenly you get a whole new world of and liqu structures even over periodic

numbers way maybe you didn't feel like you need them because the Sol Theory already works with you so I want to ask a small question just to clarify possible confusion so you have the the  $L_\beta$  for different  $\beta$  which are again I believe defined in the same way just by some of the  $\beta$ 's power without taking one over  $\beta$  just this in the same way a slight mismatch what I know if I literally specialize the definition  $OFA$  to Infinity I would be summing the supremum Norms which is not what I'm doing when I do this Infinity here the Supreme here it is the the actual there's a little bit of a mismatch notation between here and here but I think okay but what is the when you take Union over  $\beta$  less than Alpha I think usually it was a filtered Union that is is it the case because here you have to be slightly careful that the way you IND index the constants some should change when you increase ah so yeah so so so why why the the the limit the union over  $\beta$  what are the inequalities between the  $L_\beta$  here the Tak unit over all  $C$  yes and then and then there it's literally the case that these spes get larger and larger as you increase  $\beta$  ah okay braet it like this in some sense okay uh so something you can think of a whole series of series that go from zero all the way to Infinity where you put some put the corresponding Norm here where at this end you have follow modules and then at each point Alpha here you have the alpha liquid ones and uh I in terms of the class of modules uh I mean being solid is a much stricter condition than being Alpha liquid so uh um the class of modules that you allow here it becomes larger and larger as you uh make alpha closer to zero and some things as you go to what zero here you have all condensed object because if you kind of naively put some kind of  $L_0$  Norm there meaning that there there's only bounded number of coefficients that's nonzero and then you put some bound um then this actually recovers a free condensed module without any competition by variant of what I said at the beginning about the integers okay all right uh I guess it's time for the uh for the uh for the new thing um right so back in back then we were trying to look uh find natural candidates for how what the free compete modules could be and then it was some hard matter of proving that they actually Define the EXs of being an antic brain uh uh but but this in this course we have a different mindset is that to the  $F$  and the structure uh  $L$  find something natural the morphisms of this projective object  $P$  that you want to be morphisms  $B$  the free out no sequence uh that should come after completion and because in life being groups this  $P$  has these very strong properties like being Compact and internally projective this will always Define drink structure so has become extremely easy to produce analytical instrues and here's here's an example I mean so it turns out that to Define uh the taptic Curve you only really need two things um the first is that  $P$  should be topology um right it's in new and should be unit clear and both of these have clear meaning in terms of the underline cont ring right I mean so toing newent means that you have a map from like basically this guy and up with the natural natural ring structure sending one  $t$  to  $C$  um but then you need some you need some completeness for your modules you need to be able to some certain sequences and really the only thing you need is that one minus  $Q$  \* shift acting on this projective  $p$  and it's clear that there is a universal example of such an antic ring I mean you just take the free guide generate by unit that's just cond ring and then this condition put some ring structure on this ring also sign which is good to much h Ser is that there is an initial such ex okay Ian existence is easy the hard part a description hey so it's a pair of a triangle and you can actually describe a triangle and well let me describe the underlying ring um this is precisely those those those sums such that uh the PO and thus if you like the claim is that we can do analytic geometry as usual over such rings and so you can just repeat the construction to  $T$  the curve and then you will see for like geometric reasons in some sense that it must live or precisely this ring where the most and uh I can affect this SC of three the free modules so a  $s$  now already means the free complete guy right a triangle Jo  $s$  would be a free noncomplete guy but a triangle is free complete guy it is is the union overall a greater than zero sorry U yeah so as usual as is a limit of  $s$   $s$  um of yeah something also um oh limit I of those um of like the part of the free module let's say was long sering on SI but where uh the coefficient of  $Q$  to the  $N$  has a  $z$  Norm at most I don't know  $n$  plus  $n$  um yeah let me maybe in the future lecture give a more precise description but basically you're asking doing a similar description as before but now this time you're asking for some po nor gr conditions on the coefficients oh all right and uh so what you have written on the Blackboard is barely visible to in the all the indices and so on are almost invisible so you have Union over  $K$  bigger bigger than zero or bigger than or equal to zero I mean  $k$  gets large okay and then and also gets large um basically I I want to put say that the condition here is uh is of size at like the coent of  $Q$  to the  $N$  can grow at most like a polinomial in  $N$  of degree  $K$  so most  $n$  to the  $K$  but then I also have to allow the presence of negative coefficient so that's why there's some kind of index  $n$  here that allows me to put negative for coefficients so anyway then the liit  $N$  has zero Norm at  $n$  plus  $n$  to the  $K$  which is

just what you have limit of over  $M$  or  $I$  or what  $I$  in the  $I$  okay and then you had the coefficient of  $Q$  to the  $n$  as  $l_z$  Norm at most  $n$  plus  $n$  to the  $k$   $n$  plus  $n$  plus  $n$  to the  $K$  what is  $n$  plus  $n^2$   $N$  plus  $n$  to the  $k$   $n$  the but  $n$  plus  $n$  is two  $m$  is the exponent of  $Q$  ah  $Q$  to the  $m$  ah  $Q$  to the  $m$  okay I didn't I didn't okay I to the okay okay thank you um basically if if you want your free you want the ring to be the thing where you have  $\text{Co}$  of at most  $\text{polom}$  grows and you think about a way to encode that on the modules then this is what you would you would be doing um Peter do you really mean  $l_0$  Norm or  $L_1$  Norm yeah because it's it's I mean these are again it's about a free module on  $\text{DSi}$  so yeah but I mean if you want the correct answer for if you want the correct answer for  $S$  is a point I mean it seems yeah me yeah I don't know let's say yeah one yeah I think so I mean for me like on theet guy all  $L_0$  and  $L_1$  of some saying but yes let's say I one um right uh right so yeah so within like doing analytic geometry over this space you will get a like geometric way to construct the tape curve really over the kind of smallest Ringle which is actually the sun um uh this actually also means that if you specialize back uh to liopic numbers then it turns out that the Ganges series it's somewhere ah I didn't yes is yeah so this and so the the gas research it's very close but not quite at zero that and uh it is a fun exercise to take this description here of the fre modules and base change it to  $\mathbb{Q}P$  let me not write down the formula it's somewhat nasty uh but anyways okay let me startup

### 13. GENERALITIES ON ANALYTIC RINGS (CLAUSEN)

[https://www.youtube.com/watch?v=38PzTzCiMow&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=38PzTzCiMow&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

#### Unfinished starting from 0:00

e in the last lecture Peter was um well we we spent a lot of time talking about solid analytic Rings um in the last lecture Peter started to introduce different kinds of classes of analytic Rings which also work in the archimedean context so that's these a liquid and these gaseous analytic Rings um what I want to do today is I want to uh talk about some generalities on analytic rings partially in in service of the the story Peter is telling um and partially because uh it sort of should be done at some point it's going to turn out when discussing these new examples of analytic Rings it's kind of nice to have a good handle on the the category of analytic rings and how to manipulate um how to manipulate in it so I'll just be presenting some various facts about uh analytic rings in their category um so let me just start with uh let me start by recalling the definition so so an analytic ring is a pair uh uh where this triangle ring is a condensed ring commutative say and  $\text{mod } R$  uh inside  $\text{mod } R$  triangle meaning condensed  $R$  modules uh is a full subcategory closed under uh limits colimits and extensions um and such that additionally um uh if you have anything any  $x$   $i$  group in our our triangle modules from anything in our triangle uh to anything in  $\text{mod } R$  uh it still lies in  $\text{mod } R$  uh for all  $I$  and finally you demand that um that this the unit object the the uh underlying ring itself should lie in this full subcategory let's call it's called that last condition star um so um so uh now I'll make another definition so so uh uh let's say I don't know a non-complete analytic ring it's not a special kind of analytic ring it's something more General than an analytic ring so maybe a pre-analytic ring uh is a pair as above uh except we don't require star so it's the concept you get by taking the definition of analytic ring and removing this last condition that the uh unit should be the unit object should be complete um so what's the reason for considering this relaxation it kind of plays a uh a technical role because for example if you want to produce an analytic ring structure uh on a uh an interesting condensed ring uh then prior you have to control some  $X$  groups in the category of our triangle modules and that can be quite complicated to do because you would have to understand the homological a prior you'd have to understand the homological algebra of condensed  $R$  triangle modules before you can get going on such calculations and if this is for example the some liquid base ring some convergent lant series it can be kind of difficult to do that but what often turn turns out in practice is that there's a a simpler ring that you can map to our triangle um in the case of the arithmetical rant series it would just be say polinomial and  $T$  and  $T$  inverse um such that it is easier to produce a pre-analytic a pre-analytic ring structure on that simpler  $R$  triangle um and then you get the you have that this condition is not necessarily satisfied but what I'm about to discuss is that there's in a completion procedure that lets to um go from any non-complete analytic ring to a complete analytic ring and that's kind of the most convenient way to do things um and we've also seen examples uh when discussing attic spaces so recall with this uh solid  $RR$  plus Theory um when we were specifying the category attached to a rational open we took the category of  $R$  modules and then we just specified a bunch of conditions um which actually made it into a pre-analytic ring in this sense um but then uh this condition was not necessarily satisfied so you would need to change the  $r$  triangle to basically almost the value of Huber's structure sheath on that rational open in order to get an honest analytic ring so so these kinds of things arise in practice and in general what you want to do with them is you want to complete them to get something that is an analytic ring in the honest sense and that's that's the procedure I want to discuss right now um so and simil so a map so just the last Parts yeah yeah you still require that that's right yeah um uh right um so a map um a map of pre-analytic rings is is just like um just like for a map of analytic brings it's a map of underlying condensed strings uh such that uh  $n$  in  $\text{mod } s$  implies that well the Restriction of scalars of  $n$  uh which is a prior just an  $R$  triangle module actually lands in  $\text{mod } R$  and this is maybe best better thought of as saying that the base change functor um so um and then there's the completion function the  $r$  completion going to  $\text{mod } R$  and then you have the  $S$  completion uh going to  $\text{mod } s$  so this is this completion functor exhibiting this as a localization of this and similarly here and this condition here is the same thing as saying if you go here and then you complete that actually factors through here um necessarily uniquely because this is a localization so it's saying that if you have a map here which is is uh which is kill which goes to an isomorphism here then um then it goes to an isomorphism here so then you get a a symmetric monoidal base change functor um uh from  $\text{Mod } R$  to  $RS \text{ mod } R$  to  $\text{mod } s$  sorry okay um uh so that defines the category of pre-analytic rings um uh and there's a so so those so in other words there's a fully faithful

inclusion  $\mathcal{U} \hookrightarrow \mathcal{A}$  of pre-analytic rings into the category of analytic rings. The inclusion has left adjoint  $\mathcal{U} \dashv \mathcal{A}$  and the claim or proposition is that this inclusion has left adjoint  $\mathcal{U} \dashv \mathcal{A}$  moreover  $\mathcal{U}$  so the left adjoint sends so given a pre-analytic ring  $U$  our triangle  $\text{mod } R$   $U$  this left adjoint  $\mathcal{U} \dashv \mathcal{A}$  sends  $\text{mod } R$  to  $\mathcal{U}$  and some explanation will be required but so we take our triangle and we complete it with respect to this  $U$  this  $U$  pre-analytic ring structure here so we apply the left adjoint to the inclusion from  $\text{Mod } R$  into  $\text{mod } R$  triangle  $\mathcal{U}$  and then we take so to speak the same  $\text{mod } R$   $\mathcal{U}$  so let me give the proof  $\mathcal{U} \dashv \mathcal{A}$  so first we need to make sure that this is of the appropriate type I mean I need to explain how  $\text{mod } R$  is actually a full subcategory of  $\text{mod } R$  triangle  $\hat{U}$  in order to for this to parse as a claim and then I need to check that this setup satisfies the same axioms as as this setup so it actually defines a pre-analytic ring  $\mathcal{U}$  so  $U$  so first of all well I also I should also explain why this is a commutative  $U$  a condensed commutative ring I'm sorry  $\text{mod } R$  is the same yeah yeah yeah so so first of all so this our triangle  $\hat{U}$  so is a commutative ring condensed ring so this is a sort of a completely General phenomenon so we have this  $\text{mod } R$  triangle  $\mathcal{U}$  and this  $U$  localization functor  $\mathcal{U} \dashv \mathcal{A}$  to  $\text{mod } R$  this completion functor this  $R$  completion and this is symmetric monoidal and cimit preserving and it has a right adjoint which is the which is the  $U$  the inclusion in this case but when you have a symmetric monoidal functor with a right adjoint then the right adjoint is automatically lack symmetric monoidal  $\mathcal{U} \dashv \mathcal{A}$  and in particular if you take the unit object here  $\mathcal{U}$  and then you apply the right adjoint it gets a commutative algebra structure so so is a commutative algebra in this category  $U$  in this tensor category and therefore a fortiori in in  $\mathcal{U}$  in condensed ailen groups so it's a condensed commutative ring  $\mathcal{U} \dashv \mathcal{A}$  so two we should check that  $U$  and moreover yeah so not only also in this General situation with the right adjoint to a symmetric monoidal functor if you take any object any object here and apply that right adjoint it gets the canonical structure of a module over this commutative algebra  $\mathcal{U}$  again just purely formally so  $U$  so there's this natural factoring so we have  $\text{mod } R$  including into a  $\text{mod } R$  triangle  $U$  this actually factors through  $U$  our triangle  $\hat{U}$  modules  $U$  and then the forgetful functor  $\mathcal{U} \dashv \mathcal{A}$  and the next thing I should check is that this is actually fully faithful so so that we can indeed view  $\text{mod } R$  not just as a full subcategory of  $\text{mod } R$  triangle but of  $\text{mod } R$   $U$  complete  $U$  our triangle complete and so for that  $U$  so the proof is you can just ask for HS so if you take  $M$  and  $N$  in  $\text{mod } R$   $U$  you want to compare  $h \otimes m$   $U$  sorry  $h \otimes m$  triangle  $MN$  to  $H$  our triangle  $\hat{U}$   $MN$   $\mathcal{U}$  but  $U$  so  $n$  is an  $R$  triangle module so therefore this one is the same thing as oh I'm sorry I think I should give myself a little bit more room so this is the same thing as  $H$  our triangle  $\hat{U}$  from our triangle  $\hat{U}$  tensor over our  $\hat{U}$   $M$  to  $n$   $\mathcal{U}$  but now since  $n$  is complete since  $n$  lies in  $\text{mod } R$   $\mathcal{U}$  we can  $U$  complete this we can apply the completion functor to this too  $U$  and we'll get the same thing so this is same as  $H$  our triangle  $\hat{U}$  our triangle  $\hat{U}$  tensor our triangle  $M$   $\hat{U}$  to  $n$  and now we can use that the completion is symmetric monoidal so that we can move it inside this whole relative tensor product and when you move it in the relative tensor product and use the completion is item potent you find that this is indeed just a  $h \otimes m$  triangle  $M$  tan okay so  $\mathcal{U}$  it's now we have to verify all of the axioms and this is actually a little bit more subtle because we need to make sure that the  $X$  calculations  $U$  are going to work out correctly  $\mathcal{U}$  and this argument so so let me so remark  $U$  this argument for fully faithfulness  $U$  will also work for  $x$  provided that  $U$  that the derived completion so now I don't know what derived completion  $U$  on our triangle is the same as the completion on our triangle so so recall that we had this  $U$  notion of the derived category of an analytic ring which was a full subcategory of the derived cat of our triangle and we had the derived completion functor which was not necessarily on on an even some on something sitting in degree  $Z$  does not necessarily agree with the the usual completion it's something which whose  $\text{Pi } Z$  or  $h_0$  is the same as this but it could have higher homology  $\mathcal{U}$  for example in this  $U$  yeah in this situation with these weird non sheify  $U$  attic spaces  $\mathcal{U}$  but  $U$  in general we don't have that so what we what we can see what we can formally see see is that  $U$  everything works if we replace  $U$  our triangle  $\hat{U}$  by our triangle  $\hat{U}$   $U$  the derived thing and use the notion of a notion of derived analytic ring meaning that if you  $U$  if you ask not to give an analytic ring structure on this ordinary ring but an analytic ring structure on this derived enhancement then using the same kind of formal argument you'll be able to check all of these properties so limits and co- limits are actually automatic  $\mathcal{U}$  given that  $\text{mod } R$  has those closure properties but to be closed under extensions in here is is more a priority more than being closed under extensions here but if you can make the  $X$  calculations you'll see that it's the same  $\mathcal{U}$  and then this is also a question of an  $X$  calculation  $\mathcal{U}$  so  $U$  if you were to use a notion of derived analytic ring then you'd be done  $U$  with that it wouldn't prove the original proposition because I stated it in the underived way but



it would prove something um so now I think uh I will introduce a notion of derived analytic ring kind of in the middle of this proof uh so so definition so so uh an analytic e infinity ring uh is a pair um  $R$  triangle mod  $r$  or maybe maybe I'll say  $d$  of  $R$  uh where actually it's it's sort of more convenient to only specify the the part that that's homologically in homological degrees greater than or equal to zero so where our triangle is a a condensed uh e infinity ring um and connective connective thanks yeah so meaning it also only lives in homological degrees um and  $D$  greater than or equal to zero our subset uh oops  $D$  greater than or equal to zero our triangle is a full subcategory such that uh so it's closed under limits colimits and such that uh internal  $H$  uh in our triangle from anything in  $D$  greater than or equal to zero I should say in  $D$  greater than or equal to zero our triangle uh uh to anything in  $D$  greater than equal to  $z$   $r$  is still in  $D$  greater than equal to  $z$   $r$  this full subcategory um complete or not complete uh yeah that's a good thank you uh yeah so let's say a pre-analytic ring um and then uh I want to explain the relation between this notion of analytic ring and this sort of derived notion of analytic ring and then the Bare Bones notion on the aelan level that we defined earlier so the proposition is that um so for an any in connective condensed infinity ring uh our triangle uh there exists a bje between the set of uh pre-analytic ring structures uh on our triangle uh and the pre-analytic ring structures so this is in the old sense this is in the new sense um on just the condensed commutative ring which is gotten by taking  $\Pi_0$  um and also uh an antic matches up with analytic so or complete so I can also could also remove the pre from both sides so this projection on the level of pre-analytic ring structures restricts to a bje on the level of analytic ring structures um and what is this gotten by so here you have uh this  $D$  greater than or equal to  $z$   $r$  is the data and you you send that to uh just those things in  $D$  greater than or equal to  $z$   $r$  which happen to lie in the heart so uh so let's say  $d$  equals  $z$  uh  $R$  triangle which is the same thing as  $D$  equal  $0$   $\pi_1$   $r$  triangle so you have a module concentrated in degree zero over some derived ring that's just the same thing as giving a module usual module over the the  $\pi_0$  so um and in this direction it's given by sending uh mod  $R$  to the set of those  $m$  in  $D$  greater than or equal to zero  $R$  such that  $h$   $m$  is in mod  $R$  uh for all  $I$  greater than or equal to zero so is this definition that be a problem that  $D$  greater than zero might not be close on the limit limits didn't I ask didn't I ask it to be no I mean a whole category not I don't uh can you repeat your question pops Li it goes to to like the greatest than minus one means ouring here yeah I mean yeah I still have but we gave the argument fixing that uh so when you have this assumption you can you can see that that holds um when you if you take a an arbitrary direct sum of copies of the unit then that then you'll see that products of a fixed even higher higher derived products of a fixed a given fixed element will be still in the category and then if you have an arbitrary product then you can write it as you can write it as a retract of a product with a fixed element by taking the direct sum of all the elements and uh yeah so we we did this argument in one of the lectures I guess he asked if infity oh yes was that your question or or well in any case it means in the infinity categorical sense it means  $R$  Lim implicitly and  $r$  colim um right so let me make a remark um so in a special case uh where where well  $R$  triangle equals  $\Pi_0$   $R$  triangle we see that the two definitions are consistent meaning in the case where the notion of rings over lapse namely where your derived ring is just concentrated in degree  $Z$  so it is a classical ring the derived definition of what an analytic ring structure or a pre-analytic ring structure is matches up with the the the naive definition we gave on the level of a billion categories um so that's the first remark and the second remark is this finishes proof uh the proof of the proposition on completion um because what we can do is we can just um so there's no obstruction to proving the proposition on completion in the derived SE uh setting uh and then we get a um but then we can move it down to the Aion setting by going from the analytic ring structure on this derived thing that we produced to the analytic ring structure on its  $\Pi_1$   $Z$  just by applying this procedure here's a question are reallyly that greater equal to  $Z$  is limit I think claiming that you take the limit in  $D$  greater equal to  $Z$  yes that's right yes yeah so somehow there's a there's something going on here so for example being claimed being closed under the loops funter so that that that's actually stronger than you might think because well you what what what is the loops in in this greater than equal to zero derived category it's you shift it down and then you kill the bottom thing so it's really just like killing the bottom thing up to a shift um and so that kind of goes into the argument here um that's how you can see that an object in there is an object lies in here if and only if each of its homology groups lies in there and that's one of the one of the key things that goes into this proposition here um okay uh I think this condition doesn't always match up so if you have something that's complete the Rock complet complete but in the compass Direction I think might have some  $J$  uh and that's not possible right just because of this fact that if you have something derived complete the needs of each of its homology groups is also complete or am I

misunderstanding now you're saying that you have  $\text{pric}$  which on the  $\text{LEL } Z$  is already it's complete when you say it's already complete but I don't think there but sorry one more complet match up it seems to say that if you have if  $I$  is the property that only zero is complete that it's already complete but I oh you're right you're right I'm sorry yeah no you're you're right uh um um um so then let me rather say thank you Peter so let me rather say um an analytic ring here goes to an analytic ring there uh uh and the converse is true if your ring is concentrated in degree  $Z$  so in the setting of The remark um um yeah topologically as space of sorry the sound is not coming through very clearly so uh if you could mean understand topological ring I don't no I understand topolog topology space I so here's the way I would think about  $e$  Infinity is it's just it's a it's just there's a general categorical concept right if you have a symmetric monoidal category then you can look at commutative algebra objects in it and there's an Infinity categorical version of that which implicitly involves things like infinity operads or something but really I mean um and then an  $e$  infinity algebra over  $Z$  will just be a commutative algebra object in the derived category of  $Z$  in that Infinity category with the usual Drive tensor product so if you believe in this Infinity category mumbo jumbo you don't need to think about it in being built in terms of topological spaces and operads explicitly you can kind of just plug into some simple categorical formalism um okay so that's the notion of completion um yeah maybe okay I was I'm reassessing that so maybe the good thing to say is that you have an an analytic ring over here um gives you an analytic ring over here such that also the yeah so analytic ring structures here are the same thing as analytic ring structures here uh such that uh such that each  $\text{pii}$  not just  $\text{Pi } Z$  is complete in this category here so um yeah uh right um so now uh I'm not going to prove the proposition this was proved in uh in one of the older lectures so you can look there for the argument um and now I want to move on to the next topic which is co- limits in the category of analytic Rings although maybe I should make another well sorry maybe I should make another remark yeah so it's important to note that the uh the derived categories can change when you so if you take you take an analytic ring structure on  $\text{Pi } \text{kn}$  it gives you an ailan category but we also argue it gives you a derived category um that derived category is not necessarily going to be the same as the derived category you get on  $R$  triangle when you move along this equivalence because the the higher homology in  $R$  triangle can can uh can make things a little different so um so when we're doing this completion on the level of naive analytic rings sitting in degree zero it's important to note that so to speak the derived category is not the correct one um the derived category of this naive completion so yeah maybe I'll make a warning um so if so if  $r$  at triangle mod  $R$  is the completion of uh our triangle mod  $R$  then it's not necessarily true that uh the derived category of this thing is the same thing as the derived category of this thing um but only if if you use derived completion on the left hand side so if the if this ring that you calculate if you calculate this derived completion and it happens to live in degree zero then this will be true and it'll and the derived categories will pass along correctly but if this happens to have higher homology then um then you really should be considering a derived analytic ring uh instead of an ordinary analytic ring and this was the phenomena we phenomenon we saw in in adct spaces that for some rational localizations in very exotic contexts you get a derived structure sheath and then that's then you really should even though you have an underlying ordinary analytic ring by taking  $\text{Pi } Z$  you really should be considering the derived analytic ring there sorry how do the  $X$  computations differ well they're  $XS$  over different Rings right right but like they're all like I mean in the sense that like when you complete it then like you have that condition that  $X$  the mod  $R$  triangle mod  $R$  lives in mod  $R$  yeah but then so now it's still the same mod  $R$  yeah it'll live in the same category but the objects won't be the same yeah you can have like wildly different things well wildly I I I mean you can have different things these are kind of exotic phenomena I'm not saying yeah so it's I haven't actually investigated really I mean because the the phenomena like when this happens it tends to be fairly exotic so um haven't really seriously tried to do those calculations um okay so all right so now we have this notion of completion of analytic rings and this lets us discuss um uh Co limits of analytic rings so uh there's something else that I maybe should have mentioned in connection with this what I just erased so this uh analytic ring structure is on  $R$  corresponding to analytic ring structures on  $\text{Pi } Z$  of  $R$  um which is that maps of maps in the category of analytic rings in the derived sense are the same thing as maps on the level of derived Rings which are maps in the usual sense in the old sense when you when you look on the aelion categories there um so okay um right so Co limits in the category of analytic Rings uh so so the uh the procedure is is you take the co limit uh in pre-analytic rings and then you complete and formally this must be how you compute Co limits just because the completion functor was the left adjoint to the inclusion um so what are what so then what is this Co limit in pre-anal IC Rings it's actually quite

naive so if you have a diagram so uh  $r \rightarrow i \rightarrow m$   $R$  triangle  $I \rightarrow \text{mod } r \rightarrow i \rightarrow \text{oh } I$  and  $I$  then the co-limit is is given by taking the co limit of condensed rings and then so this is a yeah co-limiting condensed commutative Rings uh oh sorry uh what no sorry sorry uh the co limit of  $R$  triangle  $I \rightarrow \text{mod } r \rightarrow i$  is uh the pair Co limit over  $I$  uh our triangle  $I$  and that's the co limit in condens commutative Rings um and then you take a certain full subcategory so called mod cimit  $I \rightarrow \text{ini } I$  uh  $RI$  um where this is the those  $m$  in mod uh colimit  $RI$  uh triangle uh such that uh for all  $I \rightarrow M$  lies in mod  $r \rightarrow i$  or when you restrict scalar is  $RI$  triangle rather um then then it lies in that category there so it's just an intersection of a bunch of categories uh where those categories are just required when you restrict scalars you lie in the corresponding complete thing there um so this is quite um it's quite straightforward to from the definition of the category of analytic rings to see that this is the way to calculate co-limits um by definition a map was a map of rings uh satisfying a certain property so you certainly want um certainly want to put the co-limit of the Rings here and then this definition here is just tailored so that you have the um correct property there um um and checking that this satisfies the axioms of a pre-analytic ring is also not difficult um so well certainly closure under limits and colimits is Elementary because the forgetful functor that you're using here commutes with limits and colimits and the closure under the  $XS$  is uh also not so difficult um so um yeah uh you yeah you just you you just check it directly um so let me now take some examples so filtered Co limits in analytic rings so the claim is that here uh the completion is unnecessary so the this filtered Co limit of  $RI$  triangle um mod filtered co-limit of uh  $r \rightarrow i$  is already complete um so what does that mean that means that uh you take this you have to ask whether this ring lies in that in this category um but um lying in that category means for every  $I$  you lie in mod  $r \rightarrow i$  but uh from  $I$  on in this filtered Co limit you lie in mod  $RI$  by definition and so from  $I$  on that's a co-final collection here so you can calculate this as a filtered Coit of things which satisfy the condition so it also satisfies the condition um so an example of this we've already seen is that if  $R$  is a discrete commutative ring uh then solid  $R$  is the filtered Co limit of solid  $r \rightarrow i \rightarrow \text{ini } I$  if  $R$  is the filtered cimit of the  $r \rightarrow i$  maybe you can say what you mean by this Coit there this Co limit here yeah yeah this is in the category of analytic Rings yeah so or pre-analytic ring some um okay so if you understand filtered Co limits then the next thing you should try to understand is pushouts so uh so push outs so if you have a  $r$  goes to  $a$  goes to  $b$   $r$  goes to  $A$  and  $R$  goes to  $B$  uh diagram of analytic rings then you need to complete uh the pre-analytic ring which is a tensor over  $RB$  um and then the full subcategory of those  $m$  in mod  $a \rightarrow \text{oh sorry triangle } uh \rightarrow \text{mod } a$  triangle tensor our triangle  $B$  triangle such that uh as an  $a$  module that lies in mod  $a$  and as a  $b$  module it lies in mod  $b$  and here it is uh important to complete because um uh this thing has no reason to uh lie in mod  $a$  or in mod  $b$  and in fact so the the here's a remark the completion functor here a priori involves uh uh iterating uh a completion and  $B$  completion this is a alternating it one category or yeah so I'm doing it in the one category here but the exact same thing all the the exact same remarks apply in the infinity category uh if you know the ver the notion of derived analytic ring you just interpret everything is a derve tensor product and use  $D$  greater than or equal to zero instead of mod it's all there's no difference in this discussion yeah um okay so yeah in general you'd have to take this thing a completed as an a triangle module then  $B$  complete that as a  $b$  triangle module then a complete that and  $B$  complete that pass to a sequential Co limit and that would be the description of the completion functor for this in practice it's usually not so bad but that's um a priori what what you need to do um and we also had examples of this in the solid setting so um so example if we had one of our  $RR$  plus settings and then to a rational open and say of  $R$  uh  $r \rightarrow +$  then we get a new analytic ring um I don't know um which was the completion of the pre-analytic Ring structure on  $R$  where you enforce all of the relations descri that your rational open describes so if it was fub1 FN over  $G$  then you require that  $gxs$  inverti on your modules and that  $fi$  over  $G$  um uh is a solid is a solid variable with respect to all of your modules um and then the the pushouts so new analytic ring  $I \rightarrow I \rightarrow I$  kind of hesitate to call it  $O$  of  $U$  plus but okay I'll do it anyway um maybe um and then if you take  $O$  of  $U$   $O$  plus of  $U$  um tensor over  $r \rightarrow +$  with some other  $OV$   $o$  plus  $v$  uh then what you're going to get is the same thing for the intersection of these rational opens the analytic ring corresponding to the intersection of these rational Ops I guess in the discret case our discrete it literally is just this um um so that the notion of push out in analytic Rings is corresponding to intersection of rational opens here and that just follows from from the definition of this here um so yeah this this uh these push outs are in general kind of the most important construction because they they're geometrically what's supposed to correspond to pullbacks um and this business of having to complete them makes for an additional subtlety compared to the usual algebraic geometry okay so uh questions about that um is the procedure becomes easier when you just want

finite finite product no it's the case of relative tensor products is no easier uh the case of absolute tensor products is no easier than the case of relative tensor products so in fact I mean I could have I didn't really need in this first part I didn't really need to say filtered sifted is enough if you know what sifted means um and then General co- limits just as general Co limits can be decomposed into filtered co- limits and push outs General cols can also be composed with into sifted Co limits and and co-products so and concretely the the completion functor for this relative tensor product is just the same as the completion functor for the absolute tensor product it's shift shifted e easy sifted Co liit is just as easy as filtered Co limit yeah but to so the the only difficulty is about the finite exactly exactly so yeah a tensor  $RB$  is no more difficult than a tensor  $B$  um so okay now I want to now I want to have a little bit of fun well depending on your definition of fun uh um um right so forus so I want to prove the following theorem so suppose  $R$  uh is an analytic ring uh then the fenus map which goes from our triangle to our triangle modulus  $P$  it's induced by  $X$  goes to  $x$  to the  $p$  uh that's certainly a map of condens strings um but this is actually a map of analytic Rings um from  $R$  uh to well  $\text{Rod } P$  by which I mean  $r \text{ hat mod } P$  um and then  $\text{mod } R$  yeah so  $P$  toin elements yeah um so you just induce the analytic ring structure so the a complete module over our  $\text{mod } P$  is just a a module over our triangle  $\text{mod } P$  which is complete when viewed as an  $R$  triangle module um so you should probably I mean you can you're free to assume  $R$  is characteristic  $p$  in which case uh it's really just a map from  $R$  to  $R$  uh okay um so what do we need to do to prove such a claim well according to the definition what we need to do is uh so proof need to see that if let me let me assume uh  $R$  has characteristic  $P$  so  $R$  is an FP algebra our triangle is an FP algebra uh just to avoid the notation of them modding out by  $P$  and all that um we need to see that if uh  $m$  is in  $\text{mod } R$  are uh then uh yeah the the fenus the fenus push forward of  $M$  uh which lies in  $M \text{ mod } R$  triangle actually lies in actually lies in  $\text{mod } R$  and this is uh yeah this is not so obvious how to check um we have these Maxum of analytic ring they're all about closure categorical closure properties uh in linear algebra like limits and colimits and  $X$ s and so on they say absolutely nothing about the frobenius um they give no kind of hint as to why this should be true um however let me make a remark uh there is another perspective on maps of analytic rings um so first of all so first of all if uh if  $R$  triangle and  $\text{mod } R$  uh  $s$  triangle  $\text{mod } s$  is a map of analytic rings uh then for all light profinite sets um  $t$  uh we get a map an  $R$  triangle linear map um from  $R$  bracket  $T$  the free  $R$  module on  $t$  uh to the free  $s$  module on  $T$  Sorry by  $f$  mean  $\text{frob}$  I mean  $\text{frob}$  yeah fenus as a yeah map of rings so yeah thanks uh and the reason is well by by hypothesis of this being a map of analytic Rings uh what it means is that these these modules here are actually actually live in  $\text{mod } R$  that's equivalent to being a map a map of analytic Rings you need to check a prior for all  $s$  modules but but by co- limits it's enough to check it for a generating class in fact you know so for these guys in fact you can even take  $T$  to be the counter set if you like so that this is the thing that you act you you get is that this is um this is an  $R$  module um so that the map from the free  $R$  triangle module on  $T$  to it uh factors through  $RT$  um and this is this is functorial in  $s$  uh functorial in  $t$  sorry um um um moreover well there's so these aren't quite arbitrary there's a small extra there's an extra condition that in particular is Satisfied by them so if uh if you do have a map from  $T$  to our triangle uh so a map of condensed sets uh from  $T$  to  $R$  triangle then there's two things you can do um one you can make a map from  $R$  bracket  $T$  to  $R$  triangle um an our triangle linear map because by definition of analytic ring this thing is complete uh so you go to  $R$  triangle bracket  $T$  and then you go to  $R$  bracket  $T$  um uh but then you also have this map here that we've assumed exists from  $R$  bracket  $T$  to  $S$  bracket  $T$  but on the other hand you can compose this to  $S$  triangle um and you get uh this thing here so this is just a just the map of rings this is the thing we posited to exist and this exists for the same reason this exists and that square uh will commute when you have a map of analytic rings so I want to claim so LMA uh conversely if  $R$  to  $R$  triangle to  $S$  triangle is a map of condensed strings uh such that there exist these Maps um uh from  $RT$  to  $St$  satisfying the conditions above uh then  $R$  to  $S$  is a map  $R$  to  $s$  so this is nice because you don't have to explicitly think about how you'd build  $s$  bracket  $T$  from  $R$  bracket  $T$  modules if you just have the maps that would kind of indicate uh indicate it then then you can actually do it um let me maybe give a hint of the argument it's in the again it's in the notes from a previous iteration of this so a hint of the argument would be that so note that well would be to note that so so a hint of the argument is that uh  $\text{mod } R$  is monatic over condensed sets so you have a forgetful functor from  $\text{Mod } R$  to  $\text{mod } R$  triangle which in turn forgets to condense sets and that satisfies the hypothesis of *barbec* it's basically a localization or the right adjoint of a localization followed by a forgetful functor and so so this category can be understood as the category of what whatever they call it algebra is over some monad here which is kind of the free  $R$  module monad and then

if you want to show that every  $S$  module is an  $R$  module it would be enough to produce a map of monads from the free  $R$  module monad to the free  $S$  module monad well the very first step in producing such a map of monads would be giving the map from the free object here to the map of the free object here and then there's a condition you need to check compatibility with the monad structure and if you play around enough you can reduce the what you need to check to just uh this commutative diagram here okay uh all right so uh so let's continue discussing  $\text{fenus}$  so now what are we reduced to so we need to uh for every light profinite set  $U$  a map well a  $\text{fenus}$  linear map  $u$  from  $R$  bracket  $T$  to  $R$  bracket  $T$  satisfying well it should be functorial in  $T$  and it should satisfy this little extra condition relating to when you take a map from  $T$  to  $R$  okay so now let me uh recall uh just in pure algebra a little bit about the  $\text{fenus}$  so recollection on  $\text{fenus}$   $u$  so  $\text{fenus}$  is a map of rings so I said it's a map of rings from  $R$  to  $\text{Rod } P$  but actually in some form it exists uh for an arbitrary ailen group so if  $m$  is an ailen group uh then you get a natural map from  $M$  uh to well the zero Tate chology of  $CP$  acting on  $M$  tensor  $p$  uh so  $p$  is a prime so recall that this is the uh cernal of the norm map or the transfer map maybe is a better name for it from  $M$  tensor  $P$   $CP$  orbits to  $M$  tensor  $P$   $CP$  fix points  $u$  you said  $CP$  specific group of yes uh thank you yes  $u$  and I'm going to this is quite long notation I'm going to adopt some homotopy Theory abbrev a for this I'll just write uh  $\pi_0$  of the  $CP$  of the Tate kology written in the style of hopy theorist so it's shorter so  $u$  uh and what is this map induced by uh uh you send an element  $m$  to the diagonal tensor  $M$  tensor  $m$   $p$  times which lives in uh  $M$  tensor  $p$  and it's clearly fixed by the cyclic group of order  $p$  and then you take its class uh in the take chology okay and the point is that of course this expression by itself is not additive but if you look at  $M$  plus where  $M$  plus  $n$  goes you'll have a bunch of cross terms and you'll be able to recognize that each of those cross terms uh is a norm uh from uh from  $M$  tensor  $P$  so it's a nice exercise if you've never done it uh oh sorry continue over here so I should say you have a it's a group homomorphism that's the special thing that happens here  $u$  and what is the relation of this construction with the  $\text{fenus}$  on a ring so if  $m$  equal  $R$  is a commutative ring uh then then you have this thing which goes from  $R$  to  $R$  tensor  $P$  Tate  $CP$  and  $\pi_0$  of that  $u$  but then you can uh use the multiplication map from  $R$  tensor  $P$  to  $R$  uh to go to this where now the  $CP$  action is Trivial here you're using that the ring is commutative so that  $u$  it's a  $CP$  equivariant map multiplication is a  $CP$  equivariant map from  $R$  tensor  $P$  to  $R$   $u$  but now since the  $CP$  action is Trivial this is just the same thing as uh this is doing nothing and then the norm map is just usually it's summing over the action of the group but the group is  $Tri$  else you're just summing over  $P$  copies of one so this really is just  $\text{Rod } P$   $u$  and if you trace through this is the frobenius so just by the same formula so okay and now if  $m$  is an  $R$  module then this map uh this so-called  $\text{fenus}$  map from uh Tate  $CP$   $u$   $u$  this is a map of a bilon groups so these are both our modules uh this is a map of a bilon groups not a map of our modules but in fact it is a map of our modules if you forbus twist uh uh five twist uh on the right hand side so in other words thisan group level frobenius is kind of linear over the the ring level for obus  $u$  in the appropriate sense so what this suggests is so now we're trying to look for some uh  $L$  some linear algebra uh incarnation of frobenius on the level of these light profinite sets this suggest that we should just try to apply this uh ailen group version of frus and and see what happens so now we take  $t$  a light profinite Set uh then we get  $u$   $R$  bracket  $T$  and we can always do this in uh so now we're viewing  $R$  bracket  $T$  is just a a sheath of a billion groups on light profinite sets and we can apply this uh construction here so we tensor it  $P$  times  $u$  and we get Tate  $CP$  and then we take  $\pi_0$  so that's all just happening at the level of a sheaves of a bilon groups so so that's a very uncompleted tensor product but we can always map this to any uh further completion we like and in particular we can map it to uh  $\pi_0$   $R$  bracket  $T$  I apologize this is not the polinomial ring on one variable this is the free  $R$  module on a profinite set  $T$   $u$  to tensor over  $R$  uh  $p$   $t$   $CP$  but you have this uh sort of so to speak kunth formula so the tensor products of the free modules are just the free modules on the product  $u$  and then  $CP$  is acting here as well Tate  $CP$  and uh we wanted to make a map from  $RT$  to itself which is forus linear uh and we've gotten to  $RT$  to the  $P$  so how do we compare them well we can put  $RT$  Maps here via the diagonal embedding and that's also equivariant for the  $CP$  action if you make  $CP$  act trivially here so you get a map like this and the claim is that this map is an isomorphism  $u$  if you buy that then you're basically done so  $CU$  again  $CP$  is acting trivially here so the take construction uh is just modding out by  $P$  so this is indeed  $\text{Rod } P$  bracket  $T$  and then for similar reasons the composite is actually going to be  $\text{fenus}$  linear  $u$  and gives you the desired construction you also have to check that condition but the full argument is in the notes analytic PDF what I want to highlight is is  $u$  what's different here compared to last time this was discussed by Peter in a previous iteration  $u$  last time this had to be added as kind of an axiom on our analytic rings that well



basically that this argument Works which reduces to this claim being true um but now that we switched to the light profinite set setting it's actually you can just prove it's true uh for arbitrary  $R$  so that's what I want to explain now um so to finish we need to show so the Lemma so let's say um if uh so if no  $s$  if  $s$  is a light profinite Set uh with  $CP$  action uh then the map  $um R$  bracket uh  $s$  the fix points the  $CP$  fixed points which is a closed subset of  $s$  hence also a light profinite set um to our bracket  $s$  um induces an isomorphism on tap chology and in particular on take chology when you pass to  $Pi 0$  so just take chology and degree  $Z$  okay so this is going to use some facts about light profinite sets which Peter mentions mentioned in his first lecture but haven't sort of well they have well they have been used for example for this projectivity of this  $P$  space but well um we're going to more directly use them now so the so proof is so the first claim is that um  $R$  bracket  $SCP$  is actually mapping injectivity into  $R$  bracket  $s$  and the reason is um any inclusion of light profinite sets has a retraction this was a fact that uh Peter proved in the second lecture of this course so if you apply it to the inclusion of  $CP$  and  $S$  you find there's a retraction so then if you hit it with any funter it'll still be it will still be injective um so what does this imply this implies by the long exact sequence and take chology this means it suffices to show that if you take  $r$  braet  $s \bmod R$  bracket  $SCP$  uh that this thing has Vanishing take Construction um and the second claim is that um generally if you have  $t$  inside  $s$  then  $R$  bracket  $s \bmod R$  bracket  $t$  uh only depends on the locally compact space locally light profinite uh space which is the complement  $s \setminus t$  thank you it's not like group where you can choose yeah okay um so that's not a precise claim so what's the more precise version of the claim if you have  $S$  Prime mapping to  $S$  and then you have  $t$  including here and you form the pullback to get  $T$  Prime and if this is an isomorphism over  $s \setminus t$  so you're sort of blowing up  $T$  so to speak or you're choosing a different compactification of  $s \setminus t$  um uh then  $r$  braet  $s$  Prime  $\bmod r$  braet  $t$  Prime maps isomorphically to  $R$  bracket  $s$   $r$  bracket  $T$  and and this holds because uh this Square here is a push out in condensed sets which is a little a little uh well light light condensed set let's say a little exercise you can see using the definition of the gro de topology um yeah I wanted to assume it was a pullback yeah so um okay um and then the third thing is that um if let's say  $x$  is a let's say Sigma compact uh a totally disconnected locally compact house space and  $CP$  acts with no fixed point then uh  $X$  is actually isomorphic to some co-product of  $CP$  many copies of  $Y$  or  $CP$  is acting uh by permuting those copies uh Sigma compact means countable Union of compacts sorry what wase  $S$  Prime you can choose okay yeah you it's any  $S$  Prime you like which maps to  $S$  by a map which is an isomorphism over this open subset here so it's only it's just modifying  $T$  making  $T$  bigger yeah um okay um uh so the yeah by the way the thing is if you have a a light profinite set and you remove a light a closed subset which is also light then the complement is one of these guys um it's going to be Sigma compact because of the lightness basically there's a countable system of neighborhoods of any closed subset and then it's totally disconnected and locally compact house DWF so um what is why it exists there exists a why such that also also with these properties such that so I'm saying the action is free in a very strong sense if provided the action has no fix points then then it's free in a very strong sense it's just literally induced um so the reason is that uh well so  $X$  to  $xod CP$  is a a covering map um that just uses locally compact house DWF and freeness of the action um but then this will also be Sigma compact uh locally compact house DF and totally disconnected and as Peter explained uh this there's an easy argument that this implies that it's a disjoint Union of countably many light profinite sets um and what's that not necessarily the GU oh I'm sorry yeah oh yeah right in the in the in the application they will be light but yeah thank you I forgot I should have maybe said locally light or something like that um so there a disjoint countable disjoint Union of profite sets um and over and this covering map will have to be trivial over each of these profinite sets because the topology is totally disconnected so locally trivial implies globally trivial um and then that implies this because the the  $Y$ 's or  $Y$  yeah  $y$  will just be  $xod CP$  um and then the trivialization of the cover over each profinite set will will give you the the required decomposition there okay so if you combine two and three what do you deduce you deduce that this um  $r s \bmod ah$  you deduced that  $R$  in that setting over there  $s \bmod R$   $CP$   $p$  uh is actually a direct sum of  $CP$  many copies of some guy  $X$  or  $CP$  is acting on this set here oops  $s$  because by two um this only depends on the light profinite set which is the complement that's one of these guys that here so then that light profinite or that locally light profinite set is um yeah is a disjoint Union of  $CP$  many copies of some guy and then you compact a in a different way by just compactifying  $Y$  and then taking the  $CP$  many disjoint Union copies of uh of that to get a compactification of  $X$  and then use that to calculate the quotient here and then you'll find that the the module is is induced um and then an induced module has Vanishing Tate chology so yeah okay so this um this uh fact that fenus is a map of analytic Rings it's

not just a cute fact so I'll make a remark I don't think I'll go into the details um but uh this this theorem on fenus implies that if um if our triangle mod  $r$  or let's let's do it in the in this setting uh well a derived analytic ring or a pre-analytic ring uh and we give our triangle the structure of an animated commutative ring uh condensed ring which gives new functors  $s$  derived symmetric powers on um  $D$  greater than or equal to zero our triangle which are the things that are used to build free animated Rings or or you say used to build the monad describing analytic ring or sorry animated Rings over our triangle then these Simi descend to uh so if  $M$  mapping to  $n$  goes to an isomorphism in  $D$  greater than or equal to  $r$   $d$  greater than equal  $0$   $R$  so this is a map in  $D$  greater than or equal  $0$   $R$  triangle then so it is Simi uh which lets you define the Sim eyes on the derived category of this analytic ring so this uh the the argument for this can also be found in analytic PDF for this deduction from the statement about  $f_{\text{bus}}$  to the statement about symmetric Powers it comes from a calculation of what doen pooppy did about the so-called stable um stable derived functors of the symmetric power functors um and what does this let you do uh this lets you uh do normalization or completion of pre-analytic rings uh also in the animated context um so you get a good category of animated analytic rings so this is a pair consisting of uh an animated commutative uh condensed ring light condensed ring and then a  $d$  greater than or equal to  $\text{Zer } R$  um this is an analytic ring structure on the underlying infinity ring say or you could just specify the  $\pi$  not level thing or whatever you want um so the so the the extra uh structure you're putting on here is just that this gets an animated structure and then the whole discussion Works uh just fine there um okay so uh next so one last topic um um so maybe I'll I'll call it a killing algebras killing algebra objects so so recall for motivation that we presented the solid  $Z$  Theory uh as uh the analytic ring structure on uh on the integers where  $M$  was in solid  $z$  uh if and only if  $U$  when you take internal  $H$  from  $P$  to  $M$  and then you apply uh this uh shift minus one uh to internal home from  $P$  to  $M$  that this should be an isomorphism uh but this is equivalent to saying that say  $\text{ROM}$  um from  $a$  to  $m$  equals  $z$  where  $a$  is equal to  $P$  modulo  $T$  minus one a priori I should say derived modulo  $T$  minus one but actually it's easy to see that multiplication by  $T$  minus  $1$  is injective um and in fact it's not going to matter either way um as in the class of module satisfying this doesn't really change um so it's the things that and and this has an algebra structure uh in the ambient category um say  $d$   $d$   $DZ$  condensed because uh  $P$  itself was a commutative ring object via addition on natural numbers and then we're just modding out by a by an element of that ring  $T$  minus one okay um I'm sorry goes to isomorphism I'm sorry yeah so if this is a map which goes to an isomorphism in there then it symmetric power also goes to an isomorphism there that's the condition you need to extend this symmetric power functor from this thing to that localization of it  $D$  greater than or equal to  $z$   $r$  what would do back goes to oh you take completion it yeah exactly yeah under the goes to under under the localization functor yeah um so um so General context so oh maybe I should say yeah so General context so let's say  $C$  tensor is a symmetric monoidal uh let's say presentable uh stable Infinity category and yeah presentable let me include that tensor product commutes with co- limits in each variable um and let's say  $A$  and  $C$  is an algebra object and it really I only need it in some kind of weak sense so let's say that we have a multiplication map literally just a  $t$  or a goes to a um we have a unit so there's the unit object in the symmetric monoidal Cate category and then we require that like the multiplication is either left or right unal so um let's say one so Unit  $t$  for a and then multiplication this is isomorphism um so you said you don't require any associativity or no nothing yeah yeah it's really really weak okay um then the so then we can Define uh so  $D$  subset  $c$  uh to be the full subcategory of those  $M$  and  $C$  such that internal  $H$  which implicitly is an  $\text{ROM}$  here uh from  $a$  to  $m$  equals zero so we're we're killing a we're declaring that a should be equal to zero um but also in an internal home sense um and then the goal will be in in favorable situations uh give a formula for the left ad joint to the inclusion to the inclusion sorry is  $K$  the name of the mathematician or just killing kill killing killing killing means uh not not the mathematician responsible for the killing I'm that this kind of seem probably he didn't do it yeah yeah um okay um so maybe I should give some examples um well there was the solid example that I just described so there was also solid  $ZT$  uh which was also obtained by killing some endomorphism of or requiring some endomorphism of  $P$  to go to an isomorphism but that endomorphism of  $P$  was also just given by multiplication by some element with respect to the ring structure on  $P$  so it's the same thing as killing the the cofiber just like this um there's also kind of pure algebra examples so for example if you take if you're in usual derived category  $D$  of  $R$  and then you look at  $R$  mod  $f$  um so that's  $\text{Cals } D$  of  $R$  um then what is  $D$  well  $D$  is uh  $D$  of  $R_1$  over  $F$  so it's kind of inverting  $f$  in some algebraic sense is an example of this um another thing you could do is  $\text{Cal } D$  of  $R$  um then you could take this algebra instead  $R$  bracket  $f$  inverse in  $D$  of  $R$  um then what is  $d$   $d$  is the uh sort

of like  $f$  complete derived subcategory and we're looking for a formula for the derived  $F$  completion okay uh  
um so right so let's uh so Define a Define a functor  $f$  um from  $C$  to  $C$  uh by the following so uh  $F$  of  $x$  uh  
is internal  $H$  uh from so uh so let me write uh so sorry so let me write  $C$  for the fiber of the unit mapping  
to a um and then I'll Define this functor  $F$  by  $f$  of  $x$  is internal hum from  $C$  to  $X$  um and there's a natural  
transformation from  $X$  to  $F$  of  $x$  because this is the same thing as internal home from one to  $X$  and uh by  
construction  $c$  maps to one so you get a map in the other way on the internal home so I want to claim that  
this is a first step towards constructing the required localization so claim um if uh  $m$  is in  $D$  then uh  $p$  uh to  
 $M$  applied to this map is an isomorphism so Hing to  $M$  living in  $D$  doesn't see the difference between  $X$  and  
 $F$  of  $x$  um okay um so for the proof so it suffices to see uh that um the fiber of  $X$  maps to  $F$  of  $x$  uh is an a  
module the reason for that is to show that Hing out of this from this map to  $M$  being an isomorphism that's  
the same thing as is saying that  $HS$  from the fiber to  $M$  should be zero um but um by definition internal  
 $HS$  from uh anything in  $a$  to  $m$  is zero uh therefore so if it's an a module then it's actually a unidol yeah  
maybe I should say it's a retract of a tensor itself so any a module is a retract of itself tensor a just by the  
unity axum um and then because internal  $HS$  from  $a$  to  $M$  are zero that implies that  $HS$  any kind of  $HS$   
from  $A10$  or anything to  $m$  is zero and then you get the same for any retract and therefore uh you get the  
required claim um uh so now why is this fiber an a module but this fiber is just up to shift maybe it is just  
internal home from  $a$  uh to  $X$  because  $a$  is the difference between one and  $C$  so up to a shift the fiber will  
be internal home from  $aex$  and this has an a module structure because you can so to speak act by a here  
okay okay so now now now iterate  $f$  in the following way so you get  $X$  going to  $f$  of  $x$  and then you'll go to  
 $F$  of  $f$  of  $x$  um and one should actually potentially be a bit careful about which map one writes down here  
um so what I want to do is I want to take uh  $F$  applied to the previous map as opposed to um taking the  
instance of the previous map with  $X$  replaced by  $F$  of  $x$  okay um and then you can continue on like this  $F$   
of  $f$  of  $f$  of  $x$  um and then I claim that so uh so toine  $F$  Infinity to be the co limit of this here then I claim  
that I'll do it over here so claim so if either uh one this uh Co limit stabilizes uh for all  $X$  so if for example  
the every map from here onward is an isomorphism um or to uh internal  $H$  out of a uh from as a functor  
from  $C$  to  $C$  commutes with Co limits or really maybe only needs sequential Co limits uh then uh  $X$  goes  
to  $F$  infinity  $x$  is the left ad joint uh to the inclusion  $D$  inside  $C$  so for the proof um you have to check two  
things one you have to check that um if you map from  $X$  to anything in  $D$  that's the same thing as mapping  
from  $F$  infinity  $x$  to anything in  $D$  but um we just proved that claim for  $X$  going to  $f$  of  $x$  um and this thing  
is given by hitting uh an instance of that map with the funter  $F$  but um the functor  $f$  is going to preserve  
the property used in the proof here that the fiber um is an a module because internal  $H$  uh to an a module  
will still be an a module so the exact same argument shows that each of these Maps also satisfies the same  
property that internal humming out of them uh doesn't see the difference between one guy and the next and  
then you have a co-limit and internal hming out of that is just an inverse limit of internal humming out of  
all the other ones or humming I should just say uh so that uh formally just passes through to the co- limit  
so uh just as as for uh  $X$  goes to  $F$  of  $x$  each  $F$  to the  $N$  going to  $f$  of  $n + 1$   $x$  is ISO on cming out to  $D$  and  
then so is  $X$  goes to  $F$  infinity  $x$  uh and then the next thing you need to show is um uh the next thing you  
need to show is that um this guy actually it lies in  $D$  so that's that's and the second thing you need to show  
is that  $F$  infinity  $x$  lies in  $D$  but to check that you need to check that by definition of  $d$  you need to check  
that internal hming uh out of a into  $F$  of  $x$  gives you the same thing but by construction internal hming just  
shifts you one along in the sequence and so the co limit is going to be the same so by construction Okay  
so so in principle for example uh you can just you could at least attempt to use this formula to compute  
solidification it's all actually quite explicit this this in that case this this humming out of  $C$  is just uh yeah last  
thing I meaning  $a$  is not the same as oh oh oh oh oh oh oh wait what uh uh oh for yeah there's two there's  
two things I yeah there's two things I skipped here sorry the first thing I skipped was pulling the co-limit in  
and that's where I would use hypothesis one or two to to be able to pull the Homing from  $a$  into there into  
the co-limit um and then the second thing as Peter says I got the wrong explanation so um yeah so you have  
to be um wait now I'm blanking uh can you help me Peter I think the arent should be that uh so you want  
to know that  $a$  becomes zero and the arum should be that all the transitions just become zero ah right okay  
uh yeah okay right okay so sorry I saw I was actually using like there was question which map to use after  
the first to use the other okay well when I when I prepared this I wanted to use this map um I want you  
from  $a$  to just becom zero and you can yeah um um internal home from  $a$  yes but um but the same but you  
can use this I mean but that that also works here because the internal home from  $a$  to  $x$  commutes with the

operation of  $\varinjlim$  forming  $F$  like applying  $F$  so yeah both okay okay yeah so sorry let me try uh so so for this you check that internal  $h$  a uh to each  $u_n$   $F u_n \rightarrow F u_{n+1}$  is homotopic to the zero map okay yeah I'm sorry I I kind of okay um right so yeah so in in the case of solid you can really just make this completely explicit and this is some this operation  $f \circ x$  is some quotient of null sequences in  $X$  modulo null sequences in  $X$  via some map  $u_n$  and it's basically this  $F \circ x$  is basically you freely you look at all null sequences and you you mod out by their relations that would say that then intuitively you want to map that to  $M$  by by summing them and you what you're modding out by is just the telescoping sums and so you can kind of try to analyze what happens to that and iterating it and so so on to get a an expression for the solidification of a general module  $u_n$  it's also fun to try to unwind it in these cases and see that you recover the classical formulas so like for example uh so here The Well of course it's  $M_1$  over  $F$  but you can rewrite that as uh there's this wellknown as a as a sequential Co limit right and so I claim that that's exactly the same as this sequential co- limit  $u_n$  and here you have this formula  $u_n$  so uh here it it looks different because it's an inverse limit  $u_n$  but what's actually happening here is that this inverse limit is what you just see as the first at the first term and then all the maps after that are isomorphisms so this this is just the the first iteration of  $f \circ x$   $f$  of  $f$  of  $M$  in this setting so I don't know so it's a yeah it's fun to to use this formula as to to understand left the joints  $u_n$  maybe I should also say that this this situation occurs  $u_n$  so EG if  $a$  is item potent yeah so that's what what's going on there I'm sorry but it feels that in two and three you are ENT in two no not in two so this is not this is not item poent okay this is why you okay yeah so but if you took the completion no it wouldn't no yeah in solid solid would then but then you'd be doing something else you wouldn't be doing the algebraic inverting  $F$  you'd be doing the other thing yeah so that's yeah okay uh I went a little over time I apologize for that um thank you and see you next week

## 14. GASEOUS MODULES (SCHOLZE)

[https://www.youtube.com/watch?v=krq6jCy-dhE&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=krq6jCy-dhE&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

So last Friday I was uh discussing the the paper look the curve and uh I was um I was in some sense already introducing a new kind of analytic ring structure that we called the guess analytic R structure and so today I wanted to pick up there um and uh but some to analyze this guess and on the GRE structure uh I will need to use some of the results at Dustin um explained on Wednesday okay mod and okay so so the plan roughly for today is uh to First say something about um guess I call the guest displacement which is a certain certain version of arithmetic long series some in some sense some kind of suffering of the wrong Series in some variable that I was s called Q last time um then actually want to take a little bit of time to talk about the corresponding Ser of the real numbers guess real and finally I want to like the motivation for this was uh coming from p-tic curve and last time I someone claimed that this G structure would be good for that and I want to start a little bit explanation for that so I want to say just a few words uh already now about the construction of the table of the C um but at that point we'll have to do some more serious analytic geometry and I think that will be a point where we then we start to develop more a general formalism to really carry that out all right um okay so let's start with  $\mathrm{gu} \text{ space } l$  over here um so we call it the motivation uh was to find the minimal a triangle a um over which can find bit AER necessarily bit imprecise up the  $r$  take  $\mathrm{Li}$  the curve there's an example of  $\mathrm{Li}$  the curve and the universal one would be the the Mod Space will so this is not what we want to do we really want something that is some of the form  $\mathrm{DM} \ Q$  to the  $Z$  in some antic depending on some um and so the or following that there should be this element  $Q$  This should topologic unit in the one and the second the  $r$  that I mentioned and I think is quite natural but where I didn't so far didn't really explain and maybe I will get to the to the what the end of today why it's really the good condition is that uh if  $p$  is a free a module on a  $n$  sequence let me it let um so  $p$  is a free uh light condenser being  $D$  on a  $n$  sequence um then  $1 \text{us } Q * \text{ the operator acting on } P$  Tender Is So This is similar to how we Define the solid analytic R structure where we so head  $Q$  to one instead and of course then we didn't ask it one um uh I mean this is some weaker condition because saying that for no sequence a  $n$  and a we stand for the sum of a  $n$  right so if you think about what the inverse of this would be then be one plus two time shift plus sare time shift and so on particular Zer coordinate evaluate such we can form some kind of geometric series uh in Cube uh with the coefficients from a no sequence and we certainly I mean we certainly want some condition here right because if you didn't ask any kind of completeness this condition then we would just work with all a modes it wouldn't be completed in any sense but we do want to to able to form some kind of infinite series all right and so something that I said last time was just clear but maybe I should really say a few more words about this um is that there is certainly an initial analytic ring uh prop and um you may or may not actually work with animators or infinity or whatever Rings uh in the setup um the proof also shows that there's initial animated such uh later on we will I will actually compute the initial animated one and show that it's actually con Zer and so then agree with initial I can really the previous sense was let me introduce an notation so let me denote the free free condens strings on a topologically ne topologic ne poent element by  $Z$  ad joint  $Q$  head so this is defined to be  $Z$  joint  $Q$  infinity infinity equal to zero um so actually this is just uh  $P$  right because I mean it's just a condensing being group I'm just thinking  $Z$  join any Infinity mod Infinity which is precisely my definition of  $p$  but then I'm ending it with a spring structure where I'm interpreting this  $n$  here is small the powers of some some variable but I want to keep  $T$  sort of a module separate from from this kind of ring so so then we certainly one corresponds to a map from this  $Z$  joint you head and then and let me call this thing here a triangle zero so we want one then we want a triangle to to come with such an  $M$  right because it's a free free condensed ring uh equipped with a topolog NE poent unit this a fre guy with topolog NE poent element and this okay so then we can form uh stre all sequence and send it up to this guy and we have already here we have oneus 2 and then we can Define certain for subcategory um of all the modules over the string uh like we want this to become a nice LM so in other words whenever we map this to something over a module over analytic ring this should become anism so we can simply declare this to be all those modules over a  $z$  triangles such um  $1 - 2 * \text{ acting on the internal } H$  um from  $p$  uh this is extremely analogous to how we defined solid modules right the ring that was just the iners and then we ask a similar condition for  $qal$  to one and uh because thisal home from  $P$  such excellent property um it's immed to check and this is what I did for solid modules um that uh the



subcategory has all the stability properties that we ask for for antic gr so then um the fair triangle mod a is some what we call nothing called last time with preanalytic bra so kind long analytic bra so it doesn't satisfy I me satisfy all the properties of analytic ring except for the property that the ring itself is complete the base ring is really just this uncompleted uh thing so far um and we can see something about sh but then properties one and two they p pris mean that analytic over analytic and so now we just use uh what do pro on Wednesday that uh the inclusion from of analytic ring to panal rings has left joint so you can always complete such a panal and so yeah s also explained last time it's actually better to uh when you have a general suching to complete it in the sense of animat antic ranks um because I prior the completion uh of of the unit in this me the drive completion might not SRE zero and then the better object to consider is the drive completion in this case it will turn out to be the case that this drive completion actually sits in degree  $Z$  so there will not be a difference uh so that's one reason that uh we someh switch in this discussion about completion of analytic structures um the reason for the final stretch of the talk last time was that we now want to compute uh this analytic ring structure and for this DUS gave some general recipe for computing uh this comption factor in some some cases so want to compute um um okay so let me first give give um give some formula uh for what this comp looks like I already stated it uh very briefly last time but let me deliberate okay so uh the following grab the initial analytic animat uming is um but it turns out to look can get form us yeah so first of all a triangle but also all the pre modules all the three a modules on live Prof sets uh they sit in degree zero so they have no hom  $Z$  and there are also CA separation so we're not describing some uh secondly this a TR uh it is actually a sub ring of the long Ser  $R$  um let me first gives a formula for the underlying gr even a point so these are those sums 2 to the  $n$  such the absolute value of  $n$  i mean certainly there are zero for for very negative  $n$  very long series um but then when then go large they should have most po and so this is tying things back to discuss  $T$  of the curve has the property that when you write down the corresponding  $y$  property um and then he's completing it to get this anal but that a a question oh I'm sorry I didn't know we were miked up yeah I was I answered it I think so um happy to answer it for everyone it's yeah well the question was was whether the um whether you're discussing a specific analytic ring here or some general situation I'm I'm just I'm just here I'm describing of course the initial one coming from the position the inial animator okay so the uh right so I'm taking the underlying ring here in two levels right if you have an analytic ring that first of all has an underlying condensed ring it this has an underlying ring so this is this one um and now if I also want to describe the condensed structure then here's a way to do that uh let me try to write big so that there are no questions about indices um uh take a union over  $K$  and  $NS$  um so basically this is  $K$  is giving me the the polinomial grow and like  $Pol$  of degree  $K$  and is giving me the order of the pole of of my the wrong suit broadly speaking and so then I'm taking a product over all  $n$  which are at least minus  $n$  uh of those in which lie in the interval from  $n$  plus  $n$  to the  $k$   $n$  plus  $n$  to the  $k$  um right integer in these bounds times so so each of I mean this is always a finite set so this product here is is a like Prof finite set and then taking the Union this describes this as a light or as a light set first of all but then so subset of here and stable under the ring scure here the way what I'm not sure why start call me switch to my usual name  $s$  um so this describes this thing and then he also describes the free modules uh so for and live profile said written as an limit of fin  $s$  i um Jo  $s$  has a sar form uh to to write it um right so first of all again you can think of this as uh like one way to say it if it's Liv in like also on  $Z$  wrong series  $Q$  you can take to guide on  $S$  and you can solidify that so something we understand let me recall this is just limit over power or in other words these are some kind of  $Z$  long series two value measures on this proficient um so it sits inside there um so in particular I only just describe it as a cond set because the group structure will be the one on the target um and so uh One Way think about this is that well he have an element here some kind of measure which is a sum overall  $M$  and  $V$  of some measures  $m$   $m * Q$  to  $M$  um and then again I asked I put a gross condition on the  $mm$  um so first of all actually say the  $mm$  they are actually just some of  $D$  measures so for any fixed power of  $Q$  you cannot really take infinite sum here this really must just be sum of  $D$  measures this uh but even more than that you asked uh that some these are not too large nor sub poal grow and I guess I mean this is probably rather describing on line like not the condense structure so let mebe the cond structure now uh it's very similar to the above so it's again Union over case and ends uh play the same row as above uh but now we also need to squeeze in this inverse limit overall I to get something for for  $S$  and then now we just do the obvious things we take  $M$  greater equal to minus  $n$  of uh the measures on  $Z$  join  $s$  i which are of Norm at most  $n$  plus  $m$  to the  $K$  time right so here you're taking allow to take some some differences of at most  $M$  plus  $m$  to the  $K$  basis elements so if he did this when  $s$  was I was just

in one element would be down to do that thing there um so this is just some finite set uh this was some like Prof finite set if it takd still set take you uh Peter I wonder if might be a bit clearer to just like drop the  $n$  greater than zero only like only take the term  $n$  equals  $z$  and then just invert  $Q$  on the outside yeah but then I would have to do this funny thing where I we have to replace  $n$  plus  $M$  here by something like  $n$  plus two in order for the zero coefficients be allow oh yeah to be allowed to goow yeah okay yeah all right and then it looks artificial again so yeah or you could you could have a constant  $C$  in front like polinomial growth yeah well anyway yeah I was okay this is correct and you can certainly find many other ways to rise this in and I thought if I'm already taking a union then inverting  $Q$  at the end seems like an operation many I don't know a matter of taste I guess all right is it okay um so I don't want to give the the full argument here but I do want to provide a sketch I want to at least explain where the Pol normal growth condition suddenly appears because this maybe AOS surprising thing because when we Define this  $G$  Gus analytic  $R$  structure we didn't think of any specific grow conditions at all right we were just asking that one two time soon is convertible and then we wonder what what we're getting um and this is certainly also how we what happened to us when we thought about this business so we just well let's write down this condition it seems natural and see see what we get uh this was was what came out uh so so the way to compute is to use the uh formula from the end of St leure so there there was a formula how you compute some kind of completion uh in cases where the competence condition is given by uh asking that the home inter oral derive come from some algebra is Zer so note that um um this condition that this interal uh is equal to the point than um so that's uh let me call it  $c$  um so this is  $T$  the a triangle mod oneus but now I actually want to I do want to think about  $p$  as a ring suddenly icept I didn't want to do that and I want to keep my notation separate but at this point I do want to remember that  $P$  has a as a admits a ring structure um and so then this is actually um it's a free guy on the top logic new element that  $X$  head before and then aad was also a three new variable and then okay so I had tover  $q$  and then one minus I mean shift operator is multiplication by  $X$  by one so this is actually a commut  $r$  uh so keep uh so it's equiv to the  $F$  condition uh using this  $P$  so the condition is that the  $r$  over a triangle uh from  $B$  right because this  $R$  home is literally just the cone of the homs and and and the homes are in degree zero because of these excellent properties of  $P$  and also uh this  $B$  has a property that internal home from  $B$  commes with all limits uh I mean same property for  $p$  so uh this was one of the conditions under which uh Dustin explain the formula for comption I mean the  $r$  completion the three analytic  $RS$  a  $z$  triang a is given by um taking a module  $M$  which is in the drive category of a triangle gra your condens a  $z$  triangle modules um and taking this theal colum of the following diagram so  $m$   $m$  to the internal  $AR$  from I willot the fiber  $Co$  um can take the  $r$  home from  $I$  into and because oops this fiber this naturally maps to one so Dually on our Homestead met from  $M$  this and then there was some last time about how to per the next  $NS$  but I think doesn't really matter all that one um go to find the Alum twice which is of course just the Alum from the tender product all right so that's some kind of formula and uh as  $D$  mentioned last time I mean there are some situations where it's quite a useful formula for example if you're in a situation where you're just algebraically inverting some element  $F$  so then this would be some bring a mode  $F$  then the is usual way inverting  $F$  where this Lally just always the same module and multiplication by  $F$  and so in the column get  $M1$  um but you could also try to use this formula to compute solid confusion uh but this turns out to be extremely confusing and yeah maybe one thing that will what happens that really only the co liit that's well behave both in the Sol case and this case here um and in the col I claimed that you get something nice and degre zero qu separator and so on but each of the individual terms will actually be spread far out and have some non some higher homology which probably not separate and so so the individual terms are a bit hard to control um right so uh uh anyways so so how do we actually go about Computing this so This I uh I mean it's built from one and  $B$  one is not so hard of course and  $B$  was just built from  $P$  basically by this conone so to to understand what's Happening Here the key is to understand what uh um uh one key is certainly to compute the internal home uh from pnal home from  $B$   $CH$  from a triangle a triangle but also they trying right so basically want to evaluate this formula on the  $M$  which are some three guys on this and then see what we get and so here's the LI um maybe make let me make a remark that uh you don't actually have to invert  $Q$  in any of this um we can proceed without inverting  $F$  and then you you will have similar the completion will basically be the same as have to drop all the terms which have negative positive um uh and because yeah slightly easier to to write down the formul if I don't have to invert it would just extra operation have to do it every um so then if I don't bre you then this is really just the free di to new element which is just  $P$  so we one thing we have to understand is some

