Essentially Non-Oscillatory Methods

Nick Derr

Applied Math 205
Harvard University

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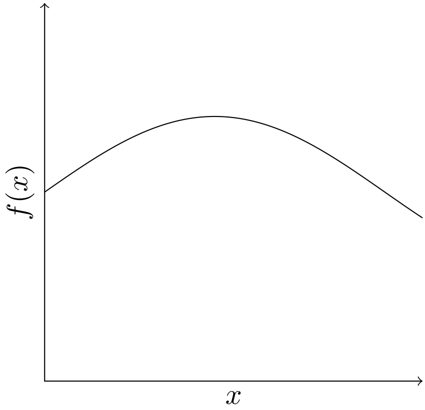
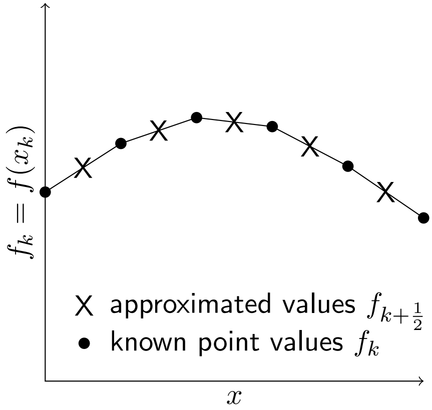
Approximating derivatives by finite differences is at the heart of many numerical methods.





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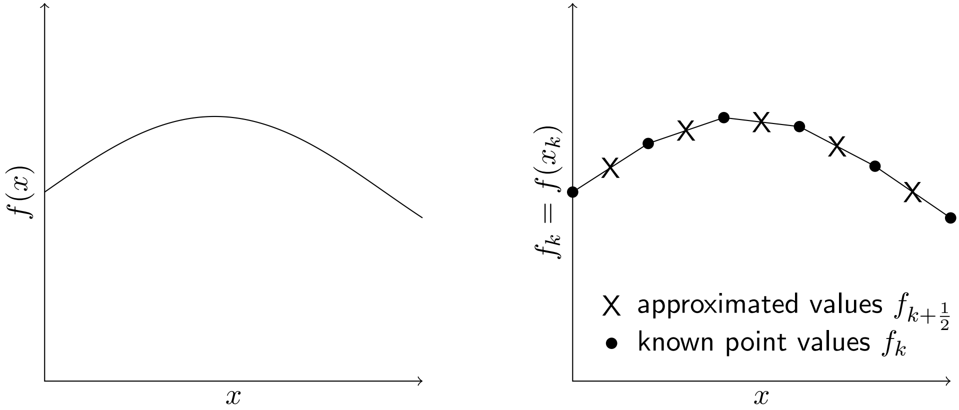
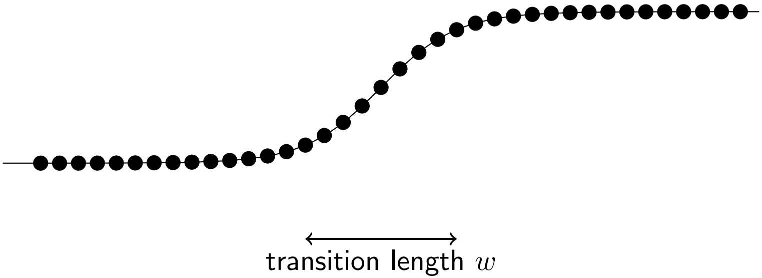


We can approximate any derivative value by evaluating that of an interpolant through a set of nearby points.

If grid points are spaced too far apart, fine features in the solution may not be captured.



