Burger's equation with AB3 advection and trapezoidal viscosity in the spectral domain

Consider Burger's equation for $u = u(x,t)$, on a periodic domain $x \in [0, 2\pi)$:

$$\partial_t u + u \partial_x u = \nu \nabla^2 u.$$

Expand u in a discrete Fourier series

$$u(x,t) = \sum_{k=-K}^{K} e^{ikx} \hat{u}_k(t)$$

where each wavenumber k is an integer, to ensure that each term in the series satisfies the periodic boundary conditions (if the domain size was L instead of 2π, then k would need to be an integer multiple of $2\pi/L$). Defining the Fourier transform as

$$\mathcal{F}_k[f(x)] = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x)\, dx = \hat{f}_k,$$

then for integer wavenumbers p and k there is a discrete orthogonality relation

$$\mathcal{F}_k[e^{ipx}] = \delta_{p-k}$$

where δ_{p-k} is the Kronecker delta. Using these, Burger's equation becomes

$$\partial_t \hat{u}_k = \hat{A}_k^n - D_k \hat{u}_k,$$

where the first term on the right is the nonlinear product is computed in grid-space,

$$\hat{A}_k^n = -\widehat{[u\partial_x u]}_k = -\mathcal{F}_k[\mathcal{F}^{-1}(\hat{u}_k)\mathcal{F}^{-1}(-ik\hat{u}_k)],$$

and the second term is the viscous dissipation,

$$D_k = \nu k^2.$$

Note that D_k can be replaced in the scheme below by more scale-selective dissipation operators, like hyperviscosity or an exponential filter.

Following Durran's analysis in §3.4.2, we use AB3 for the advection term, and a trapezoidal step for diffusion (AB3T). The result is a scheme who's stability is largely independent of viscosity, and determined almost entirely by the Courant condition applied to the inviscid case. Denoting the time-step as h and the time index with n, the scheme is

$$\hat{u}_k^{n+1} = \hat{u}_k^n + h \left[a_0 \hat{A}_k^n + a_1 \hat{A}_k^{n-1} + a_2 \hat{A}_k^{n-2} - \frac{D_k}{2} \left(\hat{u}_k^n + \hat{u}_k^{n+1} \right) \right]$$

where

$$a_0 = \frac{23}{12}, \quad a_1 = -\frac{16}{12}, \quad a_2 = \frac{5}{12}.$$

The trapezoidal step is implicit, but since we're in spectral space, solving for the future step doesn't require a matrix inversion. The result is

$$\hat{u}_k^{n+1} = \left(1 + \frac{h}{2} D_k\right)^{-1} \left[\left(1 - \frac{h}{2} D_k\right) \hat{u}_k^n + h \left(a_0 \hat{A}_k^n + a_1 \hat{A}_k^{n-1} + a_2 \hat{A}_k^{n-2} \right) \right].$$