what is internal home from  $T$  to  $P$  where we think of this here say  $Z$  joint  $X$  and just joint and uh there's a formula for this so uh so this these three condensed to be groups on some Pro Set they always given as this Union over all ends of like things which have bounded Norm end and so this will some pull through this internal home and so this will also be a union Over  $N$  bigger than zero where now I'm somewh taking the part of this algebra here where some of Norm at most  $n$  so at most  $n$  uh terms are allowed to appear but then when I do that uh the new so now I'm allowed to do a overall  $n$  which is like the power of  $Q$  dividing by and so then that's so you take  $V$  joint  $Q$  not  $Q$  to the  $N$  but then inside this you take sums of at most  $n$  basis elements you take sums  $* Q$  to the  $AI * Q$  to the  $I$  where the sum of the absolute values of the  $AI$  said most then and then you take polinomial in  $X$  um which not sure I should call  $x$   $x$  dual but never mind call um uh but it's also just a form a variable so this is the part of poal algebra where each coefficient satisfies expound okay and so there's a similar formula uh for internal home from  $P$  which again think of joint except towards uh the ad joint you have adjin some light profile sets and and uh it's just the same sort of we take Union of all bigger than zero and take an invers limit over all  $NS$  and then you take  $V$  joint  $Q$  to the  $N$  Jo as  $I$  but inside there again this part that some of at most and some are difference of at most end basis elements maybe I should put some operates um and then I take a pols in in the Dual variable um uh right for each term settis is done all right so Peter just a point of information about uh the way things look on the screen here so if you're close to the the boundary of the board uh like far right of the right hand board or the far left of the left hand board it's quite a shadow and it's we basically don't see anything it said still okay also to far in um that's better that's readable with some effort yeah anything limit over the uh uh yeah sure yes of course what is the Stop and the exponents of the  $X$  sorry  $X$  maybe I should do the  $EXP$  y it's it's just a different okay I didn't want call  $X$  because it's some of the Dual of  $X$  or now but course maybe I use the head to have some specific meaning here the star was didn't have any meaning that s uh so in particular uh one way to uh a consequence of this set this internal home from  $P$  top is actually just the Subspace of uh  $Z$  joint  $Y$  and it's a Subspace with a certain kind of uh funny uh funny condition that um s this um so some kind of let me notes as far as why power series and cubes bounded because in some sense you're asking that yeah there's some  $B$  boundedness uh in terms of the coefficients of wise and similarly the from from te to this um there also a subring of  $U$  you take  $Z$  joint uh  $Y$  and then you take  $s$  and then you take and again it's a substate of this where each coefficient of  $Y$  uh satisfies a certain boundness property so again I will denote this here by the join  $y$  join  $s$  power series and  $Q$  um Peter there's been a request to explain again what the less than or equal to  $N$  means in the description of  $h$   $p$  top uh for finite that's I I didn't use I foring here right um the join  $I$  less equal to  $n$  a certain sub space of fre three free on  $I$  and set of those  $Su$  over  $I$  of some a itimes  $I$  so that is sum of the absolute vales of the is it most all right so if you take this formula for the internal home from  $p$  and just do these formal manipulations that you express  $I$  in terms of the other thing sorry sped way too high um if you just do this manipulations uh then you for example see that the Rome uh from sorry uh I said I didn't invert didn't invert  $Q$  so let me explicitly say just for  $Z$  Jo  $Q$  head and okay so  $I$  can also be defined corresponding version inverting you have um uh so if you compute that then this is a two-term complex um but those terms are see computer up there it should take this uh this power series over the one which has some bound coefficients and previously there was on transition that was  $1 - x * Q$  uh but now  $Y$  is some kind of dual object  $X$  so now actually transition turns out to be just  $Q$  minus or  $y$   $q$  minus one and so this sits in degree zero and this sits in homological degree one and uh if you do the similar computation with more variables hey uh s effector here then you get a similar picture but just the  $K$  of the variables  $y$  so this is computed um by complex in uh  $y$  variable so inste what take is you take  $Z$  and then you join  $y1$  up to  $YK$  and then again you take such  $b$  power Series in  $Q$  and then I mean some of this was the derived potion by just one variable tus  $Y$  and now you take a derived potion by  $Q$  minus  $y1$  up to  $Q$  corresponds to Tak fil and I mean you can do the same with with a with a sets in right so if you have a profile sets somewhere then you just plug the profile sets here in the middle and then get the same for all right s so so now we have a somewhat concrete formula for these things right so if you want to compute the underlying ring we literally just have to understand the co liit of all  $K$  of these things so thus the ring we're interested in is just the co liit over all of this same take  $y$  want to  $YK$  if you found it then you take a d quent and here turns out to be a case where it's really very important that each individual level to Tak the  $D$  full  $CL$  on because there will be high homology then here plug s what are the transition Maps a transition sorry yeah I should have said so this is similar question to what the transition mat in call that  $D$  wrote down last time um you just I mean each of these is a subing of the next one where you just don't have the new variable

it's not a sorry all these things are actually not range because if you multiply two things uh what confusion but um let me not try to say whether they r or not but uh certainly there sub modules all right um so so far there was no apparent polom grow condition at all there was just some kind of boundness that appeared for more transparent reasons but now uh let's analyze uh like the of this or rather separate Co so I mean this for Simplicity you can do the full analysis but uh it takes a lot of concentration so I think there's already enough interesting stuff going on without these addicitional technicalities um so right so so what is the H zero so you really just take this ring and then what out by  $Q$  minus  $y_1$  of the  $Q$  minus  $y_k$  or if it takes a separate equion really by the the closure of all these things yeah um so in other words here I'm are formally allowed to uh replace any recurrence of any of these variables by I that it's really just set into a  $q$  right um so this uh there a question com what is the image of uh this thing uh mapping towards power Sol where all the  $Y_i$  so let me again recall what this was so this was the union over all end of of the limit over  $M$  of the part of this thing which was of Norm at most  $n$  so this wasn't literally written as a finite free module but I mean you take the basis by giving my 1  $Q$  of to  $Q$  to one uh and then you pend the variables by one but in particular it will actually be sufficient to analyze the part where all of these are say just equal to one uh  $Z$  or one this one um so here we just allowed to take some kind of bounded sum but then we have all these  $y$  ones through  $y_k$  here at the posal but for each of them individually I can take I can take a I mean just each coefficient individually is bounded so for example if I like for each end we have to analyze what the image is but then which which multiples of  $Q$  which like integer times  $Q$  can I achieve well this integer time  $Q$  like for each occurrence of  $Q$  I can decide whether I want to keep it a  $q$  or whether I make it a  $y_{12}$  YK but for each  $y_1 y_k$  i can Som use it most end times but then in total like The  $K$  uh I can I can get this variable  $q_k$   $n$   $k * n$  times because yeah there um uh and then actually maybe the nend is rather red pairing so let's ignore the  $N$  so if  $n$  was equal to one uh then we can some get a coefficient up to  $K$  here for this  $Q$  because I can split each occurrence of  $Q$  between the different variables and then if I look at  $q$  squ then I have all the monomers of degree 2 hereos in the  $y_1$  YK so there like  $k/2$  mon have at disposal each one I can use like at most  $n$  times so I get  $n * K$  two and then in degree some big  $m$  i sorry it's  $K$  plus one I guess I have  $K$  plus  $M$  over choices for for these monomials here uh so the coefficient of  $Q$  to the  $m$  can lie in likeus  $n * uh$   $n$  plus  $k$   $k$  actually same and maybe I I didn't uh okay I would have to be slightly more careful here the precise noise but I mean basically something like that um and so the question is how does this grow how does this grow in  $m$  and in  $M$  this is precisely polom of degree  $K$  right and then this just the polom function of the yeah so whenever I have something that grows in where the coefficients grow in  $m$  just like poom then because yeah I have all these monomials here at disposal I can somewh split up all the individual summons uh to make them bounded in here and and there's c a lot of choices about how you go about doing this and when you want to prove that actually this uh this complex end up ends up being in degree zero you Som have to show that a different way of arranging the terms here uh is actually uh like differs from the other one uh by uh something bounded some bounded some in the next  $C$  of the complex and this will actually not be true for giv  $K$  but when you allow your to increase your  $k$  a little bit then you can do it Peter there's a question in the chat there's a question in the chat yeah which is uh or you want to read it or uh oh yeah so there was a question about this  $Z$  join  $y$  join  $s$  whether I can interchange the order and indeed you can interchange the order so the kind of argument you have to do here to see that this is well behave this a little reminiscent of the arguments you have to prove that liquid stres well behav but it's several orders of magnitude easier right so it's again a situation where you have uh some kind of system of complexes where each term has some kind of norms and then you have to like see that if you have something with differential zero you can bounded pre image but here it's really you can really just do it I takes some concentration but it's not all all right all right so um this is all I wanted to to say about uh the proof of this box ination um uh now they I want to talk just briefly about Gaz  $G$  real Vector spaces so let me condens or light we started emitting the light at a couple of times so throughout this course we're always working light filling and it might not always be mentioned yes if I take the internal home from like  $P$  was always ining the free with integers here and then I choose one top poent unit in the real say so now you might actually Wonder uh doesn't matter that it shows a half here and it doesn't so I mean so in this Ser of liquid analytic of this liquid drink structures on the reals there was this extra parameter that you have to choose your how non convex you allow your vector spaces to be um here it seems you also have to make a choice but actually it doesn't matter the condition is independent of number like a half you could Al take a negative any top um actually why so it's easy to see and uh intuitively believable that if he makes this number smaller then the conditioning becomes weaker uh

because here some of the series you're trying to to to sum there like even faster Decay but on the other hand if you're trying to sum a geometric Series where the coefficients are like one over two to the end then you can always um the finite sum where you Su the first few terms and then have telescoped uh not really telescope but um there's another question in the chat I I actually missed it uh so I can't read it um let me see if I can read by the way general question nonl condens sets will not be used anymore in the Ser originally right so there was a question uh like originally extreme disconnected set play quite of a crucial role how set up the Ser because gave us enough projective objects and it is really quite convenient to have um so for much of what we're doing we're kind of forgetting about the full category of condensed sets but on the other hand for some arguments actually is still quite convenient to have the ambient framework of all condensed sets because I mean the light ones they still embed into all condensed sets and for some so if you want to check that something is an isomorphism you can still do that by checking the values and extremely disconnected and this is still convenient for a few arguments um so yeah yeah for and the other setting is also nice but like for for the specific application analytic geometry we found it more convenient to make this restricture because there are a couple of things in particular this internal projectivity of  $P$  that we're using all over the place that's only from this uh maybe related questions like can you Al like the Ser of liquid real Vector space that also worked not just in liing but the full setting you can ask the same question about the gasius anding structure whether you can also Define that on all condens modules I think you can but the argument we gave here does not work because we use internal projectivity of  $P$  so you would have to argue slightly more carefully to see that there is a gr structure of  $G$  real Vector spaces in the full condens setting um uh generally is onlyus yeah and this is basically a telescopic argument all right I mean if if she wanted some the you can always rewrite like at a finite sum and then new sum where you still have a no sequence of coefficients and now powers of  $Q$  to the end here uh all right so you you have let's say um and so anything that's  $P$  liquid for any  $P$  or Alpha liquid for any Alpha uh is always cases so this is extremely General class of condensed Vector spaces uh but it's still a workable class for many purposes um and so uh you can now wonder what are the three uh guesses real Vector spaces and here so first of all so here here's a proposition what does it have to do uh with uh what we had previously so our guess which notes the analytic  $R$  uh corresponding to  $g_u$  is Vector spaces so again it's clear that this defines structure because of this good properties of  $P$  um this just gotten by taking this one we had is makes a completed one and the  $g$  mod over this and then in some moding out by oneus  $2 * \text{setting } Q \text{ equal to a half}$  you take this one you complete that antic as we did and then you set  $Q$  so it's easy to see that when you do this to this uh ring uh Point number gross condition that you're getting you actually get the real numb  $H$  you don't all you don't all yeah no I mean like if if you take this completed ring and one by oneus  $PQ$  you really get the real  $I$  so these are like functions on like like the real part of this thing is like a function open unit the real numbers and then if you set  $P$  equal to a half there like this one dimensional thing you just get real numbers in particular um this tells you what the the free gas miners are you can compute them the free  $G$  your vector Spaces by taking the fre guys here then taking the poent uh but now the description becomes a bit confusing because when we wrote down the gross conditions that we allow here it was some kind of Pol normal grow condition in  $q$  but now  $Q$  has become a constant so it's not immediately C clear how you phrase these Rose conditions uh directly on the real numbers uh but you can do it and here's the description so the free guess is uh real vectors space on some light profs uh has the following crazy description we take a union over again some case which corresponding to this poal draws that we had previously and now there is a constant then I take a limit over  $I$  and then I take a certain bounded subset um in the three real Vector space on the fin set  $s_i$  uh which satisfy some condition on how large  $I$  all to be and the condition is that if you sum um Su the size and measure it or their size measure it in terms of certain functions  $FK$  that I will introduce in a second there at most  $C$  so formally this is very similar to what the three liquid Vector space look like and there the the Norms you were taking there were like you were raising to some power also or maybe Al um generally the functions that you would like to put there um they should be increasing of course because larger larger exes should be calized more um on the other hand you want the transition apps to be well defined and for this you need the functions to be uh concave turns out um so well this could be reasonable um these to  $FK$  they should yeah they should measure how lot to real number is in terms of some let it some real number greater equal to zero um uh I mean they should be increasing and conct so basically you need that  $FK$  of  $X$  Plus  $y$  should be at most  $FK$  of  $X$  Plus  $FK$  of  $Y$  she wants a transition Ms there to be well defined because the transition maps are somehow summing some of these  $X$ 's um and this is Ed by theity uh but on the other hand you



don't really care what your function is for large values of  $X$  because uh really what you're trying to think out is just which infinite sums are allowed and of course in these infinite sums almost all the terms should be very very small so when I choose such a function  $F_K$  really only the only thing that matters is the behavior near zero okay this is a Prelude to saying that uh so  $F_K$  is any uh increase in comp functions such that for small  $X$  um it's given by the following it's uh the absolute value of  $\log X$  to Theus  $K$  so I'm saying it's this way because I mean in principle I just like to take this function and in the neighborhood of zero that is a nice increasing concave function but that at some point uh it's becomes convex goes to infinity and then does some nonsense because like if you said  $x$  equal to one  $\log$  of one is zero one to the minus  $K$  is some nonsense so um so I can write down this function for all  $X$  but I can do it for small  $X$  and then at some point I just aritar extend and it doesn't do me what all right so this is maybe a little bit hard to just so let me let me give a sort of diagram uh so in the liquid in the liquid story uh there this function have like some the of  $x$  to the beta where beta is some number between  $Z$  and one so there some function so this is always increasing in concave so I can just always use it so some function like that um and then the locus where the sum of the  $X$  to the beta um in the two variable cases will be some some kind of lus like that some sling on convexing uh in the  $G$  case um it's some function that's extremely steeply ascending near near uh zero so even if it's just a tiny number it's often treated as rather large by this by the function  $F_K$  then then the actual function that I wrote there would make a turn and then go up and okay instead I just cut I can make it linear if I want to right need not be strictly um and then if I look at the set where the see this will similarly be something that extremely closely tied uh to to the coordinate  $X$  so you're allowed to take certain infinite sums but basically you have to ensure that the coefficients have exponentially  $P$  but not quite that actually so from the liquid the we can form some of some and and there exist some detail some prespecified Al such that the sum of the  $X$  the BET finite so there in principle we like to just ask usual stability but as I said this doesn't really work but when you slightly restrict the class of sums for which they ask us that they are always like part of the structure ofid re Vector space to be the speed want and this works uh in the  $G$  the you have much more spring what's uh so yeah the alha liquid Series so the liquid Ser depends on extra parameter uh Cas Theory can only form those sums or theic say but the some ni um and this actually two one to say  $x_n$  up to reordering if you sum in the order starting with the largest ones and descending uh then the  $x_n F$  uh one/  $x_n$  sorry the should have prob the exponential growth by which I mean the following so you can form such a  $su$  only if exist someone greater than equals to zero and maybe some constant  $C$  greater than zero such that the abute value of  $x_n$  is at most  $c$  times a half to the end to the absolute right so if I didn't put the EPS on here then this would be some kind of exponential decay uh uh it's enough to have some fractional power of  $n$  here in the exponent um and if they satisfy some Decay condition like this then then you're allow to  $S$  guesses so why does this expression appear here well if you take a half to the end to the Epsilon and then apply this kind of procedure here then the log turns this into to the Epsilon and then to the minus  $K$  you get one over  $n$  to the Epsilon  $* K$  uh and this becomes summable when I don't know Epsilon  $* K$  becomes very all right so some rather crazy type of grow conditions from get was real numbers but it's still a workable thing because um like do agation CA on complex geometry and uh at least all the all the stuff in complex geometry would work as well as Gus Vector spaces because the only thing that we really needed that this algebras of conver phoric functions um where some kind of thing in the theory and some nuclear uh and this is still true uh in this story because the transition Maps when you form like when you restrict all functions on one this all function on small this then you're precisely if you think in terms of their power series expansions you're rescaling all the power series coefficients by but fixed call of the number less than one so actually the transition map they have exponential decay so yeah so I think the guest uh story would be just as good for this discussion of complex geometry all right um yeah so yeah so the claim is that you can do this computation of the three gas elector spaces uh forly from the one I did over this uhing in the variable  $Q$  takes a little bit of unraveling but it's not it's not all that hard all right so uh finally let me try to do some geometry yes so the goal is to define the GU is uh okay let me again recorded and so the idea is something that we already used a couple of times we have this fixed to and like an add B some use that when a fixed to unit you can use this to normalize absolute values uh so that some of when you want to understand how large some other function is uh you can see some how large this compared to to  $Q$  um uh so we can use  $Q$  measure the size of any other function so basically you're you're wondering like how do powers of the absolute value of uh  $F$  from comp to PO of um and uh one way to organize this information um is in the following place and now I will start to use this formal of analytics STX uh that this whole course like

the main aim of introducing  $u$  and this language is really convenient to phrase before play because I could give a more down to earth version of this but it would be much more confusing  $u$  and I think it's a really neque way of packaging the information I will maybe explain a little bit about those so the exist the morm  $T$   $u$  from let's say first from the  $f$  over a  $u$  and on the fine line you have the coordinate function in particular like when you want to do something for any function you can something do for the universal function which is the function  $t$  on an  $A_1$  right so when you have any function on the  $a$  then this gives you  $M$  from the spectrum of  $A_2$  with  $A_1$  let your function  $T$  to your given one so to define something for any function is enough to do it for an  $A_1$  Anders Cas  $U$  and so then there is some kind of absolute value of  $T$  function measuring measuring against the ABS value of  $Q$  uh that goes to well in this case it can actually be infinite uh because some could be some kind of unrounded function maybe a better way to think about that you have one on the projected line and if you have the point this homogeneous coordinates he want to  $2$  then this goes abute implicit here is that something is down to earth as this closed interval uh will have a certain very crazy Incarnation and an antic  $St$  in all setup and uh this is required to make  $S$  of this mths  $u$  so simpit here again is again so if you really want to go to zero Infinity here uh  $u$   $u$  I again need to normalize the absolute value of  $Q$  and so I what I always do is to let but I'm me in my last lecture it kind of suddenly mattered at one point uh what I show there but here it doesn't matter again and always just resale but let's assume we're in a formalism where something like this makes sense where we have some kind of way measuring the absolute value on this same something there so then we can Define what I call the analytic Chay has a pre-image of the open part so this is intuitively speaking the union over all  $n$  of the locus where the absolute value of  $T$  is bounded between some Powers some powers of the absolute value of  $G$  uh right and this was something that already entered at the beginning of the last my last lecture  $u$  that this is what you just start with if you want to define the  $T$  with the curvevent and then find that  $u$  and then you still have acting on this multiplication by two and uh this map here is actually some kind of multiplicative map so if you mute value of product the prodct the absolute values where this if it so happens that one is zero and the other is infinity then there's no claim being  $u$  but in particular on this Locus I me you have this analytic here to  $Z$  Infinity you have multiplication by  $Q$  here this corresponds to by multiplication by a half here and this map here is actually this is actually some kind of proper uh that because  $P_1$  is proper zero Infinity is proper so the map is proper  $u$  proper map to here  $u$  and then if you pass the equion  $u$  then you have  $GM$   $Z$  I mean this must be totally discontinuous I mean free and totally discontinuous is actually because it's it is here right so this then is just  $F$  over  $Z$  INF half  $Z$  which copy of circle  $u$  so so the  $m$  is it's locally as is proper but now  $S_1$  is again proper  $u$  and so this means that this one is proper but of course it's also locally asmic to this one which is an open subset of  $P_1$  and  $P_1$  ought to be smooth uh so it's and a curve so this must be a proper smooth curve and this is the  $D$  right and so uh this is how we want to argue how we think construct so take a look to curve all right questions this  $Z$  Infinity is also def over this a you can Bas change it to a you don't have to write okay and this always that's you can from what is the usual there will be a fun from like condens s towards analytic steps and so zero Infinity I consider as a life set and this will have an Incarnation as analytic  $St$  this will actually mean that the theory of analytic stes will be related to the condens story in two ways one because like the analytic Rings themselves are found some on on condens things uh which is the role that condens have played so far but suddenly there will be another role that the condens  $Stu$  plays because yeah condens will also have an ination in the worlds actually two kind somewhat AAL condens Direction because the way this this thing will be realize is antic  $St$  it actually just use the sced rins it will not use any interesting condensed structure on the ring level any thanks Peter yeah

## 15. STACKS (CLAUSEN)

[https://www.youtube.com/watch?v=EEH\\_0QhrgEg&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=EEH_0QhrgEg&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

The topic of today's lecture will be stacks.

So far, we've discussed the theory and examples of "analytic rings". Next, we will explain how to use these to build geometric objects, the "analytic stacks". We already saw a clue of what phenomena we want to include in this world of analytic stacks in the previous lecture, namely the Tate curve over the gaseous base ring.

Today we won't give the precise definition of analytic stack, but will provide motivation from algebraic geometry. The paradigm here is you have commutative rings, and you want to think of these as describing some sort of basic geometric objects which are called affine schemes.

commutative rings  $\rightsquigarrow$  affine schemes

$$R \mapsto \operatorname{Spec} R$$

These two categories are anti-equivalent. Then you want to allow yourself to glue these affine schemes together to make more general geometric objects.

There are two aspects to this gluing:

- (1) What gluings are allowed?
- (2) How to identify the results of different gluings? More generally, what are the maps/what is the category we get from these formal gluings of affine schemes?

A classic example is the projective line

$$\mathbf{P}^1 = \mathbf{A}_+^1 \cup_{\mathbf{G}_m} \mathbf{A}_-^1$$

where we glue together a "plus" version of  $\mathbf{A}^1$  and a "minus" version of  $\mathbf{A}^1$  along the multiplicative group  $\mathbf{G}_m$ . But we want this object to have symmetries like  $\operatorname{PGL}_2$ , which don't respect this decomposition. So there has to be something said about what are the maps between different gluings.

It's better to think of (2) first: specify an ambient category containing the category of affine schemes, and then single out a full subcategory by specifying allowable gluings of affine schemes in the larger category. This is one way to describe a class of geometric objects in algebraic geometry.

$$\begin{array}{ccc} \text{category} & \supseteq & \text{affine schemes} = \operatorname{CRing}^{\operatorname{op}} \\ \wr & & \wr \\ & \text{single out } \{*\} & \end{array}$$

In the classical approach to schemes, you take the ambient category to be locally ringed spaces, so a commutative ring gives you  $\operatorname{Spec} R$ , and then a scheme is a locally ringed space which is locally isomorphic to some  $\operatorname{Spec} R$ . So the allowable gluings between affine opens are gluings along open subsets, in terms of viewing  $\operatorname{Spec} R$  as a locally ringed space.

There's a more modern approach which says we shouldn't try to be clever about choosing an ambient category—we don't have to find the concept of a locally ringed space. We can just formally build an ambient category based on the category of commutative rings, and then work there. The category where you're allowed to glue arbitrary affine schemes is called the **category of presheaves** on affine schemes,  $\operatorname{Psh}(\operatorname{Aff})$ . This is the universal category in which you can glue; more formally, it's the initial category-with-all-colimits with a functor from  $\operatorname{Aff}$ , and giving a colimit-preserving functor out of  $\operatorname{Psh}(\operatorname{Aff})$  is the same as giving a functor out of  $\operatorname{Aff}$ . There's a set-theoretic technicality here, so caution is needed; we'll discuss that later. The category  $\operatorname{Aff}$  is not a small category, so one has to be careful when taking functor categories out of it.

Now you've formally allowed yourself to glue, but you haven't explained how you should identify the results of two different gluings. If you pass to sheaves for some Grothendieck topology, that explains how to identify gluings; more generally, how to map between two of these formal gluings. The more covers you put in your Grothendieck topology, the more maps you're going to be making, which might not be so evident from the perspective of the cover. For example, the automorphisms  $\operatorname{PGL}_2$  of  $\mathbf{P}^1$  may not be apparent. But you don't want to add too many elements to your cover, or else you might destroy information by identifying too many things; so there's a delicate choice to be made in the Grothendieck topology.

For example, you could take the Zariski topology, and then schemes is a full subcategory of Zariski sheaves on affine schemes such that they're locally representable, in the sense of open covers.

$$\text{schemes} \subseteq \text{Sh}^{\text{Zar}}(\text{Aff})$$

You can define the notion of an open inclusion in  $\text{Sh}^{\text{Zar}}(\text{Aff})$  just by reduction to the case of  $\text{Aff}$ : a map  $X \rightarrow Y$  in  $\text{Sh}^{\text{Zar}}(\text{Aff})$  is an open inclusion if for every  $\text{Spec } A \rightarrow Y$ , the pullback

$$\begin{array}{ccc} \text{Spec } B & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ \text{Spec } A & \longrightarrow & Y \end{array}$$

is an open inclusion  $\text{Spec } B \rightarrow \text{Spec } A$  in  $\text{Aff}$  (this includes the condition that the pullback is in  $\text{Aff}$ ).

This is the perspective we're going to take on defining analytic stacks from analytic rings. However, we don't want to take schemes as a model, because more general geometric objects come up and are relevant, namely *stacks*.

**15.1. Why stacks?** Often moduli spaces in algebraic geometry have the form  $X/G$ , where  $X$  is a variety and  $G$  is a group variety or scheme acting on  $X$ . So we have a quotient of some variety by some group of automorphisms.

**Example 15.1** (Moduli of elliptic curves). Consider the moduli stack of elliptic curves  $\mathcal{M}_{\text{ell}, \mathbf{Z}[\frac{1}{6}]}$ , where we invert 6 to simplify the discussion. Then this is  $X/G$  with

$$\begin{aligned} X &= \text{Spec } \mathbf{Z} \left[ \frac{1}{6} \right] [A, B][\Delta^{-1}] \\ G &= \mathbf{G}_m \end{aligned}$$

where  $\Delta$  is the discriminant.

This parameterizes elliptic curves with affine equation in Weierstrass form

$$y^2 = x^3 + Ax + B.$$

It then turns out that the only isomorphisms between two elliptic curves given by the above equation are given by scalar multiplication with certain weights on the  $x$  and  $y$  variables, and that gives the  $\mathbf{G}_m$  action.

There's also many other ways of presenting the same stack. For example, you can add level structure and then mod out by a finite group,

$$\mathcal{M}_{\text{ell}, N} / \text{GL}_2(\mathbf{Z}/N)$$

If  $N$  is sufficiently large,  $\mathcal{M}_{\text{ell}, N}$  will be represented by a variety, and then you quotient out by the finite group  $\text{GL}_2(\mathbf{Z}/N)$ . The Grothendieck topology that we choose should be such that this is identified with  $\mathcal{M}_{\text{ell}}$  our category, so we should at least allow étale covers into the story for this, and that is indeed the classic choice.

These quotients exist in the category of schemes, and they give  $\mathbf{A}_{\mathbf{Z}[\frac{1}{6}]}^1$ , the so-called “ $j$ -line”, implemented by the  $j$  function. However, this is not a good quotient. One way of measuring that is: on the moduli stack of elliptic curves, there's a natural line bundle  $\omega$ ,

$$\omega = \text{Lie}(E)^*,$$

which is the dual of the one-dimensional vector space of tangent vectors at the origin. In other words, it's the cotangent space of the universal elliptic curve.

This means you can write down a line bundle on  $\text{Spec } \mathbf{Z} \left[ \frac{1}{6} \right] [A, B][\Delta^{-1}]$  which is equivariant for the  $\mathbf{G}_m$ -action, or a line bundle on  $\mathcal{M}_{\text{ell}, N}$  which is equivariant for the  $\text{GL}_2(\mathbf{Z}/N)$ -action. However, this line bundle doesn't descend to the quotient  $A^1$ , so it's a bad quotient in the sense that you can have equivariant objects on the top, but they don't come from something on the bottom. Even more basic, the universal elliptic curve over  $\mathcal{M}_{\text{ell}}$  can't be defined over  $\mathbf{A}^1$ .

The problem in the above example is that the action is not free. There are some elliptic curves with extra automorphisms, and because the action isn't free, the naive quotient in schemes is collapsing too much. The solution is to take the quotient in a more refined (2-)category: (sheaves of) groupoids.

Working in groupoids, there's a notion of groupoid quotient

$$X // G$$

where:

- the objects of  $X // G$  are the elements of  $X$
- $\text{Hom}_{X // G}(x, y) = \{g \in G \mid gx = y\}$

There's always a map  $X \rightarrow X // G$ , and the fibers are all isomorphic to  $G$ , where “fiber” means a pullback

$$\begin{array}{ccc} X_x & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ \{x\} & \longrightarrow & X // G \end{array}$$

So this sort of allows us to pretend that every action is free; the map  $X \rightarrow X // G$  is always like the total space of a  $G$ -bundle.

The only trick is you have to interpret fiber product in the sense of 2-categories: in a pullback diagram of groupoids,

$$\begin{array}{ccc} A \times_C B & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

an object of  $A \times_C B$  consists of a triple  $(a, b, \gamma)$  where  $a \in A$ , an object  $b \in B$ , and  $\gamma$  is an isomorphism between  $f(a)$  and  $g(b)$  in  $C$ .

So it's a way of taking a quotient such that you don't really care about the difference between a free action and a non-free action. And it's such that you formally have that line bundles on  $X // G$  are the same as equivariant line bundles on  $X$  and so on and so forth.

This leads to notions such as Deligne-Mumford stack, or more generally, Artin stack. These are all full subcategories of étale sheaves on affine schemes.

$$\text{Deligne-Mumford stacks, Artin stacks} \subseteq \text{Sh}^{\text{ét}}(\text{Aff})$$

In the discussion of schemes, we had Zariski sheaves that we asked to be locally representable, where “locally” is in the sense of open covers. We also could have used étale sheaves, it doesn't change the resulting category of schemes. For Artin stacks, you more or less require that you have a smooth cover by affine schemes, and maybe some technical things you want to put in there as well.

The basic example is, let  $X$  be an affine scheme, for simplicity let's say over some base  $S$ . Let  $G$  be a smooth group scheme over  $S$  which acts on  $X$ . Then you pass to the quotient in the stacky sense,  $X \rightarrow X // G$ .

So these are more general geometric objects, and this is really good for moduli theory. The theoretical justification for that is Artin representability theorem which gives concrete criteria for when a functor is represented by an Artin stack. But it's still constrained by the need for a smooth cover; in particular, this means finitely presented. So if you had some non-finite type group scheme acting on something, you wouldn't necessarily be able to take the quotient in Artin stacks.

Now you might say, why do you care? It turns out there are many  $X \in \text{Sh}^{\text{ét}}(\text{Aff})$  which are geometrically relevant, but are **not** Artin stacks.

**Example 15.2** (Formal schemes). Let  $R$  be a noetherian ring, and  $I \subset R$  an ideal of  $R$ . We want to consider the formal spectrum  $\text{Spf}(R_I)$ . There are different options as to how to encode this thing, and what it should mean. One we've already discussed is Huber's theory, which includes formal schemes as an example, and is based on viewing  $R$  as a topological ring. Grothendieck's theory of formal schemes is also based on viewing  $R$  as a topological ring, but localizing along a smaller subset than in Huber's theory.

Another way of looking at it is it's just the union of all the  $n$ th order infinitesimal neighborhoods of  $\text{Spec}(R/I)$  inside  $\text{Spec}(R)$ ,

$$\text{Spf}(R_I) = \bigcup_n \text{Spec}(R/I^n).$$



So giving some (finite-type) data on a formal scheme should be the same as giving compatible collections of data at all of these finite levels. For example, vector bundles on  $\mathrm{Spf}(R_I)$  should just give a compatible collection of vector bundles in the usual sense (finitely-generated projective modules),

$$\mathrm{Vect}(\mathrm{Spf}(R_I)) = \varprojlim \mathrm{Vect}(R/I^n).$$

This is a very different gluing from what you think about with schemes, and even when you think about Artin stacks. There are no smooth covers in sight here; instead, you're taking some union of infinitesimal thickenings and getting a new object.

There are other examples: for some moduli problems, you really are quotienting by an infinite-dimensional group. One example that's dear to the hearts of both homotopy theorists and number theorists is the moduli of one-dimensional formal groups, the **Lubin-Tate space**. In this case the group you have to mod out by is coordinate changes on a one-dimensional formal scheme, and then there's infinitely many coefficients you have to specify. So you have an infinite-dimensional group you have to mod out by, and that doesn't fit into the framework of Artin stacks.

This suggests using a different Grothendieck topology, say fpqc instead of étale, if you want to accommodate infinite type phenomena in your covers. But in addition to these, there's also a very remarkable class of examples started by Carlos Simpson.

Simpson: “Every linear algebra category is  $\mathrm{QCoh}(\text{some stack})$ ”

Simpson didn't literally say this, and it's much too strong to be true; this is just an slogan interpretation of his work. His work gives lots of examples of this phenomenon, where you have a natural linear-algebraic category, and then it turns out you can write down some stack whose quasicoherent sheaves are that category. There are fun examples of this already in the world of Artin stacks.

**Example 15.3** (Representations). Let  $G$  be an algebraic group over a field  $k$ . Then a special case of a non-free quotient is a point  $* = \mathrm{Spec} k$  with an action of  $G$ , so

$$BG = * // G.$$

A quasicoherent sheaf on  $BG$  should be a  $G$ -equivariant quasicoherent sheaf on the point; but a quasicoherent sheaf on a point is just a  $k$ -vector space, and the  $G$ -equivariance exactly means you have a representation of  $G$ . So we get

$$\mathrm{QCoh}(BG) = \mathrm{Rep}_G(k\text{-vector spaces}).$$

**Example 15.4** (Filtered objects). We can also consider  $\mathbf{A}^1/\mathbf{G}_m$  for the natural action of  $\mathbf{G}_m$  on  $\mathbf{A}^1$  by scalar multiplication. This is a funny stack, because there's an open locus in this stack, corresponding to a  $\mathbf{G}_m$ -invariant open locus in  $\mathbf{A}^1$ , namely  $\mathbf{G}_m$  itself. On that open locus, you're taking  $\mathbf{G}_m/\mathbf{G}_m$ , which is a point. So this has a point as an open subset, and the closed complement is  $0/\mathbf{G}_m = B\mathbf{G}_m$ . In this case, (flat) quasicoherent sheaves are given by filtered  $k$ -vector spaces.

$$\mathrm{QCoh}^{\mathrm{flat}}(\mathbf{A}^1/\mathbf{G}_m) = \{k\text{-vector spaces equipped with a } \mathbf{Z}\text{-indexed filtration}\}$$

The restriction to flat modules is just to stay in the abelian world rather than going derived.

Note that at the origin, we have

$$\mathrm{QCoh}^{\mathrm{flat}}(B\mathbf{G}_m) = \{k\text{-vector spaces equipped with a } \mathbf{Z}\text{-indexed grading}\}$$

These are Artin stacks, so there's nothing exotic there. Here's a more interesting example.

**Example 15.5** (de Rham stack). Let  $k$  be a field of characteristic 0, and let  $X/k$  be a smooth variety. Then you can form, and Simpson did, what's called the **de Rham stack** of  $X$ . There is a presentation

$$X \rightrightarrows X^{\mathrm{dR}}$$

but this is not quotienting out by a group action, it's just some by equivalence relation. The equivalence relation in question is that which identifies two points if they're “infinitesimally close” to each other. Equivalence relations are supposed to live in the product  $X \times_k X$ , and what you do is you take this product and you formally complete along the diagonal,

$$(X \times_k X)_{\widehat{\phantom{x}}}.$$

Then  $X$  mod this equivalence relation gives  $X^{\mathrm{dR}}$ . The formal completion should be taken in this sense of Example 15.2, i.e. the union of the different scheme structures that are available on the diagonal as a closed subset.

(The definition of  $X^{\mathrm{dR}}$  makes sense in arbitrary characteristic, but some things we're about to say will not be true in positive characteristic.)

What are quasi-coherent sheaves, let's say vector bundles, on  $X^{\mathrm{dR}}$ ? This is the same thing as vector bundles on  $X$  equipped with some kind of descent datum, but that descent datum exactly amounts to a flat connection.

$$\mathrm{Vect}(X^{\mathrm{dR}}) = \{\text{vector bundles on } X + \text{flat connection}\}$$

That's Grothendieck's interpretation of what is a flat connection. It's exactly giving descent, identifying infinitesimally close points.

There's also cohomology of the structure sheaf. This gives de Rham cohomology, which is the natural notion of cohomology in the world of vector bundles with flat connection.

$$R\Gamma(X^{\mathrm{dR}}, \mathcal{O}_{X^{\mathrm{dR}}}) = R\Gamma_{\mathrm{dR}}(X/k)$$

This is not even close to being an Artin stack either, and for different reasons from the moduli of formal groups. Here we're modding out by some formal scheme giving an equivalence relation.

**Example 15.6** (Prismatization). More recently, Bhatt-Lurie [?, ?] cite *meF*-gauges and Drinfeld cite *me*define stacks whose QCoh capture coefficient systems for various  $p$ -adic cohomology theories in characteristic  $p$  or mixed characteristic. For example, there's a stack capturing de Rham characteristic  $p$ , but it is not the one of Example 15.5. You have to use the divided power envelope of the diagonal instead of the formal neighborhood of the diagonal.

So there are stacks capturing prismatic cohomology, de Rham cohomology, and crystalline cohomology, as well as filtered versions of these. Moreover, the comparison theorems in prismatic cohomology between all of these various cohomology theories can be explained, so to speak, "geometrically" in terms of the stacks. (It's arguable how geometric these kinds of stacks are :)

Maybe the better way to say it is that, a priori these comparison theorems are about comparing linear algebra categories, e.g. vector spaces. But it turns out there's a more fundamental explanation, which is that you have an isomorphism of *stacks*. You then deduce comparison theorems of cohomology theories by passing to quasicohherent sheaves. So you promote a comparison of cohomology theories to an isomorphism of stacks.

I want to give another example of this phenomenon. We've seen de Rham cohomology in characteristic zero, some  $p$ -adic cohomology theories as well, but what about Betti cohomology? Here's a fun example which actually has quite a bit of relevance for the course, so that's why I'm going to mention it.

**Example 15.7** (Betti stacks). "Betti cohomology" is the algebraic geometers' term for, if you have a complex variety, then you take singular cohomology or sheaf cohomology with constant coefficients on the underlying topological space with the analytic topology. For example, if you have a compact Hausdorff space  $S$ , then I claim that you can make a stack.

How do you do it? You use the old familiar idea: you find a surjection from a profinite set  $T$ , and then you have some fiber product  $T \times_S T$ .

$$T \times_S T \rightrightarrows T \twoheadrightarrow S$$

Since  $T \times_S T$  is a closed subset of a product of two profinite sets, it'll also be a profinite set. Write  $T_0 := T$  and  $T_1 := T \times_S T$ .

So your compact Hausdorff space  $S$  is a quotient of an equivalence relation in the category of profinite sets. We can then apply  $C(-, \mathbf{Z})$  (continuous  $\mathbf{Z}$ -valued functions) followed by Spec to get a groupoid in the category of schemes, in fact in the category of affine schemes. We define the Betti stack of  $S$  as the quotient of this equivalence relation in the category of sheaves for the fpqc topology on affine schemes.

$$\mathrm{Spec} C(T_1, \mathbf{Z}) \rightrightarrows \mathrm{Spec} C(T_0, \mathbf{Z}) \twoheadrightarrow S^{\mathrm{Betti}}$$

You could also replace  $\mathbf{Z}$  with a commutative ring  $k$ .

What are quasi-coherent sheaves on Betti stacks? These are just usual sheaves of abelian groups on the topological space  $S$ .

$$\mathrm{QCoh}(S^{\mathrm{Betti}}) = \mathrm{Sh}(S, \mathrm{Ab})$$

That's a fun exercise. So, coherent cohomology on the Betti stack  $S^{\mathrm{Betti}}$  is just usual topological cohomology of the topological space  $S$ . More generally, we can embed condensed sets into stacks via the above procedure, using the presentation of a condensed set via profinite sets.

(something I didn't hear)

What you have to check to see that is that, if you have a surjective map of profinite sets, then it goes to a faithfully flat map on the level of continuous functions. That's not that hard to do:  $C(T_i, \mathbf{Z})$  are filtered colimits of continuous functions of finite sets, which as rings are copies of products of  $\mathbf{Z}$ . In fact, for *any* map of profinite sets  $T_1 \rightarrow T_0$ , the induced map  $C(T_0, \mathbf{Z}) \rightarrow C(T_1, \mathbf{Z})$  is flat, and then if it's surjective it's faithfully flat. You also need to check that fiber products in profinite sets correspond to relative tensor products; this follows again by a reduction to finite sets.

So now we're faced with this somewhat baffling array of different stacks, some of which don't resemble Artin stacks in the least. But we want them because they're convenient ways of encoding different linear-algebraic and geometric phenomena.

**Question:** how to define a reasonable subcategory of  $\mathrm{Sh}^{\mathrm{fpqc}}(\mathrm{Aff})$  containing all these examples?

**Answer:**  $\mathrm{Sh}^{\mathrm{fpqc}}(\mathrm{Aff})$  (modulo set theory)

In other words, it's not clear that there's any other answer to this question than the entire category  $\mathrm{Sh}^{\mathrm{fpqc}}(\mathrm{Aff}, \mathrm{An})$ . So there is content in the answer, but it didn't necessitate introducing anything wasn't present in the question.

**15.2. Desired examples of analytic stacks.** This is also the approach we will take in defining analytic stacks. We will define a Grothendieck topology on  $\mathrm{AnRing}^{\mathrm{op}}$  and then take sheaves with respect to it (again modulo set theory). Before getting into the details of exactly which Grothendieck topology, and these set theoretic technicalities as well, let's see what kind of phenomena we want to capture, so what the examples should be. In some sense, we've already seen some.

**Example 15.8** (Adic spaces). We certainly want that any adic space in the sense of Huber should give rise to an analytic stack. We already explained how the basic ingredient in adic spaces, namely Huber pairs  $(R, R^+)$ , give rise to analytic rings, and we explained something about how the formalism of analytic rings lets you glue. But we didn't quite discuss how you can use that to then glue these analytic rings together to get some kind of analytic stack.

We saw that at least you can localize the category of modules over that analytic ring along Huber's spectrum, but we didn't quite discuss how you can use that to then glue these analytic rings together to get some kind of analytic space. But we certainly want the kind of gluing that shows up in Huber's theory, gluing along rational open subsets in the topology defined by a basis of rational opens, we want that kind of gluing to be allowed and to give you an analytic space.

**Example 15.9** (Complex analytic spaces). We also want any complex analytic space to give an analytic stack, say over  $\mathbf{C}^{\mathrm{gas}}$  or  $\mathbf{C}^{\mathrm{liq}_p}$ . So the kind of gluing allowed should also incorporate gluing along open subsets in complex analytic geometry.

**Example 15.10** (Algebraic stacks). Another even more basic thing is we want the world of analytic geometry to be a generalization of the world of schemes, and even of algebraic stacks (in some sense—maybe not precisely the fpqc topology discussed above, but a slight modification of that). These should live over the universal base  $\mathbf{Z}$ , with trivial analytic structure ( $\mathrm{Mod}(\mathbf{Z}) = \mathrm{CondAb}$ ). So universally, over any analytic ring, if you have some algebraic object, you can get an analytic object.

**Example 15.11** (Banach rings). Any Banach ring  $R$  should also give rise to an analytic stack. This in some sense matches Berkovich's theory, in the same way that the affinoid analytic stacks coming from pairs  $(R, R^+)$  match Huber's picture. There's a small interesting tidbit here, which is that the stack that we'll assign to a general Banach ring will actually not be affinoid, it will really be a stack. So, if you take  $\mathbf{Z}$  with the usual archimedean norm, it will go to an actual stack that's not affinoid; instead, it's a stack which in some sense corresponds to  $\mathcal{M}(\mathbf{Z})$ , the Berkovich spectrum of  $\mathbf{Z}$ , so that's a fun little twist.

**Example 15.12** (Coefficient systems). As above, there should be analytic stacks whose QCoh capture various coefficient systems for cohomology (sheaves of abelian groups, vector bundles with connection, prismatic  $F$ -gauges etc.)

**Example 15.13** (Tate curve). We want to be able to define the Tate curve as well as its uniformization over  $\mathbf{Z}[\hat{q}^\pm]^{\text{gas}}$ . We also want to have machinery to prove it's algebraic. So we have this curve that we define via uniformization. We take the analytic  $\mathbf{G}_m$  and we quotient by the multiplication by  $q$ , and we get something which Peter argued was a smooth proper curve, and it has an identity section. So then, if you have some Riemann-Roch theorem, then you can see that you have a projective embedding. And if you have some GAGA theorem, then you'll be able to see that it has to be algebraic. So we want Riemann-Roch, and we want GAGA.

**Example 15.14** (Comparison theorems). On the theme of GAGA, we want that various linear algebraic comparison results should promote to isomorphisms of stacks. GAGA is one such.

GAGA is a general principle which applies in different contexts. But for example, in the world of complex analytic spaces, it says that if you have a proper algebraic variety  $X$  over the  $\mathbf{C}$ , so it has an analytification  $X^{\text{an}}$  which is compact, then it says that coherent sheaves and their cohomology in the algebraic and in the analytic sense agree.

So that's a question about making a comparison between two linear algebraic categories. It's saying algebraic coherent sheaves are the same as analytic coherent sheaves. And one thing we would like our formalism to do is to promote that to an isomorphism of stacks.

Another example, again in the complex analytic context, would be the comparison between Betti cohomology and de Rham cohomology. This should also promote to an isomorphism of stacks.

**15.3. Analytification.** Recall that adic geometry in the solid context really got started once we noticed that there's a nice subset of  $\mathbf{A}_{\mathbf{Z}, \blacksquare}^1$ , namely the closed unit disc, which we were thinking of as open. Or you could think of it in terms of its complement. or the translation of that back to the origin. In the end, algebraically speaking, this came from an idempotent algebra

$$\mathbf{Z}[[T]] \in \text{Solid}(\mathbf{Z}[T], \mathbf{Z})$$

Once we had this idempotent algebra, then we could move it to infinity via the change of variables  $T \mapsto T^{-1}$ , and then that gave us the closed unit disc, and then that let us tie into the  $(R, R^+)$  theory, where we could ask that the elements in  $R^+$  actually land in the closed unit disc as opposed to just being maps to  $\mathbf{A}^1$ .

Similarly, over  $\mathbf{Z}[\hat{q}^\pm]^{\text{gas}}$ , you can define a “subset” of  $\mathbf{A}_{\mathbf{Z}[\hat{q}^\pm]^{\text{gas}}}^1$ , corresponding to an idempotent algebra of “functions which are convergent in some (unspecified) disk around the origin”. So it's germs of functions defined at the origin, so to speak. Formally, it's going to be the filtered colimit

$$P[q^{-1}] := \varinjlim \left( P \xrightarrow{q} P \xrightarrow{q} P \xrightarrow{q} \dots \right)$$

In the solid case,  $P$  was itself idempotent and turned into  $\mathbf{Z}[[T]]$ . In the gaseous setting,  $P$  is not idempotent in  $\text{Mod}_{\mathbf{Z}[T]}(\text{Mod}_{\mathbf{Z}[\hat{q}^\pm]^{\text{gas}}})$ , but it becomes idempotent after we take this colimit to shrink the open unit disc down to the origin.

The idempotent algebra  $P[q^{-1}]$  satisfies many of the same properties as  $\mathbf{Z}[[T]]$ , and it again lets us import Huber's theory of pairs to this context. What you do is you take this idempotent algebra functions of germs at the origin in the affine line, you move it to infinity and you get germs at infinity.

Let  $R$  be of finite type over  $\mathbf{Z}[\hat{q}^\pm]^{\text{gas}}(*)$ , and assume that  $\mathbf{Z}$  is bounded in  $R$ . Then you get an analytic ring structure on  $R$ , namely  $(R, \text{Mod}_R(\text{Mod}_{\mathbf{Z}[\hat{q}^\pm]^{\text{gas}}}))$ . Every element  $f \in R$  gives a map

$$\text{AnSpec}(R) \xrightarrow{f} \mathbf{A}^1,$$

and we pass to the “subset” where all such maps land in the locus “away from  $\infty$ ”. This gives a new analytic space  $\text{Spec}(R)^{\text{an}}$ , the “analytification” of  $\text{Spec}(R)$ .

**Example 15.15.** When  $R = \mathbf{Z}[T^\pm]$ , we get  $\mathbf{G}_m^{\text{an}}$ . After base change to a ring where 2 is bounded, this is the object of the previous lecture, the thing we quotiented out by to get the Tate curve.

So now we have two different contexts (solid and gaseous) in which you can import Huber’s theory of pairs into the world of analytic spaces. It turns out that they can be glued together, and so can the idempotent algebras. First notice that the gaseous theory makes sense over  $\mathbf{Z}[q]$ : all we had to do to define the gaseous theory was we had to write down the endomorphism  $1 - qt$  of  $P$ , and then ask that it become an isomorphism in our theory. And to do that, you didn’t need to require  $q$  to be topologically nilpotent. So this gives an analytic ring  $\mathbf{Z}[q]^{\text{gas}}$ .

- if we set  $q = 0$ , we get the uncompleted  $\mathbf{Z}$  theory
- if you set  $q = 1$ , you get the solid theory,  $\mathbf{Z}^{\blacksquare}$
- if you require  $q$  to be topologically nilpotent and a unit, you get the gaseous theory,  $\mathbf{Z}[\hat{q}^{\pm}]^{\text{gas}}$
- if you work away from the locus where  $q$  is topologically nilpotent, then you are working over a theory where you force both  $q$  and  $q^{-1}$  to be gaseous, and that implies that 1 is gaseous, which means you’re in the solid theory, so you get  $\mathbf{Z}[q^{\pm}]^{\blacksquare}$ .

Then we can define a certain quotient of

$$\text{Spec}(\mathbf{Z}[q^{\pm}]^{q\text{-gas}})$$

which more or less parameterizes choices of a notion of “analytification”. So, anytime you have this variable  $q$  which you’ve declared to be gaseous, you can form this colimit and you’ll get an idempotent algebra at zero, and you can move it to infinity, and then you get a notion of analytification, as discussed.

However, different choices of  $q$  can give rise to the same thing. If two choices of  $q$  differ by a bounded unit, so something in  $\mathbf{G}_m^{\text{an}}$ , then you’ll get the same algebra. So the quotient we want is

$$\text{Spec}(\mathbf{Z}[q^{\pm}]^{q\text{-gas}})/\mathbf{G}_m^{\text{an}}$$

where  $\mathbf{G}_m^{\text{an}}$  acts by multiplication on  $q$ .

This is a different role of stacks in the theory. Here we have a stack which is a quotient of  $(\text{AnSpec of})$  an analytic ring by an equivalence relation, and which is in some sense parameterizing choices of analytic geometry over a given base ring. Let  $R$  be an analytic ring with a map

$$\text{Spec } R^{\flat}(\ast) \rightarrow \text{Spec}(\mathbf{Z}[q^{\pm}]^{q\text{-gas}})/\mathbf{G}_m^{\text{an}},$$

which we call a “gaseous structure” on  $\text{Spec } R^{\flat}(\ast)$  (and again assume  $2 \in R$  is bounded). Then we get two functors from  $R^{\flat}(\ast)$ -schemes to analytic stacks over  $R$ , plus a natural transformation.

Say  $X = \text{Spec } A$ , where  $A$  is an  $R^{\flat}(\ast)$ -algebra. We can view this as an analytic stack over the uncompleted  $\mathbf{Z}$ , and then you can base change that to  $R$  to get  $X_R := X \times_{\mathbf{Z}} R$ . But then you also have a subset

$$X_R^{\text{an}} \xrightarrow{\subseteq} X_R,$$

the analytification over  $R$ , given by requiring that all the functions land in the part of  $\mathbf{A}^1$  that’s away from  $\infty$ . Now there’s a general theorem.

**Theorem 15.16 (GAGA).** *If  $X \rightarrow \text{Spec}(R)$  is proper (and finitely presented), and every element  $f \in R(\ast)$  is bounded, then  $X_R^{\text{an}} \rightarrow X_R$  is an isomorphism.*

*some discussion between Peter and Dustin about whether finitely presented is really necessary. It might not matter for GAGA, but could for some other things.*

**Remark 15.17.** This implies completely formally that  $D(X_R^{\text{an}}) = D(X_R)$ , which is some form of GAGA. Non-formally, maybe with some more conditions on  $R$  (but satisfied in practice), this implies that  $\text{Vect}(X_R^{\text{an}}) = \text{Vect}(X_R)$ , which is classical GAGA.

Basically you can do algebraic geometry over the uncompleted  $\mathbf{Z}$  theory, and then that means you can do algebraic geometry over any analytic ring just by base change.

One of the things that maybe I should have been emphasizing before launching into this whole discussion is that over the completed  $\mathbf{Z}$  theory, you can basically do algebraic geometry. And then that means you can do algebraic geometry over any analytic ring just by base change. For example, we were considering  $\mathbf{A}^1$  over an arbitrary base ring. It’s not analytified yet, but when you have extra structure on your analytic ring, then that picks out a choice of what it means to analytify an algebraic variety.

All the classical GAGA theorems are special cases:

- complex-analytic GAGA after Serre (using  $\mathbf{C}^{\text{gas}}$  or  $\mathbf{C}^{\text{liq}_p}$ )



- Grothendieck's formal GAGA: **look up statement of formal GAGA** if you have again complete noetherian ring  $R$ , and a proper scheme over it, then coherent sheaves on that is the same thing as coherent sheaves on the formal scheme you get by formally completing or, in other words, compatible collections of coherent sheaves on all the various  $n$  potent thickenings there.

In terms of Huber pairs, you would work over

$$\mathrm{Spa}(R_I^\wedge, R_I^\wedge) \rightarrow \mathrm{Spec}(\mathbf{Z}^\blacksquare)$$

so  $\mathrm{Spa}(R_I^\wedge, R_I^\wedge)$  inherits the notion of analytification from  $\mathbf{Z}^\blacksquare$  based on the closed unit disc.

- non-archimedean GAGA over  $\mathbf{Q}_p$  or any complete non-archimedean field. For this you would take your analytic ring to be  $\mathbf{Q}_p^\blacksquare$ . But you don't put the gaseous structure on it which factors through the map to  $\mathrm{Spec} \mathbf{Z}^\blacksquare$ , rather you put the gaseous structure on it which corresponds to

$$\mathrm{Spec}(\mathbf{Q}_p^\blacksquare) \rightarrow \mathrm{Spec}(\mathbf{Z}[\hat{q}^\pm]^{q\text{-gas}})$$

$$q \mapsto p$$

That's the one that picks out the notion of analytification that corresponds to usual analytification of algebraic varieties over your non-archimedean field.

**Remark 15.18.** You could try to use the other gaseous structure on  $\mathrm{Spec} \mathbf{Q}_p^\blacksquare$  where you factor through  $\mathrm{Spec} \mathbf{Z}^\blacksquare$ . But GAGA won't apply in this setting, since then  $1/p$  will not be bounded, and if you make it bounded in the solid setting you'll kill everything.

Nonetheless, there is still a different gaseous structure on  $\mathbf{Q}_p^\blacksquare$  obtained by factoring it through  $\mathbf{Z}_p^\blacksquare$ . In a sense the above gaseous ring structure corresponds to some kind of overconvergent version of rigid geometry, while the gaseous ring structure that factors through  $\mathbf{Z}_p^\blacksquare$  corresponds to usual rigid geometry.

#### 15.4. Addressing set-theoretic technicalities. Unfinished starting from 1:29:22

things that maybe I should have been emphasizing before launching into this whole discussion is that over the completed  $\mathbf{Z}$  Theory you can do algebraic geometry and then that means you can do algebraic geometry over any analytic ring just by just by base change so that's um so that's kind of like considering the apine line like I was considering the apine line for example over an ariary base ring it's not it's not analytify yet but when you have extra structure on your analytic ring then that uh then that picks out a choice of what it means to analytify also in algebraic variety um and so uh and I'm also conf exit are fine or yeah when I was describing kind of explicitly what it is I was looking at the case where  $X$  is apine but in general both things glue and so you then then you can Define uh you can doesn't need to be it doesn't need to be Aline no yeah um so so all the all classical GAGA theorems uh are special cases so there's um so for example groen's formal GAGA well the complex analytic GAGA um so Sarah's original GAGA um take for example  $\mathbf{C}$  gaseous or  $\mathbf{P}$  liquid or whatever you like um uh there's also formal GAGA uh which is you know if you have a a again complete nean ring and then you have a a proper a proper uh scheme over it then coherent sheaves on that is the same thing as coherent sheaves on the formal scheme you get by formally completing or in other words compatible collections of coherent sheaves on all the various  $n$  potent thickenings there um then you would take a spff well maybe I should say  $\mathrm{Spa}$  so in terms of Huber pairs you would work over this um which lives over a solid  $\mathbf{Z}$  and therefore inherits the notion of analytification based on the the closed unit disc um it also includes like non- archimedean GMA GAGA uh say over I don't know over  $\mathbf{Q}_p$  or any complete non- archimedean field so there you would take your analytic ring you would take to be a spec of say  $\mathbf{Q}_p$  solid um but you wouldn't put the gaseous structure on it which factors through the map to  $\mathrm{Spec} \mathbf{Z}$  rather you'd put the gaseous structure on it which corresponds to mapping topc of  $\mathbf{Z}$   $q$  hat plus or minus one  $\mathbf{Q}$  gasas uh where  $Q$  goes to uh  $Q$  goes to  $\mathbf{P}$  say so St say standard choice of topologically mil potent unit um that's the one that picks out the notion of analytification that corresponds to usual anal analytification of algebraic varieties over your non- archimedean field um you could have also chosen the other gashes structure on this guy where you factor through uh  $\mathrm{Spec} \mathbf{Z}$  solid um and that would give you a different statement of GAGA in fact a different theorem so yeah so we're Mark and three could also choose uh the solid  $\mathbf{z}$  uh gaseous structure on uh on solid  $\mathbf{Q}_p$  that gives a an a priori different GAGA theorem oh the the analytification of  $A_1$  will then just be the open unit or the closed unit disc so um yeah so there's so let me illustrate H that cannot be is it has what sorry dation sorry1 ofp canale so they must also1 so iation of  $A_1$  in that context is oh wait ah you're right you're right you're right I'm sorry you're

right I'm sorry yeah that's that that open your dis is when you just force  $T$  to be analytic but there's no reason why uh that should also Force something like uh  $T$  over  $P$  to be analytic yeah no ah same Gaga thank you yeah I miss uh I I was not the same guys it's just no it is the same isn't it because you you would make you should always assume that your whole ah you're right I forgot that Axiom shoot yeah so I'm sorry you need ah oh boy than uh thanks again so you need uh to assume uh that every element in here is is bounded uh uh yeah so I was kind of making the same mistake again over and over um yeah I'm sorry uh so you don't get a Gaga statement at all in this context because um one over  $p$  uh is never going to be if you try to make one over  $P$  bounded in that sense then you're just going to kill everything so in in the solid QP setting but nonetheless um there's a point I wanted to make which is that there is still a different gaseous structure on solid QP obtained by taking um uh obtained by factoring it through solid  $zp$  and in a sense the difference between this gaseous ring structure and this gaseous ring structure is sort of the difference between usual rigid geometry and um and some kind of overon convergent version of rigid geometry um yeah uh okay so is there something weaker than  $n$  something weaker than  $n$  could you explain the question a result that would put uh identify  $x$   $r$  with some less uh regular could you explain uh okay maybe okay yeah this assumption that this uh bound it yeah yeah so what is the issue if it's if you don't assume that it's empty like the if you don't assume that everything in here is bounded then when you try to do the analytics that I described then in particular you're always forcing all of the scalars IE elements of of here uh you're forcing them to be bounded anyway so if you don't if you don't have this assumption then your analytics changing your your base so we have so a special case is when you take  $x$  equals  $\text{Spec } R$  and you try to analytify that what this assumption is saying is basically that then it doesn't change it like  $\text{specr}$  is its own analytify ring where it's where this thing has been forced to be bounded but in the case of solid QP if you try to force with that with that with that gaseous structure factoring through solid  $Z$  if you try to force all of the scalars to be bounded um you're just going to get the zero ring because yeah yeah um right yeah um so yeah so that was the uh yeah  $x$  equals  $\text{specr}$  was giving a problem there my apologies um okay uh right so that was um kind of explaining one example of where a linear algebraic comparison result uh can be promoted to an isomorphism of stacks um more generally we want relations um so relations between the various kinds of stacks uh various examples so for already in Peter's talk he had this map from the  $T$  curve uh with parameter  $Q$  to the topological Circle so and we want this to make sense as a map of analytic stacks so where I already explained how such a thing is an algebraic stack and algebraic Stacks will embed into analytic Stacks even over the the initial analytic ring um and then we have the Tate curve and so this kind of analytic space should map to that kind of analytic space but also if  $x$  is a complex analytic space uh then you want um  $X$  to map to uh its underlying topological Space  $X$  of  $c$  um and if say  $R$  is a banak ring then you want its well sort of its liquid  $\text{spec}$  uh to  $M$  map to the burkovich Spectrum um so uh yeah so there's all these we're going to have all these various different classes of analytic spaces and we want the category to be such that we can make maps between them when we expect comparisons um so for example it's well known that the coherent chology of a comp complex analytic space localizes on the underlying topological space um that's kind of more or less by definition in a lot of ways um and that kind of linear algebraic relation is supposed to be explained by the existence of such a map in this in this large category of analytic spaces um so all right um I think uh maybe I'll say one word about the set theoretic technicalities just to get it out of the way um so so addressing uh um so um well let's go back to our uh algebraic analog so so this was a Comm the opposite category of commutative Rings um so we have  $\text{fpqc}$  sheaves on there but again this is not a legitimate definition because this is not a small category and this by definition is a full subcategory of functors from uh commutative Rings uh to say sets or groupoids or whatever you like um and there's well-known problems with such things that the morphisms in this funct category involve a some more than a sets worth of data so you have morphisms that aren't sets but are some sort of bigger entity so that's a real pain um so what is the fix um uh consider so instead of instead of pre- shees uh on on  $F$  consider uh what are called accessible prees um which are the same thing as accessible functors uh from commutative rings to sets and what are accessible functors uh if and only if there exists a cardinal  $\text{Kappa}$  regular Cardinal uh such that uh  $\text{fun } F$  such that  $f$  commutes with with  $\text{Kappa}$  filtered coletes um so what does this mean uh the the first choice of  $\text{Kappa}$  is  $\text{Alf}$  not um um and in general when you think of a regular Cardinal you shouldn't try to think about what it is as a set what you should really think of is the collection of sets that are of smaller cardinality that's what  $\text{Kappa}$  is really indexing it's don't think of it as the cardinality of some set think of it as indexing the so this corresponds to finite sets um those are the sets of cardinality smaller

than  $\kappa$  and then the notion of  $\kappa$  filtered  $\text{Co}$  limit in that case is just the usual notion of filtered  $\text{Co}$  limit so if your functor commutes with filtered colimits you're okay but if  $\kappa$  gets bigger uh you have fewer uh  $\kappa$  filtered  $\text{Co}$  limits so to be a so for example if you take  $\kappa = \aleph_1$  then you're indexing the countable sets and you're only then a  $\kappa$  filtered  $\text{Co}$  limit would be one where every countable set has a cone as opposed to just every finite set so there are fewer  $\kappa$  filtered colimits that means there's more examples of functors which commute with  $\kappa$  filtered colimits um and uh uh  $\kappa$  um so so the nice thing uh is that every uh for every  $X$  in  $\mathcal{F}$  uh the  $\text{colim}$  uh the  $\text{colim}$  on its image under the  $\text{colim}$  embedding is always accessible and in fact accessible is equivalent to being a small  $\text{co}$ -limit of representable functors hxs so if we think of these as schemes is the basic building blocks and  $\text{Co}$  limits as our gluing procedure then we're saying kind of an obvious thing we're only allowed to glue a sets many worth of things together at a time and that's exactly what's captured by this notion of accessible functor and moreover uh fpqc sheafification uh preserves accessible sheaves or pre-sheaves sorry that was actually that's a theorem of Waterhouse um so that um you can um yeah so you can also impose the sheaf condition at will without running into set theoretic technicalities um but this accessible PR is not presentable right it's not a presentable  $\mathcal{C}$  never there AIC yes okay yeah we have we have to prove it yeah yeah yeah it has to be proved so what what makes this work so these are not kind of in some sense non trivial claims the claims made here so what makes this work so the first thing is that this category that we're working with uh is presentable in fact compactly generated uh that's what makes the theory of accessible prees work so it makes that the condition they're commuting with  $\kappa$  filtered col is the same as being a small limit of representables um and then there's something about the  $\text{gr}$  topology which is that every uh fpqc cover um uh in  $\mathcal{F}$  is a  $\kappa$  filtered is a there exists a  $\kappa$  such that um and in this case it's  $\aleph_1$  is a  $\kappa$  filtered limit of  $\kappa$  comp compact uh cap fpqc covers so every uh every flat cover of a ring is a  $\aleph_1$  filtered  $\text{Co}$  limit of countably presented flat covers of that ring um and this kind of condition having some a priori bound on the basic fpqc covers I cardinality bound is what you can use to um to check this kind of fpqc sheaf result and um so well so maybe I'll say then so then at least I we haven't explained what our  $\text{gr}$  topology is but we've least explained what the category of analytic Rings is so so there's a theorem um so analytic Rings is uh presentable in fact it's  $\kappa$  compactly generated for  $\kappa$  uh true to the AIC not plus so so so that you  $\kappa$  small means cardinality less than Continuum um so the uh that's what lets you kind of formally avoid all of these uh set theoretic difficulties involved with taking pre-sheaves and sheaves on a um a big category uh maybe yeah oh no I won't add anything that's fine okay thank you for your attention this category of analytic Rings is naturally enriched of a condensed animal right I mean mapping mapping home mapping space is a condensed animal yes so we could also consider enriched seeds instead of usual SE it it seems also natural to me but you take non-enriched we take non-enriched yeah and um it's confusing the it's confusing um what can I say um so in particular this is the category of analytic uh Stacks is not um no it's not that's right yeah um right different way because also okay but how do you use that to get an enrichment uh I mean yeah well maybe there's some adjoint functor is there I mean you're saying I mean okay it's yeah it's not even clear it's enriched over itself right well I mean I don't know yeah but no I guess I guess you're right you're you're saying you just take your profinite set or whatever and then you cross it with  $X$  and map it to  $y$  yeah no yeah that's right yeah so there is an sorry thanks yeah so you you could take take so in in analytics  $\mathcal{O}_X$  is enriched over light condensed sets because you can define an  $s$ - $v$  valued point to be just a a map like this um but uh it's different from the enrichment there's essentially no relation with the enrichment you had on analytic Rings because we imported these things via kind of very trivial analytic Rings just continuous functions and it um yeah it's all all in all it's a little bit confusing and um yeah so for example I think in some recent Works in in with p-adic coefficient systems and like like V. Vistnik was telling me that she wants to work with like sheaves of solid aelian groups on the pro- $\ell$  site and then you know on the  $\text{pro-}\ell$  site you already have profinite sets there so you're somehow you have you're doing condensed in two orthogonal directions at the same time but apparently that's the correct thing for her and it's just it's it's all a little bit confusing but yeah yeah I want to explain how this space be seen as an analytical space um well this by this I just meant the topological space um and maybe I make some assumptions to make sure it's compactly generated otherwise I mean sorry metrizable otherwise I have to modify a little bit what I mean maybe but um yeah just so the same way that uh that this thing was a a topological space and therefore an analytic stack so but this you could think of as providing an analytic stack structure on this a non-trivial analytic stack structure on this this this topological space here yeah um what do you mean by two years Bond what

do I mean by by one CU is bounded two is bounded two is bounded ah yeah so what I meant was that the  
 yeah so we have some Spec  $R$  and I was looking at the apine line and then so two is a function on this well  
 if you have any element in the ground ring it gives you a section of the projection and you can ask that that  
 section be away from the the local ring at Infinity so so we had this ring of of uh ring living at the origin this  
 filtered cimit of  $P$  along scaling the coordinate by  $Q$  then we can put it at infinity and we can ask that this  
 $F$  not meet that Locus at Infinity in the sense that well in the sense that if you that if you take some relative  
 tensor product like  $R \bmod f$  and then that that ring that lives there then you get zero um so um uh right and  
 that if that happens we say that  $f$  is bounded or that  $f$  is of analytic so um it doesn't yeah it doesn't get too  
 close to Infinity um so you have to ask that condition for the scaler two um in order for some of the claims I  
 made to hold um yeah and for the Gaga you have to ask that condition for all elements in the Basse ring um  
 yeah I feel particularly silly about forgetting that because I mean basically we we already basically wrote  
 up the argument in the previous lectures on complex analytic spaces and that aim was explicitly included  
 there so I just kind of forgot about it um okay other questions maybe at Point you said so you wrote what  
 what was the  $G$  structure on Spec  $R$  on Spec  $R$  yeah uhuh said this amount to Spec  $R$   $G$  GM did it say  
 did I say spec  $RQ$  plus one I meant spec  $zq$  plus one ah okay spec oh I'm sorry I'm sorry I meant  $Z$  I  
 meant  $Z$  I'm sorry but is it possible to elaborate or maybe or maybe well I mean the point is that if you  
 have a map to that stack then you get to Define such a Ring The Ring of germs at zero um uh which lets  
 you run this machine of of building analytic so yeah but so why you divide by  $G$  because you don't really  
 want to specify exactly you don't want to specify  $Q$  so for example you know if you have like a a Tate ring or  
 whatever an analy  $T$  Tate Huber ring or whatever then there exists a topologically no potent unit but you  
 know it doesn't play any role I mean so or at least up to some bounded difference it doesn't play any role  
 so that's exactly what this quotient is doing it's saying you have some say topologically nil potent unit um  
 but it shouldn't it should only matter up to some bounded difference and so you define this GM analytic  
 and then you say that if you have  $q$  and you multiply it by something bounded both both away from zero  
 and away from Infinity then that should play the same role in particular it should give rise to the same item  
 poent algebras and the same theory of analytic and so on by the way I don't see you multiply by something  
 about it I mean rather that you can mully by something one or you raise to raise to some power but this  
 gets confus s model is not really antic gen at all uh it is I mean you can have something that's like  $Q$  nor  
 itself but if you multip by something bound that's too large then you would actually not have something  
 anymore yeah but we that's true um that's true wait uh there but it's not really I thought I wrote the thing  
 down um but maybe we can just sort I thought I wrote this thing down um just a second ah okay sorry yeah  
 maybe it's not the GM maybe it's the uh so you can also Define the locus where uh the coordinate is bigger  
 than or equal to one or sorry some in some over convergent sense and the locus for  $T$  is less than or equal  
 to zero maybe it's the thing that's uh that lies in between there um that's not a yeah that's a group yeah  
 um we will discuss later in the course yeah but now I'm concerned okay anyway okay we'll discuss it later in  
 the course thanks everyone