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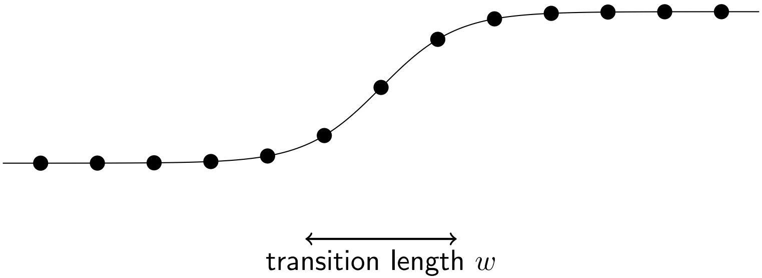

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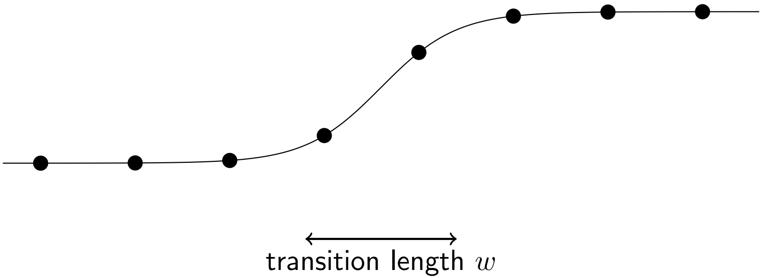

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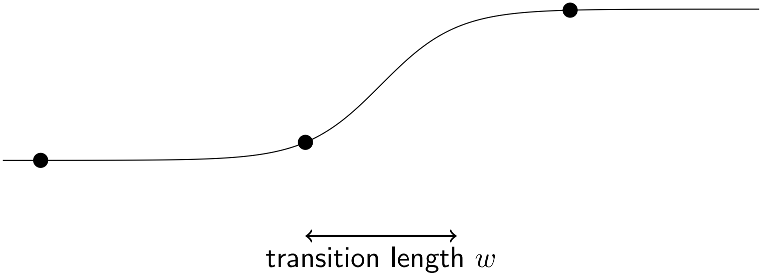

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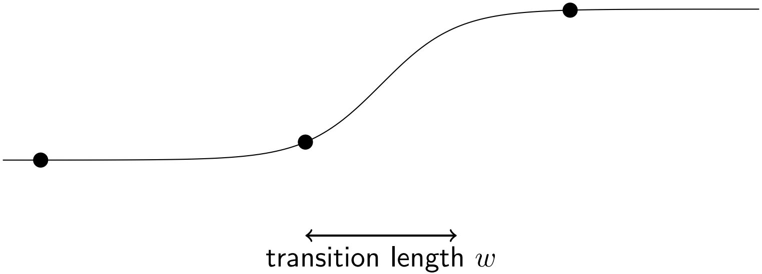
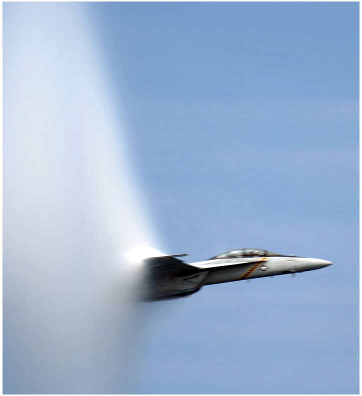


Need $\Delta x < w$ — but what happens as $w \to 0$? Does this happen in "real life" ?

Often, features that appear discontinuous are actually transition regions of some width w, but the cost to simulate with $\Delta x < w$ would be enormous!



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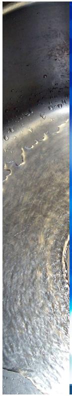

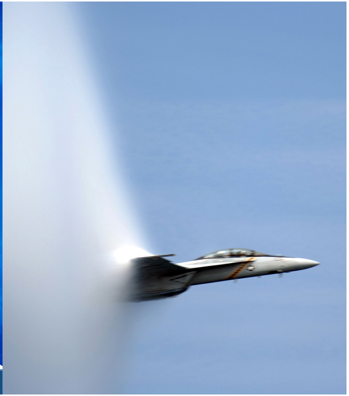

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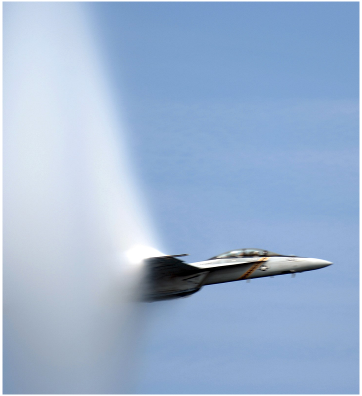

Need methods for simulating discontinuous solutions!