## 16. ANALYTIC STACKS AND 6 FUNCTORS (SCHOLZE)

[https://www.youtube.com/watch?v=BV0-dlAuS3U&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=BV0-dlAuS3U&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

So last was starting the discussion of and was really any of actual definition that we used but was trying to give an overview of what the definition should look like and what kind of examples should commodate um and so today I want go more into like I want to talk about analytic spaces but something kind thinks how we set up uh how set up this definition is actually a series of six functions so okay so okay so last time uh result and motivation for making a definition of spaces um want be some category just check this definition mod issues of set that explain um actually now and then in that Ser of CH other soci related to soal Hy she um I will probably also say something at the end of this lecture um uh so we would like to Define our geometric object by which gives us our basic fib box the spaces and then specify Somey tell us how we lose um so key question is like which the sound transmission is is is is not very good here we cannot we cannot it's hard to identify what you say but I don't know what's the problem but the problem is I didn't mic so sorry I say thanks for catching that sorry um sorry do you hear me better now yes no maybe now it's seem yes okay um sorry uh yeah so I said that I want talk about analytics spaces and six funs and that last time we saw some motivation that we should try maybe to Define our category of geometric objects these analytics Stacks by starting with the fine analytics spaces corresponding to an rings as basic building blocks and then specifying some drok topology which tells us how we allowed to L them um in setting up the precise details of this there is some setic issues related to the fact that this is a large category and consider small sheets um and another technical issue that will arise is that we're really considering sheaves of infinity groupoids or what we call sheaves of Animate here um and then there is an issue of Chiefs versus hypers sheaves and we will actually at the end consider some something strictly in between the okay let me not get into this uh but the key question really is which broque topologies are uh to to put on this there was some motivation for this definition and uh lots of examples were mentioned um that we will uh be taking up again once we have the proper definition of the category all right um be before I I get there let me remark that what we're doing here um is kind of very very Clos uh so at the beginning of the course uh we did something extremely similar we defined condens sets or live condens sets you can put or not put light on both sides of equation um sh on the category of profite sets or uh if you want Prof sets I was thinking Alas and so there we playing the similar game that we had some basic objects that we start with and profite sets and then we're building a larger category uh by some allowing ourselves to glue them in a certain way and the gluing that we allow was specified in the gr topology that we chose and there we chose is rather General Gren topology allowing all in particular all subjective maps of profile sets um and and some we're now repeating we are now repeating this game with a much more broad class of uh basic geometric objects corresponding to this antic Rings uh and these are some able to model all sorts of different geometries they can model SS of just algebraic geometry I can model uh some pic geometry some geometry some complex Geet whatever um and all these different kinds of geometry are someh built you can put them together in this world of analytics stxs and build build spaces out of all of these things together right um so uh so something that done in some areas of M also I know maybe consider instead of some nasty class topological space like Prof fin sensing rather some nice class like Smooth manifolds and then play these games and there different logical space or something like this and but all of these would be part of this world of analytics Texts now um all right so so the key question is which gr needs so here here are some things that we want um so the most important variant um so if you have a triangle a um antic GRE and let me justess that but this a really analytic animated rings again most practical examples I will not have this animated structure but the general formalism works much better uh if you allow these things in your formalism um so whenever you have such a thing uh then the primary intent that we're interested in is uh it's dve category of complete modules recall that this was defined as full subcategory of the dve category of the condens a triangle modules such that all the homology groups are complete and okay so this is the primary variant and we want uh that some a met the v a uh is a sheet but this is this it's a drive the only sensible way to phrase this is that it's a cheap of infinity category so we actually need to treat from now on always this as an Infinity category so let's s with something we want we want it this so this this already means that for any analytic stack x uh we will be able to Define uh the rough category of POS curs on XY descent so it's just limit over of a over all a that to so that's good uh but actually we want something flatly more this is some this business with the six funs um so we



also want  $X$  to  $X$  as  $um$  so let me say a little bit about what this is so there there are six funs  $um$  there is always a fender  $uh$  which has some kind of right on which is a internal  $H$   $uh$  then there is a pullback which has a  $r$  on which  $P$  forward  $um$  and then and these will be things we already have and then there will be two more that we would like to have you would have to like a  $FL$  funter  $uh$  and the  $r$  join  $uh$  so always this one is left what what's the structure here so each  $d$  of  $x$   $uh$  should be should have a temper  $I$  a symetric model and then whenever you have a tender you can ask for an internal home object which has this usual Junction to to the  $T$  end if if such internal  $H$  exists is called a close symmetric structure  $um$  then whenever you have a  $MTH$   $F$  from  $y$  to  $x$   $uh$  there should be a pullback fun it should be able to pullback she and this should actually be compatible with a tender product should be tender fun  $um$  and this should have a right so also this is something we just get automatically because  $uh$  like when you have a Mor of rings you can base change modules and also then this the same fun reality on  $D$  of  $x$  and it's the way this form Works automatically if you change the modules will be compatible space change so so this is structure that we definitely already have and yeah  $uh$  but then there there's these lower Street fun not Street fun  $um$  and so this is a more delicate kind of structure which only exists for certain labs for certain  $F$  from  $y$  to  $X$  we want  $um$   $um$  want  $um$  the function that flow straight soal formology is compa support  $uh$  and this should satisfy  $uh$   $uh$  two properties one is the base chain  $uh$  so whenever you have your morphism  $Y$  and you have any base change of this first of all the base change should be again in the star where this work is Define  $uh$  but secondly there be like taking first the floor streak and then pulling back this should be naturally the same as first pulling  $X$  and then three and another thing that should satisfy is a projection form isomorphism that for  $a$  and  $v$  of  $x$   $b$  and of  $Y$  you're trying to say how these lower Street functions they interact with the existing structure and so the first thing is that with a pullback they should just commute in this sense and then with a tender you have the following property that when have  $a$  and  $b$  then take  $F$  three offer star a  $CER$   $B$  the same thing as  $a$  this so this is the kind of structure  $um$  that arises in a lot of different contexts in mathematics  $um$  the most classical  $uh$  in some sense if you take some nice of top space something locally com Co of spaces  $uh$  find a version of the right category  $um$  then this has a structure actually for morphisms  $uh$  where and where this is literally callous compa support so you looking at the fun of sections which have comple support  $uh$  and then you can dve that  $F$  what with setting  $um$  actually the first time this was developed not for in this case but it was for a to modu schemes I think forly  $um$  but then it was realized that there's I don't know there you can also do this for  $D$  modules and you can also there lots of different settings where you can do this  $um$  one setting where it was however not so much developed is a setting Vari now is some kind of coherent settings or quality coherent setting where usually not really have a notion of comy coherent com with complex support there is this appendix of dele to Heart STS resid andity where he kind of says that he can develop some s formalism but and there there are a few papers here and there which talk about complex support but it's not really  $uh$  a series that is used much I don't say  $um$  but actually our our naturally will through a Ser of  $uh$   $C$  for compact support that behaves rather well and we we do absolutely want  $uh$  to have it  $uh$  right  $um$  but before going on maybe let me make a remark because I think govern will complain in second that there's a completely imprecise definition here of what the six fun formalism is because  $uh$  when I write this isomorphism here there is no natural comparison with between the two these are just two random funs but they should naturally be identified so you have but you have to supply this isomorphism but once you start supplying this isomorphism then you run into trouble because now I don't know you can like base change twice and then they're like comparison here comparison here and one for the composite well of course you would hope that they commute but then you start to wonder how many different such things you can write down and which kind of compatibilities you have to enforce and I I think for a while it was some kind of open question what what a really good and minimal way to encode all data is that that is present in a six fun formalism  $um$  and there has been work by Liu Jang  $uh$  where they do this in the world of like AR Dex and there's been work of Gat Ros and blo where well they actually mon Cent although the their fers have different names and so they don't really talk about complex support chology at all  $um$  but also in  $C$  setting they they set up some kind of notion of what such as things fun formalism is  $um$  that treatment is however infinity2 categorical and so  $uh$  that's difficult  $uh$  and then like Lucas man maybe really isolated a key key structure that you need so say yeah so so yeah there is a goal no such anract and this is something that I gave a course about last winter and let me  $uh$  simply refer you you there right and so so but something  $uh$  some morals that I also personally was taking away from this course is said  $um$   $uh$  if you're dis are that you really just want the six fun formalism  $uh$  then this kind of dictates

everything including the grth topology including in quotation Mar correct go all right so um right so yeah before starting the discussion uh in our our specific case let Rec call uh the usual definition of such an FL stre fun uh like usually so you do this by specifying to uh collections of morphisms so there's a class of proper morphisms and here a Flo stre will actually be just a floor star um uh but if you want to make this definition you better check that it satisfies the properties that uh that an a FL should satisfy so need proper Bas thing and the projection formula in the to hold um but this time during like better situation because when when you take this to be a flow star then there's actually always a natural base change transformation and so then here I'm not supplying data I'm just asking conditions so when when you want to declare a morphism to be proper and so a FL streak to be a FL star there's something you can simply check whether does the natural trans base change transformation is an alph morphism the case of this morphism F and similarly for the projection form there is a natural map from here to here if you think about it when when this FL start um so these are just conditions there's no data Supply okay so there there's a clause of prop morph uses and then there is a clause of open immersions where instead you asking that of nor Street I mean here he ask that it's the right joint of P for open emergency it's the left joint of PB and again uh um you need to check that this is reasonable definition so you need to check that satisfies space change and projection formula but this time again also if it's a left ad joint there is a natural uh uh comparison M between these two things this time going in the other direction uh and you ask that the these things are satisfied again base change and projection formula are conditions better and uh then the general Nets f it are low or call them streetable these these are the ones that are for which this no streak fun is Define um they're taken to be the Composites of an open merer and a proper uh so we have open called problem that that far said these Ms that you can choose are the ones which you can somewh compactify like that and then you want to declare that the street is comp streets where the case have no have L that here this left joint of P get right uh this definition comes up prior with a really big caveat that when you write down this as a definition it's not really a definition because it chose the compactification in general there are many many possible ways to comp vile um and so you have to show that this definition is independent of the choices canonically because I mean you really want to get inal structure so you don't need just need to show the some unique up to isomorphism but even the isomorphism is unique up to high isomorphism and so on and so on and so on and so on um uh so this sounds like a real pain but fortunately there is a serum that was essentially proved by and then like slightly streamlined in this formulation by Lucas man um that under really minimalistic assumptions on the classes of proper Maps open Emer and some general three Maps so essentially just what I said except like for example you want that a composite of fre Maps is still shootable so you need that a composite of two such Maps uh again can be compactified U so there are some very small assumptions but they're extremely minimalistic um uh the theem is that Z can always produce such such an abstra an autic the preise theorem is in this lecture notes on or in luas CES or whatever in particular these assumptions include no condition whatsoever onto unique of compactifications so do you assume something like the two compactifications can be dominated by a third one no no no uh is it a question there is a question in the chat I mean these are the kind of things you want to do with this yeah so the the question in chat is using can formology characteristic classes index Antics yes that's the kind of things we hope to do with this stuff um right so gaba's question was uh about um whether I should yeah assume that two complexifications can be dominated by a third um well there is something that follows from the mere existence of comp of morms um right so you you have two then he can take a fiber product which is a third thing that's still this m should still be so it might be that a little bit follows but it's it's not explicitly mentioned that you have to check something like this see you need to know that all maps which are shable are such Composites but you just ask for each shable map you just need to produce one complexification um all right right so uh so now we apply these General ideas to our setting so so start we start with the category or infin category um C which is analytic Rings up and we call them F analytics section if you want analytics spaces and uh we will generally call an object in here the analytic spectrum of of some a but the unpe does I mean it's just a symbol yeah there is no topological space now it's it's just a symbol to to say we're taking over geometric object passing through the opposite here um all right and so we need to figure out which morm here should be proper which should be open emergence uh and at this point we should forget any preconceived notion of what a prop morphism algebraic geometry is and really just look at what the formulism tells us okay uh and that F un a so in other words map of antic R from HB um is proper so usually you would expect this virtually never case or it's only for finite Maps now it will be a rather General thing okay um so this notion

of properness is different from the usual one you use in alge geometry if uh from  $D$  of  $D$  to  $D$  of a right so this is simply just forgetful fun right you morph of rings from  $A$  to  $B$  so in particular any  $B$  module is an  $A$  module is a  $\pi_*$  projection formula a prob I should ask for more like for basent and so on it turns out that if this happens then all the other good things you want are also true so you uh so to analyze to analyze this not of properness uh let me recall uh so if you have such a  $m$  uh this will always factor in the following way so you can always uh first take this thing in  $t$  analytic  $R$  let me just second so what is this this is an triangle and The Cloud of all those the triangle modules such that if you forget the  $B$  module structure down to an  $A$  modu structure so triangle module structure this is uh complete in the cent of  $A$  and so some such things we call an induced analytic ring struction so do you implicitly work with animated ring and take modules over  $\pi_0$  as the category yes yeah yeah yeah yeah um yes uh okay let me use  $D$  notation so he was always talking about in The Animated context he was talking about this he was talking about a pair of a animated condensing light condensing uh and the subcategory of the like connective der category satisfying some conditions and then I could just as well say and of course working the context the  $T$  project incly derived too um right so whenever you have any analytic green and just an animated algebra over the underlying animate cond algebra here then you can always endow this one with an antic ring structure where completeness is just completeness where you restrict the reaction to the suffering um right so whenever we have any map of analytic Rings there's an induced one and then there is some kind of localization where you're just taking the same ring but then doing a  $f$  completion so you always have the spectation um and this it turns out geometrically precisely corresponds to the the process of compactifications basically except this may may or may not be an open Emer but these are these structures they are pris prop okay but here here's a proposition as as above proper if only is um  $B$  hased saan is saying that the localization is ISS right yeah this the second um maybe before uh uh going there um the only reason this it all seem like particular if you like work with if you usual algebraic Rings commuter Rings into this condens setting then you would like do the probably by choosing the trivial antic ring structure and then any map of commu rings and usual algebraic geometry will be declared to be proper which seems very strange the only reason to seem at all sensible to me is actually because uh this did seem familiar from uh  $H$ 's tic spaces so in the Ser of  $H$  tic spaces so yeah so this matches notion what apply to uh than for the central image of like your which s yes series there's always this  $A^+$  and if you have a map of AIDS where the  $A^+$  upstairs is like the smallest possible thing it can be determined by the  $A$  plus downstairs and this is something calls already proper um and turns out that it satisfies also in the  $T$  hology all the good properties that it proper map should satisfy like probation um do you take ear analytic Huber pairs or or or your addic Maps or or um implicitly I might be thinking about the like let me say take just so that I'm completely confident uh all right um Pro okay so what does the projection formula say um um all a I want to say ask l f m tender over  $B$  with  $n$  it's the same thing as mender 8 so both sides here are okay so what does this really say here um sorry what should um so what you're doing here is BAS changing a  $m$  from  $A$  to  $B$  this is what  $F$  upper star is and then you change the  $r$  over  $B$  was say the question was say El and this seems like it should obviously be true because you should be able to cancel um but now you have to be a little bit careful about meaning of symbols and then just doing structure you want the that tole be thanks um so okay so let's analyze this so uh if has instru then all ters can be understood to be CH in in or or like the relative to and then just checkes out um so this is One Direction um but let's say in the opposite case let's assume if a triangle speed triangle um then base changing to  $B$  is just some sort of completion to and let's even forget the  $N$  like even if  $n$  is just  $B$  uh then like take and to be just each triangle oh I see under Lin say uh which is  $H$  then then on neither on the left nor on the right this tendering do does anything right you're just tendering with a unit um so then we get that  $M$  tender  $AB$  must be equal to  $B$  to  $M$  this is what this formula says uh so in other words the further completion uh shouldn't actually do anything right was already and so says that if a triangle  $B$  triangle and the map is proper satisfying the projection formula then actually thing must be the same all right and so the so in general you just have to do the same argument uh more carefully um namely I mean in general you can just take  $n$  equal to  $B$  uh then the left hand side becomes the base change from  $M$  to of  $M$  from  $A$  to  $B$  in the sense of analytic rings but the right hand side is a base change from of  $M$  from  $A$  to  $B$  purely uh in the sense of complete a module and so that these agree is precise in the assertion that uh it's in very for so you get this equation uh where the  $T$  of these meanings and uh that they agree that the base change in the  $S$  of analytic is just the tensor of a is precisely saying that this must be induc because for the IND ener structure is precisely how you comp to Bas change in general when you compute the base

change for  $\mathcal{C}$  and first you do the base change for  $\mathcal{C}$  so you do this but then afterwards you would still have to make it complete as a  $\mathcal{B}$  module but here it's saying that you don't actually have to do it right uh okay was a question I'm not sure what the question like like the map from in usual algebra geometry the map from the fine line to a point is not proper but if you like pass through this word it's counted as a prop here because satisfies the projection will also satisfy prob base change and so on I mean actually a classical base change the for  $\mathcal{C}$  geometry so you don't need to ask for prop you only need to ask for qcq estimate all right so and Dustin would be able to give to really Sol you was this this is good um right so so it's just a cloud of proper map it's on the base  $\mathcal{S}$  um uh and proper base change and the projection form proper base change same base change or it tells you that after any base change still the projection formula holds uh but also proper base  $\mathcal{C}$  and he simple check if you haven't induced any the structur then you can just unravel what all symbols mean comes out right all right so let's get to the class of open emergence and again I ask you to forget any preconceived notion of what an open open merging I will give examples in a second uh let me actually call it  $J$  just for psychological Comfort um merge and if the  $P$  map  $B$  of modules Adit left three uh set the projection from so let me give first on example and then an example so non example is any open immersion algebraic geometry that's not also a tost immersion open close immersions okay there I don't know they kind of stupid but uh like if you take I don't know the  $GM$   $Z$  joint  $t$  plus one inside of some  $Z$  joint  $t$  uh with like like triv ring structure this is not an open imersion because I mean usually in  $\mathcal{A}$  geometry pack like tendering of modules virtually never commutes with any infinite products right if you want left joint should means that the pullback should commute with products it just um uh all right uh but here's but there is a way using some uh tic spaces and like relative solid ring structures so there's a different way to  $Su$  a path from algebraic geometry to analytic spaces using solid analytic ring structures and if you do that then actually it is okay then open immersion go to open immersion so here's a key example if you take um like here's one example um the  $UN$  spec of  $Z$  join  $P$  plus minus one solid this will be an example or also um and sometimes more primitive because this one can be realized as the base change of this one not quite but essentially like the key thing to know is that if you take like these joint solid and Advance this into the induced one solid uh this is open this is something that already came up in a remark once somewhere in the lectures but that I want to spend a little bit of time on now again I forget something the definition I wanted to be an open version not just which for which I should ask that left the Johnson before he face it's aival to the rer Joint which always exists  $J$  start to be fully fix basically some kind localization here um um right so why is that two uh so what is  $J$  upar the second example uh jar modu is the internal home from uh all's let's say if I if I take a module on the base pull back and then push down again so that's some the completion let notification of an  $M$  which is in the and uh Dustin show how to compute this in terms of this from this funny object this thing should receive a map from  $M$  so the  $r$  um and indeed I mean this complex it  $M$ s to  $Z$  homologus homological degus um so this which on our homes gives me which see Junction um I think yeah if you look back at Dustin's lecture on like sertification of a  $z$  joint ke this is Formula he gave and this means that actually if you tender with this object then this would become the left joint um so it is  $m$  i in this one Str and here thiss mean the comp and so this means that this left the joint exists uh on the image of jaapa but jasta is essentially subject so this um It also says that like this left just given by tender some module which is exactly what you need to check that projection for all right uh so why is this reasonable why why does this have anything to do with the intuitive idea that these lower Street funs there should be some kind of com with compact support So in particular like of the structure sheet of ECT uh the um I the Upp speak of some so this should be like the compactly supported coherent chology in some sense of the airline line um well what should it take to someh give a like a compactly supported thing should give you yeah if you want a function on your thing which has compa support should should vanish near in so giving a giving a function well that's an element  $Z$  Jo  $T$  and now you want to say that ADV Infinity but like an Infinity is going to get functions the Infinity of this long Series in  $t$  inverse and so you would like to take functions that vanish near Infinity so that  $L$  the kernel unfortunately like any po function is detered by Infinity so map injective but as a complex still a reasonable thing to do and you have a  $CO$  which is non Tri nice it's actually just a product of  $Z$  so like a complex object  $z$  um for and so the key thing making this work here that you get some uh such uh localization with this properties and like the key thing you need to check when you want to check that for example this is what the completion does is that  $Z$  inverse is animportant algebra so it's an object if you  $T$  it with itself STS itself all right and this is in fact the uh General description of these open emergence um so given some  $x$  a um the open imersion  $J$  from some  $Y$  into  $X$  um are equivalent of such

is equivalent to uh either equivalent or anti equivalent I'm not sure now right now um to uh item potent Comm Algebra I know let me call C um Ina not all of them but those such that the internal home from C uh 551 preserves some contivity assumption so here Jay maps to um the following so you can check J freak of the unit and this will always M back to the unit and the cone of that map this you can check is always an item P algebra so it always comes so what is to find item algebra you have to only Supply little data you only have to supply the unit and then you just have to check that when you take this unit in tend we see it becomes an ni morphism but if you take this map intend it with uh C again uh then well it's some simple matter of using the projection formula for J stre to see that nothing changes the key here is that if you take jow streak of one and 10 J stre of one J streak of one 10 of one but just J so also J of it would be so it's also equivalent to item to algebras J stre of the unit would always be an it coalgebra but but I'd like to take talk about coalgebra so much and it turns out that they anyways canonically equivalent to algebra why this fun procedure taking the cone to sorry there something else such as this fun preserves all Po and some come back here H how does the one uh well it's a con right so one just maps The Cone right you have con from X to Y then always y Maps into the con okay whenever always from right I there always X to Y just this um right so uh right I mean let me just say a few words the projection formula implies that stre module is really just the streak of the unit and the rest and this here s um and so this means that this thing is completely determined by uh right and so right and so this means that uh the right on this is actually just given by uh the right fun so it's actually just the internal home from day because the unit right so the left joint tendering with some object is our home from the and so we know that the completion is really just uh given like this um and so this means that yeah like particular this this new analy green structure is completely detered by this object and then you just have to specify or equivalently by by by this Co right because then we can recover it by taking the fiber from one to C um and then then you just need to supply the conditions on this so that this completion really determines the analytic R structure and so for this you need to check two things basically that this commutes this completion commut with all Co limits uh which is one condition I just put and the other condition is that should preserve connectivity U and the connectivity comes down to this other condition I put um and yeah all the other properties of antic R structure aut it's just some formal formal procedure to check that right so basically uh when you have an open Merion then there is always this item but commu AUD which you think of as some Comm of algebra describing a space near Infinity right so in the sense of this openion you have this open imersion and then there's some kind of complimentary closed subset determined by some algebra which is functions infinity and these are other important algebra so that's the general idea that whenever you have open emion and there's a complimentary th stop at Infinity which is described by some item po algebra this and again as a call of this you can check that uh the class ofion is stable on a base change and this all right so so we've isolated what we want the proper me open measures to be or not what we want them to but the follow them tells us what they are um and then uh this leads us to a shable map if well if you fact open work and the proper map but we already know which proper map to take because I already told you about this canic characterization of any moris structure and then some kind of localization so this F it always spectors some canonical F Bar in this case the compactification is canonical which is nice uh from this so this thing get let's call it J this J always exists this will also always have the property that J does for the faceful uh so the real only condition is that yeah someone have left joint the projection formula so if and the interesting example for example here is if you take the solid structure on Z and that which is solid this what satisfies in this case is a complexification but precisely these one over so this far this that more generally like if you have any M fire type algebras and take the solid the relative solid ring structure then these map will all all beable or if you more General if you have like map of U pairs then being shable is exactly this condition that hu calls being of plus weekly finite type which just means that this the suffering of integral elements  $A_+$  is as a ring of integral elements generated by just finding many new elements so H was defining these kind of fin Notions also for six but for talk these where notion Mor for which you could Define stre fun this uh it's hard spaces but now they are also the ones where you can Define Ro Street fun this compatibilities for proper open ersion out there as well yes yes so that's next yeah so there is a little bit of check like if you take a composite of two still recable that's true um basically just by comp on the base chain and then for the six fun there one you have to check which is some interaction between the low and the low star fun for open mer proper Maps again that's very check so I didn't discuss in this lecture like all the little EXs you have to put to make theism work there are some little little bits to check but each of them is like a really simple check uh and so as a proposition basically a COR to this



General construction of success some the fall re you can someone else say that yeah this data like of proper Maps open maps satisfies all the required conditions to get enough FM get XEX on on our category  $C$  which I recall was or you want  $f \times u$  and uh so cloud of three Maps so in our the first  $C$  we gave on my condensed mathematics uh this was the aim of this was precisely to uh explain this in the case of like formology so there we some specialized to EIC spaces and in fact the ones where the underlying ring is just theet and we work with the solid modules and particular these examples and then we discussed that there should be such a thing we didn't actually fully construct it then because we didn't have the required technology really so we nicely set it up but now it's really just a consequence of these General existence results for and uh then it's starting from this it's really easy to prove and this is what's done in these Mones to prove titles of prology and uh sity and so on and so for all right okay so that's that's nice um you get this nice Ser of in particular get some nice Ser of complexing uh but uh I started this lecture by wanting to define the SP topology sorry uh so here's a general question uh that I so given the six fun of  $X$  um from some some category  $C$  towards some categories yeah on some um you may want to pass to larger category of geometric objects which are some built by gluing objects and see uh just like um schemes both from fine schemes and so on or more TT so uh you want to try to extend the uh to Chiefs Chiefs on  $C$  Chiefs of  $C$  was values in if you want what we call any um uh for some notion of for some of and this in particular was maybe the original question that Leo and Jen were interested in when they uh when they wrote their their paper some wanted to extend from the from schemes to AR Stacks or higher AR Stacks something like that and I Pro some general uh General results about how one can go about extend from some from one category to category um and then some of this was also used by Lucas Min was start rephrasing it and then when I get to SC I again slightly rephrased what they did and uh uh so yeah so in this in these notes on six funters I tried to analyze this question and try to some pin down like what the best apology is um find such discussion including the question including the general definition of growth need for did I a d there no I'm actually not like I maybe not completely happy with just a very precise combination of this topology in the most case it works but it's slightly ugly at one point um but there's still takeaway from this so that uh if you want to do the  $X$  tangent the take away it said the covers should basically be those let satisfy Universal Universal descent what I call Universal star descent and Universal and I will explain what I mean by this one in just a second this for um let's say mically to the Limit Delta of and then there's a whole simpal diagram here I guess um where this here is a pullback and then here is  $P_1$  upper star  $P_2$  upper star and everywhere you put upper stars and Universal this means the same after in Bas Universal start and then there is a similar  $s$  where you replace the oppos  $St$  by the OPP Treet so map but then of course the map admit such soakable map now the same thing holds but for the upper three so  $f$  it Street fun and any  $P$  of it should also have Street fun so all these  $s$  and again Universal means auen it's actually small sub here which is precisely why I'm not with topology um in setting this up abstractly I actually need to set ask this in the case of shriek descent I need to ask this after base change to any preceive which is a really awkward condition um I don't want to do that and I don't do it now so when I say any base change here here I really just want to B change that staying in my categories all right so don't have much time so let me just  $St$  a serum that actually uh in the case we're in actually like up you should ask for Universal start and Universal which seems like a lot to check actually you need to check much much less uh to some extent is always any sixun but I think it seems better here so them is that if a streetable net to  $X$  satisfy  $St$  this then then actually Universal to and univers in fact it satisfies something even stronger it satisfies to the end so if you're not thinking about like usually we have a ring and you think about the category of modules but now we can go one categoric level higher and think about presentable stable Infinity is linear over  $D$  OFA and it turns out that asking for Street descend is equivalent to asking descend at a two categorical level which is actually how the serum is proved but this is probably not one I want to do now in the last five minutes um one category um so let me just end by giving the definition of the  $gr$  project we that's t uh was the brok to fall well gener on the one hand just by fire just unions so whenever you have a antic  $St$  just Union of several then it's covered by those this includes the empty cover of the empty set um and these stream mtis  $F$  so this is a rather General class of things it's actually uh very very close to related to this notion of descend ability that AR defined um which is some yeah nice notion nice notion of descent that's satisfied for virtually all faceful flat maps at least all those that are accountably presented which will be another motivation actually for us to at one point switch to the lightens setting because there things accountably the Rings are accountably generated um right yeah so particular if we restrict to proper Maps then this condition of stre descend is precisely

the the condition that the map of algebra is descend um yeah have thaty and then okay they like in the beginning of my lecture I said that there are two small wrinkles one about set Series so again we should look at accessible things and for this we should check that this gr topology has some approximation properties that means that specification of something acceptable stays acceptable the other some hyper complet issues is that we actually don't want the thing that's just chaps on this because also on the light condens setting we're actually considering hypers Chiefs so we actually want to allow a certain class of hyper but uh it's it's the same ideas that go into the CL of hyper about but okay basically analytics TX will be sheets on F analytics TX forsus all right let me question what do you mean by fin joint well I mean you can have like f analytics STX they have fin unions I mean du like rings and fin products and so whenever I take such an F fin dint Union if an F analytic Tes it should stay this fin dint Union in analytics ST which like it's covered by the individual wants right no just this is a cover whenever I is a finite set and  $x \in SE$  then this is also an so like for for Prof sets we were looking at all subjective NS and fin so just include the EMP includes the empty cover of the so for for usual schemes a fine schemes so what are the the the shable maps and what are the those which satisfy the scent for shrick or for star and what is the topology I mean yes you could do the same discussion just with usual commu reins uh in that case I would declare all my maps to be proper all my maps of f seems to be proper um seems bit awkward but it works um and it's actually like in this largic category of analytics it's actually reasonable um anyway so all all maps of f are shakable the condition of satisfying shink descent is precisely the condition that the M of algebra is what Arc called descendible um descendible maps that the ra General class for example I mean it includes all accountably presented Facey flat Maps it also includes all quots by NE poent ideals maybe more surprising it also includes all H covers so this is wot's H topology um so you find schemes that covering wot's H topology that's also allow here actually when you restrict to theeran schemes uh that's equivalent so when Y and X are fine and thean then this satisfies this condition of descend ability I mean satisfying STC if and only if it's a cover and what's H do you mean V topology you have the H topology with present I restricted to yes it's there are some questions about POs to the limit which are a little bit subtle but when I in a seran setting then H and V cover are the same anyways so um so I want sorry I wanted to fin presented map of thean schemes maybe and then it's it's a cover here if and only if it's a h cover if okay and what is The Descent so when you have Shri descent do you also have star descent or those are unrelated that's what I said here right so if satisf I mean the same Ser some special case so if it satisfies descent satisfies Universal descent and Universal St descent ah this is just didn't read it properly and what is the other statement what is the meaning of in descent for for infinite two of module categories uh okay let me just very briefly do it I mean I think this will take a whole other lecture um so it's an extremely wonderful structure that Jacob L def find it this category of presentable Infinity categories with call preserving functions uh and in there you have a sub category of stable ones or also like whenever you work over some ring you have some categories which are linear over some base ceg D of a this itself so the objects are something like DG categories and which ofas vely Speaking that there's a precise definition of what this means as the funes are colum preserving fun um and this has a well maybe me I mean this also has a Tang structure um and now you can also ask the question whether the association that takes such an A or the unspe of it to this Infinity one or Infinity 2 category actually it doesn't matter which one you choose for this purpose of asking for descent that this satisfies descent so by modular categories I mean objects in this same and Jacob the is formalism so the okay so the so for example a derived category of something over a is is of some B over a is in this category derived category of an a algebra is right right yes yes yes H okay so you some of you giving such things locally then you are given the you I mean check the send at list for okay and what why is it Infinity so the home between two such things is viewed as a as a as itself most naturally would be Infinity one category actually for this I mean you can't forget about the non inable two morphisms and then you get an autic one category and actually for the question of the same turns out it's like two morm the non ones will automatically satisfy the S as well so in some sense the two categoric structure of this thing doesn't matter so much uh and yeah so uh right for these questions I would really recommend that you read this paper of AR mesu which some awward what all right questions so for for the last statement like from a to present which type of descent what does it mean it set by be set well I mean is a pre right I mean so you can just when map of range you can base change mod basically and this pre- should be a she so it's some kind of St yeah so two cat start desc is equal to one category s descend which a bit awkward all right stuff

## 17. !-DESCENT CONTINUED (CLAUSEN)

[https://www.youtube.com/watch?v=rN\\_iM7Z8vdE&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=rN_iM7Z8vdE&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

Okay so let's get started so I'm going to uh so well I'm going to continue the discussion that Peter started last time on shriek descent um so well let me recall the setup um so we're kind of well what are we trying to do we have analytic Rings um and then well we're going to build some geometric objects on them in somewhat in the model of scheme Theory where you have start with commutative Rings um so the first thing you do is you define what are the analytic things um and might as well just formally Define them to be the opposite category of analytic Rings um and then so an object in here maybe you call  $X$  and it corresponds to uh let's say an analytic ring which maybe one can denote  $\mathcal{O}_X$  comma  $\mathcal{D}_X$  so we'll that an analytic ring consists of well an animated condensed ring and then a certain full subcategory of modules over that of the say full derived category of modules over that so I'll denote the first coordinate by  $\mathcal{O}$  of  $X$  and the second coordinate by  $\mathcal{D}$  of  $X$  in this notation so here I mean  $X$  is really just a formal symbol here but okay also in the in the derived context you somehow use is just this yeah and I suppose there is a way to define the full dve category also yes so but I don't I'm not sure how I mean but is it important to just restrict to the I mean for for some purposes it's easier when you for so the the two formalisms are equivalent you can either think of this or you can think of the full thing um and you can get back and forth between them for some discussions it's more convenient to consider the greater than or equal to zero part to be the primary object but for the discussion today it's actually um much better to consider this thing as the primary object and the reason has to do with a descent so it turns out that the um the full derived category will have this shriek descent but the greater than or equal to zero will not so in other words another way of saying it is for an analytic ring we have a nice  $T$  structure on this category but for a general analytic stack um there's not going to be a  $t$  structure so um on which on the on the on the derived category of the global thing so which is obtained by gluing these local derived categories the for an analytic ring there's a  $t$  structure which is so this thing is by definition a full subcategory of you know derived category of  $\mathcal{O}_X$  modules okay this has a natural  $t$ - structure and the claim is that you mean just the yeah you just forget down to derived the condensed ailen groups so the  $t$ - structure is detected all the way down here and the claim is it passes to it induces a  $t$  structure here and what is official of the the full one yeah so me when I talk about it what I'm thinking of is the following there is a functor from simplicial rings to say  $E_1$  rings so meaning uh uh just algebra objects in this category um and then I just take modules over that algebra object in this category here so I take I use some Infinity category formalism of algebras and modules yeah so it's not defined by but it can be defined I I suppose you can take like chain complexes of mod yeah no doubt but I prefer to think of it like this I mean you can understand perfectly well like how to calculate with with this definition I mean the free you know the free objects are like things in here tens are with a and then you know you can make resolutions and you know ring is the same as  $E_1$  Rings yeah simpli Rings up to equivalent of course yeah yeah okay okay okay so uh right and um and we singled out uh um there's a there was this definition so uh so uh a map is shable if it can be Factor factored as an open immersion followed by a proper map um and uh what was the definition of these Concepts well for proper map um so let's say let's call this uh  $F$  and let's say  $J$  and then  $\Pi$  um so for a proper map we have uh this means that uh an element  $m$  in uh  $\mathcal{D}$  of  $\mathcal{O}_X$  Prime uh lies in  $\mathcal{D}$  of  $X$  Prime uh if and only if uh it's image in  $\mathcal{D}$  of  $y$  I mean image under the forgetful functor uh lies in  $\mathcal{D}$  oh  $\mathcal{D}$  oh sorry yeah lies in  $\mathcal{D}_y$  um so in other words you just take the class of complete modules you have on this analytic ring um and then inherit the notion of the notion of completeness up here so just given the data of this ring you can always do such a thing you can Define this  $\mathcal{D}$  of  $X$  Prime to be ones where you forget down and you ask it to be complete there and that always defines an analytic ring structure and those are exactly the ones corresponding to these proper Maps um yes exactly yes and did you claim things like the composition of I forgot now uh yes we claim that I think I mean Peter claimed it last time I believe didn't didn't give the argument but it's um it's quite uh it's quite straightforward so one of the one of the one just but let me let me continue the discussion um and so open immersion meant that uh well first of all this uh  $J$  upper star functor which you always have uh this should be a localization and uh the kernel should be just modules over some uh some algebra yeah it will necessarily be item potent it will also necessarily be a compact object because the um because by definition of a map of analytic Rings the right ad joint of this commutes with  $\text{Co}$  limits and then the this is not in yeah it's yeah just in the pure category Theory sense

there is a notion of like usual categories you can localize by multiplicative set for example no but these are these are like big categories presentable categories and the good definition is just what Peter said that the right adjoint should be a fully faithful functor right adjoint is fully well yes so but this this fun well we we I mean we already know as part of the discussion of analytic rings that the right adjoint exists so yeah yeah um want a left adjoint uh yeah this uh this implies that there's a left adjoint yeah um so let me yeah so I'll I'm going to continue the discussion um right um and then so we had uh we had this claim or this theorem uh oh yeah so then the remark is that for a proper map um for a proper map  $\pi$  uh the right adjoint  $\pi^*$  lower star is good uh and what does that mean it commutes with co-limit uh satisfies uh projection formula uh I e it's it's  $D$  of  $Y$  linear so if you take an object in  $D$  of  $Y$  and you pull it back and then you tensor with something and push it forward that's the same well yeah so if you take an object in  $D$  of  $X$  Prime you can either and an object in  $D$  of  $Y$  you can either pull it back tensor and then push forward or you can push forward and then tensor and those give you the same thing there's a natural map and you ask it to be an isomorphism um and uh and commutes with base change or so in other words a pullback of a proper map first of all is proper and second of all um the the formation of this lower star this  $R$  joint commutes with that base change that you can write down um and then uh for an open immersion uh  $j$  there exists a left adjoint uh I don't know  $j_!$  or whatever people use uh uh which is good in the same sense  $j^*$  I don't know I've seen I've seen this  $j^*$  lower sharp  $j^*$  lower natural yes yes yes it will be  $j_!$  or shriek yeah I mean okay I'll just yeah yeah no there is there is the other thing which is an adjoint like for sometimes for in Duality like an adjoint of  $f^*$  no no okay it is not the the bizar it is usual yeah it is lower shriek okay um uh right um and then the the theorem that was discussed last time was uh so so there exists six functor formalism on  $F$  uh such that uh the collection of maps  $F$  from  $X$  to  $Y$  for which this extra functor you have  $f^*$  lower shriek from  $D$  of  $x$  to  $D$  of  $Y$  is specified uh is the class of shable maps uh such that for  $F$  proper  $f^*$  lower shriek equals  $f^*$  lower star and for  $f$  for  $j$  an open immersion well  $f^*$  now now here's why I wanted to yeah you know what I'm going to I'm going to say left adjoints or  $j_!$  or natural and yeah just so that I can make a non- topological claim here yeah um um oh thank you yeah  $j^*$  thank you all right um so so ahm contains the notion of proper the class of proper it does not no so I'll explain in just a second exactly the manner and what like what the precise definition of this and and how it's encoded and everything um in that encoding it's not specified what is a proper map and what is an open immersion there's it's just specified what is the class of shable maps um uh oh yeah and maybe I remark I perhaps should have made before is that there's a the class of shable maps has has good closure properties so it's uh if so if we have  $X \rightarrow Y \rightarrow Z$  say  $FG$  um then  $f$  and  $g$  shable implies  $f \circ g$  composed with  $f$  shable uh  $G$  and  $G$  composed with  $f$  shable implies  $F$  shakable that's nice so that a map between shable Maps is shable that's the way to think about this um and uh shable maps are closed under base change oh and maybe I'll add one more uh so if um uh if like  $F_1: X_1 \rightarrow Y$   $F_2: X_2 \rightarrow Y$  uh  $F_N: X_N \rightarrow Y$  are maps with the same Target uh then uh uh  $f_i$  is oops  $f_i$  is shable for all  $i$  if and only if uh the map  $F$  from the disjoint Union over  $i$   $x_i$  to  $Y$  is shakable um and this these disjoint unions in this category those those finite disjoint unions in this category they correspond to finite products in this category of analytic rings and they're kind of just naively defined coordinate wise so um so for finite finite Co products um so maybe I'll do just uh I think Peter discussed it last time but just a let me do a reminder some example of this six functor formalism so so let's take um  $y$  to be sort of spec of the solid  $Z$  Theory um then we could and then let's take  $X$  to be spec of the solid  $ZT$  Theory um uh with the natural map from  $X$  to  $Y$  which is just a yeah um then we can take  $X$  Prime to be spec of the solid  $ZT$   $z$  uh Theory um so this is pro now this is a so that we have  $X$  goes to  $X$  Prime goes to  $Y$  now this is proper and this is an open immersion um and the the complimentary algebra  $a$  uh is equal to this uh  $Z$  lant series  $T$  inverse um and then for example  $j^*$  lower shriek of the structure sheaf of  $x$  uh is uh is this two term complex um so um and then that's also the formula for  $\pi^*$  lower star  $\pi^*$  lower star is just the forgetful functor just forgetting that you have a module structure over  $ZT$  um and so intuitively speaking this is functions on the Aline line uh and this is functions localized uh near Infinity localized near the missing point and so this is like functions on the Aline line which vanish near Infinity so to speak because you're taking a fiber um so that's kind of uh compactly supported cohomology of the structure sheaf on the Aline line so to speak and this is what uh six functor formalisms are supposed to be doing in general they're supposed to be specifying some notion of compactly supported cohomology relative compactly supported cohomology which behaves well in families um so the the key property of this eor shriek is that it commutes with base change in in complete generality so so this is so this yeah so the algebraic um algebraic objects are proper even if you're dealing with something like the Aline line which is not proper in

traditional algebraic geometry but it's kind of compensated by the existence of this solid Theory where you have a new version of the Aline line the solid affine line which is not proper anymore um yeah okay so um uh and then there was the definition so um uh a shable map uh satisfies Street descent um if um so you can take  $D$  of  $Y$  um  $D$  of  $x$  and then  $D$  of uh  $X$  cross  $YX$  and then you continue like this um and then here you use the upper shriek functors to Define this diagram so upper shriek being the the right adjoint uh to lower shriek in every case um uh so this induces so to the shriek limit uh of the  $D$  of uh  $x$  to the NS over  $y$  so the naive thing would be to consider the star descent you'd want to know that a an object in here is the same thing via pullback as a compatible collection we actually asked for something different uh it turns out to be stronger which we'll show in the lecture today that when if you take an object here you ask that it's you know the same as giving compatibly it's shriek pullbacks along this this same diagram um so now I can that was the recap of last time now I can explain the goal for today goal so well first goal would be to um uh uh prove some streak descent results I.E uh Identify some natural she uh natural pre-sheaves that have have shriek descent so in other words that um some pre-on aphids such that whenever you have a a cover satisfying shriek descent which by definition means that  $D$  of  $Y$  satisfies streak descent for upper shriek um we wanted to find some check what other kinds of descent properties that implies um or well maybe I should actually give the definition sorry sorry the the Groen de topology uh of shriek descent uh so covers are uh given by uh finite collections uh  $\{x_i\}$  to  $Y$  uh sorry um sorry of shable maps uh such that um  $f$  from the disjoint Union of the  $X_i$  to  $Y$  uh has streak descendency on the underlying yes so yeah so I mean or yes so I could say a sieve is covering if it contains finitely many shable Maps satisfying this condition yeah what about the composition of Shri it's involve interchanging proper and open or is it it's quite straightforward and the reason the essential reason it's straightforward is there's a canonical candidate for the  $X$  Prime the factoring because your  $X$  contains this algebra  $\mathcal{O}$  of  $X$  the structure sheaf and then you can just take  $X$  Prime to be  $\mathcal{O}$  to have the same  $\mathcal{O}$  of  $X$  and then  $D$  of  $X$  Prime should just be the full subcategory of  $D$  of  $\mathcal{O}$  of  $X$  consisting of things whose image in  $D$  of  $\mathcal{O}$  of  $Y$  land in  $D$  of  $Y$  so so you you you have a like an open immersion and then a proper yeah and then you say that this so this is like in adic spaces the okay so you take uh the ones okay you use the other uh uh yes okay so you claim that thinking about it you have a canal and you check it yes exactly yeah okay and the second uh goal for today is a well then you want to be able to give examples of Street covers so so that you then can apply these descent results I don't remember if it was defined this was a talk I saw in the way but whether it sh def what six fun formalism is exactly I think he refer to Lucas man so I'm going to actually discuss it today because the precise way it's encoded will help uh give some arguments so it's in fact the next topic that I'm going to turn to um uh okay um so yeah so so the the this theorem so the existence of this six funter formalism on apids um follows rather immediately uh from work of Li Jen as reinterpreted by man and so following Gates sorry Rosen Bloom uh man encodes uh six funter formalisms uh in terms of span categories so let me um Let Me Now set this up um so so suppose given uh well let's say could be an Infinity category uh  $C$  uh with all pullbacks um yeah um and a class of maps  $s$  and  $C$  stable under composition and pullback um then so I'm not going to give the precise uh um the precise description of the infinity category of spans I'll give a kind of handwavy description the precise description has been given by Clark Barwick um so as a complete seagull space I mean there's there's well or no he did it as an anyway doesn't matter there's there's different ways of making it precise and it has been done um so then we get a span category uh span  $C$  let's say  $s$  all is the notation so the idea is this is going to encode let you encode six fun formalisms where you have shriek Maps defined for uh maps that lie in  $s$  and you have uh star Maps like upper star defined for any map whatsoever um so this has objects the same as in  $C$  uh but Maps uh  $X$  to  $Y$  given by uh diagrams so  $X \rightarrow Y$  and then you put something in between uh  $M$  say like this yes thank you yeah or this map um let's say this map is an  $s$  or do I would I rather say the other map lies in  $S$  I don't know you have to make a choice and then the uh composition uh is given by pullback so I'll just write it down like this so you have a span from  $X$  to  $Y$  and a span from  $y$  to  $Z$  you want to make a span from  $X$  to  $Z$  you just do the pullback here um and then a I don't know a two functor formalism on  $C$  uh with respect to  $s$  uh is just a functor um from span  $CS$  all to some some category of categories so so I to every object  $X$  you assign a category Infinity category uh  $D$  of  $X$  say we can call it  $D$  um um and then to every span uh you assign a functor so well what is this doing so there's two special classes of maps there's the ones where you only have uh something interesting going on in the  $S$  part and there's the ones where you only have some something interesting going on in the  $Y$  part um and every map is a composition of is canonically a composition of uh two such Maps like this so you can break this up into well it's kind of yeah



$x \in \mathcal{M} \mathcal{M} \mathcal{Y}$  and you can write this then as a composition of two of these so if you kind of at least intuitively want to understand what your functor is doing on morphisms it's enough to know what it's doing here and what it's doing here um and here this is giving you a functor that you call uh let's say this is  $\text{map}$  is  $F$  this is giving you a functor you call  $F$  lower shriek from  $\mathcal{D}$  of  $x$  to  $\mathcal{D}$  of  $\mathcal{Y}$  and here it's if this is  $F$  this is giving you a functor uh you call  $F$  upper star from  $\mathcal{D}$  of  $x$  to  $\mathcal{D}$  of  $\mathcal{Y}$  stable under composition includes having the identity yes yes it does it's what happens when you compose an empty sets worth of composable maps um um and then if you and then it's a functor so you get some relation when you have a pair of composable maps and the functoriality um amounts to the base change formula uh for the lower shriek functors so the fact that the lower shriek commutes with upper star when you have a cartesian square in your category so two is a smaller number than six a two functor formalism versus six functor formalism but two is the essence of six in this case because the what are the other functors um so then that's two of the six functors but then there's a so if uh well base change formula and also compatibility uh of and upper star under composition and it also encodes kind of compatibilities between the base change formulas and the compositions that you have on these things so there's kind of higher order data implicit in here um right so that's two of the of the six functors you'd like to have but then so then that that gives you these functors  $F$  lower shriek and  $F$  upper star but then for free provided this admits a right adjoint then you get the right adjoint as well and provided this yeah and provided this admits a left adjoint then you get the left adjoint as well um and that gives you four functors you don't need to encode them explicitly they just kind of it just kind of follows um and then the remaining functors are some tensor product operation you have defined on each of these um and then some adjoint to it some internal adjoint and again that if you have this and it satisfies a certain property then it has some adjoint so you so the only remaining question is how to encode this tensor product and the uh expected interaction of it with uh these functors here namely projection formulas um so so uh uh plus compatibility with lower shriek upper star and for this we use a symmetric monoidal structure on this span category uh induced uh by by product uh in  $\mathcal{C}$  so now I have to assume I guess maybe I assume  $\mathcal{C}$  has a terminal object I mean I assume it has pullbacks I need I need products um yeah so let's say that  $\mathcal{C}$  has a terminal object um see um whoops uh so yeah that's cartesian product in  $\mathcal{C}$  but it's not the cartesian product in this span category anymore it's just some symmetrical monoidal structure um and then and then request that your functor  $D$  from this could you say that again Peter prize one U product break symmetry of because at least if you take for all maps then so far yeah not anymore for structure because only the yes uh yeah Peter was making a claim that the um well I I didn't finish the sentence maybe yeah uh so we request that this be LAX symmetric monoidal uh with respect to that some that uh tensor product we defined just defined here and the uh symmetrical structure here are given just by cartesian product of categories um so uh um so the um uh so what is this amount to so the basic data is that you have  $D_X$  uh  $\text{cross } D_Y$  should map to  $D$  of  $X \text{ cross } Y$  so that's the direction of the laxness if you were wondering um plus uh you know some compatibilities which are conveniently encoded in this being a LAX symmetric monoidal functor so in particular  $D_X D_X \text{ BS}$  to  $D_X$  using the diagonal yes and then you can probably reconstruct this by pulling back over  $Y$  but then one can wonder why you need the product in the category it's not enough to specify  $D_X D_X$  to  $D_X$  for every  $X$  in some coherent way well you need to encode the compatibility with the lower shriek functors somehow so that's extra data yeah but yeah indeed if it were just if  $s$  were like trivial class um then then giving this would be the same as giving a symmetric monoidal structure on each of the values  $D_X$  so yeah this so part of this uh is yeah symmetric monoidal structure on  $D$  of  $x$  and then the  $f$  of  $x$  functors are also symmetric monoidal as as of said by if you take  $x$  and  $y$  you can do this and then you can pull back along the diagonal what about object one does it come for free one in the of the terminal object oh boy yeah I'm sure it comes for free from so it lacks unital I mean there should be some unit yeah yeah um okay um so uh yeah I have to advance this story a bit so I um yeah so I said that if you just have these well really these three functors ding ding ding satisfying certain properties then you automatically get all six functors and there's a very convenient way to um to organize uh the passage from 3 to six and that's uh it's it's a bit of higher categorical magic defined by Lurie so this is Lurie's magic category uh called  $\text{PRL}$  um so I have to say a little bit about this and how it works so here the uh objects are the presentable infinity categories presentable means you have all small co-limits and um you're in some sense controlled by a small subcategory so in the sense that there's a small subcategory such that the whole thing is gotten by formally adjoining some sufficiently filtered co-limits so you should be Kappa compactly generated for some Kappa um uh and the morphisms are the cocomplete preserving functors and then that's the equivalent to admitting a

r joint um so um so let me uh tell you a bit more about this magic category um so so PRL has all small limits and um the forgetful functor uh PRL to  $\text{cat Infinity}$  uh preserves them um so that's um that's nice so limits in this category exist and are completely naive so um but also PRL has all small Co limits and this is the really remarkable thing is that you can also access these co- limits in a completely naive way um uh so so there is besides the naive forgetful functor there's also a constant version of the forgetful functor where you take  $C$  and send it to  $C$  um but then you take a functor from  $C$  to  $D$  and you send it to its right adjoint um yeah and it and you can make this into an honest functor so that's part of what Lurie does um and this functor preserves co- limits which uh colimits I mean colimits in the opposite category so really it's limits um which translates to limits uh in  $\text{cat Infinity}$  so also the co limits in Lurie's magic category are just calculated naively limits of underlying categories but with respect to passing to the righted joints of the functors in your diagram this is quite magical because in like in  $\text{cat infinity}$  or category of categories or something like co- limits can be very difficult to calculate like for example you know Amalgamated products of groups are examples you know of Co limits in categories and it's kind of a uh not so simple construction but this uh this is sort of easy um uh right um so in particular let me let me so in particular uh just to connect with what we had earlier uh this weird looking condition of shriek descent uh that we had for maps of for shable maps of AIDS it's actually the same thing as um well Cod descent so so Cod descent for the lower stre functors in PRL and that's actually a much more convenient way of thinking about it maybe I should put this down here now and then this is the limit but the limit is to be interpreted in the the Co limit in the sense of PRL exactly which is not this which is okay which is a okay um now uh there's more magic in PRL there's also a symmetric monoidal structure so which is maybe the main theorem in in Lurie's book or something uh H it's either I don't remember which book it's in I would claim the whichever book it's in it's the main theorem in that book my opinion I mean it's really yeah so um so it's a tensor product and it's it's it's characterized by Universal Property just like you would expect that of a tensor product so maps in PRL from  $C$  tensor  $D$  to  $E$  uh that is the same thing as functors it's from the product uh which commute with co- limits in each variable separately um and uh so this all so this uh and then this this tensor product on PRL also actually commutes with co- limits uh in each variable separately so you get a a very nicely behaved um tensor product on this category um so in fact uh it's so there's a there's a there's a there's also a corresponding internal Hom is given by internal Hom from  $C$  to  $D$  is just the infinity category of co-limit preserving functors from  $C$  to  $D$  so note so note uh the the mapping space in this category from  $C$  to  $D$  is what you get from this category here uh sort of by uh only remember isomorphisms so but you can get you can somehow recover the full mapping category uh by using this uh tensor structure that's that's one way of recovering it at least um so in principle PRL should be considered as an  $\text{Infinity } 2$  category because there's a whole category of map between two objects but you don't really need to remember the  $\text{Infinity } 2$  categorical structure because it's it's just also just the internal Hom um you mean equivalent isomorphism in this Theory I mean after to almost okay this is  $\text{Infinity } 2$  yeah there was a reference to a paper on  $\text{Infinity } 2$  in some places but so do they Define  $\text{infinity}$  and categories and all of this in general or I haven't kept up with with that I don't know yeah yeah SE some recent paper or someone I know about this  $\text{infinity}$   $\text{infinity}$  or yeah I uh I haven't kept up with the literature on  $\text{infinity}$   $\text{infinity}$  so far I've been fine with just  $\text{Infinity } 1$  so when the when the need arises oh yeah um okay okay I have to or I want to continue a little bit more uh uh so um so there's more so if you have a so now you can this is PRL is now a um a tensor category it's a category with a symmetric monoidal structure and you can ask what is a commutative algebra in this tensor category so meaning just a it's an object in here equipped with some kind of symmetric multiplication I mean the basic data is this but then there's some higher coherences and so on about it being sufficiently commutative um and what it what it means is that is this is the same thing as  $C$  is a symmetric monoidal uh presentable  $\text{Infinity}$  category and the tensor product on  $C$  commutes with Co limits uh in each variable um some people often call this because this is a a mouthful people often call this a presentably symmetric monoidal  $\text{Infinity}$  category so um and uh well basic example for us would be a  $d$  of  $X$  for an apoid um uh and then you can consider so can consider modules over  $C$  in PRL uh these are sort of a cinear uh presentable  $\text{Infinity}$  categories so they're presentable  $\text{Infinity}$  categories which are tensored over a  $c$  um but that also lets you say that they're enriched over  $C$  in some sense so um so what yeah what do you have you have like  $C \otimes M$  goes to  $M$  um uh yeah so that's the basic structure that you have here um but then if you have any two objects like  $M$  and  $N$  in here you can make a you can make sort of a  $c$  internal Hom from  $M$  to  $n$  uh as some adjoint to a fun you can easily write down here so so maps from an object in  $C$  to this should be the same as maps from  $C$  tensor

$M$  to  $n$  in  $m$  so it's a nice it's a convenient way of saying uh you have a presentable Infinity category where the mapping spaces are enriched to objects in  $C$  yeah can you say again if you have a  $c$  to  $M$  to how do you enrich the mapping space Oh you want to so you you characterize this by a universal property and then you prove there's a representing object so the Universal Property is that if you have a giving a map like this is the same thing as giving a map from  $C$  tensor  $M$  to  $n$  um I'm almost done with the 4 a into into PRL um I promise it's going to be helpful so when you T like category of modules over AR like Dr  $R_1$  to  $d$   $R_2$  should be so as far as I know if you T not over  $D$  of  $Z$  you get  $D$  of the the T you T like this you get something big yes okay yeah so let me yeah that's uh yeah you what you get something bigger um well unless you're in characteristic zero um but uh yeah so maybe let me give an example of uh no no sorry so that now yeah just a I I want to continue the story just a little bit so mod  $cprl$  also has all limits and colimits and mod  $C$  PRL to PRL preserves them the forgetful functor so completely uh completely naive uh very simple Theory um uh it also has has a tensor product which is a tensor product over  $c$  um so so this is kind of a standard thing with some geometric realization of of uh yeah some some some standard relative tensor product um uh and uh let me give an example of a calculation so so if um so if you have  $c$   $c$  PRL and then you have  $R$  uh and  $S$  which are commut algebra objects in  $C$  um then you can make mod  $r$   $c$  and oh let's just say algebra objects yeah and so you can consider left modules over  $r$  or  $S$  and these will be in uh mod  $cprl$  um so just  $R$  modules in  $C$   $S$  modules in  $C$  and then mod  $r$   $c$  tensor over  $C$  Mod  $s$   $c$  uh is just mod  $R$  tensor  $s$   $c$  so so in the definition of  $c$  linear  $s$  of course you say  $C$  tens to him but of course this comes with yeah yeah there's more coherencies and there is a good way to formulate this system of coherencies in this so we have to read some higher algebra you have to read higher algebra but you know yeah you you yeah it's it's not easy to read higher Algebra I know but it's possible people have done it and someone even wrote it which is even more amazing yeah okay uh so tser products okay so now I want to connect this to analytic rings so note uh so there's a functor or well maybe so yeah so now now I can say the the good way to encode or a good way at least to encode a six funtor formalism is a is a LAX symmetric monoidal functor uh from span  $C$   $s$  to PRL with this tensor product here so we're no longer using the cartesian product on cat Infinity but the this tensor product on PRL but it really just amounts to a condition on uh on the the kind of formalism we had in the other sense it's just the condition that when you look at the this  $D$  of  $x$  cross  $D$  of  $Y$  going to  $D$  of  $x$   $x$  cross  $y$  that that should commute with Co limits in both  $D$  and  $x$  and  $d$  and  $y$  a  $d$  of  $Y$  separately so um it's just a condition on this formalism here but it's best to think of it as being a formalism with values in PRL okay um um now I want to connect with our specific example so note um so analytic Rings uh maps to PRL uh by sending um our triangle  $D$  of  $R$  to uh  $D$  of  $R$  well in fact as I already said it maps to  $C$  alge PRL um but uh in fact uh it it maps to  $cge$  PRL uh over a certain other commutative algebra object namely a derived category of condensed to bilon groups so everything by by its nature lives over this derived category of condensed to bilon groups so we have a presentably symmetric monoidal category with a functor from this uh presentably symmetric monoidal category um and then I claim uh this functor commutes with colimits uh and and and detects isomorphisms under over under thank you under yes yeah we're in the world of algebra so I should say under yeah thank all so we have this we have this category here and then we have an object in this category and this notation means that you consider an object in this category together with a map from this object to to that object so it's this some slice category or under under or over category or something condense every condens every complex of condensed abil Co gives you by pull back some guy in of okay yeah I mean it's just yeah I mean the so the the initial object here I want to say it always SS to see over over IM of the addition ah okay that's that's a analytic ring to uh what's the initial object  $Z$  okay so for  $Z$  you get  $D$  of condens so you don't need them ah okay CLA this  $F$  just yeah just a sec just a sec so uh in particular um so if you want to so the derived category of a pushout so uh let's say a so this is push out in in analytic Rings which was this kind of um slightly subtle operation in this perspective on analytic Rings because you had to complete some pre-analytic ring structure and so on but actually on the level of the categories it's quite naive so it is just this lury tensor product um this is the relative tensor product uh in PRL um so the proof is not um so difficult so let me indicate what's going on um in the proof uh just in this special case which is really all we need um so in the case where  $A$  and  $B$  are both proper over  $R$  I'm not assuming any shriek ability but still let me use this language of proper Maps so in the case When  $A$  and  $B$  are proper over  $R$  um it's just an instance of this general fact here so did you did we say that commutative and now got confus all this so somehow you get you said the modules as limits and Co limits and then did you say that symmetric oh I didn't say that  $c$  alge has co- limits I should have

maybe discussed it but it does so and um as usual pushouts are calculated by relative tensor products  
 pushouts in  $\text{cge PRL}$  are calculated by relative tensor products in  $\text{PRL}$  okay and and it has limits also  
 it has limits also those are yeah those are naive those are calculated in  $\text{PRL}$  it has  $\text{Co}$  limits I would it has  
 arbitrary  $\text{Co}$  limits yeah and filtered  $\text{Co}$  limits um or more generally sifted  $\text{co-}$  limits are calculated again  
 in  $\text{PRL}$  so yeah um right so in the case case where  $A$  and  $B$  are proper over  $R$  um it's an instance of this  
 general fact in the general case um well it's not shable but you can still Factor by basically by definition any  
 map of analytic rings as a proper map followed by a localization um and then for localizations it's quite  
 easy to um just check Universal properties uh to compare the two sides of this so granting the case of proper  
 Maps um then you just have two different quotients of your category that you're trying to identify and you  
 can just identify them by uh looking at the descriptions um so for proper you say that it is because in that  
 case the uh in this in that case for proper map this  $D$  of is is the the algebra yeah are algebra over the bigger  
 ones and you said that algebra are yeah no for any analytic GRE it's not an category the  $D$  of the source  
 is not algebra over because in the calization you have the  $K$  algebra something but not yeah okay okay and  
 that was only for shable maps also um so right so let me make a corollary H uh so so no no no here no shable  
 it's not necessary I got but you said something about  $D$  of a  $10 B \text{ Ral } D$  this is for general not necessarily  
 right right but you can still use this reduction to the proper case so you you still Factor as a every map of  
 analytic ring still factors is a proper map followed by a localization but this localization won't have a left ad  
 joint I mean it won't but that doesn't matter so much for this argument okay so you don't need the  $i$  s okay  
 so so for localization in general you can somehow check it from gring all your stuff and then you yeah okay  
 um oh yeah I wanted to say uh this functor is not fully faithful for a technical reason so to be fully faithful  
 you'd have to be able to recover our triangle just from uh the data of the of of  $D$  of  $r$  with its tensoring over  
 condensed ailan groups so you can certainly recover the underlying condensed ailan group of our triangle  
 because you can take you have the unit object in here and then you're tensored over this so you can consider  
 endomorphisms kind of or the internal uh the internal the internal endomorphisms of this and this will give  
 you our triangle so over this linearity as a condens as a an object in the derived category of condensed ailan  
 groups in fact you can do more because this is the unit you actually get this as a uh in fact you get it as a  
 commutative algebra in this category um condens stab um but uh so so you recover the underlying infinity  
 ring but if you're working in this animated ring context that's not enough to recover the the animated ring  
 so there's there's not quite enough structure here in the animated ring context to to recover the  $r$  triangle  
 um there're certainly enough to detect isomorphisms because uh yeah you get the underline object um if  
 you wanted to recover the animated ring you'd have to do something like remember derived functors of  
 symmetric Powers incode them somehow uh say what kind of extra structure you have on this coming from  
 uh non ailan derived functors of symmetric Powers which is the kind of extra structure you get on modules  
 over an animated ring compared to modules over an in infinity ring which only has the symmetric monoidal  
 structure okay um yeah uh any anyway there's a corlar so let's let's say you take a I don't know  $Y$  in  $f$  um  
 and then consider  $F$  shriek over  $y$  this is the uh category of  $X$  mapping to  $Y$  via a shable map um then the  
 six funter formalism so we can we again look at the span category of this but now we don't need to restrict  
 to shable Maps anymore because every map in this category is shable by this kind of two of three property  
 of shable maps so I can just put all all um um which is a prior lack symmetric monoidal um it's actually  
 symmetric monoidal uh wait should I uh no I want to put a sorry I want to uh ah sorry I should do sorry I  
 want to yeah yeah I have to remember the the coefficient category linear structure over  $\text{PRL}$  yeah did um  
 because of what's going to come next okay but I mean the fact that actually yeah that's true that's true  
 yeah but uh yeah I wanted to yeah okay um uh right so man I had no idea it was going to take this long  
 to get through this formal stuff uh so so a corollary is that uh for  $X$  to  $Y$  shable uh then this object  $D$  of  
 $X$  in modules over  $D$  of  $Y$  in  $\text{PRL}$  uh is dualizable uh with respect to the tensor product the relative tensor  
 product over  $D$  of  $Y$  uh and in fact it's canonically selfdual and um with respect to this self-duality uh the  
 Dual of uh so if so so if you have  $X$  going to  $X$  Prime over  $y$  uh then the Dual of uh let say pull back um so  
 it should be a map the Dual of this map should be a map from the Dual of  $x$  to the Dual of the Dual of  $DX$   
 to the Dual of  $DX$  Prime but they self duel so you just get a map in the other direction from  $D$  of  $x$  to  $D$   
 of  $X$  Prime uh this map is uh none other than the lower streek functor so the proof proof is you can check  
 universally uh in any span category kind of span  $C$  all all um so we have a symmetric monoidal functor so  
 if you want to so the image of a dualizable object will be dualizable and if you exhibit a duality pairing in  
 $\text{co-}$  pairing here you get one there and so on um so so for examp so how so why is every object dualizable in

this Span selfdual in this span category well you have a unit which is uh um right so uh and then you have a co-unit um which is the same thing going in the reverse and then you just check the triangle identities it's completely completely straightforward and then you can see that the with respect to this this self-duality of every object that passing to the dual is just the same thing as transposing the span so to define the  $C$  you need the pairing from  $BX$  and  $BX$  either up what's  $DX$  TS of  $DX$  to the unit which is  $dy$  so what uh so yeah so if you unwind this then what you get is so you have  $DX$  tensor  $DX$  implicitly point is  $y$  here uh so the  $C$  in this example will be uh this  $f$  over  $y$  so then you get you have  $DX$  tensor over  $Dy$   $DX$  you pull back to  $DX$  and then you do lower shriek to  $Dy$  to  $um$  so then we have a proposition so that if  $F$  from  $X$  to  $Y$  is a shable map uh then the following are equivalent so Peter uh uh said this last time but he didn't give the proof and now we're going to give the proof so one is that  $um$   $f$  satisfies streak descent  $um$  two is that for all uh for all categories linear or over  $D$  of  $Y$   $um$  uh so you have descent for  $m$  so  $um$  so so if you take  $m$  equals  $D$  of  $Y$  uh you get star descent or in other words what what's what's that what's that Peter equival oh thank you yeah yeah yeah  $um$  so that that so that so by definition shriek descent mean that you have descent with respect to this weird pullback these real weird shriek pullback funs on on these  $D$ 's but then the conclusion here is that you get it also automatically for the star pullback instead  $um$  uh and then the third condition is kind of a a categorified version of star descent so star  $s$  it's called two descent uh studied by Gates gory uh uh and Rosen bloom is kind of a remarkable thing so that  $um$  so if you take this whole category uh then that's the same thing then that that assignment assigning to why this category also satisfies descent so in the sense of infinity one but if you have I'll explain more or less why it's the same requiring this for Infinity one and for Infinity 2 but I'm not going to try to say I know what an Infinity 2 means so but you well in in the proof that three implies two I mean well so do you have also a Shri version of two yes you have a shriek version of two yeah but that's uh that's completely formal from one so let's get into the proof and then it'll become more clear or maybe let me State a corollary first  $um$  or yeah so a corollary is that uh a pullback uh  $F$  satisfies streak descent implies any pullback also does  $um$  you can see that from Two and using this symmetric monoidal uh that we discussed but also I I think in the course of the proof a more simp a simpler explanation will arise for why streak descent is uh closed under pullbacks so that's important for using this to define a Gro deque topology in the first place  $um$  okay what why the  $P$  because the  $D$  of the yeah because the pullback in analytic Rings is already calculated by a relative tensor product so if you took if you took for example  $M$  to be  $D$  of  $Y$  Prime then this would give be give you star descent for the pullback but you could take  $M$  more generally to be any  $D$  of  $Y$  Prime module and yeah and then you get so You' see that condition two is stable under under pullback under base change  $um$  so so  $um$  so what do we have we have star streak descent but I explained that the best way to think of that is that  $um$  is that you have this co-limit in PRL uh in PRL but I also explained that co- limits in module categories are the same uh as on the underlying so that's the same thing as modules over  $D$  of  $Y$  in PRL  $um$  uh now take internal hom out uh to  $M$  for for any  $m$  in  $mod$   $D$  of  $Y$  PRL then this uh so  $um$  well on the right hand side we're taking internal hum from the unit to  $M$  so we just get  $M$  and it's being identified isomorphically with some limit so the the co- limit pulls out of the internal  $H$  uh limit of uh uh internal  $H$  uh from  $D$  of uh whoops but uh sorry what did I what did I just write  $D$  but I then I explained over here that this thing is selfdual it's dualizable and in fact selfdual uh so then you can actually move it over to here with a tensor product so so this is  $H$  internal  $H$  in  $D$  of  $Y$  uh  $M$  and I said that the manner in which it's self duel is is such that it converts shriek functors into lower shriek functors into upper star functors so then this is exactly a two I guess in fact this is a proof that one is equivalent to two course you have to verify that the the con that you get is really the expected one in some higher yeah yeah you do it universally in the span category kind of the this identification the self Duality turns this system to the other yeah to the expected one yeah but there is some delicate Point absolutely okay it's not because of the higher nature of the SC not so sure what you you have to check so yeah yeah you have to produce the data coherently ident you know you have to produce the data making this coherent identification of the Dual of this of this self-duality and and the Dual of this identifying with this you produce it universally in the span category and then you map it into modules over  $D$  of  $y$  and  $um$  I haven't actually done that but there's also another argument for this so that doesn't require this stuff I just wanted  $um$  I mean it can be done I'm 100actually done it but there's also another independent argument for this which doesn't require such things I just thought I would present this because it feels the clearest to me even if it's slightly difficult to make technically work  $um$  okay uh right  $um$  okay okay and then uh so 2 implies three  $um$  so yeah we have to check this statement here  $um$  so there is a so the well this the map has



a right joint uh which is basically you take a system of modules here and you forget them all to  $D$  of  $Y$  um and then you take the limit in this category here so it's like taking a system of modules  $MN$  and then you take go to the limit of the  $MN$  um and you want to check that the unit and the co-unit uh of this adjunction are isomorphisms so the the unit being an isomorphism uh this is exactly two um for the co-unit being an isomorphism um so it's true after uh if you base change up to  $x$  uh then the co-unit is an isomorphism um uh because so one so  $d X$  is dualizable over  $D$  of  $Y$  implies that can you can pull the limit out um and then two the cover is split uh on pullback uh to  $D$  of  $Y$  or  $D$  of  $X$  and then three uh there's an obvious base change property um so yeah um  $B$  change probity you mean yeah I mean that this the write a joint say from  $\text{Mod}$  so you have a  $\text{mod } D$  of  $Y$  PRL you have a pullback functor to  $\text{mod } D$  of  $X$  PRL and then you have the right adjoint which is the forgetful functor and that forgetful functor commutes with base change or where base change means push out in  $\text{cge PRL}$  so it's just like purely algebraic base change like no I don't uh yeah so first of all you say the co unit means the map from from something to the  $\text{IND}$  limit of the  $PB$  yes yeah yeah right so you have um yeah so you have a collection of things here you take its limit and you base change it back I guess I wasn't um yeah I yeah I I think you're right I didn't set this up quite correctly um so then you have a map of systems here and you want to check it's an isomorphism for all  $n$  um and then maybe I was um yeah there is a confusion of unit and think your unit and unch are they really oh you're wait no are they no no no no a unit is an isomorphism sorry no no no no no no no no the unit is the right one so the map from a guy to okay the unit is the right one the co me you have a system and you want to compare it to the full of limit which is okay and then you say so I was I was giving the argument when  $n$  equals 1 and I should have yeah sorry I think okay so let's go again the cor being in is is yeah okay so we we we don't say after going to  $dx$  because this is confusion with the co and we are we are completely we are completely okay then let's see because one because it's dualizable any limit commutes with no but this is initially should from  $t t x$  oh I should yeah I should Base change the whole system from  $D$  of  $Y$  to  $D$  of  $X$  yeah yeah that's that's that's good yeah yeah yeah no no yeah yeah so than thanks thanks thanks yeah yeah true After Base change to  $D$  of  $x$  yeah yeah in other words you take this whole collection here and you base change to  $D$  of  $X$  right um yeah so uh um change you tensoring with  $x$  in your yes tensor over  $D$  of  $Y$  with  $d$  of  $X$  and then the the limit can be commutes commutes so you can do it in the system after tensoring and after tensoring you get the system uh of the are categories for the fibro product space a fine so to speak spaces because we said the  $d$  is some  $D$  of the  $F$  product is  $D$  of the  $T$  of product of the  $D$  for Fine Things yes or not okay and then the covering is split and so by as as in usual the same the this means that you can play with some amot Toopy to okay then you you but and so what is point three the  $B$  change property oh that's so that you can identify the when you tensor the thing up to  $D$  of  $x$  that you can identify that system with the corresponding system for the pullback of the cover to  $D$  of  $x$  so it's it's some it's it's it's nothing I mean it's it's it's purely formal um okay so here there is no okay but kind of it's hard to how to be sensual about those things well actually yeah I mean I think actually this argument I believe is in Kil Matthew's paper yeah and maybe it's also in Gates Gore Rosen bloom and so on I mean which  $M$  you galwa galwa gwa the Tre refer to as some in the last it's the original reference for descend ability it's like yeah the reference I wrote down last time that yeah um there was something in back or they yeah then they they they characterized the descendible maps so in some type context um yeah um okay I man okay it's okay it's okay we'll continue again after Christmas um so uh right okay so then the so then this collection and then the image of this and then this those two things are the same after isomorphic after you tensor to  $DX$  but then two also implies that uh tensoring uh  $DX$  detects isomorphisms so done dets two also ah all right right all right okay uh yes yes okay yeah it's a bit of magic but yeah um and you get also now a strong form of three with the two Infinity right so let me quickly explain the argument for three implies two which is kind of more or less explaining why you get a strong form of three so if you take mapping spaces so yeah if you have let's say you have a what's that Peter said right you cies oh that's true yeah that yeah so I don't need to yeah you're right yeah unit being yeah you're right oh yeah Peter I was going to give a different argument Peter is right that this argument shows that um two is exactly the claim that the unit is an isomorphism so uh so clearly three implies two um but I was going to give a different argument which maybe also explains why you get an Infinity 2 categorical uh three two because so three means that this functor is an equivalence but this functor has a right adjoint so being an equivalence is the same as the unit and the co-unit being isomorphisms but when you unwind what it means for the unit to be an isomorphism you get exactly two so clearly three is stronger than two three as a right the right joint being ah okay the the uh being what the in ah okay so and then the being

the unit ah okay so this actually shows that you have two yeah going giving another proof yeah okay you said you have another yeah there's another way of seeing it which is I mean one way of okay so if you have just three in the infinity one categorical sense what do you get you get that if you have  $M$  and  $N$  in  $\text{mod } D \text{ of } Y \text{ PRL}$  uh then you get the the mapping space in  $\text{mod } y \text{ d of } Y \text{ uh PRL}$  from  $M$  to  $n$  uh is the inverse limit of the corresponding mapping spaces after you do the base changes and this is um this is the set of objects in some internal  $H$  so uh uh so it's a priori weaker than the claim that for the internal  $HS$  you get a corresponding limit statement which would be kind of the Infinity 2 uh version of this um but if you just you can just tensor this with like prees on  $\Delta_n$  for any  $n$  and then you'll get the um you know fund from  $\Delta_n$  to this and then if you let  $N$  vary um then you recover the whole well complete seal space Associated to this category just in terms of mapping spaces and yeah so I'm saying what you get a priori is the under the space of objects in this Infinity category that you're interested in but if you then vary The Source by tensoring with prees on  $\Delta_n$  and that tensoring kind of commutes with everything uh then you get not just the set of objects but the set of morphisms the set of composable pairs of morphisms the set of etc etc etc elaborate saying again that theu these things deter yes yes yeah yeah I was just giving it a little more explicitly yeah and so  $D \Delta_n$  is is bres on on what is it sense uh oh I mean I've  $\Delta_n$  is like a finite category and I have to put it in this big world of PRL so I just take pre- sheeps of whatever spaces I mean if or if I want this to be a tensor over  $D \text{ of } Y$  then I should do pre-s with values in  $D \text{ of } Y$  and  $\Delta_n$  is the simpli finite yeah category or opposite yeah good question actually because I want maybe I want to but they're they're isomorphic so maybe [Laughter] I yeah um okay so this was uh roughly half of what I wanted to do today so let's go for another two hours shall we no I'm kidding yeah um so yeah we're taking a little break I I think the next lecture we scheduled for the 10th of January which is a Wednesday so I'll be I guess picking up and actually finishing what I intended to do today giving so this was some consequence of shriek descent um H Maybe just mention one more Coral are of this which is that uh topology of of shriek descent is subconical so so that aphids will embed fully Faithfully into the sheath category so um yeah so those are some consequences of shriek descent you get some very strong star descent results and next time I'll talk about uh examples how you how you produce examples of Street covers so logically so just to get this so you you you said that in three you have  $St$  the world Infinity one then you tasing with  $d$  of  $\Delta$  and you get it in the sense of of infinity so at least the mapping spaces the mapping  $c$  not not the mapping spaces the mapping categories are the limits of the corresponding ones group are the mapping spaces yes this is the mapping categories are equivalent yes in in three and from this you deduce two how do you deduce from  $c$  2 oh because you could just take the mapping Infinity category from the unit object  $D \text{ of } Y$  to  $M$  and then ah okay so once you know the mapping spaces you get two another way so to speak and and Peter said that the if you have equivalence in Infinity one and Infinity 2 follows by it said something different than your argument or uh I don't know but yeah one thing to note is that you know there's there's two parts yeah if this you want this to be an equivalence of Infinity 2 categories you need it to be fully faithful and essentially subjective but now fully faithful means isomorphism on mapping category is that we just explained but in essential surjectivity um actually follows from just the infinity one statement because here you just have a bunch of yeah yeah so then fully faceful in the infinity one sense is weaker than fully faceful exactly a priori PRI but in context like this it's actually equivalent uh you mean when you have a  $T$  when you have some kind of a yeah when you can perform this trick well okay thank you everybody see you uh see you after the break