Discontinuities in solutions, or shocks, violate the assumptions on Taylor series truncation error and cause problems for "vanilla" methods.

Example: $\partial_t f = -c\partial_x f \implies \dot{f}_k = -c\left(\frac{f_{k+1} - f_{k-1}}{2\Delta x}\right)$ (centered difference)

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If we assume a wavelike solution, i.e. $f(x,t) = f(x - ct)$, we get

$$\boxed{\dot{f} = -cf, \qquad f' = f,}$$

so even with discontinuities the solution sees the function **move to the right**

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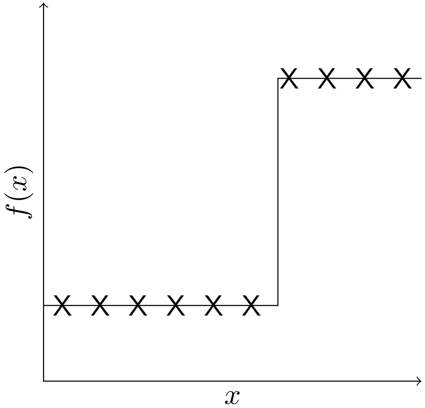

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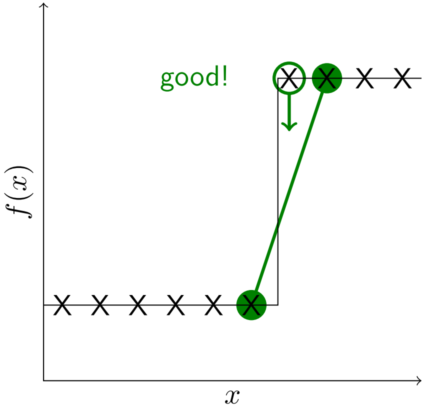
- solution moves to right
- some points move in expected direction



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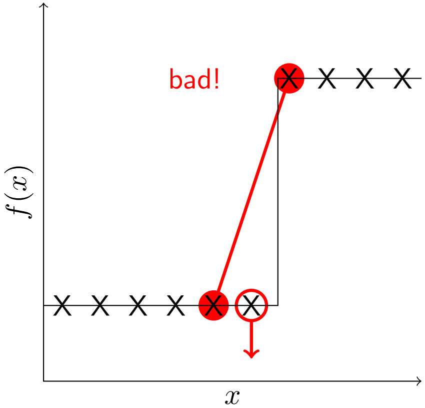
- solution moves to right
- some points move in expected direction, but others don't!



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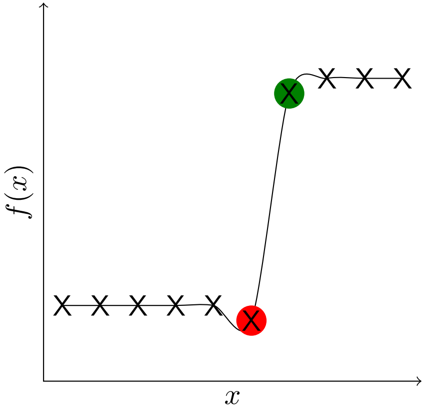

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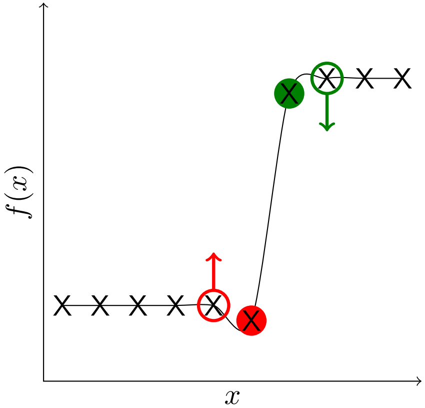

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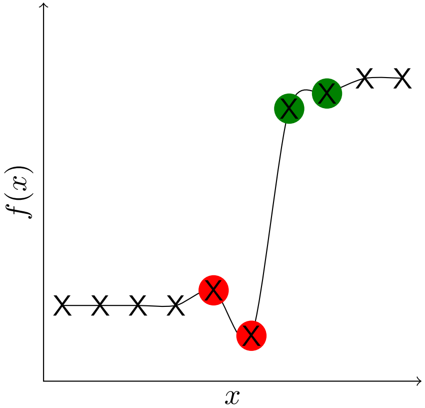

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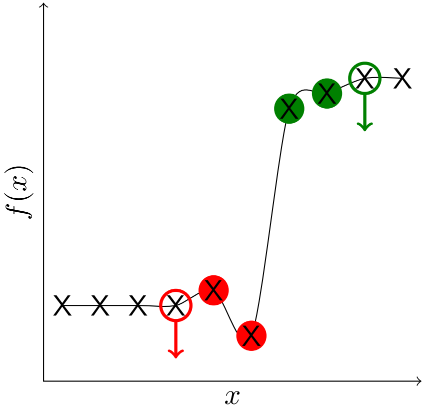

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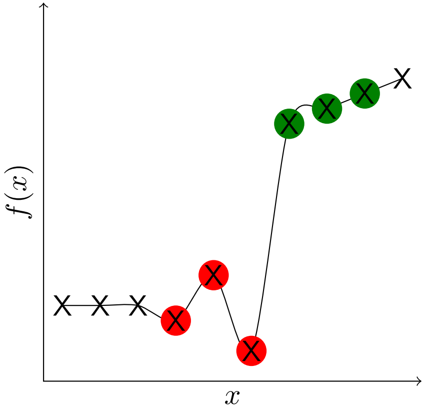

Let's re-write the problem in conservative form:

For some velocity $v = v(x, f)$,

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left(vf\right).$$

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Let $q = vf$ equal the **generalized flux**. Then

$$\frac{\partial f_k}{\partial t} = -\frac{q_{k+\frac{1}{2}} - q_{k-\frac{1}{2}}}{\Delta x}$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}

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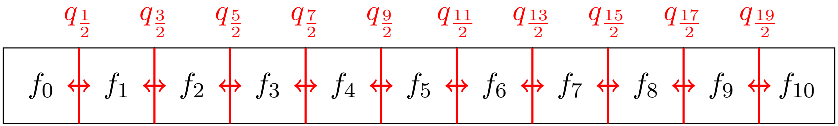
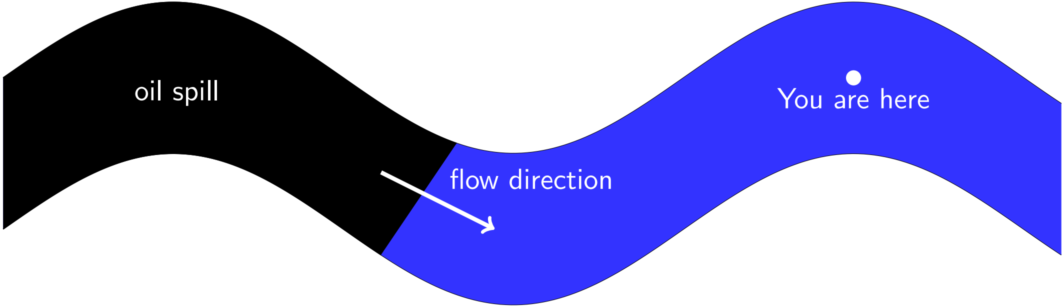


Example: centered difference on previous slide used $q_{k+\frac{1}{2}} = cf_{k+\frac{1}{2}} = \frac{c}{2}(f_k + f_{k+1})$.
Can we understand failure of the previous example in terms of the stencil $\{\frac{1}{2}, \frac{1}{2}\}$?