## 18. !-TOPOLOGY (CLAUSEN)

[https://www.youtube.com/watch?v=vRUmXU8ijIk&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=vRUmXU8ijIk&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

um let's begin so be talking more about this shriek topology um but it's been a little while since we've last met and so maybe I'll start with a recap um so so recall well let me start at the very beginning so so we have this category analytic Rings um and objects in here are pairs um there's different choices about exactly what sort of data you want to put in the second piece um um I'll take the sort of full derived category instead of some connective derived category or some aplan category um where this is a condensed um and in general we want to say animated ring so we're allowed to have some derived phenomena and this is a certain full subcategory of uh what you could call  $D$  of  $R$  triangle which is um uh are triangle modules in derived condensed a bilan groups um satisfying certain nice closure axioms for for this uh inside there which I won't uh recall right now um and then we Define sort of the category of apine spaces to just be the opposite category um and and we singled out two um well three I guess classes of morphisms of of apine spaces so um  $f$  from  $X$  to  $Y$  in  $F$  is so proper if um so the pullback map uh from  $D_Y$  to  $D_X$  so let me Define  $d$  on the level of  $f$  by composing with this equivalence and taking the  $D$  on the ring there um if this has a good  $R$  joint um  $f_{\star}$  where good means that right a joint um commutes with pullbacks so if you do the right ad joint here and you pull back to some  $D$  of  $X$  Prime that's the same as pulling back to  $D$  of  $Y$  Prime and then doing the right ad joint there um but it also means that it commutes with co- limits and it also means that it commutes with sort of the  $D_X$  action by tensor product on on each of these guys here I a  $D_Y$  sorry ah I think I I I messed it up because this is sorry you have sorry we have  $y$  Prime well yeah right good means commutes withes with pullback yeah commutes with co- limits and satisfies projection formula yeah well it's not exactly the case the for  $f$  in the other direction it is well okay so there's a map in One Direction and you ask it to be an isomorphism I mean I don't know so you you  $T$  with something in  $D$  of the  $D$  of  $Y$  yeah the base yeah uh which is then uh pull backed into  $D$  of  $X$  by the by the fun and then the  $F$  uh ah okay the right okay so this is projection ex it is like the projection for Whata okay yeah um and the pullback the kind of pullback I mean is like  $XY$  and then  $y$  Prime and then  $X$  Prime just to um okay and that was the basically the same thing as saying the the analytic ring structure is not really changing when you go from  $y$  to  $X$  it's just the the underlying ring that's changing so you inher inherit the analytic ring structure from  $Y$  when you go from  $y$  to  $X$  so that's the same thing is uh the induced analytic ring structure um it it's called an open immersion induced well on on on  $X$  uh right so you if so this is implicitly given by a map of analytic rings from like uh I don't know uh let's say  $s$  to  $R$  so you have you're given an analytic ring structure structure here and you want the analytic ring structure on here to be just inherited from this one in the sense that the  $D$  of  $R$  is just the same thing as  $R$  triangle modules in  $D$  of  $s$  so  $D$  of  $R$  is also the subset of  $D$  of  $R$  triangle well it's it's it's objects after the morphism are those which when you restrict to  $S$  are yes okay exactly yeah so it really so it looks basically like an apine map in usual algebraic geometry kind of but we call it proper because of this this uh this situation here um so it's an open immersion if if  $F$  upper star uh  $D$  of  $Y$  to  $D$  of  $x$  is a localization uh and has a good left ad joint um where good is taken in the exact same sense as before um the maps maybe Go in different directions because it's a left adjoint instead of a right adjoint but you can still always ask them to be isomorphisms um and this was equivalent to uh so uh  $L_{\circ}$  localize away from uh uh some item potent algebra uh in  $D$  of  $Y$  so such a local such a open immersion for such an open immersion there's always going to be some item potent algebra in  $D$  of  $Y$  which is somehow functions on the closed complement uh which completely determines this this situation here it's the the the kernel of this localization will just be the modules over that item potent algebra so let's say if this is a so then so the sense of this is that kernel of star is just mod a  $d$  of  $Y$  um okay and and then it was called uh shable if it factors as uh  $X \times Y$  um where this is proper and this is an open immersion um and then um there are good closure properties for all three classes of maps so closed under composition closed under base change um contains all isomorphisms um and uh kind of half of the two out of three or 2third of the two out of three properties so I mean so if you have a map like this and a map like this and they're both shakable then any map between them making the triangle commute is also shable um uh and then um so we and then we get a six functor formalism on  $F$  where the class of maps for which the the next the the important functor the lower shriek functor for which this is defined is exactly uh the class of shable Max um and then uh  $F$  lower shriek is  $f$  lower star for  $f$  proper and  $F$  lower shriek equals left adjoint to  $F$  lower star

if  $f$  is an open immersion or another way of saying this is that  $F$  upper shriek equals  $F$  upper star so that's just um few past to wred joints I think yeah should be um a map from to star in general or um yes there is so this can be gotten uh because the um oh wait just a sec No Maybe not maybe not exactly in this full generality uh oh wait uh no no there is there is yeah yeah I think it even just follows from uh so you have such a map for an open immersion uh because the it's for for any localization you have that kind of thing where like the left adjoint maps by a natural transformation to the right adjoint and then you can just compose um and you have the Cho independent of the factorization yeah there's a better way to the higher no no no yeah there's a better way to produce it which involves using the diagonal so uh the when you if you if you pass to the diagonal it's enough to show that you have such a thing for the diagonal um well I think well we'll get into this a little bit more later I I was I missed that with when I was aw because to the the closure properties are are composition and what else and oh a composition and base change they contain all each of them contains all isomorphisms and then there's this property that if you have two maps with the same Target which are in one of those classes then any map between them is also uh any map between two such guys is is also in the yeah is also in the same class or I could say that the classes are closed under passage to diagonals um uh ah this is equivalent yeah yeah yeah once you have closure under base change I think yeah I think this is some basic thing okay passes to dional ah but this is intuitively ah intuitively for an open imion usual geometry the Diagon is isor also here yes also yes also here yeah yeah so that's part of why uh yeah so that that goes into the other way of producing the natural transformation but never mind um yeah okay so uh right and then we had this definition we had this result um so we had this theorem or maybe I should say definition um I'll say uh a map  $FX$  to  $Y$  let's say a shable map in  $F$  is a shriek cover um if uh um so uh so if uh the or so if if the map from  $D$  of  $Y$  to the sorry to the limit of uh oops um to the limit of the check nerve is an equivalence where for the transition functors we use the upper shriek functor uh um which is a bit of a a strange thing to do at first glance uh but then we had this result that so for uh shable  $F$  uh the following are equivalent so uh one is that  $f$  is a shriek cover so in that sense two is that um we have kind of shet codent so if you take the co limit over  $n$  and  $\Delta$  op of  $D$  of  $x$  s  $Y_N$  um so then so then here you use the lower streek functors and take the co limit in this category  $PRL$  um or or in uh modules over  $D$  of  $Y$   $PRL$  um then you get a an equivalence so you had or in so  $PRL$  is is what you said that  $L$  introdu is is uh is a uh uh presentable Infinity category with Co liit preserving functors yes okay but not a stable any not no but I could also say oh darn but the co limits yeah the co limits are going to be the same so it doesn't matter okay and and then mode d ofy means  $PRL$  means it means yeah so  $D$  of  $Y$  is a is a symet monoidal category and it's in fact a presentably symmetric monoidal category so the tensor product commutes with colimits in each variable which makes it a commutative algebra object in  $PRL$  with respect to  $L$ 's tensor product and then you consider modules so it's a it's a a presentable Infinity category which is tensored over  $D$  of  $Y$  in a kind of cimit preserving way um so um and then the third condition was so shriek cover meant you have this shriek descent descent where you use this funny Twisted pullback um but uh it turns out that it implies that you get descent with the star pullback it well and in fact it implies you get the same result with coefficients so for all  $m$  in mod  $D$  of  $Y$   $PRL$  um so here um so if you take  $M$  to be  $D$  of  $Y$  itself then you see that this condition is just kind of usual descent that is descent with respect to pullbacks Star pullbacks but in fact you can even tensor that with any uh any module and you still have descent um and then the fourth condition was this kind of two descent this categorified version uh which says that the whole category of possible  $M$ 's uh satisfies descent so and here there's only the only thing that makes sense is star descent but I'll emphasize it it's just some naive pullback um so where the base change functors are just kind of relative tensor products in  $PRL$  so one can ask I don't know if it is a good question but instead of just like in algebraic geometry usually they Define like fpqc topology using quasi compact but you can also ask suppose you have an arbitrary collection of maps not necessarily finite you can ask whether you have descent for this can it happen with in the infinite which is not reduced to a I we think uh maybe I don't I don't know if Peter checked the details we think that is true that so you could try to make an analogous we haven't talked about the gro de topology yet so but we're making it finitary by definition but you could ask whether if you do the infinitary analog of it whether it reduces it gives you the same Gro deque topology as if as whether that Gro de topology a prior infinitary is actually finitary um and we think yeah so Peter you checked it you checked it carefully yeah yeah okay so then the answer is yes the if you you could try to make an infinitary version of the groi topology I'm about to describe and it will end up being finitary anyway so um equivalent this yes yes so every cover will have a finite subcover um the basic so I remember that in

the in the case of just what we started finite sets if you have the counter set and you take all mths from  $P$  to the counter set which is a big then this is universally submersive still one can show it's universally submersive what does that mean to there is this like in the  $H$  topology in the usual voty thing with the  $H$  anyway anyway it means when you cross with any topological space to check if something is open you can check the pull back oh right right right right right yeah so I don't know then of course you can ask whether you have the right this is not so clear I mean but at least for opens it one can check it I mean even this Universal  $s$  so this this is a kind of candidate for infinitary thing but I don't know if it satisfies the chological thing no I think uh Peter you were uh you you asked these logicians about this right uh yes yeah and they they had some conclusion so so it can happen that you have some profin set and some infitary cover by other profin sets which satisfies the  $S$  at least some example descent in of the derived the street descent because kind of full condition equivalently along the lower streets using this perspective you can show that it is fin but it's really something that only works when you ask for Street not for  $St$  so for stardent it is it is possible like the example that I gave of all all  $Ms$  from  $P$  to the  $C$  said Is it a ER well maybe it's a side question I don't know I think the specific one they looked at was like you take a really big profinite set and then you take a union of like small ones a big write it as a big Union of of small ones like I think that's the specific one they looked at I don't know if they looked at this one with  $P$  in the caner set but uh yeah um so uh okay I think uh we'll move on so um ah right so yes so those are the equivalent conditions for being a shriek cover um in particular you have some very nice descent properties for shriek covers um but now I want the next thing I want to talk about is how can you check these conditions um um so here's a little proposition so well the first part I'll say so let's say let's start with  $f$  from  $X$  to  $Y$  shable map the first claim is that if  $f$  is a shriek cover then um a certain condition has to hold so um so if you look at the image of the lower shriek functor from  $D$  of  $Y$  uh  $D$  of  $x$  to  $D$  of  $Y$  so that's a certain um full subcategory of here um since this functor is not in general fully faithful that that image doesn't have to have any closure properties whatsoever it doesn't have to be a stable subcategory or anything like that but then you can take the um so thi thick subcategory generated by it this means a um close under under just finitary operations like cones and retracts shifts and retracts ah so and uh I I didn't okay yeah and retracts I count that as finitary um um ah then then you get an A prior bigger category you can ask whether it's the whole thing or well you can ask it's actually equivalent you can ask whether it just contains the unit so let me I'm writing one  $y$  for maybe you would normally think of this as like structure sheif of  $Y$  I mean the the unit object in this symmetric monoidal category uh with respect to its tensor product um so the condition again is that this unit object can be written in a finitary manner in terms of things which are lower shriek from  $d$  of  $x$  and can it be written in an infinitary manner an infinitary man yes because of the because it's a shover it follows or not I um oh you're saying well so so you're saying if I say uh this condition but with infinitary does it automatically imply the same condition but with finitary don't know what is the relation between those yeah so I think um probably I mean this is a one maybe the key point in all this finitary is that this is a compact object in  $D$  of  $Y$  so I think uh I think indeed uh uh should be equivalent um yeah so maybe maybe it's even better to  $s$  or is maybe it's I don't know uh let me yeah I guess I guess it should be equivalent um yeah uh sorry okay oh oh yes yes so if it's a street cover then you satisfy this condition and then this the second claim is that the converse holds provided uh  $f$  is either um and then first will be the uh proper provided well let me say provided either uh one is that  $f$  is proper and uh  $B$  uh  $f$  is I guess what you could call universally locally a cyclic um let me explain what that means um so that means that uh  $F$  upper shriek is good um so again where again good means uh commutes with pullback uh commutes with  $Co$  limits and then well projection formula but really the way to think about the projection formula is it's giving some  $D$  of  $Y$  linearity and it's that condition that you want to ask so which translates into saying that  $F$  upper shriek should always be the same as  $F$  upper streek of the unit uh tensor up our star so  $T$  are in  $D$  of  $X$  so there's always going to be a natural map and you want it to be an isomorphism um yeah okay so all those properties of  $f$  are not true in general and they're in equivalent or they when you say good is the conjunction of all of these and all of them are they're cases where some of them are hold and others don't that's a good question there may be some non-trivial implications uh maybe if you yeah there may be some non-trivial implications here uh I think yeah I yeah that's true that's a that's a trivial implication yeah thanks yeah projection formula and Peter is saying the projection formula clearly implies that it commutes with colimits so I didn't I didn't need to write the second one but okay but yeah okay so I mean what how to think about this Ula condition I mean um this is kind of like the dualizing object here and the main claim is that the dualizing object is compatible with pullback



so if you're thinking of  $f$  from  $X$  to  $Y$  some family then it's saying that the dualizing objects kind of um very nicely in the family so it's kind of some kind of equisingularity property for  $f$  from  $X$  to  $Y$  um okay is there an example when the converse oh yeah so so warning uh Converse fails in general so let me give the example which will explain kind of how you should think about this condition so we could take for example uh so  $y$  to be spec uh so let's say solid  $Z$  bracket  $t$   $z$  um and then take  $X$  to be uh  $X_1$  disjoint Union  $X_2$  um where  $X_1$  is spec so spec of solid  $ZT$  or maybe I'll write it as  $U$  Union  $Z$  I want to so I'm going to write it's going to be I take a space and I take an open subset and it's closed complement that's going to be the counter example so um so solid  $ZT$  sitting inside solid  $ztz$  um will be  $U$  and then this is the complimentary guy so this will be solid uh uh  $Z$  Lon series  $T$  inverse spec so spec is just this formal notation for passing to the opposite category um uh then um so  $F$  so so let's called uh so so this is mapping there by  $J$  this is mapping there by  $I$  and that induces a map from the disjoint Union um then  $J$  lower shriek uh of of the unit is this sort of compactly supported thing so it's the fiber of the homotopy fiber of  $ZT$   $T$  going to  $zon$  series  $T$  inverse but uh I lower star well I lower Shok of the unit there uh  $I$  is a  $i$  is proper it's a kind of a closed immersion um so this thing just gives a Zant series  $T$  inverse uh and it's clear that you can build uh  $Z$  of  $T$  from this fiber in this guy by just one co- fiber sequence um and then the um the lower shriek from  $X$  the disjoint Union will just be given by restrict to you take the lower shriek there and restrict to  $Z$  take the um lower star yeah take the lower star there um and then take the direct  $Su$  of those two objects so you'll get uh you'll get this guy direct sum this guy therefore by by closure under retracts you if you allow closure on retracts you get each of them and then you get  $ZT$  individually so uh so so it is not shable sorry you said it is not sh no it's shable sorry no it's not it's not a shriek cover ah it is not a stre cover because the the the for the cover you just get youu and  $Z$  separately somehow you get the derived of youu cross the deriv of  $Z$  not a derived of  $Y$  exactly yeah because uh okay here you can you have some  $X$  or something that is not the same in yes in in  $X$  and then  $y$  you  $D$  some  $Z$  yes some particular yeah or I mean uh I mean the the the Ring of global sections on the disjoint Union is kind of it's the product of the two rings whereas yeah so yeah uh okay so so this condition is something like just set set you know surc ity on underlying sets or maybe it's like a cover in some constructible topology or something that could be how you think about this and then if you want to go from that to some honest descent you need to assume um some properties so this is analogous to like in maybe in topological spaces you know if you want there's descent for open covers obviously in topological spaces but there's also descent for say finite closed covers or there's like proper descent but you can't mix open and closed and still expect to get descent so that's so that's kind of yeah so this is kind of some cover condition some sort of set theoretic cover condition and then if you assume either that you're some generalization of open which is this Ula or some generalization of closed which is this proper then then you get honest descent but you can't mix them so what is the relation of this like when we now go to like hub's analytic spaces where there are certain topologies like the  $V$  topology the  $AR$  topology that had some descent properties there and what is the relation with the your Notions here I I'm not sure that I understand the precise sense of your question but we will discuss uh I will get to uh to discussing Huber stuff in relation to these definitions later in the lecture so let's postpone um okay right okay so the proof um so uh so let's uh let's make explicit what this um shriek descent according to the definition means so uh so we're using these upper shriek functors uh those upper shriek functors are right adjoints they they're they're left adjoint join is the lower shriek functor and there's some um some standard some uh some category Theory which tells you that this comparison functor uh uh this itself will then also have a left a joint uh which is given by sort of informally so you have some collection of objects um um and what you do is you take their uh sort of uh you take a a co- limit in this category  $D$  of  $Y$  uh of the  $G$  lower shrieks of them um okay one has to make this precise I believe Iri has done it um but that's uh that's the gist of it um so in particular for fully faithfulness this amounts to uh the claim that if you take the co limit and and by  $G$  I mean for for every  $n$  in  $\Delta$  you have a you have the map from this fiber product down to all the way down to  $Y$  and I'm calling that map  $G$  I mean it kind of depends on  $N$  but it's just yeah um so the co- limit of these uh  $G$  lower sh  $G$  upper shriek  $M$ 's mapping to  $M$  so this should be an isomorphism in  $D$  of  $Y$  uh for all  $m$  and  $d$  of  $Y$  um so uh if you take  $M$  to be the unit in  $y$  um so this is a geometric realization it's a  $CO$  limit over this  $\Delta$  op but you can filter that you can always write this as a  $CO$  limit over the natural numbers and then a kind of a partial totalization so like maybe I'll say  $d$  and then so  $n$  and  $\Delta$  op or you restrict the cardinalities to be less than or equal to  $D$  um and this is a finite  $Co$  limit that's a nice fact about the Simplex category um but then uh since the unit object is compact uh if you write it as a filtered  $Co$  liit of something huh sorry where are

we in Delta you WR Delta what is the sub the the oh here Delta yeah Delta op and I write less than or equal to  $D$  meaning I'm looking at non-empty finite sets of nonempty totally ordered sets of cardinality less than or equal to  $D$  yeah um since so since uh unit  $Y$  is compact so that's the key property here uh we deduced that unit  $Y$  is a retract of some uh some partial totalization uh where are we uh um but each of these uh lies in the image of of eor shriek from  $D$  of  $x$  to  $D$  of  $Y$  because all of these structure Maps  $G$  from these fiber products iterated fiber products they all Factor through  $D$  of  $Y$  or  $D$  of  $X$  sorry mapping to  $D$  of  $Y$  um and and as mentioned before this is equivalent to a finite Co limit this um this partial to geometric realization what's so mean by a finite Co liit a CO liit over a finite usual category no I mean over a finite say simplicial set so over a finite cimit over a finite simplicial set kind of in  $L$ 's um it it means it can be expressed in terms of iterated pushouts and yeah uh iterated pushouts and um and the empty empty the and the initial object so push outs SC here it is a CO liit over the usual category over one category that is the that's a totally just the finite sub no no no the I forgot which one the Delta is a category of opposite of finite ordered Set uh there simplicial objects or functors from Delta op so whatever whatever that so I mean yeah um uh okay so this is a I mean in this stable setting it's so the difference between this for  $D$  and this for  $D$  minus one is just given by one of these objects up to a shift or maybe some uh like there's I mean the co- fiber of the map so we have a a dindex system here and if you look at the sucess exessive co- fibers then they're just described in terms of these individual objects here so that's I mean in the stable setting some kind of do con thing simplifies all of this I I yeah and so a finite Co limit is defined in higher topos yeah yeah so this is not more General than Co liit over a usual uh well well not no you have to be a bit careful because a finite if you have ai a category a finite category in the usual sense it might not be finite considered as a Infinity category or or say considered as a simplicial set so for example like the category with one object and like a finite group uh worth of automorphisms is a finite category in the usual sense but it's not finite as an Infinity category well for example because there's group homology ology in arbitrarily High degrees so it's not finite it's not finite as a simplicial set BG it's not BG is not finite as a simplicial set um it's you need infinitely many simplies to to describe it so that's but but but this one's okay okay so you don't say exactly what okay yeah um okay uh right so um where are we now oh so that was the the implication um and for the for that was part one so that proves that proves part one um and then for part two um uh so um well so for part two you can note that the the hypothesis uh implies that every  $M$  uh are generated by things in the image of  $f$  streak um because uh we have the projection formula for lower shriek so you can just tensor this statement with  $M$  and you'll get the um desired statement for arbitrary  $M$  um but uh so then we want to deduce deduce at least the fully faithfulness this uh um we can assume by this uh claim here we can assume uh that actually  $m$  is of the form  $F$  shriek  $n$  and then and then if we have a base change result for uh commuting uh  $F$  shriek and say like or let me say some some like  $G$  Prime lower streak and and  $G$  upper shriek then we uh can uh reduce to uh a split situation what  $g$   $g$  so yeah I'm sorry it's it's not good notation but so  $G$  depends on  $N$  so  $G$  so  $G$  is just my arbitrary notation for some one of these uh maps from projection maps from  $x$   $t$  over  $y$   $n$  to to  $Y$  so what  $G$  is always going to not denote like some structure map back to  $Y$  that's uh I'm now completely do do you know uh projection formula for is part of no is part of the no I now go don't remember ah I'm sorry yes so it's part of the definition of what it means to have a six functor formalism so it it comes from the fact that we assumed have to be shable okay and then you want want to know that the limit of but I think that it was not this it was you want to represent oh I'm sorry thank you yes yes I got him mixed up thank you very much yeah I'm sorry yeah yeah um and all of these Maps  $G$  are like compositions of pullbacks of  $f$  so if  $F$  has one of these two properties then all of the maps  $G$  will as well um so in the proper case so  $G$  lower shriek equals  $g$  lower star and the base change follows from sort of the upper star lower shriek base change by passing to adjoints um and in the Ula case we have a sort of  $G$  upper shriek is  $G$  upper shriek of one tends to  $G$  upper star and the base change uh follows kind of more directly from again the upper Street lower star base change so there is some you have to make sure the two base change comparison maps that you're one you a PRI two different base change comparison maps and you have to check they're equal but okay um uh yeah in order to conclude from one being an isomorphism that the other one also is but okay all right um so that was that proves fully faithfulness for two um so that gives fully faithfulness uh in the in the shriek descent but the um essential surjectivity or so the the other adjoint or the the the the unit or the co-unit or whichever it is the one going from here to there and then back up here again um that's proved in the same way so uh handled similarly uh using base change to reduce to the situation where the cover is split So you you're pulling back to  $X$  where the cover is split so we have a functor where we want to claim is

an isomorphism we've identified if an adjoint to it and what I've just explained is that if you do the functor and then the adjoint that's the identity and you'd also need to check that if you do the adjoint and then the functor you get the identity and I'm claiming that's handled similarly using base change along  $X$  going to  $Y$  where it happens for formal reasons because the cover is split ah once you so when you are upstairs and go downstairs then you are you want to reduce the  $K$  with split cover so it's even easier even but you still need some base change yes exactly you need the same base change property that the upper shriek commutes with the lower shriek in in the for the relevant class of maps okay so we're part of the way towards so now we've uh this is kind of a more concrete thing that you might hope to be able to check so you have to be in one of these two situations let me explain some special cases um so special cases um so one will be closed covers  $F$  I should say maybe finite closed covers um so we defined a notion of a proper map and we defined the notion of open immersion but we didn't Define the notion of closed immersion um so Al so if um if  $f$  is proper and um well one and one way of saying it is the  $D$  the pullback from  $D_Y$  to  $D_X$  is a localization uh me oh yes it means the it has a the right ad joint which exists the right ad joint is fully faithful nothing nothing more no but a localization in category theory in in which context is defined now for categories of which kind of I forgot I I I know that people like in Gabriel and some ofation don't remember now what the so let me let me say that so this this is a funter which admits a right a joint um so when the wrer joint is fully faithful which is what I'm calling localization um it follows that uh you have a universal property for uh cimit preserving functors out of here namely they're the same thing as cimit preserving functors out of here which kill every object or sorry which invert every map which is sent to an isomorphism by this functor so so this is an analog of localization for triangulated categories for example yeah yeah okay and the ISS are like localization for usual a billion categories by oh but okay but then it's again the similar thing at least for Goen the categories and things which are closed and I'm not sure in which depends if you assume that all Co limits are all or not or no I remember that in some old literature category the there the it looked very hard to read uh uh-huh no I don't know so when you have these big categories the simplest thing to say is definitely just that the righted joint is fully faithful so and that's what I'll that's what I mean by localization um or the yeah you know some unit map is an isomorphism or whatever yeah okay but so since we if we we already know that  $f$  is proper which means that all of  $D$  of  $X$  so this means that  $D$  of  $X$  is just modules over so the push forward of  $1_X$  and  $D$  of  $Y$  that's the same thing as proper if you if you remember the discussion um so a proper map is just controlled by this single algebra object in  $D$  of  $Y$  that so then being a closed immersion is the same thing as saying that this this guy is item potent algebra so tens are product of two copies of itself  $D$  of  $Y$  is itself again via say the multiplication map um so this is since  $f$  is proper um ah so now let me make a warning um uh so in af so on the level of these apoid uh it's not true in general not generally true that uh closed and open immersions well are in viction uh so say with the same Target so that so an open immersion is not necessarily going to have a closed complement and a closed immersion is not necessarily going to have an open complement um it's close to being true that uh an open immersion has a closed complement the only uh so um so let's so so let me expand on this so uh given an open immersion we get uh an item potent algebra so given an open immersion so  $U$  to  $Y$  uh we get an item potent algebra  $a$  in  $D$  of  $Y$  such that um you know  $D$  of  $U$  equals  $D$  of  $Y$  mod mod  $a$  of  $Y$  but uh it's not true in general that uh  $a$  lives in  $D$  greater than or equal to  $z$   $y$  so it's not necessarily connective um and this is the condition so this is the condition uh this is the condition needed uh to get a closed immersion in  $F$  so if you start with just an item potent algebra it doesn't necessarily correspond to a closed immersion in  $af$  because you can't it doesn't necessarily have a correct underlying uh animated ring it's only the connective ones that correspond to animated Rings not the non-c connective ones so um uh yeah so in the case of usual schemes can you recall what are the open I forgot well it depends which funter from schemes to analytic spaces you're using so we'll go into it but I want to say that this is analogous to kind of some complimentary phenomenon in scheme Theory where um you know for an apine scheme uh you have a you have a closed immersion um then the open complement might not be apine right so it might not correspond to open immersion in apine schemes but it's still a scheme and it's kind of similar here even in situations where this is not connective usually you will you will get a a closed complement which is an analytic space it just won't be an apine one um but in the cas of scheme is like you taking the complement like  $a$  is qu comp spec  $a$  then you get Al GMA  $U$  which is important but and and does this correspond to a CL in this setup to like when you take  $D$  of a model this what do you get do you get  $D$  of some some which you call an openion a closed here but I I'd rather let let me again get to the comparisons with the classical theories a bit later in the lecture although

this is going much slower than I anticipated um uh so um yeah so I mean I could give an example uh so if you look at um so  $y$  equals solid uh  $Z$  bracket I don't know two variables  $X, Y$  comma  $Z$  and then uh  $U$  to be solid uh um then I invite you to do the very good exercise of figuring out what this item potent algebra is and then the the corresponding a uh has a nonzero uh  $H$  minus one um the situation on the other side um is somehow even worse so given in the closed immersion um well actually Peter described the condition required for there to be a complimentary open so a closed immersion corresponds to um an item potent algebra uh in the greater than or equal to zero derived category of  $Y$  close Merion say  $Z$  to  $Y$  um uh then uh so then you need for uh to get a complimentary open in af you need that uh that ROM from the fiber of a unit of  $Y$  going to a to this uh which is a functor from  $D$  of  $Y$  to  $D$  of  $Y$  uh uh you need this commutes with uh filtered Co limits and preserves uh  $D$  greater than or equal to  $z, y$  so that's the formula for what would be the localization functor to the complimentary open and you need that that actually defines an analytic ring structure in our sense which amounts to these conditions here internal rhom I should say yeah um okay right so as I said some of this will be fixed by allowing General analytic space is not necessarily apoid but it's probably probably good to keep all this in mind um maybe maybe I should explicitly write that uh these subtleties will be to a large extent uh fixed by allowing uh arbitrary analytic spaces or Stacks you know and not necessarily aine um okay uh oh sorry I was in I actually didn't plan to talk about this I go but then it occurred to me I should um so yeah I was talking about finite closed covers right as an example of descent um so uh suppose given uh finitely many uh so  $Z_1, \dots, Z_n$  uh closed subsets closed immersions well with closed immersions uh into  $X$  and all of these are in  $f$  um then so when do we get a cover so oh when then disjoint Union of  $Z$  mapping to  $X$  so it's a um is a cover if and only if um you just plain have the most naive form of descent for the structure sheath so um so uh so um and this is actually going to terminate at a finite stage because um because it's a finite closed cover um this is the condition that the IE the structure sheath is a sheath so to speak well well yeah just um so this is the this is the item potent algebra picking out  $Z$  and then this will be the tensor product of the two of them so uh that's what I mean by the that defines a closed immersion which is the fiber product in the category of analytic spaces or sorry in fiber product in af um and um that's what this is so may yeah maybe by definition this is the fiber product in af but then it's it's also a closed immersion and it's exactly uh corresponding to the item potent algebra given by tensoring these two things together um so why is this true that this is the Criterion for being a shet cover well um if this is satisfied then certainly uh one is in the image of the lower star functors from the disjoint unions of the  $Z$  because each of these things is so in this case lower shriek is lower star because the maps are proper um so we get this condition but then that map is proper so that implies a shriek descent um uh which is the the condition that it' be a shriek cover on the other hand shriek descent implies this fanc all this fancy star descent right um in particular it implies star descent for like  $d$  of blank so  $D$  of  $X$  is the limit and then if you just plug in uh the unit object in  $D$  of  $x$  uh and see what that um base change and then the righted joint is telling you it's exactly giving this uh sort of structure sheath being a sheath condition so yeah um so so this follows from proposition um and this uh follows from Star descent which is implied by shriek descent um okay um okay what about open covers um well yeah I guess um yeah so let let me just say so it it turns out that every open cover has a finite refinement if you say what it means to have an arbitrary open cover but let me just stick to the case of finite open covers there's in any case no loss of generality um so let's say we have a yeah  $U_1$  up to  $U_n$  again mapping to  $X$  by open immersions and let's say that  $U_1$  corresponds to so the complimentary closed is described by this item potent algebra  $A_1$  and a  $n$  sorry so these are item potent in  $D$  of  $x$  um then uh intuitively speaking um these open subsets should form a cover of  $x$  if uh you know when you take the intersection of all these guys you get the empty set or intersection of all the Clos corresponding closed subsets complements you get the empty set um so the claim is that uh is that um disjoint Union of  $U_i$  mapping to  $X$  is a street cover uh if and only if um  $A_1 \otimes \dots \otimes A_n$  is equal to zero  $D$  of  $X$  is it the case that the filtered Coit of itent algebra is zero implies that one of the yeah that's the that's that's correct and the reason is because the unit object is compact so because an algebra is going to be zero if and only if like  $1$  equals  $z$  in in it and then that's a question of a map maps from the unit object to your algebra but if the unit is compact then  $1$  equals  $z$  will be witnessed at a finite stage that's the reason behind why an arbitrary open cover has a finite refinement yeah yeah um okay um um so uh right um so um yeah and well let me just for Simplicity uh so proof for Simplicity Let's Take  $N$  equals  $2$  you can kind of reduce to that case if you well if you make if you formulate the appropriate categorical analog of this I mean the union of two of the  $U_1$  Union  $U_2$  will not necessarily be an  $A$  so to speak but

if you appropriate the for if you formulate the appropriate just purely categorical version of this statement you can run an induction to reduce to the case  $n$  equals 2 um so when  $n$  equals 2 um then uh then shriek descent well um so we need uh right so so star descent again uh gives that um what do I how do I want to say this um so um so if you look at what star descent means and use the formula for um the upper star functor which is this kind of localization formula then you find that the the claim of star descent is exactly the claim that you have a pullback of this form um uh wait I'm sorry I'm getting myself awfully confused right now uh just uh no I'm getting myself very confused this is this is the star no this is the star descent for the closed compliment I'm sorry I'm sorry um uh just oh so of course if something is is is a module over  $A_1$  10 and it goes to zero on the  $U_1$  so so it goes to zero in each stage of the simpl so there it goes to zero apparently in this limit in the infinity categorical well whatever so so you have some SC in so the condition not satisfied then there is a uh uh yeah um so um um I apologize I kind of I assumed I would be able to do this off the top of my head and I didn't think about it carefully this is um uh so uh I mean maybe I should just say uh so well okay maybe I should just say this so the so the shriek let me say so shriek descent um so the streak descent for this cover say where you have two elements and both of them are mapping by monomorphisms into  $X$  um then The Descent which a priori involves some check nerve which is some infinite thing it actually reduces to some Myer virus um as is kind of standard so it's the same thing as uh so  $D$  of  $U_1$  intersect  $U_2$  uh  $D$  of  $U_1$  uh  $D$  of  $U_2$  or maybe I'll make the maps go the other way um and these are the upper shriek maps all of them okay uh that should be a pullback um and then so and then you can check then you have the map funter from this to the pullback there and again it has this left ad joint so the uh the claim for the unit gives that the unit of  $X$  um receives an isomorphic map from uh you take  $J_1$  lower shriek of the unit on  $U_2$  or  $U_1$  um oh sorry sorry j  $J_1$  upper streak unit um okay um uh sorry yeah so I mean um I mean you have some kind of my vator sequence like this so this is a cofiber sequence um and then you have formulas for everything so I apologize for not explaining this very well but you have formulas for all of the functors involved so um in terms of the corresponding item potent algebras and I promise that if you work it out then uh it's just going to mount to the condition that  $A_1$  t or  $A_2$  is equal to zero uh meaning that this condition will be directly equivalent when you write down what everything means to  $A_1$  tensor  $A_2$  equals  $z$  um so that would be a consequence of shriek descent but on the other hand if you have this then you have this and this proves that the unit is written in terms of lower shrieks from your guy and therefore by the proposition you get a shriek descent as well I should have mentioned uh uh open immersions are so no note H sorry uh open immersions Rula so J upper sh one is one yes is one well for open imion yeah that's right okay so you have and the J of this how is it given in terms of the algebra so the J lower shriek um of one so This this term for example will be the fiber of the unit mapping to  $A_1$  yeah I mean yeah it is Elementary which is um then I made a mistake by therefore not preparing it because I thought it would just my chalk would do it but yeah I I'm sorry for messing this up um okay uh so those were kind of um special cases but we haven't gotten completely concrete yet because was still in some kind of generality of closed covers and open covers and so on um but now let's give some examples um so the first example is zariski zariski covers um so note that there is a functor um from the usual category of commutative Rings uh to the category of analytic Rings um which sends a commutative ring to um to the pair so we take  $R$  now we're viewing  $r$  as a a condensed ring but it's just discrete so it has the discret topology um and then we're actually going to take the full derived  $C$  of  $R$  but now for emphasis since this is just a discret ring I want to make sure we remember that we're taking the category of condensed  $R$  modules so quite quite large um no solid no we don't need yeah we don't need to localize to solid  $z$  um so yeah we'll just take all condensed  $R$  modules um okay so this induces uh fun from apine schemes uh to what I was calling AF which is maybe apine analytic spaces um which is just this the same functor but viewed on the level of opposite categories um and well maybe an a remark is that this this functor commutes with fiber commutes with fiber products uh commutes with finite limits in fact so it's also sends the terminal object to the terminal object or in other words yeah relative tensor products in commutative rings are also relative tensor products in analytic rings that follows from our discussion of a relative tensor product and analytic Rings um but more but then I want to say moreover the relative tens of product I mean underived or oh shoot yeah yeah I say maybe I should say derived so they say animated yeah ah and then we are much much yeah thank you thank you o Foria so now you are and then you are then of course all this notion of  $D$  are is much more yeah much more uh um so zisy covers uh map to uh shriek covers uh but now it's occurring to me I I forgot to remind what this means so sorry sorry sorry sorry just a sec so so uh sorry so so note um yeah we get a Gro topology



on  $F$  by saying uh that a sieve over  $X$  in  $F$  is a sieve cover uh if it contains uh finitely many many  $Y_i$  mapping to  $X$  now I've switched the roles of  $X$  and  $Y$  from previously sorry about that um well maybe I'll just call this  $X_i$  such that uh find it may shable uh such that disjoint Union  $X_i$  uh mapping to  $X$  is a sieve cover uh in the previous sense so and the key behind this besides kind of obvious properties of finite drint unions is the that a shriek the if you have a shriek cover then any base change is also a sh cover which um is a consequence of this discussion of co-limits in PRL what did you say now that uh if you have a map which is a shet cover then any pullback of it is also a shet cover or base change any base change of it it's also a sieve cover uh well yeah there's uh it's basically because of the fact well well if you can think in terms of this uh Co limit in PR business and then it's just because the the base change functor on the level of mod dprl the base change functor just commutes with co-limits so if you have the condition there then you base change you get the condition there it's a consequence of two facts the first was this kind of kunth formula for like if you have okay if you have if you have a base change in  $F$  then you have the  $D$ 's of all of these and the  $D$  of the base of the pullback is just the relative tensor product of the  $D$ 's of the the other three guys ah  $D$  of this is the tens of produ and this uh is and then you have that the and the descent that you had is all you formulated it in a more General way where you have a category which is a module over then you apply this somehow and check that the maps are the right ones yeah yeah somehow yes exactly exactly yeah um okay so uh right so that's that that's what I mean by there  $Z$  risky cover going to a shriek cover I guess I just mean that if you take the you know has a finite refinement you take the disjoint Union then that thing will be a shet cover um so um uh right so um the proof is simple uh indeed uh zisy covers uh go to closed covers in the sense just discussed so if you have a a principle open a principle open in uh Spec  $R$  and it's given by inverting some element and that inverting an element is gives you an item poent algebra um so uh and that that item poent algebra defines a closed cover on the level of these guys um and then the condition we had to check um is just usual zisy descent no why is the risky cover goes gives a Clos cover well the well so there's let's separate that out the first claim is that if you have a a principal open then that goes to a closed immersion ah now it is called the Clos imersion rather than ah okay so this uh ah yeah remember this in the previous talk yeah I mean I there's a picture that I would like to draw like could you see when you're doing analytic geometry you have more more more room you have more more room than in algebraic geometry because you have like nearby infinite kind of infinite decimals almost I mean I so usually you think of like the locus where  $f$  equals 0 in algebraic geometry like you think you have the locus where  $f$  equals  $z$  and then the principle open is the complement of this and then you think okay the locus for  $f$  equals  $z$  that's obviously a closed subset so the complement should be an open subset right but um it's a little bit of a lie because um I mean uh this isn't really where the locus where  $f$  equals zero it's kind of a you know could also be just a like Locus where  $F$  to the  $N$  is equal to zero or something like this um so it's really what you're really deleting is not the locus where  $f$  equals zero but you're deleting kind of the whole formal neighborhood of that and this then it it acquires some fuzz um and then I claim that what you should really think is that that fuzz is making this thing really behave more like an open subset um and that the there's a risky open it should really be thought of as closed and it should have a it should have a boundary some kind of tubular neighborhood there should be like um so it should really be a closed subset and then the formal neighborhood usual formal neighborhood is the open complement that's the picture I would like to suggest um yeah and when you go to the the solid world then then you can Rec you can again have an open version of puncturing so if you then you then you can name this boundary maybe maybe it's something like  $Z$  lant series  $T$  base changed along  $ZT$  to  $R$  where  $T$  goes to  $F$  something like that um you can name the boundary and then you can remove it so it's it's going to be a closed subset you can remove it and you get an open subset and then you're back back into usual way of thinking of having having a  $z$  risky open but then it's not um yeah you're so you take  $r$   $t$   $z$  and then  $Z$  Lan Lan uh yeah so I'm trying to I'm saying once you move to the solid framework then you can name this boundary here which what before was just heris and its name is this um so that will give it that gives an item put in algebra uh in  $D$  of  $r$   $z$  solid um and the well in fact it's in  $D$  of  $R_1$  over  $f$   $z$  solid and the complement the complimentary open is like  $d$  of  $r$   $1$  over  $F$  kind of  $Z_1$  over  $f$   $z$  so  $Z$  adjoined one over  $F$  whatever see Union one over  $F$  yeah um and but so so if you decided to if you yeah I mean there's two different ways you can embed schemes into adex spaces right one is based on sending on the level of rings to  $R$  to  $RZ$  and the other is based on sending  $R$  to  $R$  comma  $R$  and this is the one that I'm discussing right now well based change to solid  $z$  um and there  $Z$  risky opens look closed but if you use this one that corresponds to always removing the boundaries and then zarisky

opens actually go to open immersions um in the discussion yeah um so what do you mean by boundary like the foral neighborhood minus the middle I don't know what I really mean by boundary if I don't just mean this formula but intuitively I I'm claiming you're removing kind of an open piece from this chunk and then the the the closed complement should intuitively speaking have some boundary um I mean there going to be boundaries at Infinity too if if your thing isn't proper and so I mean there's there what's happening at Infinity is also important but let's let's local Let's Pretend We're in a proper thing or something um okay other questions um okay so in uh so in particular yeah we get a functor uh from like zisy sheaves on apine derived schemes um to shriek sheaves um on  $F$  um um which uh is in fact uh pullback of  $\text{Topo}$  so I mean it commutes with co- limits and finite limits that's kind of consequence of this so this is so this is one one way of embedding usual algebraic geometry into well I haven't quite defined what an analytic space is it's not exactly going to be this but the if you're it's it's close enough for practical purposes I mean the we're going to do have some kind of hyper descent condition we want to impose as well I thought I was going to get to it today but I'm clearly not um but this is basically basically analytic spaces or analytic Stacks modulo some modulo a couple of technicalities so this is shapes with values and simpli right so there's the question of yeah and that's related to whether I want to use hypers sheaves or not so I mean we could we could do either we could do we could do sheaves with values and sets and just yeah why don't yeah what I what I really will mean is she was values in what Peter and I have been calling anama which is what lri calls spaces which is what other people call Con complexes but the Quasi category of con complexes some people call homotopy types yes um and something has many names it's because probably because it's a subtle concept yeah well anyway um yeah makes sense to look just shs of sets when you have the derived things so yeah I think it still makes sense yeah still makes sense I think so yeah and it makes sense to look at the OS like shs of trated this makes sense yeah it makes sense and there the the The Descent are equivalent yes exactly then when you go to the I mean to all car complexes then you have different potentially different Ascent uh conditions and when you R this you you use the weak one with which the SEC yes covers although if you did a stronger one it still get the same claim because it would be just a further a further localization of this so and then the on the other side also yeah well right so I don't I'm not going to touch the other side one could but well the risk shes does it also depend on whether you have oh yes yes it does yeah I'm but I'm not going to on this side I'm not going to care too much but well and it was said in some talk I don't remember who that maybe Peter said that there would be something intermediate between check descent and full hyper correct yeah I I was going to discuss it today but uh it's not going to happen um yeah okay so um and then the other subtlety is like set theoretic subtlety because like the category of commutative Rings is not small so you have to be careful considering pre- shees and sheaves on it and so on and same with you take only those which are accessible yes yes yes exactly and then you have to prove sheif ification preserves this accessibility and so on and so forth but okay um I don't think uh in the remaining like 7 minutes I could do justice to the the next topic was going to be attic spaces um I could rush through it right now but I don't think that's a I don't think that's a good idea so I think I'll stop here J find that cover here by Somey and well I don't have to I wanted to know what's for example should cover oh oh oh oh sorry I didn't understand your question I'm sorry yeah whe whether in a tall cover also gives you a shet cover yes it does it does right so yeah so the yeah so yeah this is thank you thank you I I think maybe this is something I should do in a few minutes um uh yeah so so this was all I mean a lot of this was motivated this was motivated uh by Matthew Akil Matthews uh a theory of descend ability um so um so uh so we could so we were talking in af right and then we were saying we have this derived category of anything in af and it's built on this condensed framework but it's clear that the discussion is very categorical in general and we could just try to make the exact same definitions in the world of ordinary algebraic geometry instead using like for a ring you just have this usual derived category and and see what the kind of definitions give um so if you take the same definitions uh but with the pair  $R, D$  so so this is a usual commutative ring or may be derived um and then this is the usual derived category so instead of instead of analytic Rings we're just looking at commutative Rings um then some simplifications happen first is that every map is proper um that's quite clear uh well maybe that's the main simplification that happens is every map is proper because by definition you know if you have  $R$  mapping to  $S$  then  $D$  of  $s$  well almost by definition then  $D$  of  $s$  is just the same as  $s$  modules and  $D$  of  $R$  um uh so th this implies uh that every map is shable and then um The Proposition that I discussed over here um thing here there's no condensed thing here yeah what is the sh what's the what's the upper sh and lower sh oh yeah so it's not it's not what you might think so the upper shriek is just the right adjoint to lower star

which exists for General categorical reasons um it's not the same as the upper shriek in groi Duality Theory so so caution so for say for say  $\text{spec } K$  bracket  $T$  going to  $\text{spec } K$   $\Pi$   $\Pi$  upper Shake is not the usual one and  $P$  low sh is also is what pylor streek is just pylor star so that's what I mean by every map is proper okay so so the lower upper sh is like for finite map is the same finite presentation I mean Maps it's for proper yeah for proper Maps it's yeah yeah yeah but for arbitrary Maps it's just some funny thing nonetheless it turns out to be useful um exactly in this descend ability discussion because so every map is shable every map is proper um and then we deduced it so a shriek descent shriek descend or a shriek cover is the same thing as saying that the the so a street cover  $X$  to  $Y$  is the same thing as saying the unit in  $y$  lies in the image of just so  $F$  lower star  $D$  of  $x$  to  $D$  of  $Y$  and this is exactly descend this is the definition descend ability uh in Matthew's work or that's one equivalent he gives many characterizations um what about the classical grend  $FP$  you see the sand is it related is it what does it let me give some examples so so so actually so every so actually every actually kind of purely formally uh any um any descendible map uh  $R$  to  $s$  in a ke sense uh gives a a sh cover in a via this this functor here cover in now in yes exactly yes um and Matthew gives many examples of descendible maps so um so examples so a tall covers so that's more General than zisy um but also more generally than that uh countably presented uh Faithfully flat Maps so fpqc covers um so this is kind of funny you need this uh well we apparently probably for real need this countably presented hypothesis countably so so uh that means that so like so you have a map from  $A$  to  $B$  which is Faithfully flat but also  $B$  is uh presented as an algebra or maybe even just countably generated is enough so  $B$  has a presentation as an algebra I countably many generators so the so here when you work with simpli rings so uned so the condition is that that on  $\Pi Z$  it is  $f$  fully flat and that the  $\pi$  I are this is kind I did not carefully think about uh let me say for this I mean ordinary Rings because I didn't carefully think about uh the derived analog sometimes they they some references they have this derived yeah  $s$  with  $fa$  flat where they Define it like what I said oh yeah so so it is true that case you need to you need to do it yeah and and then face ordinary Rings um countably generated up because you need the to uh maybe yeah there is some uh so it's equivalent to the kernel the fiber  $t$   $m$  from fiber to a here is some  $T$  of power which is zero yes yes yes and this is related to some limit higher limit I mean so if it is a limit of  $A/n$   $s$  which that's fine too but in practice it's the same thing I mean okay but if it is only countably generated then you don't get the maybe it maybe it has to be countably presented yeah yeah because you need to yeah you need to be able to reduce to accountable ring accountable Bas ring I'm sorry yeah let's say countably sendd anyway the point is uh lot of situations where you have classical descent um you get this even stronger shriek descent which recall gives also this two descent and much else besides um but also there's some um like if uh so  $R$  going to  $R \bmod I$  or  $I$  is a nil potent ideal um and now you have to be very careful um yeah now these are ordinary rings but the shriek descent is defined on the level of the derived categories so the The Descent you get for this this does not imply does not imply that  $D$  of  $R$  is the same as  $D$  of  $R \bmod I$  obviously and the reason is that you have when you do The Descent you're doing everything on the derived level and so it's the uh you end up with terms like this um in the cimit diagram and those are those are that's not the same as our mod about go to  $\Pi z$   $r$  does it have the same property  $R$  going to  $\Pi z$   $r$   $r$  going to  $\Pi z$   $r$  um no no Peter says no you can you can have put nor algebra  $\deg 2$  generator and this has a module which by Inver  $\deg 2$  generator yeah that's right yeah yeah yeah so so if you look at like I know  $Q$  bracket  $X$  degree  $xal$  2 um then uh then there is a ring module in particular where you invert that generator which goes to zero when you mod out by  $X$  but is but is non zero so so what's the analog for simpli Ring of anent Ideal you should ask that it be truncated the ring be truncated so it has only  $F$  so you should ask that  $R$  has only finitely many homotopy groups and then yeah then going to  $\Pi z$   $r$   $\bmod I$  um yes so that's how the some kind of artinian situation you could think I don't know yeah um yeah and so more generally I think uh yeah maybe yeah I don't know who proved it uh maybe maybe it was bot schultza but if also if you have like a proper map I I'm I'm talking about the apine situation here but okay you can extend it to schemes but if you have a proper map of nean schemes uh then you also get Des sendable so that's a generalization of this this here A Proper  $s$  map yeah sir um and more generally age cover I mean that's kind of just a combination of this and and that uh let's say finitely presented um so uh and maybe I okay so anyway there's a huge class of very very much you but you do have to remember in like cases like this and in non-fat cases you do have to remember The Descent involves a higher higher higher tourus and so on so so yeah all of these kinds of things we'll go to uh shet covers in in our setting there okay so now I'll really stop sorry for going over yeah so in these two versions there's one is is taking the classical one and the other is two with condensed structures that that would not

make a difference for see whether it's a shepper or not well it's a priori possible that for some weird reason well is it uh a priori it's possible that for some weird reason you could have a a map of commuity of rings where you go into that world it becomes a street cover because of some weird condensed modules I kind of we'd be surprised if it actually happened in practice but a I I'm only claiming an implication that if you have a street cover here you get a street cover there right uh wait no no it's just tensoring yeah oh no yeah no aior it's possible it's possible that no that sounds it sounds very strange but yeah I don't know the question was whether uh whether it's equivalent to have if you have a map of commutative Rings whether it's equivalent for it to be descendible in the ordinary commutative algebra a Matthew sense or and whether the image under this funter is a shable map a street cover in our sense so I think it's equivalent is it equivalent so what's the argument well I me this important to theil this kind of relative sense but this is just meaning that The Descent and this S pro object aism but this Pro object is just an old discreet Pro well wait but uh yeah okay yeah thanks yeah so yeah what Peter was pointing out is there's another characterization of descend ability which is in terms of on the level of just the Rings like a  $b \otimes a \otimes b$  and so on you get some first of all you have to have descent you have to get a limit diagram but you but then you have to get even more you have to get a a very stable limit diagram it has to be a pro isomorphism between the the tower you get from this co-al thing the nindex tower and uh that proobject should be Pro isomorphic to the constant Pro object a and in that condition it is clear that when you uh that it's independent of which framework you put it in because the pro category um because the the usual  $D$  of  $R$  sits fully Faithfully inside  $D$  condensed  $R$  and therefore Pro of usual  $D$  of  $R$  sits fully Faithfully inside Pro of  $D$  condensed  $R$  so the yeah and so the for the pro thing it's enough to work in homotopic categor it's enough to work in weaker for the pro thing I mean I don't know so but what you can do is you can pass to the fiber and then you're ask you want to know about a tower being pro zero and then it's enough to to look in the homotopy category okay thank you again for your attention so when it is pro zero yeah then is it the case that the that it is uniformly pro zero that is because there is some finite do mean that it's enough for any stage it's enough to to add a fixed number exactly exactly exactly yes in Matthew yes and it works and it works with simping so he works in actually a very general setup just like commutative algebra object in PRL again um and then sometimes he specializes but for the general discussion yeah um but he only he only looks at the proper case so for him every every map every so he works with his base category is always some very general thing like  $C$  and  $C$  alge PRL a presentably symmetric monoidal thing but he only ever then considers algebra objects in  $C$  and he's comparing like  $C$  to mod  $a$  and he's talking about descend descend ability kind of in this sense so it's the he's for him every map is proper in our sense so object is like a symmetric monoidal category yeah and then you consider  $a$  is a is an algebra is a ah yes yes yes then you take  $a$  and algebra in  $C$  ah so you walk in this setup and then he defines the cability in this context yeah okay so it specializes also to the condensed world but it it's um it's only discussing the the proper maps not the arbitrary shable Maps all right see you yep see you Peter can I ask you something um about what you said last time okay oh I think it was last time uh they mentioned that we have this um decomposition oh yes formula yeah formula and and so it's probably not true that we have this for all analytics STS right right but um for like we have these TS of perfect do you have I mean question have similar intuition for like class of objects for which yeah I think it should be pretty similar yeah okay yeah this is something that will be relevant because we're going to talk about extending this right now we have a six funter formalism on  $AF$  and we're going to talk about extending it to more General analytic stacks and this kind of condition will show up again yeah and I also didn't discuss like fully faithfulness of was easy to settle here but for more comp in more complicated situations fully faithfulness of some funter from classic analytics basis to our analytics tax is not not at all um easy to settle but uh yeah similar considerations have to come into that as well the essential problem is that this grow de topology is quite inexplicit like usual as a risku tall you kind of have a good handle on the covers because they're you know finally generated for example and you can kind of give standard models for what they look like but in this setting it's much more difficult to get control over grot and de topology so it's harder to a prior prove that locally something weird can't happen um but okay yeah

## 19. ANALYTIC STACKS (SCHOLZE)

[https://www.youtube.com/watch?v=T9XhPCI8828&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=T9XhPCI8828&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

So today finally we will define what analytic stacks are and in fact it's not so difficult so recall that we have this category of analytic spaces so this was actually just to mention at one time it's actually presentable I think so that's prove that it has all colimits in it and actually it's generated by a set of the under all and so we can make the following definition and I don't expect it is an accessible function so the word accessible is just there to deal with some set-theoretic issues from analytic ring or formal analytic spaces so recall yeah so word acceptable here there will be a condition in just a second so this means that it commutes with how I filter colimits for sufficiently large index it's such a  $f$  such that  $uh$  so chief conditions and a preimage way I want to phrase the conditions as follows  $uh$  sorry  $uh$  we see we have some unpopulated and then some hypercover unpopulated a boards  $uh$  for which  $uh$  of a  $y$  up the streak is the spectral limit of the use of all these terms  $uh$  also  $X$  of a that's isomorphically to so by the way I write unspecified where Dustin just wrote spec I don't know I don't think we've settled on a final notations just not to get it confused with a usual spec let me write on point  $uh$  so  $uh$  this is some form of descent that's somewhat strictly between so can I ask just a minor technical question concerning the notion of hypercover so in usual algebraic geometry  $uh$  do do do you hear me no is it okay is the green they hear I ask him if he hears me do you hear me yes I hear you yes okay okay so let us consider the case in usual algebraic geometry when for example you have like fpqc to or something like this and then you you there could be two meanings of fpqc I recover so one meaning is that the map from  $x_0$  to  $X$  and the map from  $x_n$  to  $co$  skeleton are fpqc and the other meaning is that there are coverings for the fpqc topology which is  $H$  so sometimes it causes some concern what what is the meaning so in our context you use the the strict meaning I suppose that everything I mean you yes so I want all mapable but then there is no difference between like if it's refined by a stable cover then it's already a cover because of the  $S$  condition ah okay so here it is satisfied that if some  $m$  is dominated by it's already a cover so okay okay thank you  $uh$  right and  $uh$  I also need to say that commutative finite products so commuting products including the empty product I geometrically just means that  $X$  evaluated on a disjoint union is like the product okay so so this condition is so so this defines I mean up issues and it's too that's somewhat between  $k$  and three so for Street sheaves you would only consider some sheaves of covers  $uh$  and those would by definition of what a street cover is always satisfies this condition so for  $uh$  CH nerves a street covers you always have this condition so it's always a streak sheaf but and for streak hypersheaves you would ask this condition for all Street hypercovers whether or not they satisfy this condition but we definitely want to our derived category to satisfy  $uh$  to be some absolute property so we need to restrict to classify the codomains for which we have all right  $uh$  and let me just state this as a remark and not prove it so  $uh$  yeah so sheafification of an accessible presheaf is successful so this the analog of the sheaf of water house HC some particular this means that you can actually form colimits in this category because open forms the colimit in the category of sheaves so just the functors from to  $Ana$  and this is still an accessible functor  $uh$  but then you need again need to enforce the strong sheaf condition so you need to sheafify and you can check that it preserves the set finiteness condition all right so examples or remark so for any first of all for any you can look at the functor which takes you can define aspect  $a$  at a factor which takes any  $B$  to the map from  $A$  to  $B$  so again I have a small technical question concerning the sheaf which I believe you mean sheaf making it satisfying this precise condition that is not not the Shri shift and not sheaf hyper sheaf but this intermediate condition yes but then in order to construct this unification it it you need for the pullback it seems that you need to know like that the pullback of this kind of hypercover satisfying the condition on the category is in is also this you need to know that the pullback of such a thing is again such a thing but again this follows from the same argument so this actually equivalent  $uh$  to  $D$  of a being the colimit along the low street maps of the of the VA bullets  $uh$  in in there colimit is taken in this word of presentable infinity categories  $uh$  and this is a condition that base changes because the throughout categories base change in PRL and base change always as coits ah so you use all of this okay and this was Proposition in the previous talk this equates I mean this is all magic about pure okay then you use it it is the tensor product when you base change okay all right  $uh$  so first of all you have all the fine objects so for any analytic ring  $a$  you can consider on point space  $a$  which takes any  $B$  to the homotopy from  $A$  to  $B$   $uh$  climbing this commutes with this satisfies all these properties actually already so you don't have to verify so certainly commutes with products right because you're going to something here  $uh$  and then



you need to check this uh this the send condition here um and now maybe I shouldn't have called this a but so now you have a cyper cover no you have a Cy cover some B some B bullets and then you some want to map to B in particular you want to map like like particular B is like the limit of the plus start again so particular a would be the limit of a bullets so you m to all the bullets and you actually M to B and also on of the r c to again because it's limit this is um so uh think brings up and yeah I mean just by basically only this is a fully faceful uh fun so the analytic Rings up so this embeds ful Faithfully into analytic spe okay so obviously unstack is my notation for the infinity6 um and uh actually this accessibility condition is precisely it's a condition that you're a small col liit of object in the essential image I me not so why you're look at our notes on like condens sets and we do play a similar game there when we have to full category of condensed sets and things so way to think about them e just being small col from Prof fin sets or being such accessible function all right okay so so what is the SC of analytics SE well for any analytic ring there's an object which is some of the analytic Spectrum this all the others are built by some gluing procedure by some Co liit of those the airine objects and the way to think about this uh hyper condition some that tells you ways in which uh which you're allowed to glueing so someone have such a hypera satisfying this condition then the co liit of this the simpli diagram there of these unsp a bullets would be the same thing as the UNP of a no idea so theit is over okay over okay um and so as will maybe become clear later in the lecture today so we want to impose a we want to have this very general topology because it means that uh analytic states which are prior have seem completely different because they have completely different present so they are completely written in up completely different ways as a CO liit of fine guys uh often there is like a geometric picture that they should secretly be the same object and in order to say that they're really the same guys someh need to shif find rather strong topologies makes them the same thing all right so okay so uh next example that this is in some general bre geometry because of the strong nature of the gr another question oh sorry no no no just the the small in the in number two the small Co limit of objects in which category it is taken here yes in analytic stock or in in in PR no before uh limit in before the stock condition just it would also be a CO Li before the St condition because actually sufficiently filter Co limits preserve this uh this condition there because there accountable limit condition only ever so if you take omega one filter col they will always preserve this she condition but uh no no but you you certainly have to take the pament in here because like even if you GL just the few F to P1 or something like this like even the P1 wouldn't be a Coit of it sorry probably because sorry I know yeah I think two things are equivalent but okay accessible should be some condition preliminary condition yeah so so so a different way to think about this is that you can also like put a cut of condition on your analytic rings that there's some more copper small for some copper and then you can just consider such Stacks to find just a couple small analytic rings and then there's a way to extend them to like the full class but some left extension and then yeah I think this way can probably show that okay okay uh all right so there is a fun from like deriv schemes um the deriv schemes they also of course embed into like she called the Des sendable topology uh we can go to taking particular like any spec spec a for us ring on spe a Tak a condens thing which is really just the and um and you all condensed modules and if you want you can also like as Dustin discussed you can basically put the fpqc toy in here except you have to do this funny countability assumption only one FPC so basically like any fpqc stack um can also be mapped and there's really not much to show here right I mean you definitely just have to spun on rings and then you just have to show that whenever you have a des sendable cover it goes to such a street cover but that's basically by definition um and again you could also use this funny thing between sheeps and hypers sheets here in the C topology here if you wanted to um right uh so ah right maybe I can mention that serum that is actually fully faceful a ship you view is is a it's like a full back where you this actually uses tanak us T do you do you need QC Qs I think he can get rid of this because he can certainly reduce to the qu compact case and then there's this argument of offer that you can write as a limit of qu separated things you can how how do you reduce to the Quasi compact well you can certainly if you want to understand homes from X to Y you can cover X by but you can X compact but then comp wise they can definitely comp oh yeah okay there other argument that you can make the compact so can okay all right so uh you have some algebraic geometry sitting in there uh next you have Tic space is sitting in there which well these These are good from fromo one so these are a plus um like when you talk about CH Ed spaces you always assume that this is chy and so what I'm saying now let me also assume this because otherwise I would run to some longer discussion about exactly what I want to do um and so I can just s this to the onp of the analytic ring a plus solid uh so note that this way we get a different fun get two uh different

function  $u$  so there actually two ways to embed schemes into adic spaces so you can take a spec  $a$  you can also match this to  $\mathrm{Spa}(A)$  and then for there's a question in the chat Peter it says it says for derived schemes you're mapping the trivial analytic ring structure exactly here I'm currently using trivial analytic ring structure  $u$  right no  $\mathrm{Spa}$  and so you can either now take this functor and look at  $A$  modules and solid  $Z$  modules or you can look at relatively solid  $A$  modules and so now there and if you wanted to you could put it right here suitably formulated and put it then also put it derived here  $u$  so then there are like three functors from schemes or derived schemes to analytic Stacks either with the trivial analytic ring structure or with the one induced from the integers from the solid integers or with this functor of  $u$  all of them are like fully faithful embeddings  $u$  and all of them are kind of relevant to what we're doing so have to kind of start to differentiate between the different incarnations of the scheme  $u$  and particular like the one that actually the one embedding where the Notions that we use for analytics Texts actually match the usual Notions for schemes is actually this relatively solid one so Dustin introduced some Notions of like closed emergence open immersion and so on and they had a peculiar feature that like for example any map of  $\mathrm{Fes}$  become a proper map here and like in particular like open emergence become closed emergence here which sounds very weird  $u$  if you use this other embedding instead with a relatively solid structure then open embeddings go to open embeddings as you would maybe rather expect and the first one is still another one a still another one so you have three ways to put schemes well more than three really I mean the one  $\mathrm{AZ}$  is just you take that one and you base change it to solid  $Z$  so you could base change to anything then yeah I'm saying I'm saying the the most basic one is that one and then the the first one up top there with a  $z$  solid is just gotten by base changing that one to from  $Z$  with right let let me just write down what the just explaining oh oh he can hear us the green thing is open and then right so so one thing to notice that the first one first of is uh a functor from three and then you base change from the spec of just integers on spec of this and I mean this immediately suggests the generalization uh for any analytic ring  $a$  you can say the  $\mathrm{RO}$  schemes over  $a$  they can be  $M$  analytic spaces over  $a$   $u$  but just using well maybe should have  $\mathrm{WR}$  a triangle the underlying condensed Tri point so you take if you have an analytic ring then it has an condensed ring which has anline this discrete ring and you can look at schemes over there so for example I know you might have the complex numbers with the  $\mathrm{gases}$  and the ring structure and then you can just consider usual schemes over the usual complex numbers as a just  $\mathrm{fi}$   $u$  and those map to  $\mathrm{Tes}$  over there by just using the previous functor and then  $B$  changing and so right whenever you have ring over and again this actually for Yeah you mentioned something slightly puzzling about derived adic or derived Huber pairs yeah so what what does it mean does it mean that you have a derived ring with a plus in  $\pi_0$  and it is and all the  $\pi_i$  I just it because I didn't want to go into it here right now uh yes one can Define the version like some derived notion of like  $\mathrm{uber}$ ing but I I really don't want to do it right now okay and like for each functor I wanted to indicate what extent we know this is fully faithful  $u$  so this  $F$  is definitely fully faithful on  $F$  lines because then it reduces to maps of rings and we know that Huber rings and fully face the analytic Rings uh and so here we know that it's fully faithful under some  $\mathrm{CL}$  productivity assumption Pro uh we don't know a full fully face result basically because there's no we don't know a good version of  $\mathrm{tanak}$  foric spaces  $u$  I don't know if this is also like  $u$  if it's worth warning that the in contrast to the scheme case the adic  $\mathrm{Bas}$  the the functor doesn't commute with pullbacks or right so actually if if you restrict to like Tate tic spaces oric space or however you want to call them then there it behaves as expected as long as you stay in the shy case otherwise you might have to make things  $D$   $u$  yeah but then there's this  $\mathrm{PE}$  feature that like on the right hand side uh if you take a fiber product of  $f$  analytics  $\mathrm{St}$  it's always just the usual correspond to  $T$  product of rings this is not true in adic spaces so in adic space you can have a fiber product of adic space that's not itself adic anymore  $u$  so there's some subtlety there that can actually be explained a very nice way in our form in a sense that instead some  $f$  yeah let me not do it here  $u$  all right so then you can also do this is non geometry you can also do not geometry like real complex and for example you can have complex analytic spaces map to  $\mathrm{Antics}$  Text and now this some work over for example the guess is complex numbers and uh how can this be described so the one problem is that usually like in complex analytic geometry people don't really tell you what an fine space is  $u$   $u$  but they could  $u$  so any complex analytic space can always be written as some kind of Union of what we call compact  $\mathrm{St}$  subsets so these are things that really very much like behave like Eon objects so what is a compact  $\mathrm{sty}$  so for example uh the vanishing locus of some ideal inside just a  $\mathrm{PO}$  this so these are two of complex numbers all of which absolute values are most one so functions  $H$  morphic in the neighborhood and right so what we end up with with the algebra function we put on such a  $k$  is over converted holomorphic function  $\mathrm{soga}$  which is a  $\mathrm{CO}$  limit over all  $U$  which

strictly contains this complex subset of functions on  $X$  and then it is a theorem from the 70s or 80s that at least in this case where you're really close in the poly disk uh this is this uh Banach algebra it is excellent and uh if it's actually manifold it's regular and so on um so it has all the nicest properties you could ever hope for um and if you want to talk about like coherent sheaves on  $X$  then you can really just talk about like for any  $U \subset X$  Stein subset you give just a finitely generated module over this Banach ring and they can space shape so as far as I remember for some book on complex analytic geometry so there is a small lemma which is usually not very important but still for the nitty and so on you there could you could have an ideal such that the number of connected components of the zero locus is infinite and this is bad for nitty so there is some reason that's why I said that for things which are actually of this form that a Banach point there it is Banach what they think on General like come back Stein would allow something like Pro finite subsets in a pushout and this wouldn't be in a Banach uh but if it's actually it's a Banach close subset of a Banach it is in Banach algebra because it is a Banach not arbitrary compact ah okay okay okay yeah I remember that it was for another other compact yeah okay um right but but I mean if you if you like there so there is this way of looking at complex geometry where there's really a very close analog of the notion of an analytic subset and everything really has Banach algebra and everything is really very very similar to how you do in rigid geometry um and so you would send that similarly to like the Coth which is actually against of Union um over over KX of the UNP of this algebra endowed with like the Gelfand use complex norm I mean like algebra for functions they have a natural topology like uniform convergence on complex subsets so this also has a natural topology you could also define direct in condensed setting whatever it's a So-Cat dual nuclear for SP and again this is fully faithful on for the the you take this with which kind of which analytic structure okay oh just um so you view it as an algebra in the say Gelfand theory yes and then you just take the induced Banach structure so you just check on the underlying so this defines a condensed ring which is actually an algebra complex numbers obviously uh and it because it's a dual nuclear for it's actually Gelfand and so you can just induce up the Gelfand analytical structure from the complex numbers to here sorry not every dual nuclear should be Gelfand right but uh what they are it's not nuclear as a guess is certainly oh yeah yeah yeah yeah okay sorry yeah I mean these are actually even nuclear guess this here yeah what is qu no but the should be before your face for no fun it should be fully faithful General It's just tough to prove no no but a point it should give a map I mean the map of analytic it should give a map of underlying topological SP yeah that's what you need to prove yeah what yeah that needs to be proved ah yeah and what is projective it means it's like an adic immersion into projective space so maybe like an open subset of a closed so what's the question oh what is qu projective some locally closed immersion into projective space so it's closed in an open it's closed in an open after uh maybe before I go to the next example um there's actually nothing much special about using like complex manifold or here I mean you can do similar definitions work for it works also on the real analytic case manifolds or actually also for smooth or you could also I don't know take any kind of CK or just  $C^0$  topological you take the Ring of CK first Well the ring is again the slightly peculiar thing that you take like still do this Co limit of functions on neighborhoods over convergence continues functions is a kind of awkward notion that makes sense that's what you should do and each of those cases you can actually decide whether you want to take real or complex value functions and they give you two different functors where one is just the base change of the other so one you can make make an analytic St over the reals and you can base change to the complex if you want to you use you should do something at least as complete as the all right um and now comes like a more funny example so I mean this is like showing that all the usual theories like of algebraic geometry of geometry that are known and maybe okay so we should also at some point discuss Brooker spaces they would also have natural functors um uh they fit into our framework and but in any of those Frameworks you could also directly define some notion of Stack so there could be you can imagine many possible notions of a complex analytic stack by taking St with complex and spaces and D them with any Gro topology could think of maybe just open covers and then also stalks from that topology would St just because any open covers in particular is cover all right so go to um but now comes the SL M example which is some relating it back to what we started with Nam condensed sets or because everything has become a St let take condensed on and as always take the light ones so those actually they also M to to uh how does that work so this takes any and actually you could even go to like you don't really have to go analytic for this uh you can just uh this what even work for just schemes um so in particular like here there is something condensed on the right there's also something condensed namely like the adic rings that were condensed rings and so on but we're actually not using this condensed structure here uh it really comes from the speed objects yeah um how does

it work so if you have a live profile set  $um$  then you can just snap this to the spectrum of its continuous function right so if  $s$  was a finite set it was just be a fin of copies of  $\text{spec } Z$  in general as is the limit of finite set and then similarly this is the limit of These Fine schemes and so then you can access further to the  $I$  and then here's the reason that we chose this really funny version of uh between Chiefs and hypers Chiefs so con anima they are by definition hypers Chiefs of an on on life Prof steps for Theology of that we always use and so to get this fun you have to show that if you have any hyper cover here of a light Prof set by light profin set  $um$  then it goes to uh something for which you enforce the  $S$  here  $um$  it definitely goes to hyper cover and so then there's a small thing you have to check that it actually is the same condition uh on the  $D$  category holes basically the argument that faceful flat map satisfy the sent just also proof the state countably presented face yeah so I should say that here is the lightness condition is again important because uh  $um$  uh it's always true that if you have a stive map of Prof fin sets then if you look at the corresponding map of continuous functions it's always FL for any Prof only space but uh we need a descendible map for our business and so we only know that counter represented faceful that maps are descendible which is another reason that we have this Li she what is question on all right  $um$  so uh yeah so here's a remark  $um$  so so there are different ways ofing schemes into antic stacks and now there are also different ways of embedding something like topological manifolds into antic Stacks or the complex or real numbers say so either you can do the thing where you uh use the algebra of continuous fun continuous real value functions to do this or you can treat the topological manifold purely as a topological space or faceful into condens sets and then go like from condens sets to analytics STS this gives you a different thing which is actually defined over the integers right because this analytics St just integer value function these are antic the integers uh so these are completely different incarnations of a topological manifold St but there's actually again a map between them so let's let's consider the following example so you can take the two sphere as a topological space and then treat it as a topological manifold and this gives you a fun tic space which in the compa case it really is just the UNP of the algebra of continuous functions and let me work everywhere complex coefficient which which category of let's say see gashes what gashes uh the differ thing I could do is I could take  $S^2$  and treat as a real analytic man and again build build an analytic space over the guess is complex numbers  $um$  or inia you could think about all other possibilities like CK C Infinity Real analytic and so now you would have here the algebra of so what's good notation for real antic functions no no that's smooth oh Omega yeah cega sometimes it's called C Omega yeah the Omega I think that's thanks and I don't know think about all possible other algebra like C Infinity function CK function between they would all also give an spaces  $um$  but then uh you can also like  $S^2$  is like one version of  $P^1$  of the complex numbers so you can also treat this as a complex space and well then you want to get something with the guess this complex numbers  $um$  this is not  $f$  anymore right I there are not so many Global home functions here it is what it is is not  $F$  there but it's some glued from two copies of unspe orphic functions on the dis over conic functions on the dis right so you can cover this by to this meeting along the circle and similarly this would be glued but from two copies of over conic function on it this glued along the over converg morphic functions on the circle topology he's gonna do it as a scheme first okay uh so clearly like making it more rigid you can also now look at  $Al$  functions so you can also take  $c$  as a scheme and build something over the Gus complex numbers uh so again this is not  $F$  fine evidently uh but it's glued uh from two copies of the analytics spectrum of like a polom  $Al$  or maybe there should call one  $t$  and one  $t$  inverse or whatever  $um$  and I will make a comment about this in just a second but let me first finish  $um$  and then I can take again just  $s$  two and Tre just as a condens set and go to SP change to the guess this complex so here I'm using the fun from from six so basically what happens is that here you want to take the locally constant functions on  $S^2$  well there are none on  $S^2$  but if you secretly think of  $S^2$  is being the quion of a profite set by Prof inst relation which you can always do by this condensed perspective then on those profite coverings you do have locally constant functions and then you can do this bluing so secretly to evalate what this is you have to remember that secretly as two is a qu of Prof set by Prof relation and then you can lose locally con functions here you have algebraic functions here you have holomorphic functions here you have realic functions here have continuous functions all of them give you different inations of what the two sphere might be  $um$  and now there's already one really funny thing which is something that DUS already mentioned a couple of lectures ago uh that one intercation of the Gaga is that is actually an iism of analytics Tex oh so this is one kind of Gaga statement uh which is now not just a statement about some career and Chiefs or some derive category of career Chiefs or whatever uh so not just on the linear algebra but really before you pass linear algebra on the level of spaces there is an

ni particular one of the  $r$  and I mean this is an instance where Upper ear the way you build these things is completely different one is built from just poomi one from form functions and the strong work topology that we impose is precis is there to ensure that you can have the possibility of such interesting as PHS question basically the way one proves this is that you first need to prove it objective so this is covered by these two things you need to prove that those to this together to this steam guy here that they are the sendable map again then you have to unrel what it means to be the and prove it uh sorry I have a question yeah yeah uh are you able to get uh uh  $C$  Infinity schemes or from six you had point six there which it was uh schemes but was that algebraic or could you get other schemes like  $C$  Infinity schemes I'm not sure what  $C$  Infinity schemes are if I there derived menolds that's somewhere in the literature well no that that's that's uh it will it would be generalizing manifolds but it's not derived but one can one also has uh derived manifolds I guess but I was asking the simpler  $C$  Infinity scheme so locally ring spaces that are with a local model that's a  $c$  infinity ring which is not really an algebraic object but you had a six six right well six was a condensing but I think you mean five um but uh well for this I guess you would need to establish some relation between this notion of  $C$  infinity rings and analytic rings and this I'm not really sure about um there is however in practice some kind of fun from from some suitable CLA analytic Rings towards  $C$  infinity rings and that like  $C$  infinity rings that you want that like for any smooth function from the real to the real or something like this it gives you an operation on your ring oh um and after that's not at all data that we encode in our strings because they just appr some I mean just have additional multiplication you don't ask that smooth functions like uh but you can actually show that in some circumstan where you would expect it smooth functions act in a unique way um yeah so I mean so for example this Theory here gives you a completely different way of approaching the subject of deriv manifolds because like if you if you just start with user manifolds and  $B$  into our setting and then you can take intersections the intersections are derived because everything is DED for us and so you can just if you want to work with Drive manifolds you can just use our framework and just Works um but usually there was a reason that this  $C$  Infinity albas were invented because you needed that on this derived Tor product you still want the infinity functions to act which is not so clear but you can prove that also in all framework they they act I'm not sure if this was useful information uh yeah so so precise relation to like steings I don't really know and I don't really know what SK fering is also well but for like the purpose of having a workable theory of like derived manifolds and so on uh I think this also sits fully Faithfully in our framework right um um yeah maybe that's all I want these are some examples what I say next um yeah maybe one particular instance of this is like if you go from the very top to the very bottom then those functions are just have a good take us input any topologic manifold for example or any compact House of space or something like this uh you would basically always have such a map maybe you should assume it's five dimensional at some point but other than that you always have such an natur map from like this Incarnation using continuous functions to the Incarnation using constant functions but yeah the categories of modules are very different so up there the  $C$  of modules of course just the it's fine so the modules are just modules of the algebra  $C$  Vector space was an action of the continuous functions butas here there just Chiefs of  $C$  Vector space on the topological space one can show if  $x$  is a maybe locally comp sour the chat has a question Peter which is um why do we consider gashes antic ring structure on the left yeah um because otherwise this iscl for example if you want the function from complex spaces towards um towards analytics Tex you need that the the intersections match up so if you have like an intersection of compact line subsets in here then the intersections again something compact Stein and you want this to be mirrored on on the side of analytics text in our picture which means that if you compute the corresponding complete tender product of analytic rings of like functions on  $D$  and the other disc and them together you should get functions over conver functions on  $S^1$  and this is a computation that comes out correctly in the gas analytic ring structure but definitely doesn't come out for the right for the trial antic ring structure because then would just take the the usual tensor algebraic tensor product can and this is just some nonsense so you need to complete the tensor product and then it comes out right um so  $X$  dimensional sense uh then you can look at the category of like Tex and  $a$  is any and  $B$  change it to the spectrum of  $a$  and this is just Chiefs on  $x$  CHS on just  $X$  some consider toal space um those vales so yeah so you can think of yeah General CHS as being some kind of  $f$  she on an Associated St um I realized before that I should have said that uh the fun that take any on spe  $a$  towards the  $a$  or also towards uh presentable St authenticated is linear over today um satisfy satisfy the send um and hence induce funs on on all entics so for any St you can Define the direct of current on  $X \times Y$  descend um and also you can Define what is like a a chief of categories over  $X$  so here it



is again the intermediate condition the but for the yeah I for this funny version between the center typ for any this will actually both of these things will actually be part of some general sixun formalism and at some point we will uh talk more about that and basically all design Criterium for for GR topology was actually precisely that that for any analytics de we definitely want to be able to talk about the C Chiefs and at some point we also realize it's probably good to enforce r that we can also talk about Chiefs of categories and then we are basically taking the strongest possible gr topology uh where this is true what is prxl that is the thing which is gotten by descent from the association and spec a goes to pld da which is Da modules in PRL okay which is a tool yeah it's in two category yes second thing two category but we don't have to care we can just see it as Infinity one by neglecting the non um right okay so this is some general examples that maybe we want to I'm not sure we cover any all of them detail on the remaining lectures but just want to give a general picture but I did want to come back to the point where we started discussing analytics Tex which was the construction of the T of the curve and discuss this in a little more detail now because now we have the language of talking about this so let me recall what we want to do um so we have this and bring one yes so this was Universal um or and being endowed with the topological unit 2 um such that uh when you take this usual uh guy which is a free guy on a nor sequence and you tender it up to the string um then there the operator that's one minus  $Q^*$  shift so shift is the endomorph of P that just shifts all the integers on S and so this was this ring structure we introduced some Lees ago and where an on computation was that you could actually compute what's the underlying string of this was and it was this algebra of L wrong series and Q which has a certain funny to noral G condition on their coefficients um so um I'm cond string is the and the existem a k but maybe the underlying ring of the underlying condensed ring because now I'm just telling you a set maybe uh I did tell you the curent condensed structure on this okay so this is some funny ring of Lon series integral series with a rather strong on the growth of the coefficients and then we recall that the goal was to the final litic curve over a or really some over a triangle star um this would be a schem uh such that if I take the Incarnation of EQ as a scheme over a Um this can be written as a quotient of a GM mm over uh under multiplication by Q as an optim not and I already couple Lees back I gave the outline I thought it should go so the first step to see that there is a certain kind of nor um and I will talk about this more in just a second um which goes from uh the P1 array but really again the P1 Som incarnated as a scheme over a so coming in space um towards sorry toward the infinity which intuitively Speaking like whenever you have a point here tells you how large it is having a point here is basically having a point here so maybe nothing think from a to a field or something like this and then an element of that residue field and so there's something like saying that on any res fuel there there would be a nor Z to Z uh it's like the absolute but now this what this this map here is really meant to be a map of analytic stats right so the left hand side is now analytic stack we can take the P1 can make it turn it into analytic St by this General fun oper a triv ring structure but then we can Bas change it to a and maybe here should write I mean so here you're incarnating this by treating it as a condensed set and we could or could not base change it to a yeah we can you have to choose some value for the absolute value of Q like say yeah so so it should stand stand some q q is a section of this um uh and it's Mullica in fact is a unique such thing was the properties I will mention so it should be multiplicative in a sense I will make precise in just a second so I won't talk about these Norms second one uh let say sense but to make it unique I should precisely specify where where key goes um meaning like Q is actually a section from spectrum of a back into this and this should go to constant half um and then you can Define the analytic GM array as a subset of the P1 array um as a preim the open subset um sorry the pimage of open subset Z Infinity in and then you still have q to the acting there and then can take the quotient to the Z and then uh by an argument I already sketched last time this this basically a prop Cur and then you have to prove the zero some algebraization theorem that would apply to All Purpose cures with a section at least that uh it's algebraic so it is in the image found different schemes over the underlying ring towards of theity one I mean you need to make the line line okay it's not clear even geomet it's but here it's okay we discuss this you need some Gaga or exactly but Gaga and someing which is again where is a soft soft well in our in our approach to Gaga you don't really get SOA is much more General than sity but it doesn't really tell you anything like reman rock for something you don't aiori know is algebraic I mean so that's kind of a separate okay um right so one key notion that some comes into into this is kind of a notion of Norm that we want to have here so let me actually Define this in general uh let the and it's a slightly awkward notion of a normal a that is not really normal the underlying set of a or anything like that but it's rather the thing that much more geometric uh is can check the P one that's uh with which properties so such that Z to zero of

course zero section is constant zero the one section is constant one um and then there is a motivicity condition  
 uh for first Let me give it one other I want that to near poent elements have Norm most one um will also  
 follow something and here's how I want to phrase that so inside the  $P_1$  I well I can first look at the locus  $A_1$   
 where I didn't really have just function but then I can look at the locus where the function to near potent  
 and that is some are given by the UNP of a and then you join the top variable so in other words it's again  
 just this this  $P$  treated as an algebra so this maps to the UNP a which is  $a_1$  a this is just another ring I'm  
 instead of taking the next for  $S$  uh but I want to say that this factors this composite vectors over  $Z_1$  the  
 Clos interval not the half open yeah so things that you actually want to say it's less than one it's turns out  
 that the better notion is this one for reasons I can't upe already explained but it's the one that behaves  
 well I should may say that for example such a factoring here turns out to be really just a condition because  
 there's really a monomorphism and similarly for these other conditions that zero goes to zero I want to say  
 that factors over zero has a subset of zero Infinity again that's just condition all right so these are uh some  
 first properties but then when you have Norm you usually asked for some version of uh multiplicativity and  
 some also some Behavior with respect to addition turns out we just forget about addition but we keep the  
 multiplicativity condition um modli um because uh I'm not working on  $A_1$  but  $P_1$  and also the image and  
 the reason I work with  $P_1$  is actually that even on  $A_1$  I would want to allow some functions that they could  
 have infinite nor and if I'm allowing eity on the target of this map I might as well allow it on the source um  
 but now for multiplicativity I have to be slightly careful of tweeting like zero times infinity cases um one way  
 to do this is as follows uh that  $X$  inside of  $P_1$  Cube closer of  $U$  the locus where  $XY$  equal to  $Z$  and similarly  
 there are some actually smooth surface in I did similarly Define some  $xar$  inside not use  $x$  right um inste of  
 $0_1$   $C$  so there similar thing like on the open part you can look at the for  $X$  they have Incarnation on the level  
 of schemes of this clure and one Incarnation just on of real numbers and then uh you have  $X$  incarnated as  
 a scham array  $Ms$  to  $P_1$  cubed incarnated as a choay and this match to  $Z$  Infinity Cube and I on inside here  
 you have this  $xar$  and I also Peter would it be the same as um asking first that like the the map giving the  
 norm is equivariant for inversion on both and then just asking for multiplicativity when you restrict to  $A_1$   
 I think so yeah that was kind of the definition I thought we were using but maybe yeah yeah but even on  
 $A_1$  you have to be careful that even  $A_1$  can  $M$  to zero Infinity it could  $M$  to Infinity oh sorry I should have  
 said that I should have said by  $i$  instead of  $A_1$  I should have said the pre-image of of zero right you could  
 do that yeah so it's yeah like what happens on  $0_1$  and one Infinity it's just mirror images of each other and  
 inversion yeah and then on the pr of  $0_1$  you ask for extra multiplicativity that's I think that's correct uh yes  
 I think that's corage of  $0_1$  lies in  $A_1$  or not yeah lies inside  $A_1$  yeah or maybe well I because I said you're  
 EO variant with respect to inversion on both yes so so that enforces that Infinity goes to zero yeah so up  
 there I could but I didn't specify that infin should go to Infinity by has to inity go I'm sorry inity if you  
 ask it's equivariant for inversion on both sides then Infinity has to go  $Z$  he say now that the requires inity  
 going toity it follows it follows why does it follow because like  $0$  time Infinity equals one is kind of a point  
 in that closure I don't know I think it follows yeah I mean if Infinity wouldn't go to Infinity then you would  
 run to some contradiction from but and is it I didn't care it let me just specify it for saf is it the case that  
 the inverse image of of  $0_1$  lies in  $A_1$  yes yeah it's not entire I mean yeah have to show that there is no it's  
 enough to show that the compl the complement of  $A_1$  somehow which is this where you have to understand  
 yeah you have you have to see that the complement of this of of  $A_1$  is kind of generated by just the res SE  
 the infinity section world of analytics St okay so then the inverse image of the finite part of zero Infinity  
 round in  $A_1$  right right okay yeah okay so let me explicit say that there's no condition that's we don't drag  
 inequality at all no which relates to this issue that once came up that two might be unbounded like the nor  
 of two could so you usually the two absolute value two less to two implies yeah so I don't ask any version  
 uh but if absolute value of two is less than equal to two you get a triangle inequality or not probably there  
 is a theorem that it does not the absolute way of two can be infinite in this there example like on this Gus  
 analytic base like I will I will try to in the remaining minutes construct such a norm on the Gus Bas ring  
 and in particular you can then look what what the norm of two is and it's some function like from from the  
 antic spectrum of a towards  $Z$  infinity and it's s active so there is some Locus where you have to two in all  
 right so the Ser method that I already uh stated on this boardly is that uh there is unique Norm on uh on  
 let's guess his space uh taking and I mean I could chosen any number between zero and one here and doesn't  
 matter which one because you can always because I didn't use anything about addition on the target I can  
 always just rescale the Target by some exponential  $m$  and still be all right um yeah I think I don't have that

much time so let me just give you an idea um and maybe just of the construction of right so uh let me start with  $L$  let's say  $x$  again some kind of fin dimensional locally compact  $h$  and why some then I claim that there's some kind of tanak Duality describing what maps from  $y$  to  $X$  are so turns out that to give such a from  $y$  to  $X$  it's just enough to give a fun from  $\text{POS CHS on } X$  towards  $\text{POS CHS on } y$  with some property but actually you don't really need to specify the values of all yeah let me first see so this is the same thing as  $s$  the fun  $F$  from that's the  $r$   $c$  for  $B$  groups on  $\text{grps on } X$  towards of  $Y$  sorry  $\text{TOS } X$  um so this should be uh temp functors  $M$  with all poits and they should be linear over  $DC$  uh such said uh stre locally on  $y$  uh the following happens so on  $X$  you have lots of it algebras namely for any close subset of  $X$  you can take the constant she on  $Z$  whenever  $Z$  and  $X$  is closed so this becomes an unimportant algebra the con she on uh Chad has a question so Chad has a question it's about whether  $X$  is supposed to be light uh yeah yeah that is my yeah yeah the triable yeah let's assume  $X$  is really just contained in some fin di space um right so so on  $X$  any closed subset gives you an important algebra which is um the one corresponding on um he wrote whatever you need to make it work think you wrote nice um this guy should be connected this guy should be uh can active yeah negative chological contains in no higher  $CH$  just super left the thing is that I mean if this actually mod Prof set then this would some you could basically check I mean any close subset is in the section of both my close subsets and so then these would be know just retracts of the of the unit and so why would itself be fine then this should of course also go to  $R$  of the unit object but this is a connective object so uh so if  $x$  was profinite uh and  $Y$  is some fine then there must images must really be be connected um in general like giving such map you can always do locally so the condition you have to enforce just be some strect local condition so it turns out that this is the condition sh locally on why you mean the pullback or the Shri it's the usual  $P$  usual pullback sh locally for the topology yeah and then it remains connective after because of okay and note that the full category of sheves on  $X$  is actually generated by these guys on the Cox so actually to define the standard funter you really only have have to declare these item poent algebras uh so so yeah so describing  $M$  from  $y$  to some uh such guy here is completely determined by specifying for each close subset of  $X$  you have to specify an IT algebra and if they all already connected then then you're good to go and the only thing you have to check somehow is that these it for algebras the the kind of Tor prodct Behavior exactly matches the interfection behavior of the close subsets of  $x$  and that's what it means to be attens of maybe it's also nice to mention that this is actually a set even though a priori it's an ana um right okay so to so uh we have to find some other important  $A_1$  in the Der category of this1 realize that space over um and you have to typically you have to find some that should correspond to the pre-image of like some interval from zero to some number  $R$  where  $R$  is so if you know where these  $PR$  should go then because there's this compatibility on involution you also know where intervals should go that go from somewhere to infinity and then everything else can somewh be written as some kind of and the co limits and intersection some in terms of of those so really  $SC$  Norm you just have to say uh describe what is the pre-image of  $Zer R$  in other words what's the locus in here where the the norm is that most are so disc of radius  $R$  around zero which should be some subset of  $A_1$  already which should be the analytic spectrum of sum ring join variable  $T$  over phone and this you can just write down um down because and this should be naely speaking this should be certain sums  $n * t$  to the  $N$  um and greater to zero which some over convergence so that should exist some  $R$  Prime greater than  $R$  so that the norm of a  $n$   $r$  Prime to the end goes to zero or um saying it's such differently uh you would want that uh whenever what is the nor inte okay okay okay well no the are now in a um what is but but what what you do is you can when you have a  $n$  sequence in a then then you would expect that the norm stays bounded because Bound by one uh um or by some constant um and so to produce such a thing you can some stop with a with a case with the  $A$  on  $L$  sequences and then multiply by some poers of  $Q$  to make it to make it this condition set  $F$  so actually this is um some Co liit off um some  $t$  with respect to some condition on  $r$  that I'm uh not able to write down but basically you can explicitly write down the corresponding item put algebra and is well  $T$  is you just have to think about how you can phrase this idea here in terms of the gas bace ring and the idea is that the absolute value of  $Q$  should be a half so yeah starting from the idea is that I mean you can just take any  $n$  sequence of  $a$  and there's a free algebra with a  $n$  sequence and then multiply the  $a$  by suitable powers of  $Q$  to uh to use an algebra like that and then then there's a computation you have to do is with that uh if you form certain tensor products of such algebra  $BR$  maybe centered at zero in Infinity then they the tender products uh come out in exactly the way you would hope for some of that matches intersection behavior of intervals from zero Infinity this is a computation you have to do uh which really uses that you're working some over the guess space ring uh

but then uh you can produce such involment maybe I should stop because I'm over time so I mean here you it looks like that you have to consider the power of  $Q$  such that yeah so the  $T$  here secretly was meant to be a rational number um ah okay so you have to make sense of this let I say for certain for rational numbers which means that when the power is divisible by something you put the qill to this and then you you do something in between yes I don't know yeah DUS will say is much better next time [Laughter] willing all okay very nice question yeah you can ask a question hi um I was wondering is there something like an analytic space theory definition using your analytic stacks and maybe some axioms and I'm asking because I noticed like you have different analytic spaces and they're theories and you claim that they're all analytic Stacks but if we wanted to put some say additional structure in an analytic space maybe we could do this by somehow having a more general definition of uh analytic space using your language as opposed to going Case by case so I don't know if did my question Mak sense it's not stacky right it's really more like a I mean you could try to restrict the kinds of gluings you allow that they're not too stacky but we were never happy with any of the proposed definition that we could come up with um Dustin we have a better answer I I didn't quite understand the question so so like you're saying these are there are some examples so I don't know I came here a bit late so someone said there was tic spaces that someone showed though well okay not someone Peter showed that they're analytic stacks and then complex analytic spaces and then somehow if you restrict to real analytic you get real analytic spaces okay so it seems all like uh uh I understand they have different theories yeah so my question is can you Define analytic space Theory using your language because if I wanted to put an additional structure in an analytic space perhaps I want a general definition for all analytic spaces put the structure there that's exactly what we're doing that's what analytic stack is so you can say uh I have an an analytic space Theory and this reduces to each known case so well I are such analytic space analytic space Theory I don't know what yeah I have to know what you mean by that I don't know what I mean I mean like something that would generalize all the yeah that's what we're that's that's what we're trying to do with this concept of analytic stack yes you shouldn't be scared it shouldn't be put off by the fact that we change the name from space to stack that's that's basically a technicality but that that's the goal so you have a definition like an analytic space is an analytic stack that well you could try yeah for example yeah that's my question like do you have a definition like an analytic space is an analytic stack satisfying some axioms do you have this well we have too many maybe well what I mean is like do you have the ult an an ultimate one kind of like the umbrella one from we we tried but we could never I mean you could just say that the the funter of points takes values in sets no  $F$  ones are not okay sorry what if you ask that it takes than sets and even fim analytics STS are not not fine well yeah but the classical ones are so anyway the classic the  $F$  line like takes a ring to it's underlying like because the test category consist of derived things you can yeah yeah yeah that's that's true that's true no but anyway you don't have a definition of analytic but I mean you could ask that it's a CO of  $f$  along monom that's the condition sorry what's the condition you could ask that it's the Coit of Fes along monom fers okay okay thanks

## 20. NORMED ANALYTIC RINGS (CLAUSEN)

[https://www.youtube.com/watch?v=wk\\_wInYTasQ&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=wk_wInYTasQ&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

Okay so now I'm continuing the discussion from last time I want to start by maybe going over a little bit of what Peter said perhaps adding some details or some explanations so the main topic is uh normed analytic Rings um so we had this now we've officially defined this framework of analytic stacks that we're working in um and we saw uh that there was an embedding of uh well light condensed anoma uh not well at least a functor we saw that there was a functor to analytic stacks and it's induced by so these things are generated by light profinite sets so say  $t$  and to one of those you assign I'm just going to write it I know this is probably overloaded notation but you just you have this discrete ring of this is a discrete ring of continuous functions on your profinite set with values in the integers um it's a discrete ring um and you can view it with the uh maximal analytic ring structure so uncompleted uh so you look at all condensed modules over this discrete ring or derived condensed modules over that discrete ring and um that's the analytic ring uh that you associate and then its spec is an analytic stack um and uh the thing that makes this functor nice is that this uh this assignment here um this uh sends hyper covers uh to shriek hyper covers uh satisfying uh shriek  $\text{Coh}$  descent or descent uh so so apparently I don't we Pro that those are equivalent sh descent  $\text{C}$  descent this the talk oh um it's the exact same argument to you mean in the hyper case or okay I remember Peter refer to something like this in some explanation so the when you have a you can formulate one that the categor is equivalent to the one to the inverse limit by sh or the direct limit by and those are it's a purely formal fact about PRL so proved by lur that those are equivalent so it doesn't need anything on the shape of the diagram or or whatever yeah um so it's yeah uh so these were the things that uh these were the things which defined the notion of analytic Stacks um but it also preserves finite limits so for example pullbacks well let me say finite limits so if you take a pullback of light profinite sets that's a that's just a a filtered inverse limit of pullbacks of finite sets and then it goes to a filtered co-limit of uh the same situation for finite sets so this reduces to finite sets and for finite sets it's kind of completely clear um and that so and those two facts kind of imply that this functor here so that that the induced functor here that you get from this uh so this is a cimit preserving uh Co limit and finite limit preserving um which is nice um so um now I want to mention uh that uh that uh there's a paper on the archive by Rock uh gregoric uh so so he studied is the analog of this uh with a analytic stack replaced by by uh fpqc  $\text{C}$  sheaves um in in usual sheaves fpqc sheaves in in usual algebraic geometry over a field um and he wrote a nice article which he recently posted on the archive um which goes into some detail about properties of well I mean he's working in a slightly different setting but much of it is much of it is the same um yeah so I would recommend reading that uh okay um right okay and then uh so the the other example that Peter discussed was uh example of a light condensed set let's let me start with this setting so let's say uh  $K$  is a compact how is  $\text{DF}$  I think a question the how does  $\text{Rec}$  topological space from the associated antic  $\text{St}$  is it the set state of point in some suitable sense so the I don't see that there is a well-defined notion of underlying topological space of an analytic stack um so us yes that's right it's a you have you have an underlying uh light condensed set um but um I don't see an underlying topological space Oh yeah I I guess then you could map this to topological spaces okay so you have underlying so when you have a stock uh no the stock is is something with it's a generalization of ring spaces here but what is well we're not really using like a diagram You cover by  $\text{SP}$  by kind of formal spec of analytic rings then you have the the the world superal diagram things and then how do you get Space how do you get the you cannot take spec in the no although I think you can consider all open substat maybe it makes sense in your set to consider open you could you could use monomorphisms of analytic Stacks to try to there's different ways to okay there's actually different ways to try to extract an underlying topological space from an analytic stack and um well yeah I'm not sure it's worth saying anything completely General in the world of arbitrary analytics Stacks but um well once when we introduce this notion of normed analytic ring we'll see that when you have a normed analytic ring then um there's a nice way of extracting underlying topological spaces for things over that normed analytic ring let me let me continue the story having unsatisfactorily not addressed the question um so let's uh like so if you have a compact house dwarf metrizable and let me add finite dimensional and um this actually uh is the same thing as saying that  $K$  is embedded as some closed subset of uh some finite product of copies of the closed unit interval um so these things are very easy to imagine um uh then well



then we can view this as a as a light condensed set in particular a light condensed anima and then we get an associated uh analytics base and Peter made a claim about uh the fun of points for this for such a for the analytic space associated to such a k um so um well or maybe there's there's two claims so and two claims so the first is that so first of all what's the most important invariant you have if if you're given an analytic stack it's its derived category so we saw that shriek implies star descent which means that the derived category of a stack is well defined by pullback upper star pullback um and uh this um this D of K I mean the analytic stack associated to K by abuse of notation uh this is just sheaves on K uh with values in derived to bicon groups um what's that uh thank you yes thank you and more generally if you base change this stack to an analytic ring then you put uh D of R there um uh right uh and the second claim is that if you want to map say Spec R for an analytic ring R to this k um uh this is equivalent to giving a symmetric monoidal functor uh from sheaves have now not condensed aelan groups uh to D of retric monoidal cimit preserving uh funter uh such that uh shriek locally on r on Spec R what's that oh thank you yes yes yes DZ linear yeah um such that Shri locally on R we have that uh F of the connective part so she K with values and say d of Z greater than or equal to zero um lands inside D of R greater than equal to Z the connective part there um so how much the fin where is the final dimensional use is it some technicalities about I don't know you can Define the analytic stock for any case without Dimension right right so the finite dimensionality um I we're going to see uh where the finite dimensionality comes in I'm going to sketch the the argument for this um so okay so well the first remark I want to make is that both well the left hand side is a set well maybe I'll say both are sets um so we're in this world of analytic Stacks which is based on some sheaves of Ana so a priori everything is an Ana I mean your K is implicitly some co-limit of things and could be introducing higher homotopy but because of this property that this funter preserves pullbacks um so well so um so this is a monomorphism uh in condensed Ana in in light condensed Ana uh so by the property of pres and monomorphism means when you pull it back along itself you get itself again nothing fancy then you get the same in analytic stacks and that tells you that um if you have two maps can agree in at most one way so you have a set and not a generala um and over here the reason this complicated looking space of functors is a set and not anama is that um you know for well it's D of linear so it's actually the same thing as just if you take I don't know sheaves of an and you just ask for a CO limit preserving functor like this um and then this is generated by these uh like the representable ones um and uh but well if we um uh but uh so in particular over here everything is determined by what it does to the free things on these representable guys um but then we can pass to the complimentary closed and we and it's also determined by the um so by the um so this this thing is determined by uh where it sends uh you know the the the con and chief on a closed subset um and then this should just uh should be an item potent algebra in the Target or should go to an item potent algebra under f it should go to an item potent algebra in the Target um and um and then uh you just have to have some conditions that are satisfied so you just have to specify a collection of item potent algebras in the Target and then some conditions have to be satisfied so let me write down here so um so this is what implies that uh left hand side is a set but uh for in the right hand side um is just the set of uh maps from you have closed subsets of K to item potent algebras in D of R uh such that um uh inter filtered intersection uh goes to filtered Co liit um and Union uh so uh Union goes to well the analogous construction so there's a if you have a union of closed subsets you can kind of use a Myer viat Taurus to express the the uh constant sheath on the Union in terms of the constant sheath on the two pieces and the intersection and that gives kind of an algebraic formula for what the value on the union should be which you can write down just at the level of item potent algebras and you ask that that that Union goes to the that algebraic construction you have here um and then and then you still want this uh this condition here but that's just another condition um so this looks like sh this looks like Theory K the direct category she on K yeah yes this uses the fin Dimension yes that does yeah okay and then the the equivalence with sending various pieces the equivalence of giving something on the on CL on H on closed outpaces is it does it use this doesn't use no so when you work switch one with shifts or so far I haven't used finite dimensionality if you take this in the sense of lury then everything I say Works uh generally so for I think an arbitrary topological space k but um certainly for a compact house door space um uh okay um right and so the the point is that item poent algebra is form a this is actually just a post set um so there's a a prior it's an Infinity category but one checks that that it's just a postet so a map if it exists is UN you can uniquely write it down um so then we just have a the right hand side is just you have you just have to give a map of poets to specify all of this um a map of postet satisfying some simple conditions to specify such a symmetric monoidal blah blah

blah blah blah functor and so on um the next thing to note is that okay we should now now I'm claiming these two sets are in bijection and I should first write a map and the map uh the map in this direction is obviously going to send  $F$  uh to  $F^*$  where  $F^*$  is um so well a priori  $D$  of  $K$  well  $D$  of  $K$  is this thing but we can restrict to the full subcategory where um you require the values to be discrete um or I could say  $\text{dcon } Z$  linear functors from uh sheaves on  $K$  with values and  $\text{dcon } Z$  um and then I so but I should that gives one of these things but I should maybe explain why the uh that's a derived category of condensed abelian groups light condensed yeah light sorry light yeah okay so you have a uh uh so  $F$  goes to uh spec  $A$  by definition spec out okay is  $H$  is a map from the first you have to give a map on the Ring of but this is part of this functor the map on rings is included in  $KN$  the no so what is it your  $F$  from  $\text{spe } R$  to  $K$  means that you have a of analytic  $R$  well it means you have have some say shriek hyper cover of this uh and a map of that shriek hyper cover to the hyper cover of this by profinite sets oh because  $K$  is not okay yeah and uh okay um and if  $K$  is totally disconnected then it is the same as a map of if  $K$  is totally disconnected yeah so yeah so maybe yeah if  $K$  is profinite then the left hand side is the same as uh map of rings uh from continuous functions on  $K$  with values and  $Z$  to uh just the underlying ring of this uh an analytic ring that's by construction um that those are the same um okay and this actually explains why this condition is satisfied I mean for arbitrary  $K$  because for arbitrary  $K$  you can  $\text{Sur}$  from a profinite set  $T$  and then this is a surjection so by definition there will be some shet cover  $\text{Spec } R$  uh where you factor through a map to  $T$  but once you factor through a map to  $T$  then you're uh your  $F^*$  is just being induced by this functor but this is just a filtered colimit of uh of um uh yeah filtered colimit of  $C$  of like  $K_n$ 's where this is finite um and um well what do I want to say the um so the the so well what what I want to say maybe yeah I don't know if that's the right remark to make so in the case when  $K$  is profinite then so you want to prove that the image of every connective object is connective what do you know you know that the image of the unit is connective because the unit has to go to the unit in  $D$  of  $R$  which is this underlying ring which is connective by definition um but then uh on any clopen subset we then have to go to something connective because it will be a retract of that unit um but then the clopen subsets generate the connective part under co-limits because they generate all open subsets under co-limits and then everything is generated under that by Co-limits so everything will be everything uh in this thing here in the case where this is profinite everything here will be a colimit of retracts of the unit objects and therefore we'll land in here um in the case where case profinite um so in this case the connectivity is automatic um so that's why this is a well-defined map um and uh if you want to go backwards so given such a symmetric monoidal functor or such an association with its potent algebras um it's enough to go backwards uh assuming this condition because you can work shriek locally because both sides are shriek sheaves or sh satisfy shriek descent so to go backwards um uh conversely uh if you're given such a symmetric monoidal functor um let's see on the connective level um then you want to produce uh this map here um what you can do is you can take uh oh sorry this was okay uh you can take this hyper cover by take a hyper cover by profinite sets or just well I guess just a cover is fine once you start with a surjection from a profinite set oh sorry what am I doing uh uh um each of these will automatically be profinite sets because they'll be closed subsets of the product of two profinite sets um uh then um uh then uh then you get a corresponding uh diagram of rings uh commutative algebras uh in  $D$  of  $R$  greater than or equal to zero so you just take so if this is  $\pi$  uh so  $\pi Z$  and then this is  $\pi 1$  and then  $\pi 2$  and so on so you have um  $\pi 0$  lower star of the constant Chief  $\pi 1$  lower star of the constant chath and so on um have the constant Chief um and these things are all going to be connective um and in fact um you have AA formula uh which gives that this is a this thing uh is the I a check the co-check nerve I don't know of just the first map um so because these maps are proper uh any map of pro you know compact house door spaces is proper for using proper base change you can see that uh like the whole the whole diagram is just determined by the first part um so um uh and now you can apply your symmetric monoidal functor  $F$  so you can take uh  $F$  of this  $\pi 0$  lowest  $r$   $z$  and now this is a commutative algebra object um uh well first of all I should be saying  $r$   $\pi$  lower  $St$   $Z$  but it's actually concentrated in degree  $Z$  yeah yeah um um and then in fact you can say in fact animated um so it's not it's not difficult to produce the animated structure but let me not get into it um and then you take uh uh to  $R$  Prime to be the analytic ring defined by um  $f$  of  $\pi 0$  lower star  $Z$  modules uh and then um um just the induced analytic ring structure of you you just take this ring object in here and look at that underlying ring and just modules over that ring in in the thing you already have um and then you can check uh that this procedure uh induces a map from the check nerve of this  $RP$  Prime mapping to  $R$  so spec so spec  $RP$  Prime mapping to  $R$  to  $\text{Spec } R$  uh and then by the construction

you get an uh so is this  $z$  under  $l$   $z$  the same  $Z$  I'm sorry where are we on the top board yes  $z$  under  $l$   $z$  yeah  
 and then have  $Z$  right are they the same they are not the same so uh this one maps to the constant  $\text{Chi } Z$   
 and uh that one's the the the co the the co co- fiber when  $Z$  is the complimentary closed subset to the open  
 subset  $U$  so they're not the same but they determine each other in a canonical way so you said something  
 to so this is actually a single sh in any case it's a some  $f$  because it's def is a symmetric mon you get some  
 Comm object some der like in you say there is canonical way to view it is a simpl right right and but this  
 is not I think in characteristic zero it should be yes in characteristic zero it's it's clear not because of the  
 abstract way you define you have the condition it's not clear that you that's right yeah it requires a little  
 extra argument so you can see for example that yeah it requires an extra argument but so probably the work  
 that you're doing with Maxim will make it more clear because so I so what what uh what what is being pro  
 probably carried out at the moment is a kind of a categorical perspective on animated commutative rings so  
 they you can single out what properties of the derived category of an  $e$  infinity ring or the  $C$  you should  
 just say category of modules over an  $e$  infinity ring what structure do you need to put on that to promote  
 the  $E$  infinity ring to an animat uh thing and it's something like derived symmetric power functors and um  
 you have them on uh you have them on I mean maybe there's an even simpler argument sorry I think in this  
 case there might be a simpler argument so you want to produce a sorry uh let let's let's leave that aside for  
 the moment and address it at the very end okay so yeah but yeah sorry I um okay so where are we ah yes so  
 then what we get is a map from the check nerve of  $\text{Spec } R$  Prime mapping to  $\text{Spec } R$  uh to the check nerve  
 of uh  $t_0$  mapping Decay and um then we're going to be then we'll have produced a map by by by kind of by  
 descent we have produced a map from  $\text{Spec } R$  to  $K$  so then if we just so we as long as we know that this is a  
 cover okay um but note that this is a proper map by construction we built it as something where the so to  
 speak the completeness condition isn't changing when you go from  $R$  to  $R_P$  Prime um you're just extending  
 the ring so by the Criterion we had for shable uh things it's enough to see uh that the unit object which is  
 our triangle uh lies in the category generated by the image of a so in the image of the the forgetful functor  
 uh sorry  $D$  of  $R$  Prime to  $D$  of  $R$  and for that by applying  $F$  it's enough to see using yeah the lower shriek  
 but lower shriek is lower star so it's really just a forgetful functor from and the image this means the image  
 and closing by closing by say finite Yeah by cones and retracts and so finitary operations yeah okay um then  
 applying  $F$  it's enough to see that the constant chief on  $K$  is in the subcategory generated by the a lower  
 star functor so H sorry sorry sheaves of  $t$  on  $T$  values in  $DZ$   $U$  so  $P_i$  lower star sheaves on  $K$  values in  $D$  of  $Z$   
 okay um or in in in uh or in ail Matthew's language we need to see that uh  $P_i$  lower star of the structure chif  
 on  $t_0$  is descendible right um now remember that we this was kind of we could choose this arbitrarily so it's  
 actually enough for me to produce a cover by a profinite set for which I can check this descend ability um so  
 we can reduce to by pullbacks we can reduce to  $k$  equals like  $01$  to the  $N$  um and then I'll I'll discuss just the  
 case  $n$  equals  $1$  because it's simpler to write down uh so um gu you can also reduce to this case what's that  
 yes that's true by by kith yeah you can also reduce to this case yeah that's true yeah so this means when you  
 take the fiber  $P$  some so there's some yeah we didn't get into much details about how to make these sorts of  
 arguments but aille gave some tool kits which are nice um so then in this case you take the the usual kind of  
 caner set so you have the so I'm going to produce my cover of the closed unit interval by by uh by the caner  
 set so what I but let me write it in a specific way so I'm not going to not going to write so profinite said  
 a priori you want to write it as an inverse limit of finite sets but sometimes you can instead write it as an  
 inverse limit of compact house door spaces and in the limit you just happen to get a Prof finite set and that's  
 the best thing to do here so what we can do is we can take you know uh the two halves of the closed interval  
 and and their disjoint Union is a space mapping to the unit interval um and then we can do the same thing  
 again this the the base to the base to  $\exp$  of the yeah so then that's a so then we have a sequence of spaces  
 mapping to the unit interval where in the inverse limit you actually just get the caner set um which is then  
 forming a cover of the closed unit interval so what does this mean on the level of these push forwards so  
 these the push forward from the caner set will then be the sequential Co limit of the push forwards from each  
 of these um now there's a general fact about this descend ability that if you have a sequential co- limit it's  
 enough to establish descend ability with uniform exponent of nil potent for each of the finite ones um and  
 for each of the finite ones you have descend ability basically because you have a my vus cover for the closed  
 subsets and there's the point is the reason you get a uniform bound on the exponent is because in each case  
 there's only double intersections that you need to be concerned about and no triple intersections so in the  
 end you get something like exponent of  $n$  pot two for each of these individual things and then three or four

for the the co-limit or something like this each and the composition of the individual thing is no when you have the individual in because to adjacent one you have only adjacent and then so when you divide  $K$  you claim that this is a ah okay and then there I understand that when you go to Power like 01 to some power then the the the estimate will become where else yes and therefore like for the cube you cannot do it you can't expect anything that's right yes so this is where the the finite dimensionality really comes in so it plays a sort of a technical role in being able to ignore the difference between this and the derived category of shav of a being groups but um but uh this is the this is really where you need finite dimensionality okay uh okay now let's come back to the the question about e Infinity versus animated commutative so um now we have this we have this object um oh wait is this going to work just a sec oh maybe this isn't going to work uh no let's not come back to the question about e Infinity versus animated commutative with my apologies um okay let's actually um right oh I want to oh so at the end of this I want to make a remark um I maybe it's good to say that at least in this example of this  $H$  he can ow five by hand structure because you only have to do for  $A_1$  because sequential of yes that was the argument I was yeah that's the argument I was going to suggest and then I was thinking in my head um about why for item potent algebras it's automatic and it wasn't quite clear to me um but yeah but I think you can do it by hand I mean if you knew the categorical structure well we should we should think more carefully about exactly how to do this um and report back next time uh so okay um but let's move on I want to make a small remark before we move on um remark so by similar arguments or basically the same arguments um if you have a a map like this um then or sorry if our functor like this it's equivalent um a prior you're saying that you have this connectivity estimate shriek locally on  $\text{Spec } R$  um but um it's equivalent to ask uh that um so uh over a proper cover a cover by proper Maps you get the connectivity so you don't actually have to when you're wor there's over  $AR$  that mod give you an ring structure on on your  $R$  could you say that one more time Peter don't connect algebra over any antic  $R$  yeah then you can use this to Def new notion of complete modules which are modules over the sky oh yes uhuh I what you call thetic ring always I over  $H$  that that that seems to work indeed yeah yeah yeah um but are all of the intermediate guys connective as well yeah I guess they are well okay yeah so yeah um okay or maybe connectivity isn't even important there anyway all right uh oh yes um and another remark is that this will actually be a little bit useful now I've not lost ah I'm sorry oh we're over here in the setting of number two in the the the theorem here okay so so let's say you want to produce a map like this by this procedure so you want to produce a symmetric mordial funter and then you have to check this annoying condition that on some shet cover of  $\text{specr}$  you have this connectivity condition I'm saying or or let's yeah or I don't know you have no I'm no I'm making a different claim sorry that uh let's say you have a map like this then you know that shriek locally you get uh this but in fact what we see in the proof is that um after a proper map even after just a proper map from a proper cover of  $\text{specr}$  you you can ensure this condition this comes from after proving the equivalent yeah yeah yeah yeah yeah not a prior not a priori exactly yeah um another remark um is that if you have  $Z$  closed uh closed inside  $K$  or this is actually more General but um and then  $U$  is the complimentary open uh then they are also each other as complements uh in analytic Stacks so these two do determine each other the associated analytic Stacks do determine each other in the naive way I.E if you're given  $Z$  and you want to know  $U$  as an analytic stack its functor of points is you just map to  $K$  such that when you pull back to um to  $Z$  you get the empty analytic stack that's the same thing as mapping to  $U$  and vice versa so that that you can check on profinite sets where it's um where it's quite Elementary um okay uh so now let's get to sorry why did do right see toag um how else did you want me to write them only what what it means doesn't mean anything um it's just because the well I wanted to write for example I wanted to write this one this one right above this one because I'm saying the these two end points map to the same point down here so I wanted to stack them vertically like that maybe that's the maybe that's the reason does that make sense yeah yeah okay so now we get to a topic that I think is fun which Peter introduced last time this kind of norms on analytic rings so let's say that  $R$  is an analytic ring so definition uh is that a norm on  $R$  is a map of analytics Stacks um from the algebraic  $P_1$  over  $R$  which is something you can build over any analytic ring by just base change from the trivial case with algebraic geometry um to the closed interval from 0er to Infinity which is a condensed set and thereby an analytic stack um so let's call this map  $n$  and then such that and then I'm going to give some conditions um which are going to be different from the ones that Peter gave last lecture so a PRI I'm adding an extra condition and maybe slightly rephrasing um another of his conditions and we don't know whether they're equivalent but certainly uh we want all of the properties I'm about to list so let's take this as the official definition

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the one needs to justify that such a map indeed exists um let me do that now so uh so so y TS goes to TS is well defined on  $A_1$  um so the claim uh uh is that uh so  $A_1$  RN is a subset uh well sub monomorphism admits a monomorphism to  $P_1$  by definition another thing that by definition admits a monomorphism to  $P_1$  is the algebraic apine line over  $P_1$  and the claim is that this one's contained in this one ALB line is is the you take the polom bring one variable yes over over  $R$  yeah over  $R$  in which so in other words you take for the condensed you take just head po long in in a stupid way I mean without any special and the category is the the s i okay I would say yeah so it's the thing it's an it's an apine analytic stack this  $A_1$  and it's um just given by again keeping the same class of complete modules you already had in  $R$  and just adding the polinomial variable as operators H why is this true yeah so um right so uh so the point is that um so we want to produce uh so you have to check it on the two charts of  $P_1$  one of them is already  $A_1$  R and the other one is the other  $A_1$  R yeah to some Inver the operator when you know that you right so so here's what I'm going to say so we know that a norm of infinity is equal to Infinity um and uh so what does this imply so for Infinity we have um this implies that the uh infinity section of  $P_1$  uh is uh a module over or an algebra over uh is an alge albra over um so Norm upper star of the structure chief of infinity yeah one way to think about imp anal section and then there's General that many and Serv close at  $Z$  and analytic St that in the set of sches where scheme is endowed with the the trivial analytic okay so let me try doing it the way I think it's basically equivalent but let me let me try doing it the way Peter was suggesting so let's let's uh let's abstract a bit let's move Infinity to zero and Abstract a bit do it for like it's  $P_1$   $Z$  so it means you can this  $P_1$   $Z$  and so suppose given an analytic stack over  $A_1$  um such that if you pull back uh to zero section uh you get uh the empty set so it misses the zero section then the claim is that this uh F factor is through uh GM uh r and it's part of a more General claim that Peter was making about a a scheme and a closed sub scheme but okay scheme with the with the kind of trivial analytic ring analytic kind of yeah let's say usual a fine scheme is the kind of the most trivial analytic yes yes um so um right um right so fact factors oh so and the reason for this is you can think of maps like this um well suffice it to treat the  $C$  case where  $X$  is is itself apine um and then Maps like this you can think of in terms of symmetric monoidal functors so so if  $x$  is  $D$  of uh I don't know a then you can think in terms of the corresponding pullback functor from  $D$  of  $A_1$  uh to  $D$  of a um and what do we know about this pullback functor we know that it kills uh uh kills the structure sheath uh of the origin but then it's just a purely algebraic fact that um if you kill the structure chief of the origin then you factor through inverting the the parameter so so the structure sheath of the origin is just structure sheath on  $A_1$  modulo  $T$  the parameter  $T$  so if you kill that thing then you kill everything that's built from that via colimits and so you don't see the difference between an object and the result of inverting  $t$  on that object and then that exactly gives the factors through yeah yeah so thanks Peter I think that's a much nicer way of of saying it um everyone good okay uh so right so that's the claim and that implies that this map is well defined because certainly uh the multiplication is well defined on  $A_1$  but um so maybe then I could put I put the  $A_1$  here a priori uh and a priori have only the  $N$  going here but then a posteriori if I require this diagram to commute then it follows that actually this uh this lands inside  $A_1$  analytic because by definition that was the pre-image because because this ma factors through zero Infinity okay all right uh so oh man all right uh I want to get to something fun yes please no no please please uh so the fact that we denoted is and so the Norms here should somehow correspond to like correspond to Norms and analytic the notion of analy ification um what yeah so I'm going to give some of the motivation at the end so but let me finish with the axiomatics like building on that like so like I remember last time like I think R SCH said like Gaga really was like at the like he he noted down Gaga as like like as an isomorphism of stacks yeah and uh and someh that was some sort of analy ification so does that also like correspond to a norm then so this question I suggest you uh keep it for later yeah yeah um so uh yeah um let me finish with the axioms uh 1 2 3 ah so four okay so axium 4 now so maybe now's the a time to say a bit about motivation so the um so what if you have an analytic ring what we're going to try to do is we're going to try to say if you have an analytic ring you want to try to build some geometry over that ring um but it's hard if you're just if you're just given an analytic ring and you don't know anything more about it or you don't have any extra structure on it it's kind of hard to build analytic geometry over it I mean basically all you can do is you can do this trick of importing algebraic geometry for an arbitrary analytic ring that's more or less all you know how to do um uh so what we're going to be doing um and what well one what's one measure that you have some good uh analytic geometry it's that you have some nice subsets of the Aline line um and nice in the context that we're discussing here means for example shable so that the six functor formulism works and

then it kind of really feels like you're doing geometry um in some more or less traditional sense um so but uh again on a general analytic ring you you don't know how to write down any uh interesting shable subsets of the apine line so you have to give the you have to give yourself some of them and that's going of that's the point of this notion of normed analytic ring we're giving ourselves basically uh discs of certain of of some arbitrary radius inside the Aline line um and they will turn out to Define shable subsets of the apine line and then we can um we can get started on doing geometry that resembles some sort of traditional analytic geometry based on open discs or closed discs or what have you but you have to have this extra structure on your base before you can get started on that game if you don't have a notion of a norm on your ring you can't start talking about closed discs and open discs of certain radius so that's what we're doing right now but um we already have some sort of things that kind of seem to function as a we had we've already seen certain versions of the unit disk so for example we spent a lot of time discussing this basic module p uh which recall was this uh free free module on  $N$  Union infinity modulo infinity um and and of course you can base change it to any ring uh any analytic ring even um um and so well what what do we know about this guy well we know it does have a map from  $R$  bracket  $T$  so geometrically uh so let me just this is going to be not not good notation but let's let's set uh  $D$  equals  $\text{spec } P$  so it's some conversion of a dis um and then  $Dr$  will be the base change to  $R$  again your was a usual algebra structure right that's right yeah and we've we've also discussed the algebra structure on  $P$  which makes this an algebra map um so um so this uh so  $D$  does map to the Aline line by this uh by this map here um but as we've also maybe discussed uh so it's a but but it's not a monomorphism so it's not really a subset of the aine line so it's a it's a proper map in our setting so it's shakable but um it's not um it's not a monomorphism so it's not really a subset of the Aline line so what does it look like in examples um uh so in general so we have well this this map from  $RT$  to  $PR$  uh this induces an isomorphism on mod  $t$  to the  $N$  for all  $n$  so uh this this uh this map which is not a monomorphism it's it is an isomorphism on the formal neighborhood of the origin so at least in some sense you should expect close to the origin this is a monomorphism and then what happens is it kind of blows up as you move away from or it can potentially blow up as you move away from the origin what uh  $T$  goes to like the The Rock measure concentrated at at one um okay you use the multiplication on  $N$  or you use the addition on  $N$  to yeah to give the multiplication on  $PR$  yeah um so uh right so in particular you get a factoring like this so this um and so that's that's generally true that you have a diagram like this um where this is the canonical map um and in in most examples so this  $p$   $r$  mapping to  $r$   $t$  is a monomorphism is an injective so in most examples this lives in degree zero and this map is an injection so this is some kind of sequence space and um what is the condition that the sequence satisfies well uh so in the aelon category the which is the heart of  $Dr$  so in in in most examples this lives in the heart this also in most examples lives in the heart I mean I guess maybe I should be using this notation but I'm being a little bit sloppy here um yeah and then this is just an injection so to speak um so so in most examples  $PR$   $R$  is like the set of sequences  $r_0$   $R_1$  um satisfying some summability condition so such that if you termwise multiply by a null sequence uh you get a summable sequence uh where the notion of summability depends on the analytic ring structure but at the level of this discussion you could imagine for example are being the real numbers and summable means usual absolute some the absolute values you get a finite number that's um that's not actually a special case but it's it's close enough and it it serves the purposes for this this discussion um and this is kind of just by the universal property of  $PR$  that this is the the correct interpretation because  $PR$  is Maps out of  $PR$  to an to an  $m$  in  $Dr$  these are supposed to correspond to null sequences in  $M$  by construction um so um you know it's the kind of thing which when paired with a null sequence uh you get an element in  $m$  and so the idea is this procedure yeah so we null sequence and then yeah well if you well I'll let you maybe maybe me trying to explain it is not as helpful as um all that is any analytic ring I but but this is not precise mathematics here yeah but in this Norm business  $r$   $r$  is just a discret ring in this Norm well no no in the norm business  $R$  is an arbitrary analytic ring am  $m$  is then  $p$  is a okay so you you say that usually it's say in object in the category of our yeah usually that's right yeah yeah and this is the what you call  $M$  uh  $M$  oh sure well I mean yeah I could yeah I mean so so I was I was saying that this is the interpretation you get in practice where the notion of summability depends on the analytic ring structure and the way you see that this is the correct interpretation is by thinking about what it means to map  $PR$  to  $m$  and um so like for example the most basic thing was it would be if you map  $PR$  to  $R$  triangle um then that's the same thing as giving a null sequence um but then um if you think in terms of what happens when you restrict to here you have some  $Co$  if you have some coefficients in a polinomial so if it terminates if you

have zeros after while then what you're doing is you're just summing the null sequence times those things to get an element in our triangle and then you imagine that that  $S_u$  this is saying that that summing should make sense for something which is not necessarily eventually zero and so this is kind of the interpretation you should give that um right uh uh and what so so and so if for example  $R$  is  $\mathbb{C}$  with the gases or liquid liquid analytic rank structures what you see is that um you have if you look at holomorphic functions um on the usual usual ring of holomorphic functions on the uh closed unit disc meaning they overconverge on the closed unit disc um then um then every one of those satisfies this if you look at the coefficients of the power series it will satisfy this summability property because your power series has a value at one so the you know the thing the sequence of coefficients better be summable so you'll get a map like this um uh so this summability condition is weaker than the condition which defines these things but it's stronger than the condition which defines a holomorphic function on the usual open disc um so uh if you think in terms of your usual um your usual complex analysis then that's where your  $p$  is sitting it's sitting somewhere between the open disc of radius one and the closed disc of radius one um and that's the kind of behavior we're going to be aati in condition number four yeah so this is something like a what uh  $X$  exponential decay and this is something like exponential growth and this is something like  $S_u$  ability so kind of makes sense that it's it's in between the two um okay uh was a bit of a long-winded explanation um what's I haven't written it yet I've just given motivation um but yeah so what what you want to think is in these kinds of normed analytic ring settings this well okay so then condition four is uh so condition four uh is split into two parts well the the first part is that um if you have you have  $D$  mapping to  $P_1$  or  $D_r$  mapping to  $P_1$   $R$  uh mapping to  $0$  infinity um via the norm uh you want this to factor through uh the map up to  $0_1$  that uh corresponds to this um saying that this notion of unit disc is sandwiched between closed and open um um but you also want that this if you take  $D_r$   $U$  mapping via the norm map to  $0$  infinity and if if you pull back to uh  $0_1$  uh the open well a half open segment from  $0$  to one um then uh you should get an isomorphism over here so uh so uh sorry uh yeah so Norm inverse of  $0_1$  uh uh sorry ah shoot um I need I need  $P_1$  to be in there thanks yeah oh sorry Norm um so then we have um so I want that um yeah wrote an inclusion there sorry sorry oh I wrote I didn't mean to write an inclusion thank you yeah thank you what I want to say is that uh so this uh yeah well let's say this map is an isomorphism over the locus uh uh given by the open unit disc so so this is like some sort of proper map which blows up along the boundary of the open unit disc um but on the interior of the open unit disc it's an isomorphism so uh this is some something stronger than saying that you have a map like this um and the nice thing about this stronger condition is first of all you can check it in practice and second of all it's just a condition whereas giving a map like this is a prior structure um so can I think of this nor edian nor you can think of it as non archimedean or archimedean at will we haven't enforced any compatibility of the norm with addition so it's yeah last last time yeah so this is it is if you look for real numbers for example right so the um so this was the so the the cosmetically I changed the discussion of multiplicativity from Peter's lecture that's that's just a cosmetic change this is um potentially more serious Peter did not mention this axiom last time and we're not sure whether it's um a consequence of the other axioms but um yeah as Peter mentions if if you're over a base which solidifies to zero so for example if you're over the real numbers then then you can prove this from the other axioms but we're not sure whether which solidifies to zero it means when you base change to solid if you base change  $\text{Spec } R$  to  $\text{spec of solid } z$  uh then you should get the empty analytic stack if if that condition is satisfied then then this this this condition follows from the other conditions but so can you give example of saying that solidify to zero yes the real number is solidify to zero so any analytic any analytic ring where the underlying ring is an algebra over the condensed ring  $R$  uh will'll have that property so so again so you have the something going to  $\text{spec } Z$   $\text{spec the Sol}$  but all the the non archimedean seem for are not so over there potentially there so as far as we know so far there could be some weirdo Norms which don't satisfy this condition but yeah they're not any of the usual ones the usual the the ones that don't satisfy this unless it's some really extremely interesting new mathematical phenomenon which seems unlikely they're probably just very pathological we're taking this as an axiom okay and then you could ask the question whether it follows from the other AXS and um okay so um so what is a so what is a norm again so what what is a norm giving you so if you have a norm on an analy so these are the axioms um so this gives you so again in terms of the discussion of maps to uh compact Hausdorff spaces um this gives you for every for example uh for you get Norm upper star of the constant sheaf on  $z$   $r$  um so for if you take  $R$  non-negative real number uh then you get this item potent algebra uh in uh  $D$  of  $p_1r$  um uh and this course this uh the interpretation of this should be overconvergent uh functions

on the closed unit disk uh of radius  $r$  and the overon convergence it it has to be overon convergent there's no other possibility because note that the constant sheaf  $\mathcal{O}_R$  is the filtered  $\mathcal{O}$  limit of the constant sheaf  $\mathcal{O}_{R'}$  for all  $R'$  bigger than  $R$  under the Restriction Maps so the overon convergence has to be built in like if you imagined that you were sending this to some some version of functions on some version of the unit disc or unit disc sorry uh I mean disc of radius  $R$  some version of the disc of radius  $R$  then that version would be forced to be the over convergent one um because of this property here uh and because  $\mathcal{O}$  and because yes because it's a pullback functor so it commutes with co- limits um all right um and okay so and actually the data of these guys uh given the axioms this determines  $n$  because it's very easy to classify the closed subsets of the of the of this interval and um you only need to know about things less than or equal to something and things bigger than or equal to something but the axiom about inversion gives you one in terms of the other so you just have to give these things uh subject to some some simple conditions in order to specify a norm Okay so the next topic is classifying norms and uh I mean the the uh I don't mean like as in classifying space or whatever I mean how to classify Norms on a given analytic ring uh yeah so let me start with one lemma here question that the usual yes and yes there is yeah um right so uh LMA so so given a norm on an analytic ring uh so a norm maybe just say just said what data you have to give to Def find nor and you can just check that the usual algebra either geometry or solid the complex numbers with they do satisfy SE so this way you can produce by hand now such Norms over  $\mathbb{Q}^p$  right sorry but what does over convergent mean uh it means so an overon convergent function on a disc of radius  $R$  means a function which converges on some disc centered at the same point with larger radius okay yeah so it extends to a function on an open neighborhood of the the closed disc um right um yeah don't be shy about asking such questions about terminology because I know well anyway um so right so given a norm then you can look at the the locus of um of um the locus where the norm is strictly between zero and one uh and this projects down to  $\text{Spec } R$  uh and I'll say that this is a cover in the in the gro topology we've been considering so IE if we're willing to work locally uh we can assume uh we have an element you can call  $Q$  uh in the underlying ring such that uh Norm of  $Q$  uh is in so to speak lies strictly between zero and one so is it the case that when you take any real number yes just yes yes yeah yeah so actually so I'll make a stronger claim in the proof so in fact uh if you take Norm inverse of say  $2^{-n}$  uh specr this is a cover and that's a stronger claim um is a cover in the sh in this gr top the only gr de topology we've put on on our category okay but this is the strong of the is no no it's a cover in the gr de topology that it can be refined to yeah in fact this is not necessarily apine okay cover the can be refined to yes can be ref yeah um so but that said for let me so um let me for Simplicity assume that this is apine uh so there's an argument for uh getting the conclusion anyway um based on you again again resolving this by a profinite set um and using that descend ability that was proved earlier so but um let me I'm going to skip over that part of the argument so for Simplicity uh assume uh so that this Norm upper star of the constant sheaf on  $1/2$  is connective um so that this Norm inverse of  $1/2$  uh is apine and it is uh and it's actually proper over  $A_1$  it's just given by this this algebra um and hence over hence over  $\text{Spec } R$  so in that situation this is a both apine and sorry this is apine and it's proper over  $\text{Spec } R$  so by this descend ability Criterion we need that uh our triangle lies in is generated by this algebra so um but what I claim uh our triangle is actually a retract directly a retract without any cones and so implicitly I'm viewing this over the apine line instead of the projective line and then I mean I take the forgetful functor from  $D$  of  $A_1$  to  $D$  of  $R$  um when I when I write this here or in other words this is this is descendible of index zero so it's like The Descent is quite direct you just take a retract um then the intuition behind this should be clear uh this is some version of a lant Series ring it's a lant Series ring and variable  $t$  with some convergence condition um and you can just pick out say the zeroth coefficient of your laurant series that gives a linear map not an algebra map but a linear map which splits the unit um that's the idea um but we have to make sure it's it holds in this completely abstract setting here so what you do is you take uh you can take this structure chief of  $P_1$  um uh and then you can take the uh Norm upper star of  $Z$  well yeah let me well okay um so this is a pullback in  $D$  of  $P_1$  um just because the constant sheaf on this interval is the the pullback of the constant chief on zero one  $2^{-n}$  uh constant chief on  $1/2$  one glued along constant Chief at the point there um and then we apply uh uh push forward to  $d$  uh to  $D$  of  $R$  then the chology of the structure chief on  $P_1$  is just it's just it's just  $R$  concentrated in degree  $Z$  so what we get we're  $R$  triangle um and then I'm going to use the same notation or so the same just the these ones live over  $A_1$  or this one doesn't but well whatever the just just push it well it lives on the other  $A_1$  but we can just push it forward um copy the same guys over what's that first do a twist byus to get degree one why do