Big Idea 1: The sign of the velocity v tells us the direction of information flow. Look upstream! (Downstream? Garbage in, garbage out.)



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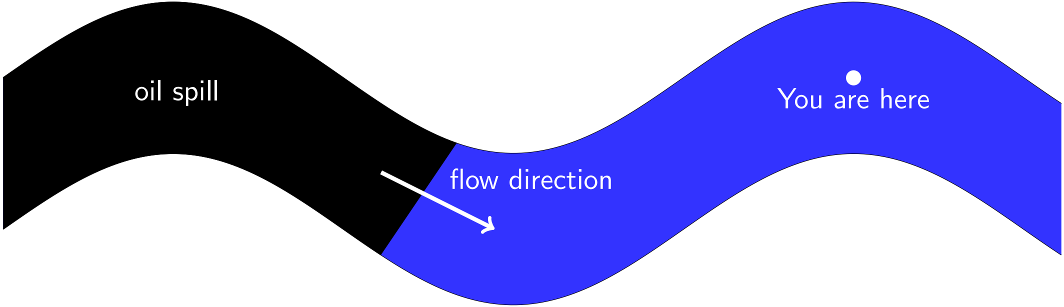

Which way are you going to look in order to understand how the local concentration of oil is going to change near you? $\implies$ Make sure stencils include upstream portion.

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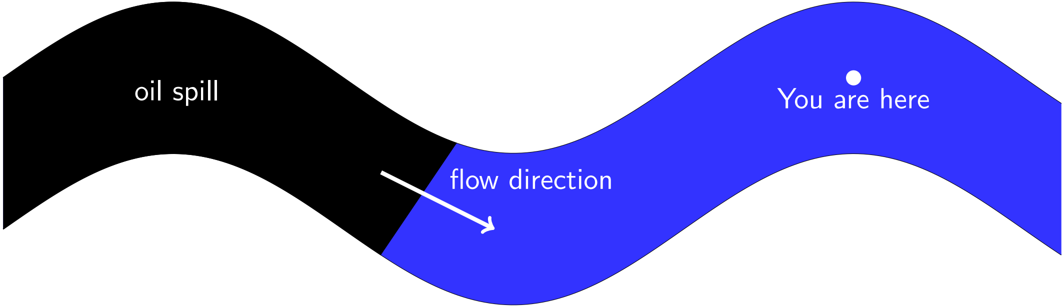

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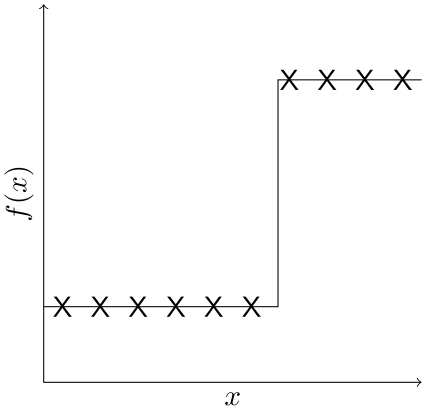
"Upwinding" is essential for shock capture

Big Idea 2: Interpolation error only converges to zero with grid size if it's with respect to a smooth function

How do various interpolation rules influence the behavior of the problematic point?



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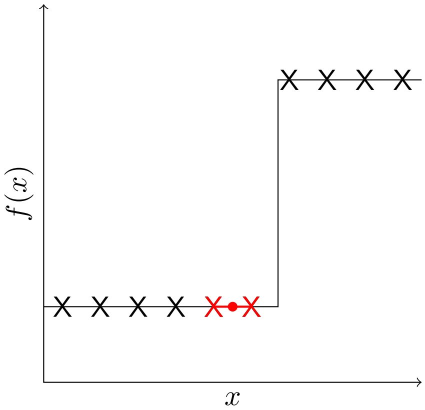
Original centered difference:

- point 1 is reasonable



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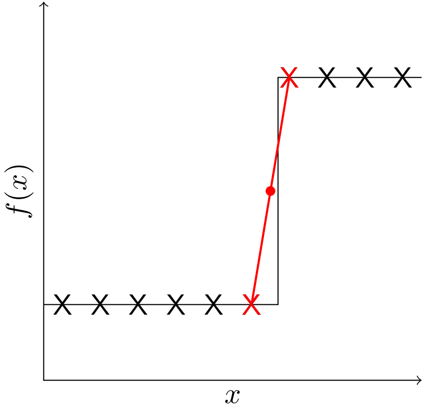
Original centered difference:

- point 1 is reasonable
- point 2 creates our unwanted oscillation because it's **interpolating points on either side of a discontinuity**



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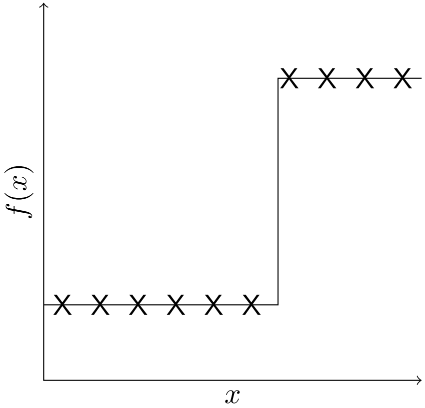
What if we ditch the centered stencil? Use a 3-point stencil to preserve 2nd order. We have three choices that upwind: let's look at them applied to second point.



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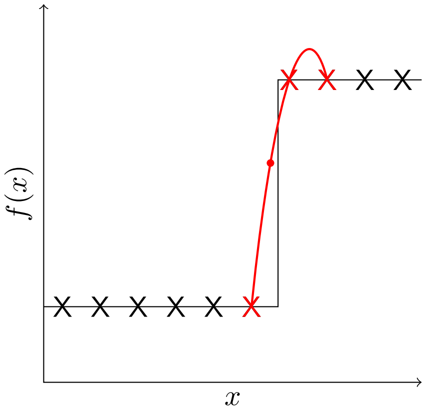

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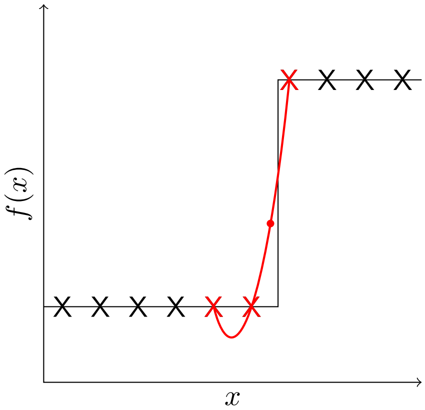

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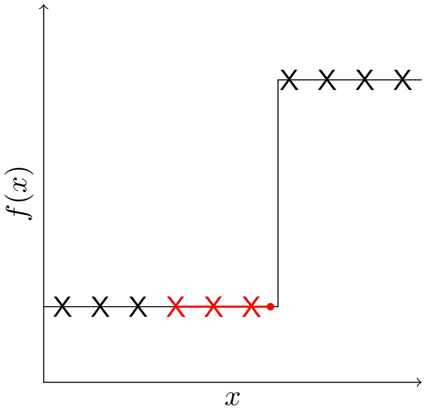
- between first two: no good
- between last two: same prob
- after last: Bingo!



Takeaway: interpolation $x_{k+\frac{1}{2}}$ from one of several possible stencils at each point

ENO Method:

Given a collection of five points $S = \{f_{k-2}, f_{k-1}, f_k, f_{k+1}, f_{k+2}\}$, we calculate the interpolant $f_{k+\frac{1}{2}}$ as follows. Following the procedure of the last slide, we introduce three stencils $S_j \in S$,

$$\underbrace{\overset{\mathsf{X}}{f_{k-2}}\quad \overset{\mathsf{X}}{f_{k-1}}\quad \overset{\mathsf{X}}{f_k}\quad}_{S_0} \overset{f_{k+\frac{1}{2}}}{\circ}\quad \overset{\mathsf{X}}{f_{k+1}}\quad \overset{\mathsf{X}}{f_{k+2}}$$