I want chology in degree one because you want a map from that's that's not the way I'm going to do the argument so um yeah so what I'm going to say is that um there's so this is then a pullback in Dr uh but it's also then well pullbacks are the same thing as pushouts so it's a push out in Dr um so if I want to make a map from here to to a certain place for example to our triangle I make a map here and I make a map here and I make sure they agree there um so what I do is here I take evaluation at Zero from our Tri uh which goes to R triangle here I take evaluation at Infinity which also goes to our triangle and then they clearly uh just both both give the identity map when you restrict to R triangle so this gives the map here which when you restrict back to here is the identity um so that uh that produces the desired retraction kind of yeah and what about the this connectivity a restriction oh yeah so then you what you have to do is you have to see that so there's then you you have some it's not if it's not necessarily apine you still have some proper cover by something apine um and then um and not not just proper but descendible so so then the unit here is generated by the image of this um and it follows then that the UN by combining the two generation claims the unit here is generated by the image here and that provides the um that provides the desired refinement by a street cover of a finds so you take any any proper map to here which makes the thing connective which I I I explained why exists then then this map will be a street cover because because the push forward will generate the unit because the analogous claim is true both here and here ah and the in the first Arrow it is true because of of the argument gave the first half of the class Yeah by this this descend ability of Canter set mapping to the unit interval which was the the uh but  $n$  inverse 12 is where is the inter no ah it's a you apply the you cover zero Infinity like you did and you look at what it gives in this uh no sorry you want to get a fine things you said that  $n$  inverse one up is not a fine in your terminology in general yeah so how do you make it find by it's not a pullback on the base it's not it's not by working locally on the base it's by working locally on on P1 or A1 it's by working locally on A1 using the previous argument so A1 maps to zero Infinity so then there exists a proper cover of A1 by an apine where the that map factors for the caner set and then that will give you the desired connectivity and also descend ability there youil yeah and you have descend ability both here and then by that argument here and then this is aine  $n$  proper and yeah okay um right so so now we now now let's give ourselves this extra data which we've assured can be gotten locally so given a norm uh  $p1r$  Norm 0 infinity and a  $q$  uh in uh such that so the claim is that um so the well we can think of  $Q$  as as as being given by a um um a map from uh  $Z$  bracket  $t$  uh to our triangle or to R why not uh so so  $T$  goes to  $Q$  uh claim this factors uniquely through this gaseous bace ring uh I let yeah so well let me not let me not name the the map let me call the variable  $Q$  here and um so a a variable which the norm makes look like it's topologically nil potent um actually has to um come from this uh this gaseous base here so the proof is so let me start with the uniqueness which has nothing to do with Norms it's actually a claim that so spec the claim is that spec  $z$   $q$  hat plus or minus one gas uh which maps to the aine line uh over  $Z$  say uh this is a monomorphism um yeah so so that's the that's the uniqueness claim or a strong form of the uniqueness claim there's a it's just there's a contractable space of factorings if one exists um uh this is not obvious from the definition of the gaseous base ring because by definition of the gashes base ring we took this uh  $zq$  hat which was just  $n$  uh which was just  $P$  um and then we inverted  $q$  and then we enforced this  $Q$  being gaseous um and this this is not item potent so not item potent so enforcing the condition of being gas what defines a monomorphism on the level of analytic Stacks because it's just some quotient of categories but because this is not item potent it's really not clear from that description that that the composite map all the way to A1 should be item potent because it seems at first you have to choose something which is extra structure namely a a proof that  $Q$  is topologically nil potent so to speak um but you can do it in the other way instead but uh uh you can also think of this as modules over  $P$  or  $P$  modules in you just take a polinomial ring in one generator uh and make that generator gaseous uh oh and then maybe I have to invert  $Q$  too well okay just um so and then the claim is that is that  $P$  is item potent uh in in  $D$  of  $Z$   $Q$  Plus orus One gas um and this actually so we want to check that  $P1$  or  $P$  equals  $P$  that's actually a claim that happens after base change to  $P$  so to speak so it actually reduces to a calculation here uh that if you take this  $p$  over  $z$   $q$  hat Plus or Min  $-1$  gas um now we have both a  $q$  variable and then we have a  $t$  variable say coming from the coming from the  $P$  here um so uh uh that if you take this and you mod out by  $T$  minus  $Q$  um you just get the underlying ring so yes yes you should still write guas because completion changes the uh oh okay sure yeah yeah I mean yeah the underlying ring of the analytic ring structure and Peter described this uh this free module and if you use that description you will find yourself is you're just you have to check a short exact sequence and you can do it um so that's the uniqueness um as for the Existence um uh well let



me just say it in words it's it's um fairly Elementary from the definition of the norm so because our Norm is contained in here uh this means that we're away from so existence so Norm of  $Q$  contain in  $01$  means we're away from Norm inverse of  $1$  infinity um and that in particular means that we're away from uh the opposite the the well let's say the the  $D$  sitting at Infinity so if you take this  $D$  the thing that was spec of  $p$  and you translate it to Infinity on  $P1$  then that's going to be uh contained in this Locus um but then saying that you're away from the  $D$  setting at Infinity is exactly the same thing as saying that  $Q$  is gaseous um but then on the other hand Norm of  $Q$  uh contained in uh Clos inter from  $01$  because of the axiom I'm I'm referring now to the axiom 4 about the placement of  $D$  in this uh in under this Norm uh this implies that uh  $Q$  uh comes from  $d$  so uh  $Q$  is topologically nil potent so it comes from  $D$  so  $D$  was this spec of  $P$  mapping to the apine line it therefore maps to  $P1$  and then you can pull back that map uh under the uh automorphism of  $P1$  which is the inversion map and you get something abstractly isomorphic to  $D$  but the structure map to  $P1$  is different um yeah yeah yeah yeah okay I'm almost done I apologize for going a bit over um so so now let me State the main result about what is comes from  $D$  um comes from  $d$  means that um it promotes to a  $p$  module so I the I mean the or the well what I mean is that  $Q$  so  $Q$  is a no no what I mean is a so  $Q$  is a section of this projection um well it's a section of this projection and the claim is that it fact through  $D$  and the reason is is that it factors through the norm inverse of this and one of the axioms guarantees that that implies that it factors through  $D$  yeah yeah have you stated what the classification is of I'm doing it right now so theorem uh so for given an analytic ring  $R$  um there is a bje natural bje or the the map um from the set of pairs well not not really set but okay n uh this is a norm on  $R$  and then a  $q$  uh and in  $R$  uh such that Norm of  $Q$  is contain in  $01$  so up to a a cover of the of Spec  $R$  um this is you know we can we can assume we have this data so up to some cover we're classifying all Norms um so but then from each of these I've just uh I've just said uh you get a map from Spec  $R$  to to well first of all uh you get a map to spec of  $Z Q$  Plus orus One liquid uh but then you also get a map to the open interval open interval from  $0$  to one um yes it's not oh thank you gasas yeah thank you uh and this is by recording  $q$  and the norm of  $Q$  um uh this map is an isomorphism I.E giving a norm on an analytic ring plus an element of Norm strictly between zero and one is the same thing as specifying what the norm of that element should be which is which is this function here and requiring that that  $Q$  actually come from this this gases theory that we discussed earlier and both sides are sets no almost well except for the annoying fact that  $A1$  is not a set so a yeah so so the the norms and  $Q$  the first the first thing we said is a is the set the nors really is a set because there is no and the  $Q$  of course is something in some set no no no the  $Q$  is not in a set because this could be derived so so both of these map to  $A1$  and like sort of the fibers of these maps are sets so there there're as much sets as each other um but they're not each individually sets ah so  $Q$  is a derived ah okay is but you still think of  $Q$  as as choosing something in the Zero part of simpli uh Zero part yeah yeah yeah yeah implicitly the higher  $s$  the higher maps are there also yeah so it's not you have to construct them as well they come automatically from the for I don't know so let me give the proof um so uh how non acate  $L$  not now not now uh so the proof uh so um so after a descendible cover after a cover uh we can assume uh  $Q$  admits all power admits and compatible  $n$ th roots uh for all  $n$  so IE so this is because the uh this is descendible so it's countable in fpqc um so we're free to because both things are sheaves uh in our topology both sides are sheaves in our topology uh we're free to work locally so we're free to assume  $U$  that that the  $Q$  has all all  $n$ th Roots um then um then um uh then the claim is that so then so then so write uh  $DQ$  uh for the pullback so we have  $D$  going to  $P1$  or let's say  $A1$  uh and then we have multiplication by  $Q$  on  $A1$  um and then we have  $DQ$  here um so for this would this makes sense for any  $q$  and  $GM$  uh um note that if if um  $Q$  equals  $Q$  maybe I maybe sorry I apologize I apologize maybe I say  $d$  alpha for Alpha and  $GM$  but note that if Alpha equals  $Q * beta$  or  $Q$  is uh top ically nil potent then we get a map uh from  $d$  alpha to and I have to get it going the right direction so  $d$  alpha to  $D$  beta is one of the two which is also induced by multiplication by  $Q$  so here use that this  $p$  is a hop algebra encoding multiplication so um um right uh I may have gotten the ordering wrong so this is uh this is some disc now um this is some disc of radius  $Q$  inverse or some version of a disc of radius  $Q$  inverse it's not the correct one because it kind of it's not a monomorphism and it blows up along the boundary but um but then we can fix that um so then so  $DQ$  so or  $d$  alpha sorry gives uh a disc of radius a kind of disc of radius Alpha of radius Alpha inverse maybe um and the the the maps above uh give give the inclusions between such discs what about this discuss I'm sorry or the identity so say what do you mean the identity you have to  $A1$  right yeah a bu of want consider yeah I think I I think I'm saying the correct thing but um yeah um okay but now we apply this apply this uh to

like  $\alpha$  equals  $Q$  to the uh you know  $m/n$  so  $m/n$  uh in the rational numbers and we get discs of  $R$  of radius uh  $m/n$  of  $Q$  to the  $m$  ah sorry so sorry we can we can assume sorry I should the first thing I should have said is we can freely assume that the norm of  $Q$  is constant equal to 12 by rescaling the norm by  $1/12$  I feel like I'm not I Peter what just happened this is crazy I I I literally just repeated your lecture with more detail and I didn't get any farther I thought I was going to get farther um I'm sorry for that um so we have we have to do a better job with the organization I don't know um but okay so you get discs of this radius and inclusions between them uh um but then you uh you do this over convergence so uh then you use these to make the over convergent versions um for an arbitrary real number you can look at all the rational numbers bigger than it you have these kind of fake discs of that rational radius and then in the co-limit all these problems about blowing up along the boundary disappear and it turns into the thing which has to be specified uh if you're given a norm on  $R$  so when you when you make the make is over convergent it will be forced to be equal to the thing that comes from a norm if you have a norm so that's more or less an argument why this is uh a monomorphism and then if you want to check that it's a bje you just have to produce uh a norm on here and you do it by following this this procedure or Norm on here and you can even adjoin all the roots that you want um and then you do it by following this procedure so uh yeah so you take this fake disc you translate it around by multiple by powers of  $Q$  and then you build the inclusion Maps between those discs uh I don't mean translate I mean expand by multiplication you build the inclusions between those discs and then you make them over convergent and then you check that you have all the axioms so uh yes sorry for going over time thanks for paying attention so in this inclusion of this using we also add an analog and discuss rubber rings of to power wounded oh yeah yes yes if  $Q$  is power bounded yes does it still follow that you have an inclusion or not maybe you need the to work in more solid context there but no it's yeah it's a good point um yeah it's not I mean it's not entirely clear in this in this general setting uh what the class of  $Q$  for which you get um such a map is yeah let me see because I mean for example this argument here doesn't prove that you have an identity map when when when like I wasn't allowed to take  $Q$  equals 1 here but but obviously you can take  $Q$  equals 1 so I obviously haven't made the um the best possible claim but um yeah but for example if  $Q$  is a root of unity okay for roots of unity we know that the norm is one I guess we we said it from plus one I did do do we know it for did skip a step that you're exactly pointing at which is that let's say we've said that Norm of  $Q$  which is a map from  $\text{Spec } R$  uh to  $0$  Infinity let's say that we've arranged that this is 1/12 then the claim is that if you have a  $q_1$  Over  $N$  then its Norm is has to be  $2/N$  yeah so one inclusion is obvious uh by just writing down the diagram with the  $n$ th powers and then you can prove the compliment you can prove the other inclusion by passing to the complimentary opens and running the same argument there um what's that oh yeah whatever the right thing to write down is yeah thanks yeah yeah I mean in both real numbers you canally right yeah but you have to but I do think you need to invoke this fact that they're closed and the open are the complement of each other to like with with just with writing down diagrams you only prove one inclusion but you have the nend SP up on P1 I guess and then you prove that the diagram commute because you use the both alapes yeah and then you consider the pullbacks and get it directly but I'm not sure this would work does it is it oh because this is a terminal object or I don't well anyway it's true well okay you're probably right but let's let's try okay Norm Norm maybe you're right yeah I mean I was I was thinking about yeah I don't know yeah it keeps the one intersection upstairs yeah it goes to yeah so this over a half but then it take a on the right of a half since is just yeah that's true that's true yeah yeah yeah yeah yeah okay um yeah okay good so there's no no subtlety at all but you just directly check that the  $N$  through of  $Q$  has to have the norm uh given by the  $n$ th root of the norm of  $Q$  okay yeah that that implies in particular that roots of unity go to one yeah okay yeah see you on Friday so what's the intuition for um like not being able to do geometry unless you choose a  $n$  no I I don't want to make such a strong claim but the I mean if you want to do some geometry what would happen uh like if you if you didn't fix wait but can I try to answer the question cuz yeah so cuz what I meant was CU I have to clarify what I meant so that if you want to define a notion of geometry that somewhat resembles usual complex analytic geometry which is in the end based on open discs closed discs and so on then you need to have something that measures the size of a radius of a disc and that's that's what the norm does but but isn't it kind of like packed already when you fix the base field or like it will change or are there like is there more than one complex analytic geometry based on the norm like basically I think this is related to the question about the equivalences like equivalent is up to like the geometry that will arise or I mean you could CH I mean you could you could choose different Norms I mean

obviously there's a conventional Norm on the complex numbers for example but but the fact that there's more than one at least in principle would mean that it's it's not the usual complex analytic I mean you get something basically equivalent so I mean you could scale the norm is something you can do like scale the norm by some Alpha it's you you get basically the same geometry so actually one thing we're going to do is we're going to mod out by at some point we're going to mod out by these sort of exponential rescalings of the Norms um when you do that you can't talk about a disc of a fixed radius but but still many other things work okay yeah we disc this many times so if you have  $\text{AUD uni}$  you choose let's say absolute Val  $Z$  and one then you get a norm on this and in particular in each residue field you get a norm a non archimedean Norm yes and this this is like choosing this collection of  $\text{ofan}$  yes yes so it is not at all the same as like so so let say for another feel is related to the previous National Norm but for a more global object it is very far from like the boovich space on just Cho one of quite different a way to uniform think of nor this is the inition yes and so you and and presumably what they said is an equivalence to take Uber Rings the national norms and your sense is the same as a way to of course you can should it continuous yeah yeah exactly continu on the back of rescale a given one so it is a torso under scaling of the fix one that's precisely correct and it it sort of uh sort of follows from this uh ah okay this okay so in this example this is what you get exactly and of course then you can uh uh and of course for other analytic Uber just locally you can do the I mean you can and for nonanalytic Uber FS this is something different well it's kind of it's ruled out actually I mean you have to be Tate you have to be analytic I mean we we were saying Tate instead of analytic in this class but you have to be Tate by this result if you have a norm I mean then locally for for so for discreet guys you are not having any non-rival any of exactly that's right but what we're going to see is that if you if you mod up by rescalings then there's a way to there's a way to extend this so when you mod up by rescalings on so Norms form an analytic stack by the way so like it's a it's a it's a it fits the definition of an analytic stack so sending  $r$  or  $\text{Spec } R$  to the set of norms on  $R$  that's an analytic stack uh and you have to put covered by  $S$  yeah I just did yeah okay okay okay yeah okay so yes and but and uh it is such that like discreet Huber things can't map to that stack but um but there's an enlargement of the stack which also uh accommodates solid  $Z$  and anything living over solid  $Z$  and so on which we we'll probably discuss um so there was this thing like in the theory of diamonds and so that that you want to get from an analy like in the pic setting you get a diamond from analytic but if you have a non analytic still you can look at maps of of antic  $a$  and get and then get that kind of there was this uh non quas I forgot now how maybe it's not related to this but somehow looks I I I I'm not sure what you're referring to actually because there was a diamond only allow taste test object so even if you start with theen one I still only remember how tast things than okay it's a not just Gally using this of restricting nor could you repeat that comment Peter the sound wasn't so great um yeah okay I'm not sure if you hear me better absolutely uh I was saying that uh in my work on diamonds I um like I only look at how  $t$  ring so per pict take ring snap to something and even if I start with a discret ring I only like remember these kinds of things and so using this notion of nor analytic ring uh one could now similarly restrict to non analytic rings and look at things only from their perspective uh which may or may not be something want do but it's kind of analogous right yeah so like yeah so I guess in this diamond setting Peter proved some results saying that even for like a discrete ring or something like the knowledge of how it maps to perfectoid Rings gives you maybe quite a bit of knowledge about the discrete ring there's a similar phenomenon here where actually yeah you can actually use normed analytic rings to you can even yeah you can basically recover everything just from the funter of points on normed analytic rings but yeah

## 21. 6 FUNCTORS FOR ANALYTIC STACKS (SCHOLZE)

[https://www.youtube.com/watch?v=R5JNomeHjtI&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=R5JNomeHjtI&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

So the main thing I want to talk about today is the form of six functors for step 1 but before I get there let me try to say I mean something that kind of Dustin kind of said last time and that I kind of also said last time but we both didn't quite really do is to do the construction but first um the construction of the norm on the SK fa um and so uh conveniently uh Dustin kind of already explained that we can assume uh that we actually uh have all the roots of  $Q$  as well um and so then we want to define the algebra that corresponds to the locus uh where the absolute value of  $T$  is at most  $R$  and uh or like like the over convergent functions on this disc and as Dustin explained the ex of Norm some uh tell you that this has to be um like there's an if guess for what this might be so this might be like this this free algebra on the to on the top element  $t$  or scale suitably so that um the norm is equal to  $R$  uh but that's not quite right but it's almost right but when you pass the over convergence thing then it actually becomes exactly correct uh so there's a CO liit over all um I know it's called um such that uh ah so I will use I will construct the norm for which the absolute value of  $Q$  will be a half and so I want to take this here  $A$  and so I want to take a  $p$  a module on a top  $l$  element where I rescale  $T$  by some power of cube uh and so this should be such that this definitely converges here so we want that we should secretly think that maybe the norm of  $t$  r um and then we want the norm of  $Q$  to the  $T * T$  this should be bigger than one but okay so this is the nor of  $Q$  is half so a half  $T * R$  should be bigger than one and now you can do the algebra so this means that  $2$  to  $T$  must be less than  $R$  so you take the co liit of all  $t$  for which two to  $T$  is less than  $R$  uh yeah so this is a condition that sorry am I doing this right sorry this should be less than this should be less than one sorry right so this is a condition that like if  $R$  is very large say then sorry am I doing this right um then we should multiply by a large power of  $Q$  to make it somehw small yeah this seems right okay was a disagreement or no no just uh just commiseration okay thanks um so so uh this has is the only possible choice is what you explained last time um and uh so what we have to see is that uh um these unimportant have all the required properties to Define uh uh well first of all you have to see this  $A$  item and they plus the anog for  $G$  inverse and inin uh Define map uh so analog infty okay okay now you can replace  $T$  by  $T$  inverse and then the  $T$  supp Infinity um all right so uh so here there are some ways to and I mean all of these are some kind of simp computations um uh let me just check that uh the locus where uh it's at least  $R$  and the locus where it's bigger than equal to to  $S$  where  $R$  is less than  $S$  this would be empty and for this you have to compute uh something like you take  $a$  and then join to  $T * T$  head um and say noers uh and then you can draw in  $T$  and the  $S$  time  $T$  head inverse um he want that sorry this um you want to show that uh such a thing is zero for if say um and one situation says  $R$  is less than one after resaling you can assume that some one is squeezed in between them um and so then you can assume that  $t$  is negative and  $S$  is also negative um but uh um right uh but so in particular here you have map from two toping elements which are uh so  $X$  goes to  $Q$  to the  $T^3$  and  $Y$  goes to  $Q^F$  and uh say satisfies relation that if you multiply the two you get something the negative power of  $s$  so  $Yus Q$  to the  $I$  don't know some axy is equal to zero where  $a$  is some positive number sorry this supp to and so but uh and in turn if you just remember the product so then you also have this is an algebra over a  $Jo C$  Aus  $a * C$  but this is precisely I mean a join  $Z$  head this is precisely this projective generator uh that we always use and then here you're doing oneus  $Q$  to the  $a * shift$  and if  $a$  was equal to one then we precisely declared this to be an isomorphism on the projective generator in the gas is ring structure uh and I also explained somewhere that the power doesn't really matter when you do this so This implies the pr fractional power so this is actually zero and this is really by the definition of the right so so like the minimal thing we really need to to get this map is that know this which shouldn't intersect do not intersect uh and this really exactly comes down to uh the thing we enforce and the guesses and then there are some other things you have to check and none of the computations is hard so those and then you have to check they cover that the guys like the they need to check the cover but again for this you just have to show that um yeah actually I will probably come to this later in the course again but I mean basically you just have to like if you say we SCT to the Fine Line then you just have to show that I mean that then you expect that there is some they form a cover and so they expect that there is some  $St$  like sequence like functions on the fine line just pooms going to functions on a dis and functions on functions on the opposite dis and then functions of the overlap and for all of these things you can just write down what they are and

then just check that you get an exact sequence and this implies the covering condition because it implies that the polynomials are generated by  $u$  in a finitary way by modules over  $R$ . Okay so you just have to check a cyclicity like the analog of data cyclicity you just have to check that  $I$  is enough to cover the covering conditions by the way the reason that we really want to use this  $u$  is that it grows topology that we chose  $u$  so I think there's another perspective here that makes some of these things easier which is like instead of using this  $u$  free thing on a topologically nilpotent it's like the free thing where you make  $t$  topologically nilpotent which is not really like a subset you could use the subset where  $t$  becomes gaseous like an analytic ring that's not induced but they're all sandwiched between each other so for the overconvergent thing it doesn't matter and then  $u$  and then  $u$  at least you have honest subsets and then it's like kind of easier to check which subset  $u$  like I'm saying there's a instead of using this thing I was calling  $d$  as your basic  $d$  you could instead use an analytic ring which is the analytic ring gotten by forcing the coordinate  $T$  to be gaseous in the sense of like  $1 - t$  is invertible what was  $D$  oh  $D$  was just the spec of the  $p$ -adic ring which is this  $U$  let me so I keep my notation keeps this notation  $u$  which to the line and then projective line so that's one thing you can consider on the other hand  $u$  you can also  $u$  Define the following funny analytic ring so you can take  $u$   $u$  we can start with the polynomial and then make  $t$  guessers  $u$  in the sense that you enforce  $1 - t$  is invertible oh sorry  $u$  yes and for  $1 - t$  is invertible as usual on Projective generator  $u$  to be Aus what is  $sh$  here I mean  $u$  on  $u$  on this  $t$  base change to this ring  $u$  and so this is also an analytic  $R$   $u$  and you can also take the aspect of this and this is actually a Subspace of  $P^1$  over  $u$  and it's again a Subspace  $u$  that will behave like  $d$  behave so it will be training Norm it will be contained in the locus  $u$  so  $u$  is contained in Long inverse of  $0$  and it contains Norm inverse of the open relation don't each other well what one you can't say contain and you could even consider a third version which would be the intersection  $u$  so you can also intersect these two  $u$  and then something that Dustin mentioned last time is that actually if you intersect these two this map becomes  $u$  an embedding also so this becomes an embedding but also this one becomes an embedding you can also consider a  $J$   $T$  head guess  $u$  and  $u$  this is also true here so all of these three things  $u$  there are some slightly different versions of what  $d$  might be  $u$  and there also sandwich between like the overconvergent close unit disc and the open unit disc so in particular like one way to verify the item potency Without Really sort of without doing any calculations is to see that the  $d$  and this  $n$  spec with the  $T$  gases are sandwiched in between each other in this co-limit and then it formally since for one of the terms you always have a monomorphism it follows for the other term you I mean there one thing you really have to check which is this that these two do not intersect I think this is the one have to do yeah the others I think you like a clever way to arrange the argument you can avoid them but yeah yeah so so if the radii are but if the radii are the same they always intersect any  $d$  when the disc  $R$  around zero any version and one around infinity they will intersect okay for all versions of the of the story okay yeah I mean it well it depends on which base you're working over okay but over the universal base yeah  $u$  okay so all right  $u$  yeah so so let me start by talking about six functors  $u$  so let me first recall  $u$   $u$  that on analytic  $RS$   $u$  we had first of all we had to fun that was taking a  $d$  of  $a$  and  $u$  we also considered the different function which was to look at  $u$  presentable stable Infinity line over  $d$   $u$   $u$  so this  $u$  was a functor from  $u$  from on ring towards symmetric monoidal categories so in particular  $u$  like the commutativity of algebra structure here has insisted some  $T$  fun  $u$  and actually there's also always in both cases they're close so there's an internal home and it's also functorial in the ring so  $u$  functors in the ring and so this gives you in particular for any map of rings you have a base change functor which  $u$  think geometrically is always upper star and Joint star so these are four of the six functors and then  $u$  we defined this class of functors and for these we had also a fun of lower streak and upper streak why does he move discretely because of the the yeah something  $t$  one picture instead yeah another high quality camera usually like you don't see right exactly and so we right  $u$  so one convenient way to package all this data  $u$  is in notion of an abstract  $u$  formalism which  $u$  I call the following so  $u$  whenever you see any category category  $u$  and  $E$  some class of morphisms so let me always assume that my Infinity category is all finite limits  $u$  and here is a class of morphisms stable under composition and base change where composition includes the empty composition exclude  $u$   $u$  then we can define a new Infinity category  $core$   $u$  where I mean it takes a little bit of effort to  $u$  write down the official definition in terms of simplices but I don't want to  $u$  write down something in simplices on the board so  $u$  the object  $u$  the object to see morphisms were these  $u$  spans so morphisms from  $X$  to  $Y$  were given by some object  $W$  on top where the second  $n$  here was an  $e$   $u$   $u$  the composition  $u$  was given



by taking two such guys and following the fiber product so then if you have two morphisms and these are both in  $\mathcal{E}$  but the compatibility and the base change the base change is an  $\mathcal{E}$  but then also the composite is an  $\mathcal{E}$  and it's a symmetrical structure uh is the product of varieties and there's an obvious way of taking a product of two morphisms okay and then we have this definition that uh well first of all three fun formalism where the three funs here that I'm talking about are  $T$  full and a fiber streak when  $f$  is an  $\mathcal{E}$  um or  $C$  is  $\mathcal{L}_{\mathcal{E}}$  from this correspondence category with it end of product towards Infinity categories with Aion Sol um in PR I would like right PRL here uh I don't because in a second I also want to discuss this example where these categories themselves they are inte but they're not in this version of them is not a presentable thing so um maybe for the moment I don't need it uh and there's a six fun formalism what is the the just cian product and the means in which direction there are two ways like  $d$  of  $x \rightarrow t \rightarrow y$  right so yeah so lights here means that you only roughly speaking you only have maps from  $d \times x$  times  $Dy$  to  $T$  of  $X$  for but they need not be equ they just have maps and if you apply this to the empty product this also means that  $D$  of the final object has a distinguished object um uh it is a six fun formalism uh if like the joints are home the upper so in  $LX$  is there a condition on on the on the identity sorry in the notion of  $LX$  symmetric monoid yeah is it also a condition on  $D$  of the identity yeah yeah so yeah you can apply yeah so there also like um from the empty product here so from the uh final category so the point there's a map to the drive category of uh uh the empty product so  $P$  of the empty set and from there from the empty set you can go further to anything else by a pullback so some particular like all the all the you have acces here are pointed which is giving you the unit objects really all for point I think it's of the empty it's sorry it's yes think it's than you show all right and so uh a this is uh very difficult amount of data that's it's not so clear how to construct it but conveniently like Le Jen proved some very difficult comori recipe well the recipe is extremely easy the proof that it works is rather difficult um but that uh makes it extremely easy in practice to construct these things um so in particular like for  $C$  brings up and  $E$  the streetable NS um sending a either to  $D$  of  $a$  or to  $crl$  of  $a$  um uh which I'll probably discuss in a second because okay let me just Dove this one for now um uh is an example all right um and so uh but now this and then some to the category of analytics text and we would like to extend uh this maybe to also some larger class of fre Maps well not just I mean the fre maps of aine certainly you want also base changes but maybe much more than that um and so uh yeah so here's the here um there is a minimal cost of morphism um unique um morms  $\mathcal{E}$  the following property you mean there is the smallest because any class satisfying the condition contains this yeah any any other class that satisfies uh the following properties will contain this class yeah so unique minimal is slightly different in some conventions because it means that the POS it as a as a unique minimal thing it doesn't mean that it is the smallest because so it's but this depends on your your Notions but it's it's slightly According to some definition better to say smallest than unique minimal but because minimal doesn't imply that it is smallest in for parti other sets unique minimal doesn't imply it smallest um uh properties so uh first of all the the class uh a class of morphisms of maps of analytic Rings Define Maps that's something we want um also we want that if we have just a disint union of fre  $m$  recable so um stable under unions in the following sense so if uh you have some  $f$  from some  $XI$  to some common  $y$  uh all of them then  $F$  the dist Union of all the  $F$  from the over  $y$  so from the of all  $x \rightarrow y$ int thisable i it's clear how you would Define a  $l$  stre function in this case I mean the drive car of the dist a product and then you just take the direct sum of all the streaks um it's this class is local on the target uh in the following sense so if if you have uh the morphism of analytics text such that the exist some cover sorry I wrote  $X$  before so stick to this um that exist some  $y$  Prime subjecting onto  $y$  for which uh  $F$  Prime which is a base change of  $f$  uh which fine isn't class uh the class say uh I'm not quite sure whether I need to explicitly say it so I definitely want this class in the end to be stable under compeition and base change and I'm not sure whether I should whether this follows at some point but always assumption um all right so uh right so this says that to check whether something is triable you can uh localize on the Target and by the way I mean if you wanted to find the lower streak funter for such a guy it's actually quite clear you would do it because you already have this ler streak fun for this one and you know that streak fun they always come with base change so to Define lower streak on  $y$  Prim on  $y$  you can do it after this base change to  $Y$  Prime but then all the further base changes they must uh I you must get it send data because uh these  $L$  streak functors come with Al base CH as  $M$  uh all right but now you also want to simplify it on the story so you might imagine this is just a point but this is some complicated arting s or something like this and then you want to be able to compute streak uh streak map here by taking a smooth cover and using the street maps which you already have in the smoth cover uh and this is included in the

next which is that in the suitable sense is EAS Lo on the source um so this says that if you have a map from  $XY$  such that there exists a cover  $um$  next to on to  $x$  uh  $G$  here uh uh what the following properties that uh  $G$  already lies in each other and is of universal stre in the sense that I will explain in a second uh and the composite and uh and the composite MTH so you should think that this is rather stacky but now you've covered it by something more reasonable um for which you already know that the composite map is is uh is needer then the original map license and maybe I should make some remark now um did you say what Universal stre descent here I mean stre descent after fullback to any fine so because this map already has street maps you can always ask whether the like the  $r$  of  $X$  Ming street maps to the  $r$  ex and Al fiber prod an equivalence and I ask this not only over  $X$  but after pull back to any but Peter it doesn't it follow from the assumption that it's a cover in the  $gr$  topology um how do you know is uh very very true thanks thank you Dustin oops Yeah what is this now so how do we know that this that how do we know that there is a sh for those kind of oh it's the assumption that  $g$  is in this uh well  $G$  is an EA no if you mean no no no but EA we don't know that there is the theorem say there is a minimal class and wait wait maybe maybe I need the next condition to to justify it um um if you have a  $m \times$  to  $Y$  is an eer and why is that fine then I actually ask that you can always find um uh a cover of eggs by things which are over  $Y$  which I F and fin then all the fiber products are also okay because and all the also  $x$  i is covering over  $X$  and then you need the full simpli so if I if I if my base is fine then I make a strong restriction on anything that's possibly allowed then I always ask that I can always further find such a shable I mean such a yeah shable  $c$  um where all the  $X$ Is themselves they lie in this original class of recable maps Peter don't you also need a variant of four with disjoint unions like or would this be strictly more General like doesn't two say that I could always no but two doesn't say that  $x$  i to the Joint Union is in  $E$  is in in  $e$  in other words if you want to localize by using a family instead of one object and then you need some argument to reduce to the  $c$  one object and it seems convenient instead of two to have also the fact in addition I guess I suppose in some places that  $x$  i to the of  $X$  I like  $X$  to  $X$  un  $y$  probably it should be in your class because it's just yeah but this is just because it's locally in the class  $X$  to  $X$  Union  $y$  why is it in the class because it's locally in the class no but in locally you assume okay okay okay okay okay okay okay just sorry and then the empty to the something is always in the class because because why because it's why empty to anything is in the class may you have to add it no this might e the empty collection ah okay okay okay no the empty to something  $X$  is empty and then going to  $Y$  this would be in the class yeah you can reduce to  $Y$  being Ain by by yes and then then it is e then it is in  $E$  okay okay okay okay okay okay all right I I hope I'm not scre up but um uh right so to you see this Universal streak desend after pull back to fine for such a map then um after pullback to fine you're in the situation of this one but then you always know that you're can refine by cover by stre find for which you know stre um um and there's one last condition which is really one we're interested in that um the six fun or the three fun formalism than you Neely uh from CE I mean from analytic rings and one we have to analytic STS uniquely in the space this you have to put it in ER to the prev yeah so uh for some of these Surs I'm implicitly using that uh I mean okay let say that okay um uh yeah so uh so this follows from results uh from extra I mean this has nothing to do with a specific six dealm this just an is all about six formalisms uh and uh and I've given some account of this in my notes on six funes uh using Min different setup but uh I think the same argument works here all right so that's an extremely convenient result um because it gives us this extremely structured formalism of six operations um I mean actually whether I say three or six fun doesn't really matter here let me say six because for this these are all presentable categories and the fun  $f$  is all Co limit and then as Dustin explained he always have the right joints anyways um so you have these uh six funs now uh for some classes of stacks and if you want to check whether uh and it's in practice extremely easy to check whether any map um has or in practice basically all morphisms have three funs and using some of these criteria that you can localize on  $S$  and Target it's also usually rather easy to check that this is the case so you can first simplify the target as much as you want um by localizing and then you just have to find some kind of presentation of your guy which makes it simpler all right um let me say a few words about how it's done so uh kind of hard to explicitly describe but it's easy in practice you can check that the given map belongs to this class so let me just give an example to illustrate this little bit what Happ must this must be covered by but then is it enough no then you iterate you do that again and again and again youate so then yeah you iterate working locally on the source and the Target and you have to iterate and trans find this I mean because once you get new shakable maps then working locally you get new things that can be covered by those uh so let's assume you're in a situation where you have some EXC  $Y$  which is

where maybe the  $Y$  is something simple just a point and the  $x$  is something spey but then you can find the cover  $um$  where this for where you already have stre Maps here so then uh the  $D$  of  $X$  because of the cover it's a limit of the street maps of like  $exer$  and then all the  $f$  but this is also the same thing as again the  $co\ liit$  along the lower stre funs now taking in  $PRL\ um$  of of the same of of the so these are the upper streak fun but you have low streak fun in the opposite direction and for all of these things you already have uh lower stre funs and they are compatible with the transition Maps uh and so thus because it's a  $CO$  limit and this guy is presentable all the  $M$  preserving uh you get the unique  $L$  stre fun here that's  $um$  so uh let me actually give uh even more specific example  $um$  let's consider uh just the interval mapping to a point so this is not something which happens in condensed sets but as we said condensed sets can be embedded into analytic stxs  $um$  uh as  $s$  explained last time there is this descendible cover uh of this by a counter set and this map here this is stable because when you uh face change to some cover here so can check it locally and if I base change this to the  $cont\ Set$  uh then the fire product here becomes itself some live profile  $cont\ um$  and this map here is just a  $faul\ flat\ map$  I mean it corresponds to  $Facey\ flat\ map$  of fine thingss and so this this here is an  $e$  so this map here is an  $EA\ um$  and and also I mean this map here is already in  $E$  and so up pror in our world we've only defined stre Maps here and we didn't Define what the street map would be for for the general compact House of space  $um$  but then turns out that in fact at  $L\ star$  and then one can observe that actually  $F\ noise\ 3$  happens to it also be a  $FL\ Star$  as you would actually expect because this looks like a proper map uh let me do yet another example uh so this way you can I mean some it automatically extend from profile sets to compact house space at least the five dimensional ones it also gives an extension to all compact house uh no let me not say that  $LE\ def\ find\ dimensional\ ones\ um\ uh\ uh$  but then you also have locally compact out of guys so for example I don't know you have you have the reals and now maybe this is a point where Dustin said I should have taken this  $J$  somewhere oh there it's okay  $um$  so this is covered by the  $distr\ union$  over all I don't know  $n$  of like the interval from minus  $n$  to  $n$  and uh because what why is it subjective because uh yeah I me just from the fun points like I by definition like as like  $cond$  said this is a union of of these things  $um$  so this is definitely subjective it's also uh it's also again and one way to see this is again to cover this by uh well maybe Again by some  $dist\ Union\ of\ Life\ Prof\ Finance\ set$  and then this is a just  $Union\ of\ close\ subset$  so it's  $um$  and so this is a  $dis\ Union\ of\ things$  which you already know is  $thiser\ um$  so also this guy here and so now the formalism automatically gives you an a flow stre fun for this kind of locally compact House of  $gu$  so I mean this argument will generally work for any kind of locally compact  $supp$  to maybe light and five dimensional thing and uh so we get some  $FL\ streak$  and you can actually check that it is a usual stre you can observe the  $comp$  and why uh well I mean just be I mean you can look at this definition here like the drive category of uh of  $R$  can be written as a  $Coit\ of\ the\ graph\ carries\ on\ these\ minus\ n\ NS$  along the lower street maps  $um$  and then to describe at low stre it just it means the  $col\ presing\ fun$  so it's enough to describe what it does on all these things but here it is just  $chology$  which is compact for  $chology$  and so I don't know uh I mean so this very abstract procedure actually recover it's a lot of like  $upior\ geometric\ interation$  about what a low fre function should be of  $um$  right uh yeah maybe so when I discuss the  $t\ a\ look\ curve$  I was mentioning that some should be proper and that it should be smooth let me say a few words about uh smoothness  $um\ um$  so here is one possible definition uh so those an inductive definition uh a morphism and why  $um$  actually wanted to be in the  $um$  is and then let me say proper now uh maybe weekly proper it's not it doesn't precisely match one something I call proper in the context of an abstract let me just say proper and maybe Dustin will complain that he wants this for something more specific  $um\ uh$  if one of the war  $H$  uh there's a question in the chat Peter yeah it says do we have any explicit description of  $f$  upper shriek of the unit object for  $FX$  to a point in the category analytics tax or at least in some special cases very good question uh let me refer to the discussion of Smooths coming up in just a second  $um$  so if you for example look at like  $f\ low\ streak$  in these examples here it was just usual  $F\ low\ streak$  and so then the  $F$  upper streak is also usual dualizing complex so in virtually any situation where you already have a dualizing complex uh our up streak will agree with what you say is dualizing complexes  $um\ um$  except like one thing you have to be careful about is that if you embed schemes into analytic stats in this way in this kind of with a tri analytic ring structure and all maps become proper as we said right and so the  $f\ l\ streak$  is always  $F$  lower star and the upper streak is then this funny  $R$  joint of a lower star that is sometimes considered but is not so well behaved in some ways  $um$  so if you want like the usual dualizing complex on a scheme of finite typ or something like this you should rather use this other embedding using relative  $s$  with this caveat I think anyways so let me try to describe find prop morphisms  $um$  so one case that I definitely want to be

proper is that it's relatively representable on  $F$  lines so meaning that after you pull back to any fine becomes fine uh repres FES uh uh sorry the following and proper you want toor for all  $\text{Prim } y \rightarrow y \rightarrow \text{um } X \rightarrow \text{Prime}$  which is the final product  $F$  and  $X \rightarrow \text{Prime} \rightarrow Y \rightarrow \text{Prime}$  is proper in the sense that we used before for uh  $F$  so that it has structure sorry Peter we were just discussing a remark that um in the setting of aine analytics Stacks the prop you know this the proper Maps satisfied descent for the gro and de topology um uhu so what someone pointing out that whether I say well they don't right because there have some connectivity issue uh right that's true but I think I pass once more to diag or something I can avoid this so let me make this so so therefore it's not the same not the same as as so as being locally no yeah we should so we really shouldn't ask for all why Prime mapping to we yeah yeah yeah discon yeah they will be a second part of this definition I would hope that um was the second part it wouldn't matter what I said here but okay I didn't carefully check it so let me maybe not however not that one thing that um this implies is that uh if this so happens uh in case one then you already know that  $F \rightarrow \text{stre}$  has a canonical equivalence with  $f \rightarrow \text{star}$  um that's given to you because in for these Maps we Define the  $F \rightarrow \text{stre}$  to be the  $F \rightarrow \text{star}$  then because you can Constructors ISM locally d canonically uh get it so basically the class of program that's be the one for which a floor speak is a floor  $\text{St}$  but but say it like this it's not a condition uh but I want to say something which is really just a condition um and some one way to do this is to oh yeah like for for these guys here have such an identification uh and in general um you can ask a diagonal  $\Delta F$  from  $X$  to  $x^*$  proper this is why I'm saying it's an inductive definition um so this might might be proper in the sense of one or it might also be proper in the sense of two in which case it pass the diagonal again then hope those proper um and um so once the diagonal is proper uh um you get an identification between  $\Delta F \rightarrow \text{stre}$  and  $\Delta F \rightarrow \text{star}$  um and this allows you to construct a canonical comparison map from a floor  $\text{Street}$  to a floor  $\text{Star}$  as Maybe recalling the second where it comes from and induced does it follow after and actually you might think you want to also ask this after any base change it actually follows and actually something even weaker of FIS  $F$  said speak of the unit unit and then Sol it after the new so does this lead to a trans possibly transfinite process namely suppose you have you use can can use fly many times and then you can say no I I want us to to determinate to find many things yeah there's no way to get it to go let us say suppose that you have a cover  $y \rightarrow \text{Prime} \rightarrow Y$  by half ah in this case one is one is one is terminating condition and the other is just you can apply fin many times but at one point you should run into to one condition one okay so the joint Union of countably many guys  $X_i \rightarrow Y_i$  which are which are proper in the in the ey sense will not be proper if when the sense goes to Infinity no no sorry again this when I said there is a cover by such I was allowing that it's a cover by a collection of such no no no but I'm asking suppos  $X_i \rightarrow Y_i$  is proper yeah then the disint union of  $X_i$  to the disint union of  $Y_i$  or is not this is also proper and I think it follows no because if if you have a sense one because of ah because okay I see what you're saying but maybe you could just take this as the terminating thing as a definition but uh okay so maybe it's not proper in my sense right now okay yes okay what's the problem it's not the problem there a variant where you because this is just simple induction yeah without going to Omega yeah so suppose you have  $X$  so yeah so maybe I should really make this this uh I should say the proper is first Define proper both both of these things are like qqs because I think this should be the case uh and then in general if it's true after pullback  $JY \rightarrow F$  or something no no because suppose you have qcq things  $x_i \rightarrow Y_i$  which are proper in the but in higher and higher senses you need higher and higher see take this will not all right but I'm not exactly sure in this Theory how much you can reduce to qcq because anything is a cover but this yeah the the equivalence reg itself is not qqs in general so it doesn't you don't have good reduction because I the proper case these things I think should be qqs because here the basic ones I CPS and then the diagonal is and I think from this you can deduce that  $F$  must be qu compa at least because it implies that the floor is talking to schore limits um so these things should be qqs in a very strong sense these maps all the maps of  $q \rightarrow q$  so in this sense uh maybe I screwed up the definition I should have given this inductive definition only this qqs case and general say locally sorry but thanks for the correction uh um right so um let me just recall where this where this is coming from so uh let's look at this diagram so then uh this map giving such a map is the same thing as giving a  $m$  from uh upper  $\text{FL} \rightarrow \text{stre}$  towards the unit but  $F \rightarrow \text{upper star} \rightarrow \text{lower streak}$  is by this diagram the same thing as  $P_2 \rightarrow \text{low streak} \rightarrow P_1 \rightarrow \text{upper star} \rightarrow \text{um}$  but then can always map uh towards uh  $T_2 \rightarrow L \rightarrow \text{Street} \rightarrow \Delta F \rightarrow \text{Star} \rightarrow \text{upar} \rightarrow \text{upar}$  right I mean here I'm just putting in the the unit of this adjunction for  $\Delta F$  in the middle um uh but by induction I have an equivalence between  $\Delta$  and for stuff so the same thing as as this one maybe just put the dial in diagram um but then both these Composites here they just like  $P_1 \rightarrow \text{up}$  and  $\Delta$  is just the identity upar

and this is the identity lower streak so this is just the identity okay so I can uh this thing here is giving me a natural transformation from  $F$  upper low stre towards the identity then by a junction and uh a par sensible thing would be to ask that this is an isomorphism of funs also after any base change but it turns out by by some funny diagram trase that actually if you check it once for the unit on your guy itself then it's Asis funs and it's true of Base change and again this is something you can find in my notes on six fun EX for and so uh so in particular with this definition of what proper is then yeah so then this map here is actually proper in this non generalized sense to any careful however that if you have some kind of H Cube mapping to a point then I'm not sure whether it's actually in uh DUS do you know in any case it's not proper because there is some the lower star fun doesn't actually con with col because of some infinite ises um right so you can uh Define this things and uh and you can also Define smooth so for smooth uh for smooth system I want that the upper streak is basically the same as the paper star but in this case you don't actually usually expect isomorph on the nose you rather expect that uh there's a same some twist and also for for smooth Maps you would never uh uh assume that this is a condition that's stable under passing two diagonals so uh um so the definition of SMU is for this reason is a bit different and here I us call the things loic um and this now really matches the abstract thing defined for any six fun form is if um um and now I really want to ask after there any base sorry this CL maps on Bas change um so there's the natural transformation from pullback Stander with f upper streak of the unit what upper streak and uh of unit which will be some object of  $X$  is actually an inversal object tender and commute to space change and so I mean you can form this  $F$  upper spe to dualizing complex for  $x$  or  $y$  but you can also do it for a base change but it should also be an invertible object and I ask that the formation actually commute to space change again you can somewhat reduce the amount of data you actually have to check um not me um let me rather just briefly say where the center comes from and so again I'm mapping to write a joint fun so what you try should try to do is in produce a  $m$  from the corresponding  $L$  stre um but then this is precisely a projection form your situation here pull back of something and so then this thing but then uh it's just by a junction so this class is  $St$  on the base change uh composition I mean also problem  $St$  in the base change um composition um it's in some sense local on the source if you smooth cover which smooth and um okay so so there are some ways like that to be able to talk about um yeah some Notions of proper smooth maps and I mean they some have the expected consequences on the rough categories and so you can uh yeah execute many arguments as usual uh so I just want to briefly now come back to ttic curve again um all right so so a is now again this guest displacer and so now we have the norm and so we can def find the analytic over that's a of so we have multiplication by two uh so all right so uh now we can make some assertions um so first of all this  $St$  here this is proper uh that's well it's not representable in Fes clearly but the diagonal is uh because the it's just a diagonal  $P1$  f um and you can check that  $p$  stre and  $p$  star the unit just agree and this is actually just the I don't know you can check that in algebraic geometry um proper um and it's also smooth come um and for this one should I mean I you can just really just check that directly it's not hard um you can also do the following argument that uh whenever you have like the morphism of six fun formalisms so for example like schemes mapping to analytic Stacks then any such morm of six fun fors will always preserve comically smooth Maps because there's some kind diagramatic way of encoding it sixun formalism um and so the smoothness that you someh know an algebraic geometry automatically implies it the algebraic version implies anal question um this  $J$  here uh it's actually also I mean it's also shable and it's actually somehow I mean I didn't Define it but I could have defined also in an open imersion um so this just means that uh see identity and again this is something you can basically reduce to just zero infinity and like the open guy the close guy um very some everything produces to the usual six funs on locally compact Al spaces um but uh so these These are in particular these are also like  $J$  low streak so  $J$  upper streak in this case Al jerar particular is also um so so this guy is it's not proper just like this I mean it's not  $F$  compa inter okay um but now again we can take since  $T$  the curve which is quotient um ah but but one thing that is proper right so the proper Maps they satisfy the two out of spe property because they were Som defined by passing and this map here is actually proper and so it also follows that if you pass to this and  $GM$  then it's still proper over zero Infinity there's a question in the chat Peter yeah are open immersions of analytics Stacks representable absolutely not representable by  $F$  I mean if they were represent by  $F$  in particular would have to be qua complex but this  $m$  is very much not quic complex so we get much more General open Emer um right but now the state litic curve um it's a of analytic  $z$  um and so thiss to  $Z$  Infinity mod rescaling by po of two which is which is a circle and so like the base change to  $Z$  infin you see that this map is actually proper but if you project as



one to the point this is also proper um and so you see that the ttic curve is proper or the point the point is now like a everything a um so you see that this constru gives you an antic St that's proper but also I mean this map here is locally split and so this means also this is here Lo local is the same as uh this analytic GM which was smoo so also the sky here isical smoo and I find it a bit remarkable how you can like combine intuition of about the properness of compa spaces with some really algebraic geometry kind of properness and so on and it all works really well and um right so uh yeah also it comes with a section as a unit section I given by one and GM and so uh so there some of the claim that uh this eqa curve which is analytic St over a limit alun from sches schemes over the underline nice ni essential image of the fully faceful embedding from schemes over the underlying ring and ring ration uh so to to execute this argument in a really nice way uh I should prove a little bit more about this ring a here in particular classifi like the dualizable objects in D OFA that they are just perfect complex which I think is true um let me do it a little bit more by hand because it's fun um okay so so here some kind of General setup so let's say a okay and the X some take over eight and we want to find and then let me assume proper certainly necessary I want to find some proper scheme and n morphism to the anal notification and as usual I mean the suggestion one can do this in a nice way is when you have an amp line bundle um so let's analyze it a little bit so you have line bundle L on X with a find properties such that some T power l or by replacing L by T power L itself L is globally generated so there is some X some some n subjecting onto to L this globally defined um for greater equal to zero um if you look at locally next I don't know sorry this is not an a billion category because of the derived all everything is highly derived here so this means that there still well notion of line bundle which means that on FES it's locally it's automorphic to the structure sheet um and you still have a t structure on these things right I mean this also just F this means it's have structure and the subjective means that the H zero is of the co is z i yeah yes and for all zero and bigger than zero uh there are no higher there's no higher chology of these guys and let's assume that the h z x the end uh let me save for some yes so I think I really just need that it's the module over the underlying mod next than for for any analytic ring a like the abstract modules over the underlying ring edul Faithfully into the complete modle and I asked that it SL on the essential image of H me what uh sorry uh I because it might be derived Ian my in the situation I'm in everything there's no derived structure so it doesn't really matter I definitely want that there's noology and um assum have this situation um and it's not clear the difference between Poli I mean this it's just I'm just trying to say you have an an line bundle in some approximate sense so for example I want that it's Global it or repl by some tens power it's globally generated uh and there's no higher chology of tens Powers which you can arrange and I also want that the global sections are not something some just to the SC module because I want an algebraic schem in the end so then we can define something algebraic to be just the project of the is this AR GMA is it just H zero or it have a positive yeah I mean if some X is actually derived it could have positive stuff let's not worry so much about it so you can Define this thing here is a direct scheme over and you automatically get a comparison that uh from this algebraic guy analytic fied and because uh globally generate you can actually check that this is really a qu comp guy okay this this is the stand but it's not enough to it's not L to I hope so um so uh so something one can say in the situation is that uh so anything comes from algebraic geometry as we said it's automatically improper so M actually automatically improper meod but also uh a simple observation is that in this abstract setting automatically it will be the case that aest of the unit is the unit structure she is a structure she um why because you can check this uh we can check after the pullback uh to f f x of x uh but there on global sections because I'm the fine guy like of CH is just of the global section because it's e um uh but uh but this inclusion here is really just given by algebraically inverting something and so then this a star here is then just some some filter po Lim of inverting some function f so this actually just reduces to uh to to the isor on sections that we know I mean so this is like the plus of some f where f is some fun some Global section of and then uh you're really just getting Co liit over multiplication by F of global sections sorry the Al I mean either on the algebraic guy or on the on X but some the AL guy was ripped so that like the go section of the line bundle are the same on X and the albra what K \* the co liit over K transition modication by some section of l f section of L the K or n maybe I don't know uh F was an unfortunate Choice actually because that was your map from X oh sorry yeah thanks thanks sorry G did I use G let all right so uh right so you have a proper map of analytics F where FL St unit is unit uh this actually implies that it's a s active net because like yeah this this is a SE true after any pullback because proper St stre any pullback and so this means that after pullback to any Fon you can get the algebra uh in terms of the image and so this actually means that f is subjective all right and so what does it now take to

show that  $F$  isomorphism uh see I mean this generality it won't automatic the case that it's an isomorphism but um to check that isomorphism uh it now suffices to show that the diagonal of  $f$  is an isomorphism right because if you have any subject  $M$  of sheets it's  $\text{Isis}$  only diagonal  $\text{Isis}$  uh but this the diagonal of  $X$  here he want just this a pull back here uh actually this also implies something else I should have said also implies that the pullback map from the garage category of this guy is actually inly Faithfully into  $X$  because um because if I look at low star upper star then by the proection formula this is just toring with a star of the unit  $C$  identity so the unit of thejunction is equivalent which means fully faceful as it holds for  $f$  but also for any base change of  $f$  so in particular also here um these pullback Maps here they are fully faceful um and so if it so happens that there a  $DI$  of  $X$  is  $Fon$  and  $\Delta x \times 1$  of one is in the image of of  $f^* F$  St discuss to show that to give an example of a proper map explain the Cur algebra so I claim so okay so I only need to check that uh this this diagram gives a diagonal commutes but in practice like the diagonal like I don't know some kind of complex or something this this some risky Clos imersion in particular some kind of  $f$  math um and the  $X$  Stu of one is really just some coherent she uh on on this product and so you definitely yeah and I claim that it's actually surprises to see that this lies on the image  $F$  time uh um then we win because of it LIC image of times upper star and it's actually the same thing as applying  $f f$  lower star and then upper star again um but if you apply the low stars and you run through this diagram you see that it's actually the same thing as a pullback of actually implies that it's equal to timeos of Del algebraic but then this means that yeah but then this means the some of  $P$  diagram all right so this upshot if you want to prove that some such analytic space proper analytic space is algebraic you have to produce an amp line bundle in that sense over there then you automatically get this kind of comparison  $M$  to some uh algebraic something algebraic uh and the only thing that remains to check is uh to see that the structure sheet of the diagonal uh lies in the the subcategory that comes from this algebraic and uh so my time is up but these are things that are simple to check in the case of stactic  $C$  so you can for for this  $L$  you just take uh the amp mod given by the zero section and then you can actually just compute what all the global sections are just inductively so the global sections of the structure sheets you can just compute the AR easily and then the rest you just acquire copies of just the zero section in addition which are just your base ring so this is okay and well and for the diagonal you can actually use that this is a group so you can move the diagonal to the  $Z$  zero section and the zero section is easy all right uh my time's up to let me stop here so uh just uh I want to clarify just from the Viewpoint of General scheme Theory some some point here so you consider Pro of rings which are not assumed to be finally generated that us suppose that we don't have a derived ring we just have usual ring so for example the graded ring of  $H$  zero on some scheme of powers of a line bundle so we just interpret what you said for uh for normal schemes for Simplicity because I'm not sure about so now when you look at the notion of proper like rotend defines finite type separated universally close but of course you people thought about the case of nonf so you can have a universally closed maybe separated but then this turns out to be projective limit in the QC Qs Cas a projective limit of usual proper things so the my question is suppose you have a appr of the type that you consider so what we know is that it's covered by Finly many opens a standard basic principle opens whatever but is it the case that it is proper well in this universally closed sense so this means that the the problem is that of course in the T litic  $C$  I think the algebra will be fin gener but that I allow the case that the great Ring is really just to start with any ring in degree zero and then just repeat it what the great thinging might just be any ring you start with degre zero and then you just repeat it in all the other in in so all all degrees are the same ring and the multiplication and multiplication is multiplication in the ring and so then then then you just get any  $F$  scheme in this way wh wh why um why you get a pro no no but this this this uh ER this will give you Pro of a which is which is no I mean I mean I wanted to proper over the right things that is over the AG zero I mean over the the I mean I I have I have the age I have Mak the degree Zero part as small as you like you can okay you can make the degree Zero part  $Z$  and then the other thing oh of course then then then of course it's not it's not uh okay thank you okay but at least in in the Tate case I think one one will have also that  $H$  zero is a ah okay so so your notion of proper so how do you know all qu contact qu separate def are prop in your sense in your sense ah in your sense not in the usual sense of okay okay so it's it's not the usual but it's funny that that usually Gaga is for proper for proper things not for oh I mean I mean this can only work if the thetic space here well all schemes anded  $F$  Faithfully inals St world so there and Al the career Chiefs and so on so this there is some kind of but for some kind of funny analytic spaces but uh if the  $X$  here is some like more usual kind of I don't know comp space or something exist and such an algebraization can of course only work if the  $x$  is

proper  $Z$  like also you like complex analytic spaces they basically embed our category and if you specialize the notion of properness I defined here in that setting you get the usual notion of properness oh okay because it implies POS compm okay all right

## 22. BERKOVICH SPACES (CLAUSEN)

[https://www.youtube.com/watch?v=fnEPiDIF9\\_k&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=fnEPiDIF9_k&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

So we're nearing the end of the course actually and um at the very beginning in the first lecture I kind of gave an introduction to what we wanted to have out of this theory of analytic stacks and in particular I was putting some emphasis on the fact that we want to know that traditional Frameworks for analytic geometry can fit into this perspective uh of analytic stacks and we've sort of already discussed um adic spaces and little bit of geometry over this um the little bit of things like like tape curves um but what I want to add spaces there was this issue of shifting and then the idea was that you have to Define some kind of derived adic spaces this will not explain in full what is the set up and in your formula yeah yeah yeah maybe there are different ways to I'm not sure it's possible there are different ways to do it yeah so this is yet in progress already yeah you yeah let's say that or something the it's in Juan esteban's paper on uh the analytic the ramstack that's okay so there you go um all right um okay so but what I want to start moving towards today is uh relation with Berkovich spaces um um but before discussing that um I want to just do a little bit of setup um so but first I want to talk about different Notions of how to localize uh uh an analytic stack uh sort of over a topological space um so well we've already actually seen one way so if um so let's say well let's give ourselves an analytic stack um and if we have uh  $S$  which is maybe let's say a yeah metrizable finite dimensional compact Hausdorff space or it could be a maybe a locally compact Hausdorff space which is locally one of these um then we've seen that that can also be viewed as an analytic stack so we can also view it as an analytic stack um and then we could just ask for a map uh  $F$  from  $X$  to  $S$  in the category of analytic Stacks um and we discussed uh kind of what what structure you get on  $X$  basically you get a bunch of uh item potent algebras in  $\mathcal{D}$  of  $X$  corresponding to the closed subsets of  $S$  um or you can think in terms of the complimentary opens as well and you get sort of some item potent coalgebras which would be like the  $J$  lower shrieks of the constant sheaf on the open subset as opposed to the like  $I$  lower stars of the constant sheaf giving you these item potent algebras but in any case you get a whole bunch of algebra objects in here which kind of let you localize this category um over the topological space  $S$  um but um we're going to want to when discussing the relation with Berkovich spaces um it's going to be too annoying to require uh these kinds of conditions even though they're satisfied in practice so I just want to discuss something uh slightly more General um some more General notion of how to localize an analytic stack along a topological space and the relation with this one in the case where you do have a metrizable finite dimensional compact Hausdorff space um so so what's another what's another thing you can do well if you have an analytic stack so we have  $X$  um well to  $X$  we can assign a local  $I$  me just call it  $\mathcal{O}_X$  which is just uh so given by the um so monomorphisms uh in analytic Stacks so this is kind of a general thing you can do whenever you have a topos or an Infinity topos it always has an underlying local um where you only look at the monomorphisms and if you look at the topos axioms then you know the yeah you can see the unions here the POS unions and intersections exist yes but uh yeah so I so I I I should say that potentially there's some set theoretic difficulty so that this yeah potentially this is not a set I didn't I didn't investigate seriously but it doesn't matter you'll see that it doesn't matter in a second when I get to the definition mean when you allow larger and larger sizes you could have more and more sub objects in principle I didn't yeah I didn't I didn't I didn't spend much time trying to prove that it's a set because um about the local axiom like of course you have finite you said that there is a way to you have to produce finite binary intersection unions but also you have to produce infinite I mean no but I mean this is this category has all limits and Co limits this category has all limits and co- limits so I mean we don't have to do anything special here to make this definition but why the co limit is a sub monomorphism so the co limit ah okay you take the co limit and then you quotient by the is it any  $Z$  to  $X$  as an image here uh yes yes that's right yeah yeah you take the you could take that's a general Infinity topos thing for example you could take the check nerve of the map and take the co limit and and this is what Infinity to the analytic Stacks modulo set theoretic technicality so it satisfies all the same exactness properties as an Infinity topos but it's not presentable so it's okay so the still could be some okay up to okay yeah so but we're going to so so maybe it's not a set I don't I don't I don't care and you'll see when I make the definition that I'm interested in that it doesn't matter whether it's a set or not um sorry yes question the monomorphism if I only consider an that's like taking connector component yes yes yes yes yes exactly yeah so yeah it just means

that if you do the fiber product then the diagonal I mean the diagonal map Associated to this inclusion is an isomorphism um okay right okay so and then of course a topological space also gives rise to a local but let me actually change perspectives and just say  $s$  is a local instead of saying  $s$  is a topological space um then we can ask for a map of locals uh  $\text{Loc } X \rightarrow S$  and um whether or not this is a set uh this is some data that makes honest mathematical sense because you're just saying that for every  $s$  so a local is by definition the collection of open subsets which is a set and you're just saying for every open subset you have to give such a monomorphism and you know unions have to go to unions and finite intersections have to go to finite intersections in the literature about local I don't remember I didn't don't use much but so this is like the direction of  $M$  of too but I don't know when they Define  $m$  of local is it in the other direction or in this yeah yeah I don't know either so I'm I'm I'm doing the the like geometric morphism let's say a geometric morphism of localis yeah morph of SES well morphism sites goes the other direction you the morphism of SES goes the other direction then I mean yes so I mean I'm talking about a geometric morphism but also of goes in the direction oh maybe oh I forgot yeah yeah yeah okay yeah okay go in the same direction okay not I don't it's not used much by okay well yeah uh anyway um okay um but we can also ask for a stronger property uh so that each uh inclusion of open subsets so  $V$  inside  $s$  um uh so that this maps to so it's supposed to give some monomorphisms so maybe I should give some name for this like  $f$  or  $\Pi$  maybe so so this goes to like  $\Pi$  inverse  $U \subset \Pi$  inverse  $V \subset X$  we could ask that uh these inclusion maps are actually open immersions from the perspective of the six funtor formalism that uh that Peter discussed last time the extended six funtor formalism on analytic Stacks so we could ask that each of these inclusions be shable um and that they be well chromologic smooth but in the case of a monomorphism then it's quite easy to see that the dualizing object is canonically just the unit and it's the same thing as an open immersion um yeah uh so right so a general monomorphism I mean it could look like could look like anything really it could look closed it could look open it could look like some mix um but um it's it's reasonable for to ask for this stronger property that what looks like an open immersion on the level of the topological space is also is looks like an open immersion from the perspective of the six funtor formalism um and then uh in the case where  $X$  is a metri uh finite dimensional compact haus door space can ask for as before  $s \rightarrow s$  thank you thank you can ask for a map uh an analytics tax um so um yeah sorry just a clarification So when you say  $s$  is a local like what's the definition sorry oh the definition of a local yeah so like I'm just like what does it mean for  $u \rightarrow v$  to be an open inside right so right I'm using a bit of loose terminology here but um so about the definition of a local is it's a certain postet okay and the axioms that you impose on this postet are the axioms that are satisfied by the postet of open subsets of a given topological space but by abusive notation I'm kind of using this symbol  $U \subset s$  to mean that  $U$  is an element of the postet which  $s$  is secretly but it's because I'm thinking of it as a topological equivalent local can be sort of the category of sub object or the final object in some topos yeah yeah and then this defines a kind of localic reflection topos is the about yeah there's another perspective is it's like it's like a it's like you take the definition of a topos but instead of sheaves of sets you look at like sheaves of truth values I don't know if this is helpful or not but it's kind of yeah um so it's it would be so if the usual topos is a one topos then maybe a local is a zero topos I don't know or something anyway uh um okay uh oh yeah right so um if you have this kind of structure then you get this kind of structure and if you have this kind of structure well kind of topologically you get this kind of structure note also that if you fix  $X$  and  $S$  then the possible such this guy this guy or this guy they're all just sets um they're not Ana or whatever because here well we're just asking for a map of locals it's just a map of post sets in the other direction no but when you say set yeah it also means in this sense of the CTIC size difficulty oh that's true that's true so up to size difficult I mean I mean there are no automorphisms of any fixed one of these let me say I mean there no non-trivial automorphisms yeah yeah that's a that's a good point okay so yeah so so this is tological um so yeah why so why is this what does this imply this well basically you can just you can just check on the level of the analytic stack Associated to such a guy that every open inclusion um actually is an open inclusion from the perspective of the six funtor formalism Peter more or less discussed this last time that our six funtor formalism on analytic Stacks when we restricted to this case recovers the usual six funtor formalism on locally compact haus dorf spaces and then this property of being an open immersion is is stable under base change so that's um uh and uh you get the yeah so can you say so you know that open immersion holds for ah it is stable under  $M$  of analytic Stacks or well uh pullback yeah it's stable under pullback yeah yeah um so yeah so this is the strongest condition so if you have this you can ask whether this hold you just check whether certain inclusions or open immersions if you



have this you can ask whether this holds that's as we discussed uh the last couple of times that corresponds to some connectivity condition on the on the itm potent algebra as you see um okay um Let me give a small example the second one is just given off by  $\text{Lo}$  is just one of open opener yeah that's that's true yeah so yeah so right so you could think of this in terms of there's  $\text{Lo}$  local  $X$  and then there's some quotient local which is like the local of just monomorphisms that are opener verions with respect to the six funter formalism and then um the second bit of data is a map like this um yeah thanks for that remark Peter uh and is it the case that the union maybe I go confus Union of open immersions is an open imersion or to make it an open yeah no no that that that's true that it's an open immersion yeah which is needed to well it's needed to yeah it's needed to have this loal I mean yeah so or well remote I mean depends on how your whe right right right no but it's true so that if you have a yeah a union of open immersions then it's still an open immersion and that's an important point to check when you're discussing these things um and it follows from this extension procedure for six funter formalisms that Peter discussed so if you have a union along open immersions then those are shable maps and but it's also a cover and so you can you can get a a lower shriek map defined on the union and then you can actually check locally that it's comically smooth um okay uh so let's give an example um so well remember way back when when we had Hub Pairs and stuff like that so let's see um then uh we assign to this an analytic ring  $r$   $r$  plus solid um then can take its Spectrum um and then we can look at the local Associated to this and what we essentially already saw was that this this so This analytic stack localizes along the usual um usual Huber topological space of continuous valuations no you you ah now you consider the spec in your in your yeah this is this is what this is spec  $n$  so Peter yeah I I'm lazy so I yeah I'm lazy so I just say spec and and then uh spies in the old sense yes it's a topological space yes and then okay the local okay and then you have a map of oh yeah this lock this means for every open in the add space you give a a monomorphism of analytic stocks and uh this of course this involves passing to some derived pairs in some sense yeah so because you are not but your original pair is is not right so if on the level of rational opens this is just going to give another apine uh analytic stack which is the one where you enforce uh that  $F$  invertible just in the world of a analytic Rings you enforce that  $f$  is invertible and that  $G_1$  over  $F$  Etc  $G_N$  over  $f$  is solid solid um and that defines some uh that defines some analytic ring uh under this analytic ring here and passing to spec it's actually a monomorphism so on the level of uh the derived categories for example it's a localization um and that yeah that gives some inclusion of analytic Stacks here and um what we argued um when we were proving that so we proved a a less um a less precise version of this claim early on when we argued that the derived category so this um so this um refines the statement that uh that the  $\text{drr}$  plus solid uh localizes on  $\text{Spa } \text{rr}+$  so recall that we argue that you you have a sheath of infinity categories on this topological space whose Global sections is equal to this and whose sections on a rational open is the analogous category where you impose these conditions um and we had several discussions about when this is the same as when this is also of this form for some non-derived Huber pair and so on so if you're if you're a tate hu pair and shifi then then in particular this um this again is just of the same form as before um right and then so it refines this statement and the proof of the proof uh of this St statement that we gave is actually shows this stronger claim so what we did was we argued that um you can always refine any open cover here so that uh so in fact you get a an open cover so in fact we showed without having the  $L$  without having introduced the language for it we showed that any open cover or cover of a rational open uh in  $\text{Spa } \text{rr}+$  uh pulls back to an open cover uh after can be refined to uh uh pulls back to an open cover of uh you know the pulls back under  $\text{Pi}$  uh to an open cover uh in the sense of the six funter formalism no so let me make a a remark so let me make a open cover to cover cover by monomorphisms yeah so let me let me make a remark um so in general uh this uh yeah it's not true that a rational open uh pulls back to an open immersion  $R$  is there a proof of this of this oh yeah I'll give an I'll give an example right now so um so yeah I mean you can do something like I don't know see yeah so I don't know have two formal variables maybe  $p$  and  $X$  um oh sorry I should say just  $\text{zp } Q$  uh yeah that's yeah I don't need two variables thanks uh so just one formal variable and then invert it um so what is this correspond to uh so on the level of analytic Rings we get this and on the level of derived categories uh so this is just solid  $\text{zp}$  modules and then uh this is just a um well this is just algebraically inverting  $P$  so so the left adjoint here is just um inclusion of full subcategory or the right adjoint here is the inclusion of the full subcategory where uh  $p$  is invertible so this passing from  $\text{zp}$  to  $QP$  is just algebraically inverting  $P$  yeah  $p$  is a prime yeah yeah yeah um and that actually uh from the perspective of the six funter formalism is a closed immersion not an open immersion that's a proper map because the uh this hasn't the second variable hasn't changed so we're just changing

the underlying ring and not the analytic ring structure um but it's also a yeah proper monomorphism it's a um it's a closed inclusion it's closed not immersion not open H so so I think we treated the case of some uh covering given by a function like  $f$  let  $F$  like two Lan like the lauran cover yeah the one where  $F$  or one minus  $f$  is less than equal to mean this kind of basic cover and this for those it was they were is it the case for those show that if it's anal what's that I mean if it's anal yes so then let me make another so however yeah uh if  $R$  is Tate um then then indeed then this doesn't arise so in general the problem is exactly these o Open covers in hubber sense which are just given by algebraically inverting a function without enforcing any any inequalities but in the setting of a a tate Huber pair then you know if you want to invert something it's you can always write it the the the subset where you invert something can always be written as a union of subsets obtained by forcing inequalities so like like the set were some you Union of like well I I don't I don't know yeah but it is not quasi compact in general right right right it's not going to be quasi compact dealing with something which is in the sense of of the adct space is qu compact because it is a rational domain I think in the yes but I mean it is Rush so it is qu compact in the yes in the so this also has to do with the fact that we didn't directly I was SL sliding one little thing under the rug here which is we didn't really directly Define this as a map we didn't really directly Define it like this remember we actually had this valuative spectrum of all of these guys um and we actually defined this map and then we used Huber's retraction here to so this was this was not quite accurate um because that was that that would be accurate on the level of spav but that's not exactly how we get it for spa a so instead we what just assume that theal sub of SP ah yeah okay right right right yeah okay yeah theidea is open and then it's okay yeah so then the point is that then um right yeah well yeah the point is that the rational opens in the the rational opens in the Tate case can always be described by forcing inequality uh among the valuations um okay uh yeah let me just try to finish what I was trying to say it's like like I don't know this is very um I just just to give an idea of what's going on um I don't want to get too precise about it but it's this kind of phenomenon and where where these guys correspond to um this kind of thing which does correspond to an open immersion so this is open in general this is closed and in complete generality your subsets are going to be a mix of open and closed but if you only need to refer to these things and they're always going to be open so sheets on this  $F$  is not there in fact you not algebraically invert  $F$  but but using the right hand side that's right that's right yep um okay and also so also uh there is another topology on the same space uh with the same constructible subsets so where um where uh open pulls back to open so another another fix even outside the Tate case is to um to choose a slightly different topology from the one Huber described uh where or like kind of both of these things are open um that's that's not the constructible topology or no it's not the constructible topology no yeah so is it a topology Which is less it's incomparable to the Huber topology so like for this one so like for this subset here like  $P$  not equal to zero it's going to be closed from one perspective and open from the other perspective so it's open from Huber's perspective but will be closed from the perspective of this this other topology is it the spectral space it's a spectral space yeah to be open now so this is some what he say sorry he's saying basically you redeclare zariski open subsets sorry yeah you redeclare zariski open subsets to be closed which the risk open they not they're not open so so for example inverting  $P$  here inverting  $P$  okay is now closed in the new yes we haven't discussed so far that element now thetics actually have compl that's true we haven't talked about that yeah but let's leave that aside I think there's already enough information being um is it the the opposite spectral space there is a con spectral space no it's not the opposite spectral space because these ones are still the ones where you're enforcing inequalities like  $f$  less than or equal to one that's still open that's open in both cases it is offet of of I mean of of some something a paper about he def a lot of topology and one of them op it's opposite of one of the ones Huber describes in one of his papers yeah okay yeah yes can make this more precise I'm going to hesitantly say yes it's obvious if you remember the definitions of everything um maybe we need compact or not because because because I mean open subset I mean I mean you want you want an item put in  $A$  correspond to it but I I'm not sure whether if it's not quasy compact you you have yeah let me yeah let me say quasi compact uh yeah yeah no I mean I I mean it's something we've actually discussed before in the in the course I drew some pictures trying to explain why it makes sense that  $Z$  risky opens should be closed um I don't know yeah so um oh okay um everything is defend all rational Subs are defend by inequality is  $f$  i less equal to  $g$  n  $g$  not equal to zero yes and this one you replace by we say that that is a closed condition for any  $G$  or any yeah yeah well I mean you I mean well let's say you have a rational open so the conditions are satisfied like the ideal generated by all of them is open um so I mean so you you can't just invert  $G$  if that doesn't correspond to a a rational open in Huber

sense but in situations where you can it's uh it corresponds to a closed conclusion I mean we we actually discussed this topology in the notes on um comp in so I think it was Gaga Redux the chapter called Gaga Redux um we discussed this this modification of Huber's topology um and yeah and apparently it's also in one of Huber's papers just passing to the opposite uh okay uh so one ah yes so one last remark um about this General setup uh so if so if um SI is some uh is some inverse system of compact haus door spaces then giving a map from local X so then giving compatible Maps uh from Lal of x to SI for all I is equivalent to giving a map um from Lal x to the inverse limit so the only claim I'm making here is that like inverse limits in topological spaces are the same as in locals in this specific situation which is something that you can uh quite easily see um and also if if uh and it's the same for Lo up so if all of those ones Factor through L up then this one will also Factor through l up um look the one we still the one yeah um and then um so then there's a remark that every compact house door space is a limit of metrizable finite dimensional uh uh so potentially potentially in many different ways but uh in many situations there's kind of a natural way of doing it we'll see this in the setting of burkovich spaces and then this gives you a way of uh going from the case of metrizable finite dimensional compact house store spaces where you can sometimes go from this perspective and you can go to this perspective and then you're closed you know the situation kind of passes to inverse limits in a nice way and um but now you no longer have to worry about things being metrizable or finite dimensional okay that was my little um just uh is a map to s the same thing as a map from loal no because you have this connectivity condition so it's the you still have to impose this locally that locally all the item poent algebras are connective so yeah I mean as far I it's not like I produced a counter example but I I I kind of believe that it's not the same uh okay okay so always confused is there really a t structure def over D of that object and no no there's no T structure so what I didn't I don't have to say what I mean by connective I have to say what I mean by connective locally and what I mean by connective locally is that there exists a cover by apine things such that it's your your object when you pull back to each of those apine things is connective now once you're in the apine world connectivity is a pullback of something connective is you have a t structure and the pullback of something connective is connective so it's kind of and also if the pullback is connective then it's connective or not no that that is not necessary neily true okay okay okay so it's not like so it's not sufficient condition for having a so this is like the for schemes that you have the the non aine orens but still if you use as the flat or pqc still if it's up upstairs it's up downstairs so that's but in your topology of course you have much more stuff yes yes so you can have these wheel we things where yeah you have a map which is apine locally but with an with a map with apine Target which is apine locally but not globally and this kind of yeah um yeah that's kind of unavoidable when you're doing analytic geometry so uh okay um where are we okay so yeah so um so burkov spaces so let me just start with a reminder on the definition so we know where we're going so so so a this is a bon ring so what does that mean it means that a is a commutative ring um and this Norm map from a to the non- negative reals um uh satisfies so first of all uh Norm of 0 equals 0 second of all uh the triangle inequality Norm of  $x + y$  is less than or equal to the norm of X plus the norm of Y and third of all um it's sub multiplicative so um and uh probably have to put the norm of - one is equal to the norm of one which is e z yeah I should probably put that yeah so uh either a equal Z or Norm of 1 equal 1 and Norm ofus one also do you need that it does not follow uh I'm not sure how much it follows but it's usually they want the triangle inequality Al with xus Y okay I think you can you can get it you can give a crazy thing like the negative multiply by two you can do something that still satisfy the a without but of course it would be equivalent something saf the a this is probably not how to show oh okay uhhuh yeah um yeah and then so let's say that and then a is a complete uh with respect to the nric uh um so an example yeah there's so example for say so say s is a compact house door space uh you could take for a to be ous functions say I don't know complex ah uh complex valued continuous functions and take the norm to be the soup Norm where you have the usual absolute value um and this maybe this is kind of um uh right another example would be if uh if R is a a tate Huber ring um and if you choose a pseudo UT forer um topologically nilp poent unit uh then you can Define uh Define a norm uh Peter actually wrote down the formula before but uh basically it's like the ptic topology um so yeah you can find Norm uh on R uh with say Norm of pi equal 12 and you want and it satisfies also that the norm of  $\pi \cdot f$  is always exactly the same as the norm of  $\pi \cdot$  Norm of f um and you need a ring of definition for the norm you have some ring of definition where the you declare the norm to be less than or equal to one yeah you can take one plus P 1 plus P 1 plus P what's P well uh I don't I don't understand in zp you could take P for for yeah and also with this there was this again any or no not zp sorry QP yeah QP take P yeah I mean zp is not a is not a

tate hubber ring what  $Q$   $QP$  in  $QP$  bar you can also take you can also take  $P$  actually so first of you must have complete because you assume it's complete oh thank you yes yes I mean yeah thank you and also if it is zero then certainly Norm of  $P_i$  is not 1 half so except when it is zero the norm of  $P_i$  is one up and and otherwise you have to to to uh to say thank you thank you Yep this have is this nor multiplicative it's say it's it's multiplicative when when one of the guys is  $p_i$  it's not not multiplicative in general um okay uh so uh right um so uh right and then so to this data so so this to a and the norm Huber assigned uh the space um which has a set burkovich thank you yeah we're switching over now thought burkovich um which as a set is given by the um ah ah yeah okay I guess completeness includes that um Norm of  $f$  equals  $z$  if and only if  $f$  equals  $z$  um right so uh so I'm just using this notation  $X$  so kind of it's just a decoration to uh so and this is now a multiplicative seminorm um with which is bounded by the given Norm you have on the the bonac algebra a um so um yeah so well um yeah so for example in and you can take uh and um if you require that the in this example if you require that the norm restricts to the usual Norm on the complex numbers then in fact um these are all the examples so that's maybe gon's theorem so Bas so those are basically the only examples in this situation and multiplicative includes the of Zer element that is the norm of one is one yes yes yes yeah this is otherwise we will get trivial thing out except but this is maybe um sorry zero mul multiplication of zero numbers yeah we don't want that so we want these to be like points so we want them to be non-empty uh so we want one to be different from zero I don't know I mean yeah so yeah um okay but do you allow a  $z$   $r$  no well wait I allow okay I allow a is the zero ring when this will be the empty set yeah um okay so uh yeah so there's the so this is actually a compact house door space so I didn't describe the topology but here it is you can view it as a subset of the product over all  $F$  and a uh um of the interval from zero to the norm of  $F$  and it's actually a closed subset and this map just takes a a norm and Records its value uh on  $F$  uh and the topology is just the Subspace topology so it's a a compact house DWF space Okay so so what we're going to do is we're going to um move this definition we're going to try try to uh we're going to try to make the same definition to make the same definition uh but in the world of analytic stacks uh instead of topological spaces so we want to take the same idea which is that we want to look at the set of all Norms multiplicative seminorms on a um bounded by the given Norm on a but we want to say that in the language of analytic stacks and that we've discussed um so using using the notion of norm uh on an analytic ring uh we discussed earlier yes yes yes yes yes that's true um okay so let me let me remind you about this notion of a norm so so the definition of so so um well we could say even more generally in an IC stack so uh um a norm on  $X$  is a map from the algebraic  $P_1 / x$  uh to the closed interval from zero to Infinity including Infinity um uh which is  $U$  multiplicative uh so away from Infinity so let's say on Norm inverse uh  $0$  Infinity uh and let me just write it uh so sort of suggestively like this you really should write down the commutative diagram with inversion on  $P_1$  inversion of the coordinate on  $P_1$  and inversion of the this extended real positive non- negative real axis here um um and some condition on how  $P$  sits  $p$   $p$   $p$  uh yeah so this is the  $P$  oh  $t$   $t$  is the variable on  $P_1$  yeah yeah right um um yes what do you mean by how  $P$  sits yeah so so remember  $P$  was this um uh this this this ring you have over any this ring object you have over any analytic ring which is the free guy on a topologically nil potent element so and it's some version of a unit dis um and what we ask is that it but this Norm function gives you another version of the unit dis which is like the inverse image of closed interval from  $0$  to one or you also have the inverse image of the open interval from well half open interval from  $0$  to one and you want to say that  $P$  sits in between the two of those yeah uh okay uh where am I yeah a nor okay and then recall so so write uh for the set of norms on  $X$  um so recall um  $n$  is is an analytic stack that's correct yes yeah it's a set because it's a we I mean we it's just given by a map in our category and our category is locally small a map in the category of analytic Stacks yeah so these objects are fixed right when you fix  $X$  this is fixed and this is fixed yes that you know that there is a uh why was it uh well we basically by definition every analytic stack was a small co-limit of representable analytic Stacks okay you have a because there was a notion of analytic ring where there is the category yeah okay there there you you you handle the problem of large sizes I mean because it's enough to check the condition some smoke but then you have the analytic stock where you cover but you don't know which covering you need to give aim so you have need all possible covers could be covers by bigger and bigger see but you say the category is accessible yeah so the the the main technical result you need to prove is that the the sheaf ification of an accessible prief is still an accessible prief that's that's the main technical result you need to prove we did not discuss this at all but that's what's what's underlying the the resolution to these issues it's like in Waterhouse uh yeah um okay so so this um um in fact so there's a cover so this uh this gaseous base

uh maps to  $n$  um in other words you can write down a norm on this gous space Tack and it's uh and this uh so this uh and this is actually the universal Norm on an analytic ring uh uh with a variable an analytic ring now with a variable  $Q$  in  $R$  such that uh Norm of  $Q$  is precisely equal to  $1/2$  um what did you write  $N_X$  is the analytic Stack  $n$   $n$  is an analytic stack yeah so yeah so the so  $N_X$  the set of norms is actually just the set of maps from  $X$  to some Stack  $n$  um right so yeah so if you ask for a norm on an analytic ring and an element whose Norm is exactly equal to  $1/2$  which is kind of somewhat stringent condition because a prioria again the norm map is Norm of  $Q$  is a map from  $\text{Spec } R$  uh to  $0$  infinity and you're asking that it Factor through this a prior its image could be some interval or something but you're asking that its image be exactly this Singleton um uh the universal example of that is this guy and moreover the map to the base stack is actually a cover in the sense of our Gro de topology on analytic Stacks because every Norm on an analytic ring locally uh you can find such an element we had this argument we we discussed this two lectures ago um right everything here was like  $X$  was a spec of an analytic rate um yeah I mean I I made this definition for a general  $X$  but it I mean it doesn't matter because the the condition I mean the Norms on uh an analytic ring they satisfy descent basically I mean it's kind of follows from General nonsense and so it automatically glues to say what a norm is on an arbitrary analytic stack and it just unwinds to the same thing um so the cover also exist because it exists locally like you GRE it well the map the map no the map exists because you can write down this Norm um and then it's a cover because given any Norm on an analytic ring uh after a cover you can find a  $q$  that satisfies this property sure do you need to do some base tank I'm sorry do you need to do a base change falls back to  $Q$  gu a base change what do you mean to is it okay or you still have a question okay um so um uh I want to before before finishing the discussion of or introducing the definition of this uh enhanced burkovich Spectrum as an analytic stack um I want to um I want to explore a little bit about uh so so what does  $n$  look like um so uh so well what do we have on  $N$  if you have a norm on an arbitrary analytic ring um so if you have a a norm  $p$  goes to Zer Infinity um so as mentioned for any section here uh you get a function from  $\text{Spec } R$  to  $0$  to  $Z$  Infinity but over an arbitrary analytic ring the only thing we know exist are the integers um so given  $n$  in  $z$  uh we get a map which records the value of your Norm on uh on the integer a little  $n$  um so what does this give this gives a map from  $n$  to right so now I should maybe pass to the local perspective uh to product over all integers of this extended half real axis so this um this stack of norms it localizes on this this product here um and uh we can ask for so we want to know what the what the image is so to speak so and let me say what I'm what I mean by image I mean the complement of the uh largest open subset uh so let's see with uh  $\Pi$  inverse of  $U$  equals mty set by definition yeah um and but but we'll also see that we'll also see that the fiber over any point in the image is non-empty so so that at least in this case maybe it's a general fact I don't know at least in this case it's kind of a theorem that the image is closed so to speak um okay uh right so now note that uh this so so uh the more traditional thing is this burkovich spectrum of the integers so that was also by definition a subset of uh this product uh going from zero to infinity and it was given by those uh Norms yeah um so multiplicative uh avoiding Infinity um satisfying triangle inequality uh let me remind you what uh what this thing looks like in in case people haven't seen this before so recall uh MZ looks as follows um you have a point at the center so to speak um which corresponds to the trivial Norm um meaning uh zero Norm of  $Z$  is  $z$  norm and Norm of everything else is equal to one and then you have several branches so you have um an aredian Branch so at the end of the archimedian branch you have the usual Norm uh the usual archimedian Norm usual absolute value um but then for each prime  $P$  you have uh and maybe uh yeah so for  $H$  Prime  $P$  you have another branch which also uh ends at some point and to get the correct topology on embedding it into  $R^2$  you should probably make the branches get shorter and shorter and shorter but okay that's um yeah but it's a the situation with these ptic branches is a little bit different so what's going on here here you have the usual absolute value and here at the halfway point you have the uh square root of the usual absolute value and here you know then you know you can put any Alpha between zero and one and you can kind of see from the intuitive perspective that this interpolates between the usual absolute value and the um the trivial absolute value here what goes at the top is not the usual ptic absolute value the usual ptic absolute value sits somewhere here so normalized to say so that  $P$  equals  $1$  over  $p$  and now you can actually scale it to any uh any uh positive real um um and then there's also a limit as the scaling goes to infinity and what that gives is the trivial Norm or the pullback of the trivial Norm on the residue field  $FP$  so in other words the norm of any multiple of  $p$  is equal to zero and the norm of everything else is one um okay what of course no I yeah this is great way I think this was something like this is the first talk there



was a talk in this course it was in this course or another probably in this course there was some discussion of but maybe I could do this another thing yeah so I'm I'm I'm yeah I'm I'm trying to recall something which is well known indeed um in order to set up the discussion of uh of What's following here so but but in particular I want to emphasize this is a really big space but the Subspace uh  $MZ$  is quite small you know onedimensional um yeah um so now what we're going to see so there there you allow allow value to be in but here you don't that's correct so um so here's going to be the the claim so the image of Norm is a is a larger subset uh so it looks like this um so you have all the same ones as before uh sorry I'm I'm trying to say that it stops there yeah trying to draw like a closed interval sign but then also at the archimedean place um it gets extended so here now the usual archimedean absolute value is also in the middle of of the interval and you can take arbitrary powers of it so for any  $\alpha$  in  $\mathbb{R}$  greater than zero you can take powers of it um so it'll go to there in one direction and to the other direction you get some really strange Point uh so strange point where uh which corresponds to uh so it's a subset of there so it's a it's given by some maps from  $Z$  to the extended real line there um and it's given by Norm of  $n$  equals infinity if uh if  $n$  is not equal to minus one 0 or 1  $z$   $z$  yeah yeah so um curly end here like Curly end like in this so it's like solid  $Z$  no it's a different kind of Base because I mean solid  $Z$  if in over solid  $Z$  for example the real numbers are equal to zero so you can't um I ask question uh you can ask a question yeah is the picture accurate in what sense I mean I don't know the price should be closed or not you mean this picture or this picture this one here I don't understand uh uh I mean I I want to say that this is the same as this on all the non- archimedean branches and on the archimedean branch it just gets extended past yeah so I sorry if that wasn't clear yeah it gets extended all the way to infinity and then compactified at the end yeah I guess I guess what I you can try to write a formula for it it's like image Norm is like you take the burkovich space of the integers ah no let me make a before I say this so in particular so uh the triangle inequality can fail well that's quite clear here you have Norm of one equal one but Norm of two equals infinity okay that's pretty drastic failure of the triangle inequality but also like for the square of the usual absolute value which is a new thing you have then the triangle inequality fails as well um so yeah I guess another way of saying this is like image of Norm you can get it from the burage spectrum of  $z$  um on the burkovich spectrum of  $Z$  you have an action of um uh the real number is bigger than or equal to one wait did I yeah or well yeah yeah um or maybe yeah I don't know but but on the on this other thing you have an action of the positive real numbers you can kind of do this and that doesn't do anything on the non- archimedean branches and then on the archimedean branch it extends it all the way and then you compactify it one point compactification so I don't know um okay uh right so let's explore this and let's see what kind of what what is kind of going on so uh let's so to to explore this uh consider just the map what's that is the one one point yeah the one point yeah yeah one point Yeah Yeah question picture you seem to see this image gen topological space but this is supposed to be something as a sub associated with  $Z$  wrong uh that's a yeah so that's a good question so by definition I made it a uh I was saying it's a closed subset of uh of the topological space um now you can view this as an analytic stack we we saw how to view this is an analytic stack and you can take the product in the category of analytic Stacks that's perfectly legitimate it's no longer metrizable so you know why is it no longer I mean no longer finite dimensional sorry it is metrizable it's no longer fin I said the wrong word it's it's metrizable it's no longer finite dimensional um so we didn't quite talk about this thing but but you can I mean you can still view this as an analytic stack and you still do get a map of analytic Stacks from end to this product and this closed subset does correspond to a uh a subanalytic a monomorphism of analytic Stacks here and the map from  $n$  there does Factor through that closed subset and so you can view it you can view it in several different ways um as usual okay um to explore this so cons fix a prime  $p$  uh and consider just uh uh as a start um Norm of  $P$  um so so here's the first claim so the claim is that we can understand uh so um so if we look at the um the locus in this Universal space of norms where the norm of  $P$  lives between zero and one we can understand this um so this is just the same thing as so maybe maybe uh just as notation I can call this  $n$  and then  $Z$  less than absolute value of  $P$  less than one so it's the stack parametrizing Norms on which the variable  $P$  lives between 0 and one um this is equal to uh  $\text{spec } \mathbb{Q}P$  the gashes version of the pic numbers across this uh stack Associated to the open interval from 0 to one um so yeah well in fact this is quite easy to see because um we know that the universal analytic stack equipped with some variable whose Norm is between zero and one is uh so is  $\text{spec } \mathbb{Z}q$  hat plus or minus one gases uh cross cross 01 um and then you just have to impose that that variable becomes  $P$  so you just set qals  $P$  or you mod out by  $Q$  minus  $P$  um and then that as Peter discussed when discussing this gous base stack gives you some analytic ring

structure on the pic numbers and then the second variable doesn't really change uh so um and so yeah and so and then if you look so so I'm claiming in particular that if you look at the the the universal uh so let's say we take the so let's say we take the fiber over a point  $\Lambda$  and 01 uh um then the the in particular you get a normed analytic ring structure here um and it is so what does that mean it means that for every radius  $R$  you have some notion of overconvergent functions on a disc of radius  $r$  uh and then notion of on a disc of radius  $R$  uh is the usual one uh from non-archimedean geometry uh where uh you take the normalization of the absolute value on the pic numbers for which the absolute value of  $p$  is equal to this  $\Lambda$  so um what we're seeing here is the the kind of the the uh interior of the  $P$  Branch here is kind of uh fairly straightforward to understand um so next uh let's look at the locus where another Locus that's fairly easy to understand um is the locus where  $p$  is between one and infinity strictly between one and infinity because that's the same thing as saying uh that uh well first of all  $P$  has to be invertible um because  $p$  is away from zero the absolute value of  $p$  is away from zero um and then it's the same thing as saying that the absolute value of  $1/p$  is between 0 and 1 um and then we again we can use the exact same argument to understand what this is and what you get is you get spec of uh the gaseous real numbers cross 01 and the argument is the same and that as Peter explained if you take this uh this gaseous this this ring here and you mod out if you set  $Q$  equal to  $1/P$  then you actually get the real numbers you get a certain analytic ring structure on the real numbers which is uh yeah this one here um and again if you look at the universal Norm with a fixed value of  $\Lambda$  um uh fixed value of  $P$  yeah sorry sorry what for example two  $P$  could be two yeah yeah um the universal Norm here is the usual uh given by usual overconvergent functions uh in archimedean Geometry yeah say complex geometry but with respect to the norm uh the which is a power of the usual um a power of the usual absolute value where  $\alpha$  is such that uh if you take the norm of  $1/p$  uh uh you exactly get  $\Lambda$  so so I'm not quite sure usual overconvergent where does it show in the uh where where the those uh overconvergent so you had some some formal series with  $Q$  yeah but and there you can speak about  $\text{Loy}$  I mean the the part where the overconvergent part I mean at least there are some algebras corresponding to but then you specify that it is okay you can walk over no I'm not I'm not because the formal variable was already replaced by  $P$  so it's yeah  $B$  lost in this yeah usual yeah you have to do a calculation so for example yeah you could take so the first thing to understand of course is why when you take this ring and you specialize to  $1/p$  why you get the real numbers um so okay that has to do with some kind of Base  $P$  expansions um and then you have to understand and say if you take this module  $P$  this basic module  $P$  you want you want to know what that base changes to it should be some module over the real numbers so it should be some sequence space with some summability condition and you can see that the summability condition is plus or minus  $\epsilon$  maybe some little fuzz it's basically just um well I mean Peter actually described what it was it was some kind of exponential decay thing thing but the point is that it that thing does sit between the usual ring of overconvergent functions on the unit disc uh and the Ring of holomorphic functions on the interior of the unit disc um so when you do this overconvergent business uh remember when we did the yeah when you do this overconvergent business it it it it that that subtlety of exactly what ring that is exactly what some ability ability property you have it doesn't matter anymore and it just Returns the usual ring of overconvergent holomorphic functions uh on the disc and there's some scaling of the usual absolute value which comes in because when we were building the universal Norm uh we had to take a fixed value of the norm of  $Q$  and and kind of yeah use that to write down the answer but it's such that when you specialize to normal situations you get the usual thing okay um so in particular uh this this Locus um this is independent of  $P$  um because in any I just I just told you what the what the what the universal guy was and I didn't have to make any well I mean there's some rescaling property of the of the norm but um that that kind of uh yeah it it doesn't affect this Subspace of the of the of the universal space of norms so this is the kind of thing that you need to see in order to to see that you're getting the Burk of age space you have some infinite dimensional space but actually the conditions on the various prime numbers are not at all they're very tightly related to each other this is like  $\text{os}$ 's classification so if you have a if you have a norm on the integers for which the norm of  $p$  is equal to  $1/2$  then it you know it kind of it has to be the pic norm and in particular its values on all the other integers have to be determined by that um and you can see that also in in our situation as well um that when you force yourself to live in this Locus which is only a condition on  $P$  that automatically tells you what the norm of everything else is because you can just do calculations with these usual uh rings of overconvergent holomorphic functions in usual non-archimedean geometry so on here uh so the norm of all other uh and is determined each is

set of norms on  $z$  uh yes uh not up to equivalence no yeah question all the point right yeah yeah why that would be is there some explanation why that would be for me it's a bit of a it's just a calculation I mean um in our in our yeah so that's a good point so in our axium for normed analytic ring we had no version of the triangle inequality whatsoever we just had this that the norm should be multiplicative but then you can look and see well if you believe what I'm claiming then kind of almost all of it does almost all of the space does satisfy the triangle inequality because you can just check on in terms of the the rings of functions that are being assigned and you can just verify that the triangle inequality holds so whenever you're in the ordinary burkovich space of  $Z$  you actually satisfy the the triangle inequal your Norm actually satisfies the triangle inequality and then okay it's a you get some sort of quasy Norm if you move out in this direction and this this part is a little funny but if you throw that away so there's always some version of the triangle inequality that is automatically satisfied just as a consequence of multiplicativity and that's that's kind of funny yeah in the usual the like in fact was consider earlier then I mean it's I'm speaking now about Theos classification the or maybe so you can put the following condition I don't remember which refence instead of inequality you can put like  $X$  plus  $Y$  Val less equal to some constant time  $x$   $y$  and then one can prove that after a normalization by some power you have the triangle inequality or even the yeah and so is it the case that here you can is related somehow where yeah in the locus where so if you know that the priority that the nor some integer like or three not Infinity then you can get the triangle in quality by scaling it yes exactly exactly yeah so if um well I have on everything now I'm speaking about Bigg like in the big let me make some further claims so claim um uh so yeah if um so if you have an normed analytic ring uh with the norm of two say doesn't matter less than or equal to one um automatically satisfies uh the non- archimedian triangle inequality um a normed analytic ring uh with Norm of two uh less than or equal to two satisfies the usual triangle inequality uh oh I sorry let me move this is the that no of two two is equivalent to no of three three yes yes yes two is an arbitrary prime number here two is an arbitrary prime number here about to six is it that's all that's also equivalent yeah yeah so I guess Prime is not so important yeah two yeah two is an arbitrary integer bigger than one yeah um okay so uh and then um where am I now oh yeah so a normed analytic ring uh with Norm of two less than infinity always uh there there always exists a constant  $C$  greater than zero such that the norm of  $x + y$  is less than or equal to  $C * \text{the norm of } X \text{ the norm of } Y$  but this is interpreted not in the sense of normal functions but it's not yeah it's some Universal thing like uh so you know you write down you have whatever you have yeah you have  $P_1$  like yeah the locus where the norm of the  $T$  variable is less than or equal to a cross  $P_1$  you know Locus for the  $S$  variable is less than or equal to  $B$  um yeah that this this maps to so it's  $a$  and  $b$  are less than infinity here so it's this maps to  $P_1 R$  and then this maps to  $Z$  Infinity but this should Factor through zero and then  $C * a$  plus  $b$  so yeah so and this is addition  $A_1$  yeah they it's well defined because this happens to live inside  $A_1$  which we already already argued earlier yeah yeah so there's always some um some version of the triangle inequality um yeah uh so how oh yeah so how do you prove these claims by the way these claims imply the claim about what the uh image of uh of  $n$  is because um so yeah so and then this uh this Extended burkavage space um the one I mean the specific one that I that I wrote down uh the reason is that you can um you can look at you know the norm of two uh which goes to zero Infinity ah yeah if you want to if you want to know the the image of something something um you can actually work uh you can actually work on stratifications of your topologic you don't have to work in closed covers or open covers you don't have to work locally in the sense of closed covers or open covers it's enough to work locally in the sense of stratifications I mean you can stratify the base and look over the different strata because you you have to see when the pullback to an open subset is empty and while like a closed and an open Don't form a cover they still uh they still detect ISO morphisms in particular they still detect whether an analytic stack is empty or not so so we can can look over uh the locus where two is less than or equal to one the locus where one is less than two is less than infinity and then the locus where 2 is equal to Infinity um and then we just have to take the union of the the images we see over there and here uh if you believe this claim then uh where am I uh yeah if you believe this claim then there you have the non- archimedian triangle inequality so here it's automatic uh automatically a subset of just by some Universal argument the the non- archimedian burkovich spectrum of  $z$  um so we're contained in the in the claimed Locus there um this thing we already classified uh this is contained in the archimedian locus the uh archimedian Locus um and so the last thing we need to do is to see that uh this Locus consists of just one point um so need um for all  $n$  different from 0 1 and minus one um and that actually doesn't follow from the claims well yeah well I'll what's that it does follow from was

arbitrated two was arated yeah that's right yeah yeah the claim that the these conditions are independent of two  $u$  does give  $u$  does give this as well so  $u$  last thing you say plus be yeah that's true I mean you could also yeah you can directly argue like if three for example wasn't sent to Infinity then you'd be in one of these locuses and then two wouldn't be equal to Infinity so I mean it's kind of yeah  $u$  okay so how do you I'm going to finish up so how do you prove these kinds of claims well you just calculate so  $u$  so to prove them wait but for the first part we only know it's not lgers than once and it's not impli directly from yeah then you also need to see that uh if you take any point say here and then take that there exists some normed analytic ring which has those values  $u$  and we already saw it for the points in the interior of the the Rays in the Burk ofage space and it's also quite easy to like hit the center because you just do some non-archimedian geometry over some lant Series ring with  $Q$  with the trivial Norm or you can hit the points at the end for example by doing some non- archimedian geometry with with  $FP$  so you can you can see that so this gives a containment on the image the image is contained in this this argument gives the containment but you can actually see the other inclusion by just exhibiting uh I just yeah  $u$  right okay so uh where am I oh yeah so to prove  $u$  we  $u$  it's enough to we we can assume uh we have this  $Q$  in addition we have our arbitrary Norm dialytic ring satisfying this property but we can add in addition assume we have this  $Q$  with a norm of  $Q$  between 0 and one  $u$  and then we're working over so uh so then we're over this Gast to base  $u$  and then for the first claim here this non- archimedian claim  $u$  what do we need to do so well then the UN Universal case I mean given that we fix this data and then then we we we just asking that Norm of two is less than or equal to one so it just lives over uh an item potent algebra so why said pass the this to it was because of this claim that you have a cover okay yeah and you can check things like non- archimedian triangle inequality after passing to a total space of a cover okay  $u$  and what is this item potent algebra  $u$  this is just you take the you know you take the over over this  $Q$  Plus orus One gas uh you take the Ring of  $u$  holomorphic functions so to speak you know this overon convergent version of the unit dis  $u$  and then you just mod out by uh  $T$  minus 2 and what this gives  $u$  and you calculate  $u$  and I won't get into the details  $u$  what this gives is like a on the level of underlying Rings uh so you have  $u$   $u$  so kind of yeah so you have usual so you have a power Ser or lant series with integer coefficients which uh  $u$  over converge to radius zero so which are which are converge on some unspecified open dis uh around the origin  $u$  and yeah in this way you see the that this already shows you that the condition is independent of two  $u$  because this this description here is independent of two and then you can just look at the universal then over this you can also calculate these what with the base change of the you know these kinds of overon convergent rings are and you can just check by hand very brute force that the non- archimedian triangle inequality is satisfied that these it's just about about a certain there existing a certain map of item potent algebras uh lying over the addition map on on the polinomial ring  $u$  so in particular this is the formal the convergent formal Ser not well we have we have we have we have negative powers of  $Q$  here what does it mean oc0 oh I'm sorry sorry sorry sorry yeah  $Q$  inverse yeah thank you thank you yeah miror morphic at the origin yeah thank you  $u$  okay so uh that's that's how you prove this claim  $u$  this claim actually follows because uh the only other Locus you need to worry about is the locus where you're between one and two and then you're in this non- archimedian branch and we saw you get the usual thing you have the triangle inequality there  $u$  and  $u$  and then  $u$  yeah so to well actually yeah so then to prove this claim  $u$  again it's it's actually enough to it's I think it's easier to prove well I think the the root is it's easiest to prove this claim  $u$  and then that implies that claim because this means that  $u$  again that the only other possible point to consider we're living in the archimedian locus so we're just some rescaling of the usual absolute value and then this uh weak triangle inequality this quasi Norm triangle inequality is actually satisfied  $u$  so  $u$  and to prove this you again just do a calculation it's sort of similar except you  $u$  instead of setting  $t$  equal to 2 you're setting  $t$  equal to 2  $u$  or well or you're setting yeah oh there there's a similar or  $t$  inverse equal to yeah what what you end up getting is some weird version of functions convergent uh uh on the open dis like this  $u$  and again you just observe that it's independent of two  $u$  yeah so in this Locus your convergence property is shrinking down to zero uh in this Locus the the the locus relevant to this claim  $u$  it's kind of it's going in the it's the laurant tals that are getting  $u$  you know forcing the convergence out to the boundary of the unit dis  $u$  yeah yes please yes yes yes sure go ahead yes okay I don't know does the a can you uh uh explain a little bit more by what you mean by that maybe by giving an example okay okay yeah  $u$  yeah okay so as mentioned uh this Friday there's no course but please come here and Peter will be here in person and so will three other people giving talks  $u$  some of which are relevant to the material here  $u$

and then next week I'll continue this discussion of Burk of geometry next next Wednesday and then that next Friday is actually going to be the last class so we're really almost done here um okay thanks everyone over it's over so yeah can I just ask I mean what I think I'm what what what it means if you have an analytic stack which sits over like any of these classical regions of but but what does it mean if it sits over infinitely really extra point I mean do you have some it's weird I don't know it's it's a bit weird the norm is quite different the norm there was this Norm on a ring in the sense but here you have an analytic one so a norm is something quite different I mean it's as we said that for for like the  $\mathbb{T}$  ring it's somehow you choose one nor and then you can rescale by continuous exponent exactly on the to around continuous functions from the  $b$  space to  $\mathbb{R}$  yes I mean this is actually true uh in the the set of norms is it also like this was so it's not actually statement is that related to this SP that just mentioned that um I mean on this stack of norms mention of by scanning the norm and you can with both and then any space has a unique map to the sky any analytic space yeah we renamed it Tate to not conflict with uh yeah so an analytic space  $Z$  by yeah exactly which is rescaling the norms and govern mentions to any I'm sorry no no this is a quotient of the I mean well except except for this AR it's a quo of the burkovich line by rescaling but but there's more structure on this because it's an analytic stack and not just a topological space so umst action is free which is not uh the action of our positive is because it's as a set say is a set is free so it yeah so the yeah so I I so relatedly like when you in some sense uh so I I discussed it what happens in the open parts of these things but actually if you try to move towards the the boundary points what about the infinity point in this question yeah it just uh it's Trigg well no because I mean you so from if you only look at the perspective of norms of integers then it might look like it's a stabilizer but um well I yeah um so uh right where oh yeah so yeah so I said that so on the interior it's kind of this this this uh analytic stack and it feels kind of like a space like on the interior you're just seeing that unit interval but base change to some gaseous version of say  $\mathbb{Q}_p$  here or the real numbers here but as you move towards the outer points or towards the inner point it's kind of it's it's more it really is more like a stack so it's not that you know the the fiber over this point is not uh not apine for example neither is the fiber over this point and um that the boundary is fixed on the component um well from a geometric perspective no because over this point you can have something like FP luron series  $q$  and then you have the Notions of overconvergent functions on a disc there and then when you rescale it's really changing which disc you're labeling by a given positive real number so it's not universally true that like yes yes yes exactly yeah oh according to the website of I we have two more weeks yes uh we may have to change that because there was some uh yeah apparently on the IHS website it says we have two more weeks so but I'll I'll get I'll get it fixed because the number of talks was originally more than the actual yeah yeah so we yeah I mean we did decide that next week is the last week right Peter yes okay what's that class here yeah I see yeah yeah in in bond classes officially end that's why we're stopping



## 23. BERKOVICH SPACES II (CLAUSEN)

[https://www.youtube.com/watch?v=vXZC3WzKZgo&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZ0](https://www.youtube.com/watch?v=vXZC3WzKZgo&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZ0)

### Unfinished starting from 0:00

So this uh this lecture is about Berkovich spaces which I um topic which I kind of began discussing uh last time and um so well let me remind you kind of the classical setup you have a a Banach ring so it's a ring equipped with a norm which is sub multiplicative satisfies a triangle inequality uh maybe I'll I'll do it do it better this time Norm of one equal to zero or one unless  $R$  equals  $\mathbb{Z}$  um and  $R$  is complete and also of okay yes uh yeah um okay uh and then to this uh Berkovich assigns the compact Hausdorff Space  $M_r$  Norm which is a subset over the product over all elements in  $R$  of the closed interval from uh  $\mathbb{Z}$  to the norm of  $F$  and it's the set of all uh the notation will be uh that a point in here will be denoted  $X$  but what it literally is is a evaluation uh a map from  $R$  to the non- negative real numbers which is now multiplicative so um and again satisfies triangle inequality and now one is one yeah now it's it's kind of very it's strictly multiplicative yeah so um yeah um and we and uh you know  $R$  is not complete with respect to this Norm  $X$  in fact there could be many elements of Norm zero but uh so remark if you complete uh complete  $r$  with respect to uh any uh Norm  $X$  sorry you get a complete valued field  $K_X$  hat I don't know which is sort of the residue field in the sense of you know since of Berkovich theory and recall that there's basically kind of three cases uh there's one is an archimedean and that's the same thing Bosschi is saying it's either  $\mathbb{R}$  or  $\mathbb{C}$  so in the archimedean case there's no variety really in complete valued Fields I will see but the norm could be a power to  $\alpha$   $\alpha$  One  $\alpha$  bigger than zero yes uh yes uh um the second is a it's non- archimedean uh but discrete um and then you have the trivial Norm so uh so and then the third case is a it's an honest non-archimedean field non-discrete I mean I mean the the induced topology on the field I mean it's non- discrete um the Valuation could come from a discrete valuation um and then again so then uh so so you could have some normalized version if you choose a on discrete there is an ambiguity yes it's not yeah it's not about discrete non-isr exactly that that was the parenthetical comment I was making a non-discrete topology it depends if you call a discrete field a non archimedean field I think not but you you wouldn't you wouldn't even call this a non- archimedean field yeah okay so the the norm is not maybe the yeah I it's not the most people call archimedean yeah okay the norm is non archimedean but but but the field is discrete so yeah it's just a discrete just a discrete field um so anyway these are the the three kind of cases of possible residue field behaviors as you see um uh right and then so here are the  $\alpha$   $\alpha$  okay um there is no normalization no I mean yeah you could choose a normalization if you if you fix a pseudo uniformizer you can  $\mathbb{C}_h$  and a and a real number you can choose a normalization yeah yeah all right okay uh yes um yeah so these are kinds of kind of points and uh so what we're going to do now is we're going to promote this so the goal is to promote uh  $M_r$  Norm to an analytic stack uh using the stack of norms  $\mathcal{N}$  so we'll so a map from a map from an analytic stack  $X$  to  $\mathcal{N}$  was the same thing as a certain map from  $P^1$  of  $X$   $P^1$   $X$  to uh  $0$  plus infinity uh so and uh satisfying nor maximum uh the most important one being multiplicativity which you have to phrase a little carefully but in the end it's just a it's just multiplicativity um okay so so and we also we kind of investigated the geometry of  $\mathcal{N}$  last time so we saw that um  $\mathcal{N}$  lies over this extended Berkovich spectrum of the integers so um so that was this this space for um so I mean we here we have a a triangle inequality you know the usual triangle inequality and that that's what made that the difference between like the reason you couldn't take an arbitrary power  $\alpha$  here but I mean if you take an arbitrary power  $\alpha$  you get something that basically basically behaves like a norm but it's really only a quasi Norm so you have to put some constant in front here um plus the limit you have also the kind of the limit at in yes there's also the limit at Infinity yeah so so that that so that makes uh yeah so and in this stack of norms uh in this stack of norms there's no triangle inequality imposed and it turns out what you get is this non-strict triangle inequality where all powers of the usual archimedean norm are allowed and then also there's a some kind of very strange limit point um where your Norm takes infinite values on natural numbers uh and then you have the uh the archimedean ones so two  $\mathbb{R}$  absolute values which end and here in a point which has both characteristic zero and characteristic  $p$  Behavior but where the norm of two equals zero um uh and then then for the other primes  $p$  um and we also kind of saw what the on the interior of these line segments you were getting the some uh the gaseous  $R$  theory on the interior here you're getting the gaseous two EIC numbers and then as you move along the normed the norm is changing but the kind of so to speak the analytic ring is not um and  $\mathbb{Q}_3$  and so on and then here you have things living you have  $\mathbb{F}_2$  living for

example um you have things living in characteristic two there living in characteristic three here and here you have things uh living in characteristic zero so in some sense uh um you can imagine that the points of this stack correspond to something like these complete valued fields or or the minimal choices of complete valued Fields like you have real numbers piic numbers you have discrete FP you have discret uh discrete  $\mathbb{Q}$  um uh so that's kind of a substitute for the notion of multiplicative valuation but then we still have to input our um our Banach ring  $R$  into the construction in order to get something non-trivial so here's the definition um and I don't know what good notation is I I'll write I'll write it like this so um so this will be an analytic stack which will be a substack sub subset of uh stack of of norms cross uh and then some apine analytic stack which is just we take  $r$  with a trivial analytic ring topology uh I mean sorry the trivial analytic ring structure So What by this what I mean is that you take a  $r$   $r$  is a Banach ring so it has a topology but you can also view it as a light condensed ring so the um so using the topology you consider those condensate yes um and then trivial analytic ring structure that is all modules are allowed yes even not okay it's the full you know condensed derived category of this condensed ring so that means that uh if you want to give a map from  $X$  to this  $\text{spec } R$  Tri that's exactly the same thing as giving a map of condensed rings from  $R$  to the the value of the structure sheath on on  $X$  uh right yeah so uh right so this  $U$  this is going to be a subset of this product consisting of those so such that  $M$  well let's say maps from  $X$  to here these are going to be by definition in injection with um so well it's going to be a par consist ing of a norm uh and then a map from  $R$  to the Ring of functions on  $X$  uh such that and then we impose a condition oh forgot to say the the thing that ties this this to that so uh you require the the uh multiplicative valuation to be bounded by the given norm and that's exactly what we're going to do here so such that for all  $F$  and  $R$  um so when you have the norm here and you have an element in  $O$  of  $x$  uh you get a section of  $A_1$  and you have a section from an element of  $ox$  you get a section of  $A_1$  and in particular a section of  $P_1$  and then you can compose with the norm and you get a map Norm  $F$  from from  $X$   $X$  to  $0$  plus infinity uh such that this map lands in uh  $Z$  and then Norm of  $f$  okay um so um well I guess the theorem first theorem would be that um uh so this is an analytic stack uh which localizes uh along uh the burkovich Spectrum  $M_r$  so in the sense that I discussed in the last lecture so that if you look at this local of all open substacks of this  $\text{spec } \text{Burke } R$  Norm uh this Maps naturally to um this compact house door space which is the Burk of a Spectrum so uh you have the local of open in the usual topology and the local wait usual topology oh oh on the right yeah on the right yeah but the on the left open substock means what it means monomorphism it means a monomorphism which is shable and uh and chromologic smooth and this was disc discuss in one of the talks okay and you you could even ask for a map of antic step right that well when when this is finite dimensional and metrizable yes which is basically all cases but I mean I yeah yeah yes yeah so again um so in in the case when this is metable and finite dimensional to verify that you have a I mean it's really just a condition to say that You' get a map of analytic Stacks to that thing and it's just you have to check some connectivity properties of of the item potent algebras assigned to close subsets that are kind of implicit in this in this map so it's a condition that you can check in practice and so is it the case that you have since  $R$  is cool you can think of it as a limit of  $\mathbb{C}$  sub Rings which are account many element yes so you can probably reduce to no but it maybe it's not Dimension well I'm not sure but no no but you can look at as a  $\text{CO liit}$  over finally generated sub rings and then and that's always embedded in a finite I mean okay okay okay uh yes yeah so that's one way so there is indeed a canonical way to write this as an inverse limit of finite dimensional metrizable spaces but um let's not do it let's just let's just work with this uh this setup here and is it true that you have inverse limits in analytic stat yes you have you have all limits you have all limits in analytic Stacks yeah ah okay so you can syn of it as a limit over the stack is a limit over the well this isn't a stack now this is just I'm just viewing this as yeah when you have those nice sings you can construct uh okay so it's not uh no but then you can take for the nice subing you can take the stock Associated to  $m$  in those limit you get something which is yes you get something yeah then then you get something the only problem I have with that it's not a real problem but just is that if you started with something which happened to be already be finite dimensional and metrizable then you'd be non-trivially writing it as an inverse limit of other finite dimensional metrizable things and I mean you know so let me just not let me just not get into it okay yeah one other question when when  $R$  is State then we we also show that we can localize over the spa and then there's a map from the spa to the I mean that's right that's right yeah there's a commutative di there's a yes there's a commutative diagram that you can write down where here you have map from you know the Huber space mapping to this and then you have a some this thing and then you have the the solid

guy here mapping to that and this this last Arrow I I mean no no yeah when I I I'll discuss in more detail what this looks like in the Tate case and then um and then you'll see yeah uh okay so in particular just I want to uh highlight that uh we get a structure chief on on this topological space and even a structure chief of you know of infinity categories so a theory of Quasi cerent shes on on the usual burkovich space um so um I'll explain in basically the Tate case how it's pretty easy to calculate this structure chath and see what it's doing in the case of uh Rings like the integers well the integers you can kind of do it by hand but it would be interesting to compare so uh uh uh so it's we I'll explain basically how you compare to the cases burkovich discussed but it would be interesting to uh uh to compare uh to puu who kind of more or less by hand described structure sheaves in certain cases uh over the integers um so this is a different different approach where you define something which is a prioria structure sheet and then you have to calculate it which can be done in principle but you know takes a while um in Poo's case he explicitly assigns the value and then he has to maybe prove some yeah prove some descent results so here we automatically get some sort of infinity descent but then you have to calculate the value um okay to like I know this and should be easy to say what I on a disc the structure sheet yeah it does reduce to seeing what goes on on a dis but um to see what goes on on a dis um yeah so I I mean I I I do believe that it yeah so I mean I I I'll explain what you need to do to do these calculations and I think you'll see that it is like but probably there are like in the case of ho Rings there are probably some derived phenomena because when you want to look at the things like the the algebra functions on close or open this anyway you you quau something by certain I mean you have non closed idea so probably you need to to work in some derived sense to get the right to get what you get from your fan Theory you should probably have the drinks which are complete we do we do we do okay yeah I mean that's what yeah um yeah so I do believe that in in cases like uh what poo considers where it's you're starting with a a discrete ring like say the integers um that that the calculations are quite feasible um but if you started with maybe a a more arbitrary Bond offering it's it's maybe not so obvious how to do the calculations um okay so where are we ah okay uh I want to explain okay so why this is true so I guess proof um so well that it's an analytic stack I mean we defined it as a a funter a functor of points so I the only thing to make it an analytic stack is some accessibility but let me let me explain how to give charts for it so let's let's give charts um well so recall that we had charts for the Norms the space of norms so we have uh we had this uh  $\text{spec of } \mathbb{Z} \hat{q} \text{ plus or minus one}$  um uh this gases Theory and this we can view as the universal um sort of uh so it it sits inside  $P^1$  is kind of the universal Locus where I say absolute value of  $T$  is equal to 2 um uh right so uh so if we pull back along this uh if we pull back along this uh then um so so so I mean we look at uh so we look at this  $\text{spec } \text{Burke } R \text{ Norm Norm}$  and then  $\text{spec } u \hat{z} \hat{q} \text{ plus or minus one gas}$  and then we get this uh the UN some Universal disc over here some Universal well I don't know anulus over here uh let me just call it um let me just call it um uh  $y$  um uh then so for this  $y$  thing um we've already got the norm we've kind of artificially adjoined an element of  $\text{Norm } 1/2$  um but the only thing we need to ensure to to go from uh this to this is we need to give the second map we need to give the map to um to  $\text{spec of } r$  with a trivial thing and we need to see that they agree and we need to enforce the condition that um this Norm condition that NF lands in inside that that part there um so we just so to get  $y$  you just take uh  $\text{spec of } \mathbb{Z} \hat{q} \text{ plus or minus one gas cross Spec } R \text{ TR}$  and then pass to the closed subsets uh uh given by the item potent algebra um obtained by uh uh taking the norm  $F$  inverse of of these closed subsets here so uh in other words this  $Y$  is our is aine so this uh this burkovich spectrum here has a a fairly simple cover by by an apine and to calculate this apine the most difficult part for like a completely General  $R$  the most difficult part is already in the first step that is making this product and uh calculating what analytic ring this is and in particular calculating what the underlying condensed ring is because what this tensor product involves is um you have to calc calate the the gaseous localization of of like  $R$  yeah so you have to then apply this gaseous localization again and recall from Peter's lecture where he described the gaseous localization that it's given by some sequential Co limit there are some coo complexes and it's a little uh yeah then a um but once you have that I'm going to I'm going to explain that calculating these item potent algebras is not that hard so once you know the the the ring you have here then you're passing to certain item potent algebras over it and that's that's not the hard part of the calculation but so in particular if you're in a situation where you can do this calculation then it's it's quite feasible to to calculate  $Y$  which is giving a presentation for this stack and um and we also saw that like The Descent for this cover um uh is quite simple it it happens at the the zero stage so like the structure sheath here will just be a retract of the the structure sheath here so the Ring of functions on this  $y$  so what did you say The Descent oh yeah when

we argued that uh that this map was a cover we did it by saying that it was uh proper and descendible and we even showed that like the unit object the unit the the structure sheaf here is just a retract of the push forward of the structure sheaf there um so that makes The Descent uh quite straightforward basically when you're passing from here to  $Y$  you're adjoining some extra variable  $q$  and it has some analytic properties but you calculate some ring up there and you just look at the zeroth possible zeroth coefficients of that ring and that um that tells you the value of the structure sheaf here um look at ring is a direct factor one yes yeah exactly not as a ring but I mean it's yeah it's a it maps and then there's a linear retraction yeah just like if you had Laurent series like constant coefficients inside Laurent Series yeah yeah mod yes exactly yeah um right okay so uh and the yeah so the the the map to the Berkovich Spectrum is it's kind of uh well so um so on this spec Burke so the map to uh so on spec Burke you have the universal Norm um let's call it  $n$  and and then we also have a for every element in  $R$  we have by fiat a map to the structure sheaf here so uh then we get a map from spec uh spec Burke  $R$  to product overall  $F$  and  $R$  zero Norm of  $f$  um but since we enforce that every element of  $R$  for every element of  $R$  the norm of that element is bounded by the prescribed Norm on our bounding  $r$  that in particular implies that the norm of two is uh less than or equal to two which means that by last time the triangle inequality holds for  $n$  and that means that this this map lands in uh inside the Berkovich Spectrum indeed I'm sorry the image the image mind yeah  $M_r$  is by definition a subspace of the product yeah this construction we could also keep some of the which does not satisfy the triangle inequality right yeah I mean I um yeah it's it's natural from the perspective of this stack of norms that we've been discussing as we've seen to relax the triangle inequality and it's and you can do that I mean you could I mean the the formalism is quite General there's I the reason I I stuck to the classical thing is just because it's the classical thing you have for the triang in quality you assume that yeah you could require that there exists a constant such that uh you know this that's that's one thing you could do okay but then if you want to uh ah so this still defines a uniform structure so you can say it's complete and uh and it is not equivalent by fix changing things slightly to a actual Norm if you have this so well I don't know because for Fields just by power  $H$  is that true this wouldn't be I think this is something z uhuh may maybe it's true yeah yeah but you could also conceivably allow the norm to take infinite values and try to um tried to build that into things as well yeah I just I wanted just wanted to stick with the classical thing um so there is a a theorem about the scaling no if you have a norm with the kind of I don't know how it's called with the constant yeah so so so the theorem that for Fields I think you only get the Pu classification with an alpha I think yeah yeah I think you're right yeah yeah yeah but for Rings you don't know that because it's it's delicate because if  $Y$  is a small this doesn't imply this condition doesn't imply that the norm of  $X$  and of  $X$  Plus  $C$  is yes yes yes I I agree it could be subtle for a general ring I I I don't want to make any claims I actually want to uh stick to the classical setting um okay so this was me explaining the general case um but there is um and in the general case uh why oh sorry this this this thing despite the notation with the spec this um is not going to be apine so for General um this spec Burke  $R$  is not apine so it's not the spec of a of an analytic ring um so for example well if we look at spec Bur uh  $Z$  and then the usual archimedean absolute value which is the um kind of the maximal the every every Norm you could put on  $Z$  will have to be less than or equal to this one so this is kind of the the choice that gives you the biggest possible Berkovich Spectrum um uh this is a this is just this uh Locus where yeah two absolute value of two is less than or equal to two inside this stack of norms and it really is a stack as you can kind of see at the at the points that live at the boundaries um uh okay so however uh suppose so let me make a assumption star um that uh so there exists a let's say say a  $\pi$  in  $R$  uh such that Norm of  $\pi$  is less than one uh  $\pi$  is a unit in the ring and I want it to be that it strictly multi like the norm strictly multiplies when you multiply by the norm is multiplicative with respect to multiplying by  $\pi$  um so uh so this uh well this condition is obviously not satisfied here but it's satisfied quite broadly so so example so any non any non-discrete yeah I apologize again non-discrete I'll put it way over here so you don't think I'm saying non-discretely valued field but non-archimedean valued field uh uh yeah admit such a norm sometimes they say nontrivially valued field okay yeah yeah um so you there you have multiplicativity for all elements and then uh if it's not discretely valued then there's something with Norm between zero and one and it that'll be a unit and yeah of course there is a slight ambiguity in valued field because sometimes it refers to absolute value sometimes to cruel valuations it could be higher rank and then you have to say I mean it's not it's here is valued in the sense of of uh of yes real value yeah exactly exactly yeah um and then so if if um if  $R$  satisfies star and uh  $F$  from  $R$  to  $R$  Prime with which is a map in the appropriate sense so so um so contractive homomorphism uh then