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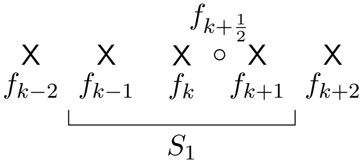

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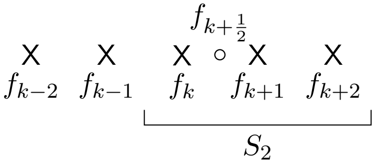

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Let $f^{(j)}_{k+\frac{1}{2}}$ be the interpolant constructed with the points in stencil S_j:

$$\begin{aligned} f^{(0)}_{k+\frac{1}{2}} &= \frac{1}{8}\left(3f_{k-2} - 10f_{k-1} + 15f_k\right), \qquad X\ X\ X\star \\ f^{(1)}_{k+\frac{1}{2}} &= \frac{1}{8}\left(-f_{k-1} + 6f_k + 3f_{k+1}\right), \qquad X\ X\star X \\ f^{(2)}_{k+\frac{1}{2}} &= \frac{1}{8}\left(3f_k + 6f_{k+1} - f_{k+2}\right), \qquad X\star X\ X \end{aligned}$$

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How do we choose one?

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If the underlying f is smooth over the region covered by S_j, then

$$\left| f^{(j)}_{k+\frac{1}{2}} - f\left(x_{k+\frac{1}{2}}\right) \right| = \mathcal{O}\left(\Delta x^3\right),$$

so the answer is pick an S_j over which f is smooth. But how do we know that?

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In the literature, β_j is often used as a measure of the "sharpness" over the stencil S_j. Multiple approaches exist. Simplest: let $\beta_j \propto |f''|$ as calculated with the points in S_j,

$$\beta_j = \left| f_{k-2+j} - 2f_{k-1+j} + f_{k+j} \right|.$$

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Let's see it in action!

How can we maximize our use of information?

We can use all the points in the stencil S to construct a fourth-order interpolant!

$$\underbrace{\underset{f_{k-2}}{\mathsf{X}}\quad \underset{f_{k-1}}{\mathsf{X}}\quad \underset{f_k}{\mathsf{X}}\ \overset{f_{k+\frac{1}{2}}}{\circ}\ \underset{f_{k+1}}{\mathsf{X}}\quad \underset{f_{k+2}}{\mathsf{X}}}_{S}$$

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\begin{array}{ccccccc}
 & & & f_{k+\frac{1}{2}} & & & \\
\mathsf{X} & \mathsf{X} & \mathsf{X} & \circ & \mathsf{X} & \mathsf{X} \\
f_{k-2} & f_{k-1} & f_k & & f_{k+1} & f_{k+2}
\end{array}
$$

There is one fifth-order interpolant of these five points, which we'll denote $f^{(S)}_{k+\frac{1}{2}}$. We can write it as a weighted sum of our third-order interpolants,

$$
\boxed{f^{(S)}_{k+\frac{1}{2}} = \gamma_0 f^{(0)}_{k+\frac{1}{2}} + \gamma_1 f^{(1)}_{k+\frac{1}{2}} + \gamma_2 f^{(2)}_{k+\frac{1}{2}}} = f\left(x_{k+\frac{1}{2}}\right) + \mathcal{O}\left(\Delta x^5\right),
$$

How can we maximize our use of information?

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$$
\begin{array}{cccccc}
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and in general any weighted average $f^{(w)}_{k+\frac{1}{2}}$ such that

$$\boxed{f^{(w)}_{k+\frac{1}{2}} = w_0 f^{(0)}_{k+\frac{1}{2}} + w_1 f^{(1)}_{k+\frac{1}{2}} + w_2 f^{(2)}_{k+\frac{1}{2}}} = f\left(x_{k+\frac{1}{2}}\right) + \mathcal{O}\left(\Delta x^5\right),$$

is another third-order interpolant.