RP Prime also satisfies uh so with respect to  $F$  of  $\Pi$  um so uh well the reason is uh yeah the uh yeah you take the same  $\Pi$  its image will be a unit and it'll be the norm will be uh less than one but also the norm will have to be the same because uh yeah this condition is equivalent to uh unless unless maybe  $R$  is zero I don't know um is equivalent to saying that the norm of  $\Pi$  inverse is actually the inverse of the norm of  $\Pi$  uh as you can see by applying the multiplicativity prop the sub multiplicativity property of the norm um so then uh and that condition is kind of more obviously preserved by okay so actually in the belovich the I think some is natural to consider morphisms which are not the norm is on  $Fe$  effect is less say a constant nor effect this defin still upap on ver spaces I suppose that your definition does not depend that is if let say you have two noes which are equivalent in this sense by constants then all you all of this will be the same for the two NOS I suppose but well is to check something not quite so I mean let let me let me say what I can say and then yeah sorry I just ah okay yeah but in in under this assumption you mean or in general I think yeah yeah yeah I I agree with that so that's why I want to postpone the discussion yeah uh a bit okay so in particular I mean uh the classical settings in burkovich Geometry you work over some fixed field which is not well often non-discrete um and uh and you're working with Bono algebras over that field and they will certainly uh satisfy this condition star but if you're working over a discret ring you know you won't have this uh condition star um so so then I then I claim uh so ah well let me give yeah we're also uh yeah so also any any uh any Kate Huber ring uh has a norm defining the topology uh satisfying star with  $\pi$  a pseudo uniformer so also some mixed characteristic examples um exist um uh right so um where am I oh yeah so then claim uh so if star holds uh then spec burkovich uh  $R$  Norm is apine and corresponds to an analytic ring structure on the condensed string  $r$  um so moreover uh this analytic ring structure only depends on the condensed ring  $R$  uh not on the norm satisfying star yeah so I hope in some sense I'll be addressing your question I mean no but I see I think that just for your definition if if you have two equivalent Norms then you you by constants that simplest case then you just look at the condition you impose apply to oh yes yes yeah I think you're right I think you're right I think you're right yes yes yes so similarly for a morphism brings if if it is only with a constant you could still apply the same thing getm of  $St$  I think you're right ofer yeah so this because in some I remember that in some text I don't know if the book of ver in some place they they consider such things which is more yeah apparently more natural because I don't I don't know that the why but it's it's it's probably you can't I think you're right that because our Norms are by definition multiplicative I mean these geometric Norms that you can uh you can argue exactly as you suggested yeah um yes that's a good point thanks um okay right um so so what doesn't depend on uh what what depends on more than  $R$  well on this thing uh you have this Universal Norm  $n$  um so the the universal norm and on spec  $b$   $r$  uh does depend on the norm you choose on  $R$  um but any two choices are uh equivalent under uh so Norm passes to Norm to the alpha so for some map Alpha from spec oh boy sorry let me continue over here and I could I could remove the this the second factor from the notation here because as I said that the stack itself the stack itself only depends on  $r$  but okay I'll keep it um so there's a what I'm referring to here is there's a scaling action uh so I'm referring to the fact that there's a um there's a scaling action which sends a norm uh and a continuous function Alpha to uh the norm you take the norm and you compose with the alpha map on exponentiation map on on zero Infinity exponentiation by Alpha on Zer Infinity so what the here you wrote uh spec no no I I'm sry the what was the the the notation for the classical B spectrum is  $M$  of  $r$   $r$  m yeah  $M$  okay so the alpha is a map to positive realse but does it factorize through  $M$  of  $r$  or is what is it is it no just right um okay no that doesn't wouldn't surprise me if it did it's just the argument I doesn't doesn't quite show that but U probably if you look more closely it will I let's give the argument and then uh and then maybe try to address the question I'm going to give the proof of this and then maybe it'll be easier for us to answer the question after having seen the proof I'm not claiming it right now but uh I'll give the proof for this claim and then maybe at that point we'll be able of course you can just change the the norm on Itself by taking a power and then this will be the alpha if you just change the nor by a power yeah if you change the  $N$  by a constant then what I said before it does this doesn't change but if you change it by raising to a power then you you probably prob be constant yes but I'm not it doesn't no I mean so the thing is like uh there's two choices we need to take account for one is like a choice well let me let me give the let me give the proof and then and then we'll be better equipped to try to answer questions such as these um okay okay so uh so the proof um um so I'm going to explain how to produce uh this analytic ring structure on the condens ring  $R$  um so take take  $\Pi$  uh as in Star um then we get a map from uh  $zq$  to the condens spring  $R$  which sends  $Q$  to  $\Pi$  um but uh  $\Pi$  is of Norm less than one which implies that it's



topologically nil potent it's sequence of powers tend to zero that implies that it factors through this this ring here and it's also a unit um by assumption so we get a a factoring through this ring here um and then uh so uh we need to check or we want to check um that uh  $R$  is gaseous  $Q$  gaseous as uh so recall that this gases theory was a non-trivial analytic ring structure which was produced by taking uh by realizing that that the the category as a full subcategory of over this ring um and then there's this completion procedure which changes the underlying ring to this gaseous thing but the the category of modules was just described at this level um so uh so and and this is something very uh this is something very straightforward because the definition of gases was that some some map from  $p$  to  $p$  being the universal null sequence so to speak namely  $1 - t$  um  $Q$  uh should be an isomorphism on on on map uh on maps to  $R$  but um when you map out to  $R$  from this null sequence space to your Bono space um you're just getting the space of null sequences in the Bono space so that's equivalent to saying that if you look at the space of null sequences in  $R$  uh and then you have some  $1 - Q$  shift this should be an isomorphism of condensed uh of Bono Bono aylan groups say um but this is uh but it's easy to see what the inverse is supposed to be and to write it down you need to just you need that that if uh sort of  $FN$  is a null sequence uh then then you can sum uh uh  $FN$   $\pi$  to the  $n$  and still get an element in  $R$  um and uh the condition on  $\pi$  is and the usual triangle inequality stuff uh lets you write this down it's just the the the limit of the you Koshi sequence um so it's a it's quite straightforward to check that  $R$  is  $R$  is liquid um and then we can just take the induced analytic ring structure so what's that what oh I said I meant to I keep saying liquid instead of gases yeah it is liquid well anyway this Al it's gash this uh take induced uh analytic ring structure uh from uh  $Z_q$  plus or minus one gas just um so now recall that on so we're call that on on spec of uh  $Z_q$  hat plus or minus one gases we have a universal Norm uh with the norm of  $\pi$  strictly equal to uh this okay now maybe let me say  $R$  is not the zero ring it's the zero ring I leave the claim as an exercise so then if it's not the zero ring then this  $\pi$  will have to have Norm bigger than zero and and less and less than one um and then on uh this we have the universal Norm where the the norm of  $\pi$  lands inside this Singleton Subspace maybe I'll write it like that to remind you that this is kind of a subset and not a value um uh right and then we can then then by by functoriality of norms but because Norms pull back we get a get a norm on uh on what is  $\pi$  what ah  $Q$  uh  $\pi$   $\pi$  is  $Q$   $\pi$  is our we're fixing one of these guys which exists by hypothesis the  $Q$   $Q$  oh thank you thank you thank you I'm sorry yes yes thank you yeah uh right so we get Norm on um  $r$  with induced uh analytic ring structure um and the claim so this normed analytic ring uh is uh is is spec um spec Burke  $R$  respect to this normes I'm sorry represents yeah uh represents is yeah yeah it's the same thing right um well from geometry okay uh spec of this um okay so what does one need to do to um to prove this claim so so what have we done so well maybe I need a name for this normed analytic ring let me call it  $r$  liquid our gaseous although in the so so far I haven't shown that it's independent of the norm on  $R$  but we'll see that in just a sec uh but but let me let me hide that in the notation um right um uh where am I oh yeah so so so far so what have we so over spec uh our gas we have the map uh to Spec  $R$  triv and we have the norm uh  $n$  with uh Norm of  $\pi$  exactly equal to uh so um the Singleton value  $\pi$  and uh and it's easy to see just by tracing through the construction that this is universal with respect to that or with respect to those that structure um so what's the what's the difference with the thing we're trying to compare to uh it's that in in instead of having a condition on just just the norm of  $\pi$  uh which we have here we have a condition on the norm of every element so need so we need that uh uh this condition on a norm is equivalent to the condition that Norm of  $f$  uh is contained in zero  $f$  uh for all  $F$  and  $R$  so we need this equivalence okay okay so One Direction is quite easy so uh if you have so so this is easy so if you have this then you apply it to  $\pi$  and to  $\pi$  inverse and you deduce that uh so so so the key is to see that just telling you what the norm of  $\pi$  is then I know I've constrained the norm of every element in our in our Bond offering um so for uh uh and also for for the the other oh well sorry that's ambiguous right so um yeah so for that direction and and because it's good to know uh we will calculate uh so we calculate this Universal Norm so more precisely um so so what is a normed analytic ring structure recall that it amounts to specifying some item potent algebras over  $P_1$  um so so we'll calculate the algebras um so uh so so for all less than  $c$  um so what so a norm in our Sense on an analytic ring is implicitly uh just telling you what the overon convergent functions are on a a disc of arbitrary radius centered around the origin um um and uh yeah so we just have to specify this and my claim is it's going to be the usual thing from burkovich Theory so so claim so this is equal to filtered Co limit of let's say yeah radius bigger than  $C$  of um you take the so I'll explain what this is afterwards but kind of you can make a universal Bono ring where the norm is is less than or equal to  $R$  um and uh so uh

is this or now I'm sorry is this local or now could you make your question more precise uh I don't know if I have fin time how much time you need well you take the time you need and when it's precise ask again yeah okay so this is like the formal series that when you replace each coefficient by the absolute value and replace  $T$  by the  $ab$  by  $R$  you it converges the sum is finite yeah so this is the set of uh set of well it's just the coefficients but let's say  $R_N T T_N$  such that some absolute value  $R_N$  oh boy I shouldn't use  $r$  let me use  $F$  I was using  $f$  for elements in my bonering so um and then this is the norm so this is the on um oh wait wait sorry sorry yeah no yeah yeah that's right that's right uh okay so um I was also making claims last lecture about calculations of the what these overconvergent functions were in various cases so I'd like to explain how to make these calculations and it turns out there's a trick where you really don't have to do anything it's just kind of purely formal uh so uh so there's a trick to calculating course this is a in the condensed now you have to view this as a condensed yeah this is bonak and then it's uh yeah so it's condensed yeah cond yeah and then this is this filtered column it is taking place in the condensed category yeah um oh yeah so well what what is it we need to calc what is it that we're calculating here actually so we're taking what we're doing is we're well we're trying to calculate we have the universal thing over this gaseous base uh which we uh more or less wrote down uh and then we have to tensor it with the gaseous tensor product so over this analytic ring um uh with  $r$  where here  $Q$  goes to  $P_i$  and we have to I mean actually you know a priori it's a derived tensor product but this is the kind of thing we need to do and if we're being too naive about it it can look kind of tricky because um naively what You' do is you'd write this as as we've explained is some filtered  $\text{Co limit}$  over copies of  $P$  so that's kind of over convergence uh and then you'd take uh you'd first calculate  $P \otimes R$  and then you'd pass to the filtered  $\text{Co limit}$  but actually it's not so easy to unwind what  $P \otimes R$  is in particular it's not so easy to see that it would be concentrated in degree zero so let's use a trick so so let's not use this approach that was how we produced this thing in the Universal case so recall the idea was that the this  $P$  was some version of functions on the open unit disc and then when you have this  $q$  and maybe all of its fractional Powers you could scale that open unit disc and get some version of uh functions on an arbitrary disc and it wasn't the correct one but when you make it overconvergent it doesn't matter it'll be by some kind of sandwiching argument okay all right so trick to calculate is some general category Theory fact so so LMA is so if so so  $C$  symmetric monoidal and's say infinity category it's not too relevant um but then uh so if you have a tower so  $X_1 X_2 X_3$  uh in  $C$  where each map is Trace class so so  $X$  to  $Y$  is Trace class means it comes from a map  $X$  so from the unit to  $x$  dual tensor  $y$  or  $X$  dual I'm not assum assuming  $X$  is dualizable this is just the internal  $H$  uh from  $X$  to one uh sorry yes us closed thank you m there's probably a way to well never mind um uh where are we um ah then then for all  $Y$  in  $C$  uh we can calculate the  $\text{co limit}$  Over  $N$  of  $X$  and dual uh tensor  $y$  so we pass to the we have a tower here we pass to the Dual thing which gives a sequence and we take the  $\text{co limit}$  over that sequence um then this uh is the same thing as  $\text{Co limit}$  Over  $N$  of the internal  $H$  from  $x_n$  to  $y$  so uh this is Elementary um I'll leave it I'll leave it just like that without giving the proof you just have a two systems and you make two in systems and you make maps backwards that go up a step using the trace class hypothesis okay so again you how do you know there is a the limit makes sense okay well this this is actually an equality of IND objects so uh I mean it's an equality of end objects does it doesn't it doesn't matter okay as a in object and you have to know what is the end for for uh this is end for for  $C$  in in  $C$  which makes sense is an Infinity okay um yeah so in particular if  $C$  has  $\text{co-limits}$  you you can just you can remove the quotation marks um if  $C$  has  $\text{Co limits}$  in tensor product commutes with  $\text{co-limits}$  which is the case in our examples then you can remove the I mean you can yeah okay uh right um okay so yeah so so what we're going to do so yeah in particular well yeah so well so what we're going to do is we're going to recognize uh so if you take  $y$  equals the unit then um you know and yeah that this object here um  $\text{coim}$  of the internal  $H$  ask a technical question yes about the definition it's nice what do you mean by it comes from a map it's that the is given by this ah right so if you're given a map like this then you can tensor it with  $X$  you get a map from  $X$  to  $X \otimes x$  dual tensor  $Y$  and  $X \otimes x$  dual has an evaluation map to the unit so you then get a map from  $X$  to  $y$  yeah yeah um right um so so we're going to we'll so we uh will recognize uh  $c$  as a  $\text{CO limit}$  over now  $P$  du um and apply this with Trace class transition Maps so once we do that then we reduce to uh reduce to um uh some to looking at null sequences in  $R$  again and then just some so some filtered  $\text{co-limit}$  of some space of null sequences in  $R$  and um you can actually modify this to the thing where you require these to form a null sequence and they wouldn't be the same at each term but it's quite easy to be the they're the same when you take the filtered column again any two versions of the unit dis are kind of the same after you make

them overon convergent um so uh yeah and then the calc the calculation is very easy uh once you once you do this and for this uh uh for that we can use S Duality on  $P_1$  uh so over well it happens to be over the liquid base but it I mean the gaseous base but it doesn't much matter um so St Duality on  $P_1$  this if you so and the and the six funter formalism so um sidity on  $P_1$  implies that the well so we have this proper map to the base and algebraic serid Duality uh implies that the this map is smooth smooth and proper um and that the dualizing object is just this  $\Omega_1$  shifted by one um and then you can calculate the then it's easy yeah it's easy to do calculations and I'll just tell you what the conclusion is uh the conclusion is that if you take the Dual of the um uh this identifies with uh the Ring of functions on the open complement so um so the DU of in the sense of uh uh a linear dual in the category so this is dual in uh in there I'm sorry so this is an object in this category which implicitly there's an underlying condensed ailan group and so on so yeah yeah do do you know that you deal these object which are sufficiently f so like the double du right so you need to right so we would want to put the Dual on the other side but we can do it with a trick more or less because um so this gives also so this this over con this by the over convergence um you get that the the the single Duel of the end object sorry sorry we can then we can write this uh this is a pro object we can we get so yeah we get we can get this we can view this over convergent thing as an end object and it's dual will be a pro object and it will be the pro object given by this thing where you increase the radius as well um but then now we can view that Pro object as an inverse limit of the overon convergent things with the the non-strict inequalities over there and then um and then use The Duality result in this direction on each of those and uh um so there's some trick trick with overon convergence uh uh this implies that if you if you take the the Dual of uh this so if you view this as a proobject um because it's an inverse limit of the things where you um we have a yeah greater than uh then the Dual of that Pro object is the end object uh that we're interested in and that gives a um that gives an expression exactly like this that this guy is a CO limit of uh of du of P's um we still need the trace class claim but I claim that that also holds for soft reasons um so the trace class so there there's a general topology fact that if  $x$  is a topological space and  $z$  and uh  $Z$  Prime are closed subsets such that there exists an open  $U$  uh which lies in between them so two closed subsets which are separated by an open subset uh then uh the map from uh the push forward of the constant chath uh the Restriction map um is Trace class in the drive category of shes okay or really I should say maybe I should say she's on  $X$  Val um Shi on  $X$  and then oh with values in  $D$  of  $Z$  yeah um which is not the category of  $X$  in general because it's right um yeah uh so so then uh on the level of just this closed interval from 0 to plus infinity then any of these transition mths will be Trace class for this reason and then you can pull back that's a symmetric monoidal funter you get that Trace class you get a trace class map in the derived category of  $P_1$  but it lives on  $A_1$  and then there's another trick to see that it's image under the forgetful funter um to the base is also Trace class so there's so so pull back to  $A_1$  use another trick and the conclusion is that the Restriction map say from uh any of these overon convergent guys uh is Trace class um and then also for pres yeah then again by sand sandwiching different versions of the discs and changing the Radia you also get the and using the trace class Maps as a two-sided ideal in all maps then you get the presentation like this which let you calculate do it by hand do what by hand you want to write over functions on this disc I mean the de clance is to use pH piece yeah you can just system and see that transition has really dis yeah that there's um yeah I I believe you can do that so certainly I remember doing that in the complex case and I assume it works over the gashes base too but I thought it was nice to be able to do it without doing any calculations um yeah okay St you still need to know that the way you presented is the way you scene you presented well yes that's true that's true yes way what was the remark the the remark was I I wasn't being very careful here about writing what the filtered systems are and all this okay you have to show that it is there yeah um okay so what that was a that was a a bit of a uh um a bit of digression so what were we doing we were um we were we I said we were calculating the normed analytic ring structure and the conclusion was that the so so the norm on uh Spec  $R$  gas is given by usual over convergent functions on diss over  $r$  so that was the the conclusion um so then um then you need to so to see what did what were we trying to show we were trying to show that uh for this Norm here this Universal Norm that we produced by fixing the norm of  $P_i$  that automatically the norm of every other element is is correctly bounded um so to show Norm of  $f$  uh contained in zero  $f$  for all  $f$  it's it suffices to show it trans it algebraically translates into um oh no oh no now we have bad notation uh it's not clear a priority that it is finite sorry you have to know even the finess is not is a statement uh that's correct yeah yeah um so we have to show that if you take this and you mod out by  $T$  minus  $F$  uh you just get uh just get  $R$  this okay this this is elementary and

indeed this is Elementary so the map the map giving this is of course setting  $T$  equals to  $F$  and um it's quite Elementary as ofer says to see that you get the correct short exact sequence of uh of Banach spaces so that the kernel of this map of  $T$  minus  $f$  is of the what's that mere existence of the  $m$  is mere existence of the map enough oh that's a really good point that's a really good point Peter yeah thanks yeah so what Peter was saying yeah so what Peter was saying is that uh right we know a priori that this thing is an IDM poent algebra uh over  $A_1$  so over the polinomial ring on one generator um therefore when you base change it um along here then you get an itm potent algebra over  $R$  bracket  $T \bmod T$  minus  $F$  which is  $R$  so this thing is an item potent algebra over  $R$  and so is  $R$  itself and if you want to show that two item poent algebras over  $R$  are equal it's enough to Just Produce an algebra map between them thanks a lot that that indeed makes it very me very no just uh we already have the unit map yeah we a map in both directions indeed but we already have the unit map uh so so so uh let's see uh because when you do it analytically with this uh instead of over converion just converion on the Clos disc like before and of course you have a a map when  $f$  is less than or equal to  $C$  you get them up but it is but to prove  $V$  division you still get to some conver is not okay in the aredian case and so it is but it's not item potent it's not item potent in that case it's not item potent in that case if you don't do the overc convergent one if you just do the the one without the overon convergence it's not going to be item potent as argument would not work exactly so somehow the proof of item potent is you must have already done work similar to showing that you know but in the non archimedean case the the thing with theid algebra does work and in this case aoid algebras are important or not they are they are yeah yes yeah when you do it with the with the the non archimedean I mean the everything yeah everything non- archimedean yeah yeah then then why is it it ah because okay I mean we basically we basically proved it when we discussed the solid Theory but maybe okay if everything is not so there you can use the yeah and solid then then it is uh right so okay but yeah so in any case um yeah the map exists because you can evaluate at tals  $F$  and as Peter points out that's enough thanks Peter um so uh right okay so that was the that was the first part of the claim um so that was the claim that this thing is um this B burkovich Spectrum this analytic stack which I'm calling the burkovich spectrum is um is apine when you have that assumption star the next claim was that the analytic ring structure is independent of the the choice of the norm um so so so proof continued uh need that uh it's spec our gas is independent of norm and  $\Pi$  say um no and that is if you have two topologically isomorphic does depend on the on the on the condens the topological exactly exactly and uh without so you could have one nor with this one  $\Pi$  which is for one  $\Pi$  another Norm is  $\pi$  Prim yeah yeah exactly exactly yeah so let but let me just give the descript the independent description so uh so claim so so our gas uh is the initial analytic ring uh with a map from  $R$  triv to  $R$  gas such that and then we just have a condition such that for all uh topologically nil potent units  $\pi$  and  $R$  it's a condition that only depends on the topological ring  $R$  um uh so the our gas is uh is  $\pi$  gases I.E that if you the map from  $z$   $q$  hat plus or minus one to  $R$  uh  $Q$  goes to  $\Pi$  um and  $R$  lies in the gases modules what's that oh I'm sorry thank you thank you thank you thank you thank you yes uh yeah yeah yeah thank thank you Peter yeah not just  $R$  but is it enough to check for  $R$  the trivial module or no no because the I mean that then you you wouldn't have to change the analytic ring structure at all then yeah  $D$  of  $r$   $d$  of  $R$  is meant is  $d$  as a condensed  $D$  of the of the condensed uh  $D$  of  $R$  means box no it's the analytic ring it's the analytic ring the  $R$  so yeah oh this should be yeah this should be um yeah so I want this to yeah so this so this map uh gives a map of analytic Rings the condition is that this map gives a map of analytic Rings uh to  $R$  and what that condition means instru on  $R$  was what was I oh sorry uh so sorry our I mean I was I'm sorry let me say  $R$  gas is the initial analytic ring a with a map from  $R$  Tri to a such that for all topologically nil potent units  $a$  is  $\pi$  gases IE uh this map here uh yields a map of analytic Rings like that so there is an easier probably much easier question in this context which if you have two Norms the same then the boovich spaces are the same is it true or not exactly because if you have two Norms so since the the condition is is less or equal to some some you don't if you're not a topology you don't control things far away from zero but because of the unit you can scale so but still I wonder about at  $Le$  in the another comedian case one can compare to Uber and get that it's the same do fication it is theic of some you can but in the aredian case of course there is this in the boovich Spectrum there is this condition that I mean I think if you take the um you could also like try to make a modification of the burkovich spectrum only thinking of our is a topological ring where you ask for these seminorms to be continuous and then I think if you take that space and then you mod out by this exponentiation action by positive real numbers um then that will be the same as the burkovich Spectrum for any fixed Norm uh satisfying condition star no no no

because when aredian the the non aredian things yes uh you don't because in the  $b$  space you don't identify no to its power to its powers so I don't see how you no but yeah the identification is maybe not so obvious it's kind of well I I mean I'm not I'm not sure I'm not sure but so you because you you you want to claim that your your your below spectrum of course it maps to the space the  $B$  space as we said yes and but you don't claim here that the map is because if you want to  $CL$  the map is the same you have to compare the verage spaces and and this looks like a little bit tricky at least in the away from the non archimedean  $Cas$  we can understand it anywhere I'm not sure well yeah I don't I don't know um right um so um yeah so let me uh okay let me let me give the the proof of this claim which is kind of giving a intrinsic description of this gases analytic ring structure so um so note that if  $\Pi$  is topologically nil potent then there exists a  $n$  such that uh uh so we um um and after passing to some power you have small Norm in particular you have Norm less than one and um let me note that this condition here uh this is invariant under uh replacing  $\Pi$  by any power power this is actually a remark that that Peter made at some point um some point early on uh so so we can assume uh that  $\Pi$  uh is Norm less than one um but then uh but then uh for the universal Norm we built over our gaseous uh sorry well sorry I need to fix Maybe okay so so my claim is going to be so so certainly uh this  $R$  gas that we built we built it so that it uh satisfies a weaker version of this property where you only demand it for a fixed  $\Pi$  uh satisfying this condition here and what we need to show is that so let's say so we so let me say so our gas was built to satisfy star just for uh some fixed uh  $\Pi_0$  uh and now we have to show that it's satisfied for all choices of  $\Pi$  um but uh so yeah so um but then uh so yeah so so can assume uh  $0 < \Pi < 1$  and then that implies uh that the norm of  $\Pi$  uh is in this interval from 0 to 1 um which we already showed implies that  $\Pi$  is gaseous just from the axioms of a a normed analytic ring okay uh what is  $\pi \circ r$  for  $R \in Z$  what is it doesn't make sense because I've only defined this when  $R$  is a bonering satisfying this condition star well sorry the the other the other condition star about the existence of a  $\pi$  Norm between zero and one Etc  $\Pi \circ Z$  no fixed  $\Pi \circ Z$  is a is a it's a fixed I the way I built the way I built this was I I took my norm and I took a fixed uh uh  $\pi_0$  satisfying condition star and then I built my analytic ring and I built my Norm over it and now I want to check that that thing satisfies this Universal Property which means that so it was universally built to satisfy that for just for a fixed one um but then um and to have the correct Norm on there but and then I want to argue that it um or well I want to argue that automatically all of the other possible pies are also gases and we can use the norm to prove that because um the norm is such that it you know yeah well such that we have this chain of implications okay um so uh right and then the last part h so the last part is um uh right that the norm if you have so that that was the analytic ring structure being independent of the choice of Norm on  $R$  but then the um so the last part um so this Norm on uh spec our gas which this a priori depends on um is independent of the norm up to exponentiation by a map uh spec our gas are greater than zero um so um yeah right so so let's say we have two different Norms uh we have  $n$  given by this here and we have  $n'$  given by this here um um so and both Norms have to satisfy conditions star but possibly for different choices of  $\Pi$  and so this one uh satisfies star for say  $\Pi'$  um then um okay what can we look at um so again uh we can assume that the norm of  $\Pi'$  is less than one um so now  $\Pi'$  had some nice fixed properties with respect to this but now with this one it's just some arbitrary map to  $01$  but that implies that with respect to the norm  $n$  uh that if you take  $n$  of  $\Pi'$  this is a map from Spec  $R$  gasius uh to  $01$  um and then uh it follows that but there exists an  $\alpha$  such that  $n$  of  $\Pi'$  raised to the  $\alpha$  is equal to just a constant uh value the norm of  $\Pi$  because uh exponentiation acts transitively so or acts simply transitively so uniquely transitively on zero um so um yeah so this is the this is the thing we have to um yeah let me finish the argument then we'll I think you're probably right um so we'll address o first point at the end of the the end of the argument um so then um but then  $n$  and  $n'$  are two Norms Norms on spec are gaseous uh uh both with uh uh um wait sorry uh normal denoted by double double bar double bar  $\Pi'$  I think  $n'$  should be double bar  $\Pi'$   $n'$  should be double bar I don't know um sorry uh so  $n$  to the  $\alpha$   $n'$  yeah are two Norms um with uh sorry sorry I sorry I want to actually make this equal to um uh wait sorry yeah right okay okay are two two Norms with um uh same value uh on  $\Pi'$  so then by the classification of norms uh they must be equal on  $P$  some value so same fixed value the the value is a fixed real number between zero and one and we we show that such a norm is uniquely determined uh when we proved this classification of norms okay uh and maybe I wanted to do this or something I don't know we can do whatever we want I I forget to yeah yeah I don't know I think that's right I think that's what I want to write okay um so now to address ofer's question about whether this map  $\alpha$  um is pulled back from



the burkovich space so first question is Alpha Spec  $R$  is  $R$  greater than zero pulled back from  $u$  so this is the canonical map and this question is whether there's some map there  $u$  to answer this question we need to answer whether this map is pulled back from the burkovich Spectrum so true if Norm of  $\pi$  is but that's true by construction because the mapping the Burk of dis Spectrum was exactly recording the Norms of all of the elements in  $R$  and in particular we're recording the norm of  $\pi$  so so the answer to your question is yes it's okay okay now I see I understand the  $u$  I understand and I think it should be also true that any two choices of NOS on your ring are differ by exponentiation to and up to con equivalent in the  $S$  of to constant by any two any one is equivalent to some constant I mean equivalent in the sense of bounded ratio from two side to the power of the other in the in the under conditioned star you mean under if you have a  $r$  topologically important unit first of all if you have a topological ring complete let me say you don't need complete this topologically important unit then you can see the topology is given by a norm  $SA$  in what you what you want oh is that true this is something in general topology where they for a uniform space they construct metrics that give it but if you have an ailan group it's enough if you have got a a system of neighborhoods of zero fundamental systems that  $u$  plus one plus  $u$  plus one containing  $u$  and let us say symmetric I'm not sure then you can decide to make a metric by deciding elements of  $u$  of Norm it's most say one two 10 you check the metric condition here you can do something similar you because if you have topologically important unit yeah and if you have a fundamental I'm not sure about the accountability okay the idea my idea is to well I need to know that it's not only topologically important but there is a neighborhood of zero such that  $Q$  to the  $\pi$  to the  $N$  time it is Convergence to zero so some boundedness some conditional terminology yeah okay then you take  $U$  and  $\Omega$  you rep  $\Omega$  by power so that well okay I I don't do you work with two equivalence or no no he's not asking a question he's telling me something and I want to hear so I I am I am afraid that I don't know the conditions that I need but maybe there are some bound but some natural condition in the sense of topological ring that I have to add I have to take a small neighborhood open neighborhood such that the power of the that it times small neighborhood converges to zero so  $\pi$  to the end it goes to zero then like for some end  $\pi$  to the end of it plus end of it is containing in and then you can construct the metric okay  $u$  using a replacing  $p$  and then you you will construct at least  $u$  by construction I think somehow I force the  $P$  to to have no one enough roughly I can probably make it Global okay one is to work it out the details and then it will turn out that if you have a continuous map to a valed field which is continuous and the norm of my  $\pi$  yes is less than or equal to is one half anyway then it will be also less than or equal to this no probably well it will be equal to 1 half yeah okay then it will be less than or equal to this yeah and then okay this is one idea but then if you have got a but by the way for with this construction you'd get only the triangle inequality maybe up to some constant again no no no no no if I have what I claim is this this is something that I check so if you have a non-commutative forgot now if you have got a in general for uniform spes they they have three three  $u$  you need to they work with three but if you have an Community grou you can do it with two so I just claim first that if you have if you have a topologic cian group with topology is defined by sequence of a symmetric neighborhoods of the origin  $u$   $u$  plus one plus  $u$  plus contain  $u$  then you you just get a metric by this by imposing that the guys in  $u$  know at most one over two to the end so you have a metric in a generalized sense it could be plus infinity for something then I will change it using the unit my my  $u$  I will correct it using my sud uniformer but at least I will get  $u$  okay maybe what I'm saying is a bit  $u$  maybe I'm thinking too fast with some mistakes but in any case I think it will also come out that any two Norms are  $u$  in some sense  $u$  yeah but this I am not sure any to okay maybe we can discuss later let me let me just finish I have only one more thing to say and it's quite short so yeah  $u$  so  $u$  right  $u$  the last thing is when I talk a bit about just say Global globalization only to say that it's trivial  $u$  so we could make a definition I don't know I mean a definition of a burkovich analytic space I don't know is a pair  $u$   $X$  or triple  $x$   $s$   $u$   $\pi$  from  $u$  local of opens in  $X$  to  $S$  where  $X$  is an analytic stack  $u$   $s$  is a locally compact house door space  $u$  and  $\pi$  is a map of locals  $u$  such that  $u$  such that  $u$  locally on  $S$  for the topology the open section topology or the the local section topology  $u$   $u$  it is isomorphic to spec Burke are  $u$   $u$  sorry  $R$  Norm  $Mr$  Norm  $u$  and then this canonical map  $\pi$  for some Bing  $R$   $u$  okay so this is completely trivial now  $u$  to globalize and yeah so the only point to note is that  $u$  working locally on  $S$  you're also automatically working locally on  $X$  and that's because these by by definition of a map of local is a a cover and the open cover topology gives a cover in the sense of open covers of analytics stxs here and those are covers in our Gro and deque topology that we use to Define analytic Stacks they're even open covers in the sense of the six functor formalism for the what do

mean local section to I mean the Gro topology on locally compact Hausdorff spaces which is generated by open covers okay yeah it is isomorphic to this basic object space so you locally yeah so recall that these these are always compact Hausdorff so you're not going to kind of if you if you say locally in too naive a sense you're not going to get any examples because you know open covers open subsets of say  $R$  are usually not compact right so but if you do this usual thing of having a compact neighborhood of every point then it's fine but I I this looks it I wonder about the derived nature of the of the  $r$  when you localize because it seems to me that of course you can make this definition but then you can ask whether for example what does mean locally onness if it is true if you take  $D$  do you have like for sufficiently fine for small open it is let us say that is if it is true for some cover it sufficiently fine one then probably you have to pass to to to the ring Associated to some small distance those are derived in some that's it's we have the same problem we have in general with the you know the H discussion of the Huber theory in the solid case it's the exact same issue and it's the exact same situation that yeah you can fix it by starting with some more General class of derived things here probably I didn't think of the details and but also in practice in most examples it doesn't show I mean in in Practical examples it tends not to tends not to matter I mean that in the sense that these local the you'll have a neighborhood basis that you know when you calculate these neighborhood bases they will just be in degree zero and the Rings will just be in degree zero and and they will fit back in the framework so you're right that there's a bit of a fly in the ointment in terms of the way we're presenting things and that we're starting always with classical objects like when we talked about Huber's Theory and when we're talking about this Theory here we're starting with classical objects instead of derived objects and a priori when you work locally on our class on the Spectrum attached to our classical object you see some derived objects instead but in practice it doesn't tend to happen and in any way you can modify the definitions of it to accommodate them so it's not such a big deal it's slightly more sub here that the basic building rings but then sheaf of overconvergence so it's actually never B sheaf of kind disjoint between the global A1 start with and no but still that doesn't I mean that doesn't obstruct the claim that that there's a neighborhood base you know yes I mean Ofer's question was about a neighborhood base yeah yeah that's that is a good point and I mean you know you can modify these you could also you know from instead of instead of these guys you could also pass to inverse limits so you could start with these apine guys you could pass to arbitrary inverse limits for example like so inverse limits in the compact Hausdorff space and just filtered Co I mean inverse limits in the category of analytic Stacks which in the Aline case is just you know Co filtered Co limits of analytic rings and then then these overconvergent things would also count is apine and then maybe that's a little nicer uh to work with and there's no harm in in doing that um but say will not be B rings but they will be condensed rings with certain yeah they'll still they'll still correspond to analytics an analytic stack with a structure map to a compact Hausdorff space and so on okay so you probably instead of B ring you can have a condensed ring with  $s$  proper yeah yeah with with some Norm satisfying some properties and so on yeah I mean we didn't we didn't try to give the best possible formulation was just a just wanted to connect to the classical thing yeah okay so that's all thank you sorry can you again with it local section oh yes so on the on the category of locally compact Hausdorff spaces you can define a Gro topology where um a you know a set of maps I mean it's a set of maps like  $x_i$  to  $S$  forms a covering if uh for every point of  $s$  there's an open neighborhood of that point and an index  $i$  and a section of the pullback uh uh you know you pull back  $x_i$  to that open neighborhood you should have a section there um and the map can be arbitrary the map doesn't have to be an open inclusion but it's also the same thing as the Gro topology generated by the the covering families which are just the usual open covers so if you look at just the usual open covers and say that you want sheaf condition for that you automatically get sheaf condition for anything any map that has local sections so that's a so if you want if you yeah so the sieve will always be the same as the sieve generated by some open cover of  $s$  and so yeah but it's convenient when you want to talk about the sense in which a locally compact Hausdorff space is locally compact Hausdorff DWF because it's not true in some naive sense but it's true in this sense okay other questions another question it seems to me if you take at least naively you take another  $p$  as the norm you're supposed to be a constant one uh  $n$  Prime of  $\Pi$  Prime was supposed to be a constant but  $n$  of  $\Pi$  Prime can vary over the Berkovich Spectrum but uh I think with supposed to be a constant because is small than the normal Prim Prim inverse the same was it to be actually no see the norm  $\Pi$   $\Pi$  Prime was adapted to  $P$  Prime was adapted to um uh absolute value prime it wasn't adapted to absolute value so it's not adapted so this this  $n$  here wasn't such that it satisfies that property with respect

to I so how did we built this  $n$  from this absolute value here for which  $P_i$  had this property that  $\text{Norm}$  so we had in other words we had the norm of  $P_i$  inverse equals  $\text{Norm}$  of  $P_i$  inverse here right and then we built this  $n$  using this so that the the norm of every element would be bounded by the norm prescribed here that implies that  $n$  of  $P_i$  has to be this fixed value but it doesn't imply anything about  $n$  of  $P_i$  Prime because  $P_i$  Prime doesn't NE  $P_i$  Prime doesn't necessarily satisfy this property for this  $\text{Norm}$  it only satisfies it for this  $\text{Norm}$  always you okay thanks guys

## 24. OUTLOOK (SCHOLZE)

[https://www.youtube.com/watch?v=YKw1XaueLJY&list=PLx5f8IelFRgGmu6gmL-Kf\\_Rl\\_6Mm7juZO](https://www.youtube.com/watch?v=YKw1XaueLJY&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO)

### Unfinished starting from 0:00

Okay so so so welcome to the last lecture um so so so today I want to give some kind of Outlook uh right I mean so with with Dustin's lecture on Wednesday we kind of we kind of finished what we promised in the first lecture um and so today I want to talk about some directions one could go in with with the kind of Machinery we developed um uh so maybe yeah so so some I don't know five or six years ago um uh like I mean I did all lot of pic geometry and like I always wanted to have a way to do this not just periodically but also with real numbers and over  $\mathbb{Q}$  but it was always clear to me that I really needed a completely new language to talk about these things um and so this is I mean as I said already in my first lecture this kind of the reason that uh I was really putting a lot of effort into this project um and so I finally have the feeling that we're basically have now the language that we we um we always wanted and then now is a sensible question to just try to uh really use it to do a lot of things um and and then so it appeared to me that actually I mean there was this original goal that we I maybe had in mind for what the series should do um but on the other it's also good to look uh in other areas of mathematics what how the theory might be useful and so I and there I'm not really really competent but I was still want to give some wake ideas that I think might be worth looking at okay all right so so here yeah yes some possible directions and I will start somehow from the most welldeveloped to the most specul um all so something then something this is not just a possible Direction but that not just works I mean so we do have now a general theory of and shes not just no finds conditions imposed on the modules uh in analytic geometry in all labers of analytic geometry and all of these different flavors of analytic geometry are also unified into this one thing which just um and what's even better is that this is not I mean this is really a full six fun form  $L$  of sixun Street uper Street  $T$  and uh and this just lets you play a lot and uh in particular like some things we kind of looked at into this uh using this formalism is that well actually this sounds stupid but it's actually for example as Dustin explained the last lecture it's actually non-trivial application of this first of all a general the of structure sheet without any Mysterion or otherwise hypothesis um where in particular like for  $B$  spaces even those kind of a finite type over the integers  $J$  had to really work a lot to Def find subject just saying um and even for edit spaces there was this kind of nonch issues which are resolv things so you have to Define what it derived well but in our I mean we just have antic  $St$  they come with the structure period okay um uh then there all sorts of Gaga theorems that you can prove but you can also prove various new sorts of of Gaga serums actually um uh there are various resolv about like the sand of vector bundles and again the songs  $B$  silly but um and for example there's this famous paper of greenfeld about like infinite dimensional Vector bundles or something like this is called but in particular he proves that if you send a ring  $R$  to the like vector bundles of all the wrong series  $T$  So something that  $R$  is natural if you think about Loop groups and so on um then he proves some theend results for that and I mean it's a difficult paper and uh AR Messi was recently revisiting this uh using some of this descendible ideas um but just a few weeks ago chich was asking me about yet another variant of of of that kind of result of dril and whether all techniques could be useful for proving it and I think they are and that's just a very simple proof using this kind of uh General formulism that we have that it's very easy to prove such statements um uh and then we also like know complex course on complex geometry we discussed like off for complex man and and such things are kind of one kind approach using this um uh one thing which we in some sense still haven't quite figured out but we're quite optimistic that in principle could be done is also to prove uh uh the sing index theum using using our technology um related to these last points on I mean they are of course very very closely related to the notion of CAS Theory um and uh right and so one thing that was kind of missing for a while is the notion of the cas of Analytics spes so like an algebraic geometry this defined first by toal and then for General schemes by Thomason and it really use it that you have a well behaved category of coherent Chiefs on it and then maybe actually stable Infinity category of per complexes and then you can Define the case theory of that but going to analytic geometry there was the issue that there is not a good enough category of modules of which you could then take uh apply these categorical techniques and get some kind of case Theory um and using our technology can you can actually do it you actually don't use exactly that those CL Shields as there but some varant of nuclear modules uh but this has not been analyzed to some extent so this definitely uses uh the work of Sasha eimo to Define Cas of dualizable categories um and then if you your analytic space are actually Ed space tic spaces

this was worked out in the PHD Ser of andf um and there we had some problems at first to the find the complex numbers but Al some ideas how to do that um the gas here might actually also help for this there actually also some other relations to to to the work of sasim of so he has these very strong results about the category of localizing motives so proving that it's a rigid category particular dualizable itself um and using this you can actually Define certain refined variant of Clic hology topological cycology and so on that um that are actually not taking values in some kind of complete category as usually when you take as one6 points you get modules over some power Series ring which are complete but instead you can go to nuclear modules in our sense again and so this ties back in this uh that the again to all kind of an geometry um let me mention maybe that if it's okay Dustin that Dustin has a joint project with brund well they use some of this nuclear Cas Theory to settle some like questions uh like froma homotopy Theory oh yeah but we figured out how to avoid it actually ah okay yeah too bad yeah I was I was a bit disappointed too but um all right so that maybe where where I'm kind of coming from and where there had a lot of applications and where really a lot of work has already been done is in the area of like thetical modulation and so on um so um and there there was a spetic hology and then which was defined from former schemes and one question that people had was how to really Define it for uh rigid generic fibers um and this then very much is related to analytic uh stats in our sense so in particular there is no what's called the analytic R st uh defin the work of what gu um uh so which gives you way to talk about uh six fun formalism on what's classically known as like dcat module so there's a certain completion of the Ring of differential operators giving some kind of differential operators of infinite order um defined by aov and one SL and then they suggested that when if you Works someone getting anal geometry should really look at these uh modes over these dcap modules and try to get a six formalism for that but as usual I mean you run into something of function analysis issues doing so uh yeah but but using our analytic geometry can really handle this and Ro G was able to write down I in some sense by specializing the six fun formul to specific St get a very general six fun formul for these uh DK modules um uh so this is related to D modules there's also in the pic world the analog of like hi hi bundles and so some kind of Simpson correspondence uh the first incarnation of which goes back to Dinger and fings um and so these can also be interpreted as like in terms of a stack and so that's there so called Analytics St St um so this you can find in particular there like okay maybe not yet using our technology just uh essentially work of aner and and H in particular recently obtained really strong results on the P correspondence and uh this work on these hotate staks is also very closely related uh to what's known as geometric 10 the uh so something that originally arose from Sensi which is something about P go representations um and which has been used to very great effect uh by Lou pun and also ralo um for applications to P point so also last Friday Vin Pon gave a talk about his joint work with boxer uh Kary and G I believe um where is a uh um prove uh modularity of Genus 2 curves of a q uh in many cases and and some of the key technical part of this proof is really using this kind of Technology um and what also is pointing to is really that there should be some uh version of like theal for loc representations yeah in terms of some kind of geometric Lance of the center so some version of what I did in my paper was long thought mention here I there something that's in some sense combining or just different the and St St so is also from antic citation and now was all I mean all these people that I mentioned um let's hope that one can uh formulate uh such thing um uh which maybe a proposal for something this direction I this proposal by Helman by um and by uh and G in this direction and so I mean this of course maybe what my original interest is that there was this work with spk on deration of local Lance and I would like to formulate all inations of local Lance uh in such terms and eventually then also Global LS um and so this is just to a large extent I mean at least work in progress and this is probably already very speculative but this is something that's very actively investigated um and so actually a lot of progress on this was made during this uh house trimester that happened last summer here in bone and in particular we discussed a lot about these things and then at some point we realized that now that we understand the pedix story really well and that we understand uh the kind of correct geometric language to phrase these things in we can just basically one-on-one translate all the ingredients in the pedig world to the real real world so there there's a real analog of virtually everything that's happening three so so there's also an analytic St and that's actually a funny version that is actually isomorphic to B St uh as it so happens in this casee where this is some ination of a re H correspondence uh you can also Define an an analytic P spe which in this case actually maybe there's some kind of here it's this non-trivial JP I think it's a trial J okay and again there's some this analization of 10 curve also here you can find I don't know yeah so there's also some some analytics stack mization in some sense encodes the vector BCE on it encodes some kind of



peric variations of  $f$  structures and so there's also an analytic stack including variations of twist structures  
 um so I mean of course this ties in very well with all the work of Simpson in this world and then mauki  
 developed this really developed the Ser variations of  $f$  structures um which are generalization of variations  
 of structures there's a certain action of  $\text{new}_1$  on on these things and you want objects um and and then it  
 seems to be possible so synthesize everything and get also get a formulation of real local Lang lines local  
 correspondence for real  $\text{Le}$  groups um uh for some kind of locally antic representations good  $C$  can I ask a  
 question yes uh can you produce the weight filtration too in the analytics stack um it's a very good question  
 let me comment on this in a second um not yet but okay some kind of geometric Lang on twist $_1$  which is  
 uh a kind of real analog of so I give a talk in Minster three months ago so um where I uh was outlining  
 the general form that we should take um and I will give some uh three lectures actually in Princeton uh  
 a months from now or might say a bit more about how's this supposed to work um actually part of this  
 again it's maybe a small thing but maybe not I mean um so usually when you talk about representations  
 of  $G$  of  $R$  you run into all sorts of functional analysis issues again and usually you replace them by more  
 algebraic notion of thek modules by passing to the uh vectors finite vectors and some compact subgroup but  
 then the theory somewh less inv variant because you need needed to choose this  $K$  and well it becomes more  
 algebraic but um in all worlds there's really not much of an issue of really uh encoding representations of  
 a real group Direction but we don't need we don't need those but you have you have many representations  
 which have the same Chandra modules because you can use different function maybe maybe this question  
 comes up so so what we actually do is we will look at so the real group it's a real analytic it's an group object  
 in real analytic manifolds and so you can do the kind of real analytic ination antic  $St$  and so this a group  
 object in or analytic  $STS$  and you can take the classifying space of this so let's say  $St$  to guess this complex  
 number is uh not that um oh real uh and so I mean as usual like she on classifying spe there are something  
 like representations of of the group uh but to realize them representations you need to go to back to uh the  
 point where and there two funter so there's a projection from the point to here and you can take either  $P$   
 or  $P$  upper streak and like for any of these usual GK modules there's a canonical object in here actually and  
 then the  $P$  upper star should I mean this is something we haven't checked yet but the  $P$  should produce  
 a minimal globalization the  $P$  Street should produce a maximal globalization so what  $C$  you take you take  
 $C$  with the gas yeah you can take the gasas it's enough and so you have the the of course this is closely  
 related to some results on existence of analytic vectors in in representations it should you should otherwise  
 you would not do you do do you use such I mean the the fact there enough analytic vectors let me not try  
 to say anything precise about this relation here because something I still need to uh think much more about  
 um um right uh so so there was this question about waste structures um and I believe it's related to the  
 following so so now we run into the speculative Reon really uh so for these things I'm pretty confident that  
 some version of this will work out uh this this is at this point more of a speculation but I have a very strong  
 belief in it but it's a speculation um uh so classically in all sorts of questions about function analysis and I  
 don't know compex geometry and everything uh you often you really need to put metrics on something like  
 if you really want to prove the  $H$  decomposition at some point you need to do some  $L_2$  stuff put metrics on  
 stuff and so on um this is something that we cannot yet Incorporated all or what we yeah that we cannot yet  
 translate into our our world everything that's related to metrics at this point but I believe I it's clear way  
 to look um namely you should look at this extended ver  $F$  that we had um I will connect this in a second  
 to this question about weight structures okay so uh so weall this bur space and maybe the way do was  $WR$   
 so you have some kind of Central Point related to like  $Ral$  numbers and then there was a  $r$  let's say for  $Q_2$   
 and then at the end of this you have kind of  $F_2$  and then there was a rate for  $g_3$  at the end of the  $G$   $F_3$  all  
 the other primes and then there was also theum Prim corresponding to the reals um and now usually like in  
 the bir space this kind of ends in the middle because you asked for a triangle inequality but we don't have  
 to do that and so there is no Point add infinity here and there's this point at infinity and I mean our of  
 geometry kind of tells you that this point must be related to metric and uh to some extent you can already  
 see that like when you but just just some fragments of it but also by analogy here like like if you work two  
 rically then like extending over this point precisely means that you put some kind of like a vector bundle  
 over this thing like a  $Z_2$  mod over the integral toic numbers and so extending over bun pic places is definitely  
 putting some pic metric on things and so it's very sensible to think that whatever exactly happens here it  
 must have something to do with putting metrics uh um metrics on some kind of real stuff um and uh yeah  
 so so some question we have in our mind is uh whether there is a way to like prove the object composition

like for complex  $K$  manifolds or something like this using some geometry that will involve this extra point here um in this abstract language this kind of difficult for me to think about but um we actually have a very good analogy uh again I mean we can again use this analogy between three and four um so periodically um so I said that there's something about Lo representations and so if you work over this kind of per part of this picture um then over this part of the bur space um like over the opening part you can Define complex space our sense called QP locally analytic um this corresponds to the what union over Pusa where  $Z$  is really just the analytic spectrum of the locally ENC tun from DP to GP so ones that are locally developable you can locally develop into Power series expansion um and this kind of comes up very naturally in all these investigations there and we know that this analytic space which lives over the open part because naturally has  $K$  coefficients this has Canal extension um what was the F2 Point uh and so this is very much related to a theory of locally antic long series two representations uh I mean these are these are the coefficients of the groups of some PP um some at least some fragments of which you can find someone on the literature um so periodically there is some kind of yeah so this Canon is actually a little bit subtle to write down it of some divided Powers um uh but exist is very important for this uh P story and one thing this suggests and which I don't yet know how to do um is that if you look at the the real part of the picture real and then close on Infinity um then over here again you also have the real numbers um like as a real antic space so the thing that's covered by yeah so there  $r$  as a real antic manifold and again you locally uh the thing with the functions are the real analytic functions so the ones that are locally developable and locally be developed to power series expansion um something that match to the base and I mean it match to the to the open part of the base um but like each each fi over a point is this one um and it suggest that this should this should extend canally over the function yeah canally over close Point Infinity um in a way I don't yet know how to really think about um but this also suggest that if you have some like I mean if this could be done and this would be some kind of ring object then also this real antic group would flips over like oh yes this should also should also expend so so here you put you use the  $G$  at all points you use the Gaz structure or use Liquid for different uh no we don't I will come to liquid structure later today for now it's not needed for anything so you just use the same the same at all points of the yeah so it yeah so acting on this again you have the rescaling action so there's a lot of different copies of the real numbers now sorry for that um you a raling action of the nor every single  $V$  but but and it maps to the belage in in your sense to the boovich space I mean to the analytic stack of the so this yeah so this this maps to the space of norms and it's an open Subspace of the space of norms and over there you have this ST which is like real of locally antic antic thing which lives up over the open part over the open Ray but then there should be a way to canonically extent so but properly speaking you mean rla cross 01 right or  $Z$  and so I expect that whatever kind of group that is in maybe representations of this have some kind of metric structure attached to them I don't really know like that if you want extend representation over the open part of the punct I would expect this something to putting a metric on it but I don't know these are some objects that also exist uh that I don't yet know um all right maybe let me mention ah I mean also I mean this some like and you can also just just try to understand like we can look at analytics text over the space of Longs that really map to the close Point infinity and try to understand what the kind of geometry is there and this is some very peculiar geometry where you is it's still about some kind of real complex manifolds or something like that but you are able to localize much much more you're able to really zoom in inally into your space um and so I think it's very interesting to try to investigate what geometry this point looks like I can all agree so we can zoom in and some it's a very different so so um so I mean you have to for example you have to us REM sphere and then the bounded part of the REM sphere is actually just what seems to be just the Clos unit disc the bounded part is some kind of weird overconvergent minimally over convergent neighborhood of the uh of the close unit disc is essentially just the CL so any real number that's bigger than one is an unbounded function on this thing so I think it will take a while to figure out how why do you picture the sphere sorry Peter why do you picture the sphere um I mean actually mean we were always looking at this this normal like T1 right T1 Infinity yeah and and the P1 over like this whole part this whole part still lives over thetic spectrum of our gas so whatever happens over this point it's still happening also over the gas is reals but then can do some kind of final localization there so particular can Tak a usual keyb over gu his wheels and base change it to here and then look at what the bounded part is it's prob that but if you remove the color part is it confirm equivalent to the plane does come from any sorry I didn't catch the question if you remove the color part is it confirm equivalent to the upper half plane half random question uh it's not really base change from

anything over here okay okay okay uh um all right so uh I think that's a extremely fascinating Prospect to investigate this type of geometry and yeah as I expect it will be it should be related to all these questions where usually you put metrics on stuff all right so that was five uh six um so like one key thing that our formalism allows you to do is really mix situations where you have to do both function analysis and use like fancy category Theory and but at this point you can go completely fancy so I mean so there's a theory of PS of higher categories Antics text so like to any fine guy I mean we first associate just the drive category but then we could also associate uh module categories over it but then as I was explaining last Friday a Ser of present Infinity n SP due to stanage um and so you can just so you can do two PRL just continue forever and I mean so you can Define n PRL on X for for any X and you cannot just Define it but again there's some you have all the kind of six fun on it and so you can just play with it so the existence of this is not a speculation but what comes next is definitely a speculation so okay so I have a strange project with stars and Z where they're Computing some Quantum CH Sim Varian and I don't know what they are but they seem to get certain power that seem to be very related to the kind of periodic structures I'm seeing and for a long time I'm trying to make sense of whatever they're doing and uh but now recently I realized that I'm able to send about higher categories and that well the C hypothesis tells you that this kind of to fi series I should really just be certain higher categories and then try to understand which one they should be and you can essentially say it but it doesn't quite work so so here's what's called Quantum transus or something this I you must be aware that I'm saying words uh that I don't understand um uh okay so so so extent that I understand anything um you have maybe start with a comple group and maybe forly simple or something like this relevant um and then you fit what's called level I mean so there a funny computation that if you look at like maybe I don't know let's see Simply Connected so then the first interesting formology group of G is a third formology Group which I mean a simple assumption z um and yeah so let's fix the level which is just an element um and so here G is just considered a topological space but then you also have kind specifying space of G um I mean gmap to be Cub so this if you want the net from G cling space of z um this actually D Loops it's map of groups automatically obstru manag so M from BG to B4 to the z z um and then but you can also restrict that to G as a real antic thing right because in general like I had to think that whenever you have a real analytic manifold it maps to the Incarnation of M like is a condensed set kind of incarnation and this basically what I'm using here uh and so but then there like okay so then there's also like the exponential sequence analytic yeah thanks um have exponential sequence so everything's living here over like let's say guess is complex numbers um and then a composite map to be to the four of GA venes because for here chology has trivial and so this actually lifts to be cubed of the analytic GM and of course the analytic gmf us so now this sounds really like the enet um all right but so so so what does it mean to give a map to be cubed here well this is too hard for me to think about but um so map to bgm well that's just the line model right so bgm that's just a classifying space for line bundles then b square GM uh this is some giving you somei algebra or something so this giving you a Twist of the category of modules and then the Cub GM this means that you give a Twist of the category of categories is over over it yeah so so this snap called Alpha from BG a real grp qm specifies a Twist of like prlg I some let's call it just l or invertible it's invertible object is a 2pr this all right okay so have this uh and you have the projection just to the point or the point is like the guess complex numbers and I think what people do is like okay so they they they want to look at like some family of G tsers like bundles with a flat connection or non flat connection whatever um and but sometimes Twisted by this Alpha and so I mean this is somehow more or less governed by pulling back this L under Alpha but then they want to integrate over the space of all G ches so basically they're taking this and actually g a finite group that's literally what they're doing as I learned from paper of three topkins l T um and so I mean this just makes sense and gives you an object in two uh and then you can Wonder like so this this this guy here it's some symmetric monid ceg very and categories from n plus one category um and I mean Quantum trans S Series should be a threedimensional to Quant F whatever that is so it should be a three by the cism hypothesis it should be a three dualizable object somewhere uh in a three category so now we have one so the question is is this streetable and uh as any expert would immediately tell you this there's no chance of working uh um uh because the space of conformal blocks I believe it's called um there you must put some holomorphicity constraint and I'm kind of not doing that here um but I mean in some sense it's not so far so but it is true duable I think that's that's you don't don't need need much uh okay so I didn't carefully check it but I believe it is to dualizable and um and to check that it would be S realizable and for suiz Ability you would need the following thingss uh you would need that pie is kind of

from proper and smooth so diagonal of  $P$  proper Smo and the diagonal diag of  $P$  proper Smo so there are six conditions to check and only one of them fails okay so so more concretely this is like the point the point this one reduces St change to the group being proper and smooth and this means that the inclusion of the point into this so for all of these you would need that they proper that they smooth smoo I mean yeah Comm smooth uh any guesses for which of the six fails actually the one you probably would expect the least is the smoothness of this map the smoothness of the second yeah only just a digression the map to be you said that it lifts to to a map to from  $B_4 Z$  to  $B_3 GM$  analytic but you want to BGR locally analytic instead of BG so the obstruction is something having to do with chology with coefficient in something like continuous chology but it is in your language so I'm not sure what it corresponds to but it's just aent chology of the stack basically what basically just Theology of the stack but it is not true for BG itself it lift or I mean here would be the chology of kind of condensed set ST which would actually be amount chology of this funny St is singular chology so if you would go here to be for those complex numbers you would just T the this is a singular commod with a complex number so this one wouldn't lift you need to go to you you the obstruction is is has to do with with maps to to right yeah so it's some kind of I mean up to this analytic it's basically coherent but on this classifying St as a condens set the coherent chology is kind of singular chology ah okay okay and it's because it's a compact group that it vanishes on the go to the lotic yeah but the compact is much more crucial for for these kind of problem sols we can hear um okay so this doesn't word but in some sense I feel like it was so close to working so maybe you just need to tweak this this here a little bit and as I said in five uh there's actually canonical candidate where you kind of go to the Point Infinity maybe that one works very naive question uh doesn't word how replacing by fiber fiber of the extension first and it seems weird that this should put the kind of correct holomorphicity constraint into the picture but I think the formalism might just work out to do that and there's also some structure you're not using with the like like um like I think this map to  $B_3 GM$  that you build it should really have a connection I mean like I think that there you know could kind of be going to define chology and weight too uhuh yeah so so maybe this is not yet completely correct I just want to point out that I mean you can just play with these things and you can hope that using if you would actually understand what you're really trying supposed to do uh you could write down something that would actually produce something spe Lal so I mean usually what you would try to do this you would run to all sorts of issues that you always want to do functional analysis and high categories and I mean people manage to do a lot of things but here you can just very naively try um all right so this brings me to the last thing I want to talk about a little and this one I actually want to uh go into a little bit more details and actually I mean this was very fancy hyro and now it becomes bit more concrete again um so this is about a theory of uh where fundamental proces now it is we don't see very well it's condensed yes cond it'll resolve um so yeah so let me ask a precise question and then I will actually answer um I so this is a question that comes up in particular like in sleed Geometry for GR of vit variant and things and that if you are in some more algebraic situation there is a known solution since some 20 30 years uh but if you're really in the analytic context it's much more subtle um and one reason I looked into this is because it's a natural I mean these kind of things they're doing syic geometry they're naturally some kind of spaces that are simultaneously kind of analytic derived stacky and so on and so this is the kind of theory where our situation where our series should be useful and so I just try to see to what extent it is useful and uh yeah I think it can actually do something um okay so let's consider fiber like a model situation that want May interested in the following consider a fiber product uh of manys  $M_1 M_2$  next um of let's say compact Mo say oriented comp  $M M_1 M_2$  feel free to assume that these Ms are closed immersions I don't think that's relevant um but I mean so yeah uh consider but which is of expected Dimension zero in other words like  $d M_1$  plus  $d M_2$  so then if the transl intersection turned out to be transverse this  $x$  would just be a finite set of points oriented points actually and so you could count the number of intersection points so this intersection transfers Fin and actually also oriented so any any point knows whether it should be count as plus one or minus one and so so you get a well defined uh counted sign count of the elements of  $X$  this is a variant under under so long as you stay as intersection stays transverse and okay so then that's a setup so for always a question the question is can you uh Define this sign count of  $X$  in general I mean without transfers intersection purely intrinsically on  $X$  of course every connected component will we have a well defined like in Fulton's Theory every connected component will have a well defined number which is the part of the intersection number coming from this connected component yeah sure but but even these local ones you need to Define right um right so so the good situation I know mean you have two circles meeting

transversely and then this plus one this minus one intersection zero um but now this might generated I you might have two circles that I don't know some situation like this where this might be I know this this might situation it can be infinitely conic and Stu I don't know I mean this might be tangent to infinite order somewhere might be the same for then cross I don't know all sorts of really funny behavior um and then right so if instead of compa move manifolds you have some kind of smooth projective varieties there then in this case it's known how to do this but also like the intersection cannot be as bad as for smooth manifold where intersection basically as a topological space has basically no structure whatsoever okay so let me first uh State Z yes compute the F product in our C so in particular X is some kind of deriv um and this time it's actually somewhat critical to do this not over the reals real numbers but over the liquid real numbers for some choice of structure depends on parameter that doesn't actually matter for this application yeah so for everything else we did in our course this kind of liquid structure that we once produced was very much effort not so relevant but here I think it actually really is relevant for reason I will explain in a second so what's the so yeah so we can Define kind of notion of we can look at analytics text in our setups that just have the property that locally it is possible to write them such an intersection and then uh one can only produce a virtual fundamental class on okay so but uh so there are some symplectic experts present at NP in particular Nate Botman and Ken Barheim and uh I kind of discussed this a little bit with them already and I want to continue those discussions um but in particular they made me aware that like even in this kind of simple space it's kind of surprising that this should work uh so so so let me say why this is surprising so let's consider consider the case where like those M1 and M2 are Cur there's my picture over there um uh and then just assume you're in local situation where you kind of have two things meeting at just one point locally intersection with one point and let try to analyze what happens in the neighborhood so in the best possible scenario this intersection is transverse you just get a point counts I don't know plus or minus one depending on your orientations um I don't know then the next worst scenario is if it's a square function then okay this to be zero and if it's like a cubic function there should be again first one and so on um and so this tells you that if it's Vanishing to finite order uh then basically it's clear that we're okay right because we just need to remember the vanishing order of that function and this will tell you what the this function crosses the line or it doesn't uh but no uh but now let's consider a situation as you can have SM manifolds but there more tangent to infinite order something like  $x - \text{one } X^2$  but then you can have a very similar function which is uh  $s$  of  $X$  time  $x$  and so what I'm claiming is that the locus where this function is zero whatever that means uh determines whether this function crosses  $A_1$  line or not the claim is Vanishing locus of these functions just the Vanishing Locus uh determines I mean distinguishes these two cases in particular determines whether the function crosses or line in which category in our category okay so let me make it slightly more explicit that me like like in classical geometric picture is quite unclear what kind of structure you to give the anything I mean the intersection point which knows it it must in some sense must know more than all the derivative of this function because it will never be able to double from AO derivatives uh but it knows much much less than the germ of this function it really only knows eventually so classically would maybe try to use the whole germ of this function in order to remember whether proc or not um uh right so I so what is locus of say some Infinity function from  $R$  to  $R$  uh let's assume it really just has an isolated zero  $Z$  so this Vanishing Locus general form this just the analytic Spectrum  $\text{alra}$  Infinity functions from  $\text{modul}$  generated by  $f$  where is this this thing here I mean it is a liquid special type of condensed are uh um with the analytic ring structure  $\text{antic ring structure}$  doesn't matter um it is a very funny one though I mean if  $F$  would only manage to find out order then this would just be TR poal algebra finish in and so this is some nice algebra but um if vanishes to infinite order then this is some funny non-separated thing and so technically you would have trouble and this with a topology but in the condens world I mean has a natural cond and so so here's a proposition that tells you that the Spanish loc someone knows about this so if you have two functions at above and the exist in isomorphism of liquid algebras between their CS you mean analytic rings with the liquid well I mean I don't need to put the anal structure on them because it's just induced one so I can really just look at them as liquid albas also I mean  $f$  is still non zero divisor here under this assumption I made so this really still concentrating degree zero just not house  $T$  so I don't have to say um so assume that these are isic so the condens set of  $C$  Infinity  $RR$  means that you you have to Define smooth functions from  $R$  cross a profite set to  $R$  you do it in the usual way yes IND construction just completely internally in the condensed world and it will produce the right condensed structure or just remember that it's a  $FR$  a space topological thing just pass to condens after ver those give you right and the



liquid thing does it doesn't change the condensed liquid is just a condition right I mean any of so it satisfies the liquid condition I mean just say condensed be okay um so assume this then uh actually  $f$  is  $G$  times  $U$  for us is an function and so in particular being in veral function from  $R$  to  $R$  must either be everywhere positive or everywhere negative and so multiplying by such a unit cannot change whether the function crosses the line or not no no no but what happens if you rescale the function you take the exponential  $e$  to the minus one  $X$  squ and then you take there say instead of  $x$  constant times  $x$  so those things have different grow rate it cannot be that there are there are you use different coordinates on  $R$  it's it it doesn't seem to be it seems fishy because the isomorphism is abstract of condens  $AR$  is abstract right right but still it's true but  $e$  to the minus1  $/x$   $S$  and  $E$  to Theus one over let us say maybe it's true in this example but  $IIIII$  is true the offer it is true so let me give a pro um so first of all why why do I stress the liquid so much um uh the liquid means that if you take  $CN$  functions on  $M1$  and tender them in the liquid sense with liquid real numbers with  $CNP$  functions on some other real manifold them too and this definitely maps to  $C$  functions of the product but by computation that we spelled out uh in the complexometry not I believe like on nuclear nuclear fet spasis um the liquid tensor product is doing the expected thing and so it actually is producing the  $C$  functions on the product um but it's not true uh for the overall  $G$  there would be some smaller thing but concretely what does it mean a a liquid is a is a  $r$  right  $h$  a set set I mean  $r$  is just the usual condensed ring yeah real numbers but then you put different ring structure there which is much closer to asking that like the building blocks are some kind locally convex things they're not quite locally convex they're just Alpha locally conx versus Alpha might be slightly less than one but you're very close to locally conx setting and so when you form this  $T$  products it basically allowing all locally comx combinations and at least if you kind of have nuclear transition Maps then the precise kind of complexity you need actually it doesn't matter one limited works okay so this is the deriv  $T$  of product it would be works so this is a non computation that comes out right in the liquid World um and so from this you can deduce that if you take  $C$  Infinity functions from  $R$  to  $r$  that this is actually item poent over  $RT$  uh is a usual coordinate yeah so maybe theomorphism should be condensed our algebra with the coordinate in the proposition otherwise I don't believe if you don't have the coordinate so Ian you can just no I mean I I said that both of them have the isolated zero at zero Peri okay both of them just have  $0 \in Z$  that's what I use but I don't use anything more than that let's see what he's doing but he's not all right so let's analyzation so let's assume we just have an abstract  $Asm$  uh so for both of both of these admit unique map to the real numbers because you can classify all the maps from  $C$  Infinity  $R$  to  $R$  they just given by real numbers uh a varation that's some real number and only one of them one of them  $f$  is equal to zero so so what is definitely true is that uh they have this unique evalation to the real numbers and this must commute because in those cases um all right and now uh I have  $R$  joint key uh equal to zero nothing to here why is the usual coordinate um right I me maybe I should Infinity  $r$   $r$  here and so uh I get there and so where does this  $T$  go to well I mean let's assume  $G$  is not just also uh Vanishing first order then like I mean you have this unique map and you also have to Unique first interal neighborhood so this must be a function that it's not managing to first order here do you actually care about this maybe I don't care um all right so I have this map uh but okay I also have this compos map and of course it has a quotient here and if I want I can also lift to here right because I just need to lift my element  $T$  from here to here and I can do that um yeah so you get it  $f$  is equal to  $G * \text{unit}$  up to a change of coordinate so it's it's  $G$  times the change of coordinate compos a coordinate change times the unit change okay let me to this algebra here just I mean let call this here  $A$  and this here  $B$  um so one I want to CLA that this met from  $R$  joint ke be um so this is a usual um there exists the unique extension which is just a condition okay and uh so is a module same and there is a unique extension in which sense in the sense abstract Rings or or I mean this sense here right so it can be at most one extension okay saying um and I claim it exists and for this I can it's enough to check that it exists here but I can classify all  $M$  from joint  $T$  to here and they all all of them do expand right okay they do extend by just comp by comp okay all right and there it's okay and you really just need the extension is necessary you need to just need to prove existence but this map is given by some  $C$  Infinity function from  $R$  to  $R$  but then any other  $C$  Infinity function from  $R$  to  $R$  can just be evaluated there so that works okay so uh so this means that this map from here to here it's factors uniquely to here so you actually get this map such a map here where this commutes but this is to this as this is also exposure right um okay so so maybe this a real parameterization that  $g$  wants me uh to incorporate all this time uh  $t$  sorry um and at this point basically like both of these are compatibly equion of these function from  $R$  to  $R$  so and then the ideal must so this presentation must be the same uh

uh anyway I so I think from there you can conclude at least that this property about  $F$  crossing the line is only deep crossing the line because uh so there is  $n$ 's presentation not saying it right but um let make a point here so there are some existing theorems series of um like derived manifolds uh but they proceed very differently and particular they consider some kind of algebra of functions where by the definition whenever you have a function then uh you can kind of compose with any  $C^\infty$  function from  $R$  to  $R$  this is some kind of data I put on the Rings but this observation is telling you that these kind of funny derived notions that we're getting they have just have happen to have the property that whenever you have a function you can always canonically extend uh to  $C^\infty$  function so they acquire the structure that for any function you can compose with a smooth function um but you don't need to put it have to put it in in the beginning try to see if can Solage was trying to save um this is the canonical put that um that you use that probably and I mean those things can be classified these are just given by usual new function from  $R$  to  $R$  um and so then okay so has a  $\pi$ -ization and so maybe you have to put a reparameterization in uh then then just makes a similar sorry I want to say the same thing it's faithful embedding  $H$  so this just pullback by right so yeah okay maybe up okay so right but so some of the thing that makes this work is precisely this thing that you do get these unique extensions to  $C^\infty$  function because otherwise uh you cannot control the kind of  $\pi$  from here to the  $\pi$  section from here you wouldn't be able to compare all right um all right so so so let me know State some more objects here um let's say as locally compact space and let's assume also it's a five dimensional which is always satisfied when it's intersection like that um uh plus achieve of animated all together oh so that is local  $e$  intersection sly isomorphic but not with a given data so this is just a condition uh intersection um then one can see so this is a family depending on on  $S$  of smooth manifolds or just constant intersection of smooth manifolds the intersection of smooth manifolds is constant locally or is it varying continuously no I mean locally you can write open substance of  $s$  can be written as the  $\pi$  where the that's so coming from the point yeah yeah ah okay  $s$  itself is okay  $s$   $s$  this is okay then one can define define uh an expected Dimension um there some  $C$  uh it's a locally constant function this integer coefficient um and orientation  $\Omega s$  so this is a zero persistent on um right and uh okay so let me do not by  $\pi$  from  $s$  to  $Star$  just the projection of locally compact spaces and virtual fundamental  $T$  of like  $s$  equ structure sheet  $\mathcal{A}$  also this this depends on  $S$  all all of these depend on  $S$  um which is a global section on  $S$  of the dualizing complex  $T$  this this makes sense yes this makes a lot of sense and is because in particular if the expected diine is zero um as is Compact and uh and it choose an orientation oh my God fix I mean assume exist and fixed one um then you can Define the Fundamental class Global fundamental class as some of the integral of the virtual fundamental class Global I know the count signed count of  $SOS$  as as the integral of the virtual fundamental class over  $s$  okay so I mean under  $G$  equal to zero in this orientation this goes away and then you just and as compact you have a trace map that goes back to the integers  $SC$  all right um so let me sketch the construction um and so basically this is just the direct analog uh of the algebra geometric construction um so I think this was pioneered in the work of uh and um uh then many many people worked on this and I'm I'm not an expert on this at all um the paper where I learned it from some kind of algebra I mean lination I know by a um but the ideas I think will maybe um so so what they Define first of all is this uh what they I think called the intrinsic normal cone um and so this realize uh so there some kind of intrinsic normal cone over  $s$  um uh that describes so the fibers some of the fibers some of this corresponds to the cotangent complex um of like over  $R$  um and uh so in our world I mean first of all you can somehow Define such a derived intersection of like smooth things but also if you compute what this kind of cotangent complex does that can like abstractly defined for  $C^\infty$  functions basically for the same reason that the  $T$  products came out right um also the Cent complex comes out right for  $C^\infty$  functions on a manifold and then for such dve  $T$  part of course you get  $S$  so this is actually a perfect complex so what what's the superscript on the  $L$  uh liquid and so uh here some kind so actually I just want to Def find this here as a locally compact space uh sorry  $St$  let's say it's a condensa so some kind of stacky version um where uh some tangent directions give you  $st$  directions I mean it's not really don't need to go all the way only group you don't need High I don't need High yeah so um so four for ah because the  $L$  itself is concentrated in the okay okay yeah sorry perfect amplitude uh amplitude 01 so the already did it okay uh for compact space or profile set if you want um  $T$  um well first of all from  $T$  to  $S$  uh you can just regard this as a as a map from aspect of the continuous functions from  $T$  to  $R$  to some let me call it unexpected  $SOS$  um so that you can by locally the spectrum of this this algebra you can build an analytic stack in all sense and then if you evaluate this on just the continuous functions um you just get um  $M_s$  from  $T$  to the underlying set um but now I

want to Define what what are the maps from  $T_2$  this intrinsic pH and this can actually be defined as a map from a funny algebra it's you Tak the this guy and you model than by zero um in other words this is a TENS product The Continuous function from  $T$  to  $r$  t over  $R$  kind of  $R$  join Epsilon where the degree of Epsilon is equal to one and some particular Square to zero but where the structure shift comes in ah okay it's so is there so here it doesn't really matter because always Factor over  $\Pi Z$  continuous functions but here's the WR structure that is because that to Something There is ah okay so in the first thing you can replace it by the first thing yeah you can replace by the kind zero and even by the house of because see here here not and just by De the you can analyze what does it take to lift the map from here to here and it's just some thing in terms of the cotangent complex and so you can realize that uh the fiveways here have some kind of uh some kind of geometric Incarnation you also cotangent complex so all the tangent directions uh they give you kind of stacky directions and uh yeah and if there some kind of actual obstruction uh this is non smooth part and there's actually Al some Vector bundle directions yeah some particular some kind of smooth map um or can understand it and so uh and yeah right so there a cent complex and like the the perfect complex so it has a kind of Dimension which is the difference between the vector B dimensions and so this already gives you the expected dimension some look at the rank of this guy um and this kind of situation here this geometry here will also produce the orientation feet for you think about it so actually what you really need to produce is need the glob section so element in this intrinsic normal cor uh with values in the that's dat you really data you really need to produce um and now there's this really funny uh thing that let's take  $r$  as a cond set here down here and then one can write down something down which everywhere except at zero it's just a point but in this  $F$  at zero you get this funny fullness there some kind of con as the form which lives over the real numbers and which everywhere except at zero is just point but then as you degenerate to the point suddenly there some very sticky thing and you see this Con here fine can I ask question yes is it the same phenomenon as you described before in the analytic stack of  $P_1$  where you compactify over the fixed point at Infinity I don't understand the question let me just continue with this discussion yeah sure a so so let's define what what is this formed con uh so mapping  $T_2$  there this corresponds to but first of all uh let me use  $G$  for safety because I'm not sure if I've used  $F$  before uh corresponds to a map from like this should map to  $R$  so first of all correspond to a continuous map from  $T$  to  $R$  but once you have that you can look at maps from uh The Continuous function from  $T_2$   $r$  modul  $g$  right uh to um where where of course if  $G$  is the zero Maps then this is just recovering what I told you before if  $G$  is non zero well then if equ equ by an invertible function this is just zero so you're not getting this is no data um so you guess just just get a point but then if  $G$  is a function that's somewhere invertible somewhere zero then suddenly you have this kind of thing where is producing some kind of doing these things together okay uh and now it's just some simple game with six fun just to produce a class because basically generically you just have a canonical section here it's just a point right and then you just degenerate that uh section uh let me try to do it so so consider the sheath  $F$  which is form so so this guy here is streetable and that's the only way I currently know how to check it is to use it I assume that a der section of smth manifolds and for manifolds it's true and then it's disable on fiber products in principle you could probably produce this canonic virtual fundamental class with much small assumptions the only thing you really need to that this map here isable um but then you can Define this thing which is a sheet on followe and uh and then okay so you have an open subset  $J$  which is the cone where from zero and then you have  $I$  which is exclusion from the cone and then you have some excision sequence for  $f$  uh  $F$  and here you have which is just  $I$  the of the  $p$  stre and then in particular you get a boundary map here and the boundary map just gives you the CLA you want and that's always a degree shift I'm getting confused about but it did work out um because the ididentification the  $i$   $f$  when you yeah's a shift by one appearing here because I'm pulled back from uh yeah so there there's shift by plus or minus one here minus one because uh I take  $Z$  here on the real line and not not pulled back from from the point and so then you get a  $m$  from the  $X$  zero of  $r$   $z$   $z$  towards the  $H$  and okay this and then you take a class here which is zero on one connected component and one on the other and the image is what you're looking for by the way is it is it true that this pie cone def is not just triable but smooth no I I don't think so I would have guessed okay I don't think so I so this cone of  $s$  this is some crazy thing right I mean it's some kind of fiberwise it's some kind of vector bule like thingy but over this crazy  $s$  mean this this  $s$  is completely nasty in general so if this map was smooth and also like the  $F$  was smooth but this has just no way of being smooth I don't think okay so I don't think this this thing that appears here is in vertical at all but you I mean you can still can still degenerate the topological fundamental class on a point towards

this cone using a little bit of six fun and then yeah this Con difference from the original space just by some kind of fiber BS which you can analyze and which give you both the dimension shift and the orientation I'm over time I want I don't know if it obstruction the well the TR obstruction series isn't that just that there this question complex that behaves right okay that's just completely infil this six any further questions the the what was the the exponential shift sequence in the previous Point uh for where you have GA local analytic and GM local analytic in which and Z so in which sense so I believe that those exist Stacks maybe with symmetric monoidal structure so this is like an exact sequence of symmetric mono analytic stack so what in which sense is it something like this or or is it yeah a group figurative groups right this St with the community structure and yes I'm not quite sure what the question is I mean it's like I mean all STS they are the definition sheets and then all these are sheeps of a being grps or the shifts in the higher sense okay okay exact sequence of which probably stly speaking would again be a triangle because okay language so it's just but this is an unstable being group Str kind of puts it into a stable context right okay all connective and is an exact sequence in there well but it's also the right so it stays triangle on St okay okay okay okay thanks Peter yeah all right