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What do we want from our weighting scheme w_i? What does $w_i \approx \gamma_i$ imply? What does $w_i \approx 0$ imply?

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Then we let our interpolant be

$$\boxed{f_{k+\frac{1}{2}} = w_0 f^{(0)}_{k+\frac{1}{2}} + w_1 f^{(1)}_{k+\frac{1}{2}} + w_2 f^{(2)}_{k+\frac{1}{2}},}$$

which is fifth-order where f is smooth and third-order near shocks.

The Inviscid Burgers Equations, Non-Linearity, and You.

In general, we have assumed a constant velocity c so far. ENO and WENO methods have been able to capture discontinuities in functions whose shapes are not evolving; what happens when we make things more complicated?

Let the velocity be given by $v(x, f) = f$, so

$$\frac{\partial f}{\partial t} = -f\frac{\partial f}{\partial x} = -\frac{\partial q}{\partial x}, \qquad q = \frac{1}{2}f^2.$$

Let's take a look.

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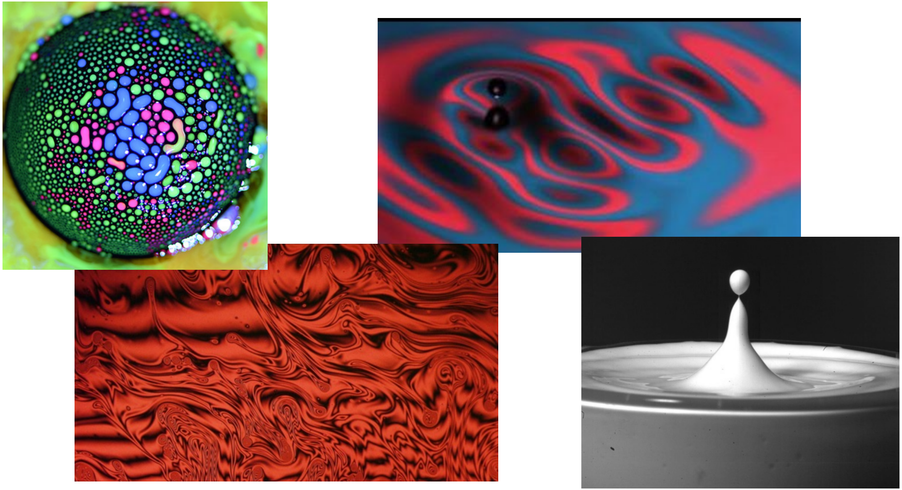
Why do we care? This $f\partial_x f$ term is essentially a 1D version of the $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ advective term in the Navier-Stokes equations.



Another application with shocks: traffic!

Let $f \in [0, 1]$ by the density of cars. A common (dimensionless) approximation to the velocity is

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(vf), \qquad v = 1 - f,$$

i.e. you drive fastest when the road is most open, and you come to a complete stop when everyone is bumper-to-bumper (e.g. at a red light.)

Let's see this one in action!

Group activity write-up:

1. Adapt the code in the notebook to produce a WENO method which upwinds in both directions and use to simulate 1) traffic approaching a red light; and 2) traffic coming out of a green light. Each of these will include a jump condition in the initial condition. Choose the left/right boundary conditions $f = f_l$ and $f = f_r$ on the sides of your domain and explain your choices. How is the behavior different? Is there a shock in both cases? One case? Neither?
2. (Optional) Use the WENO method to simulate 1) a smooth function and 2) a discontinuous function moving to the right with constant velocity. Make a log-log plot showing the convergence of the error at time $T = 1$ as you refine the grid. Do you recover the higher convergence in the smooth case as advertised?

Extra Resources:

- WENO methods description: the basis of the treatement here
- Astrophysical Fluid Dynamics lecture notes: excellent treatment of shocks, Rankine-Hugoniot relations, and compressible fluid dynamics more generally.

Thanks a lot, all! Questions?