Math 6316, Spring 2016
Due 2 May 2016

Homework 8 – Boundary Value Problems

1. Consider the boundary value problem

$$-\big(\alpha(x)\,u'(x)\big)' + \gamma(x)\,u(x) = f(x), \qquad x \in (a,b), \tag{1}$$
$$\lambda\,u(a) + \mu\,\alpha(a)u'(a) = g_a, \tag{2}$$
$$\eta\,u(b) + \theta\,\alpha(b)u'(b) = g_b, \tag{3}$$

where $\alpha(x)$, $\gamma(x)$ and $f(x)$ are continuous functions on $[a,b]$, $\alpha(x) \ge \alpha_0 > 0$ and $\gamma(x) \ge 0$ for all $x \in [a,b]$, λ, μ, η and θ are nonzero real numbers, and g_a and g_b are real numbers.

Following the approach in class, derive a $\mathcal{O}\big(h^2\big)$-accurate, centered finite-difference approximation to (1), along with $\mathcal{O}\big(h^2\big)$-accurate approximations to the Robin boundary conditions (2) and (3). For the left boundary condition (2), derive an approach that use the "ghost point" approach for enforcing the boundary condition. For the right boundary condition (3), derive an approach that uses one-sided differencing with an appropriate number of points to obtain the required accuracy.

Write the system of linear equations that must be solved to obtain the final finite-difference solution, $\{u_j\}_{j=0}^n$, where $u_j \approx u(x_j)$ for $j = 0, \ldots, n$, with $x_j = a + jh$ and $h = \frac{(b-a)}{n}$. This system need not be in matrix form, but all equations must be clearly specified.

2. Matlab: create a function of the form

```
function [u,x] = FD_solve(alpha, gamma, f, lambda, mu, eta, theta, ga, gb, a, b, n)
```

that implements your finite-difference approximation above. When creating/solving the linear system, it is strongly encouraged that you use a *sparse matrix* to store and solve the linear system. The inputs to this function are:

- `alpha`, `gamma` and `f` are the function handles or names for the spatially-varying input functions,
- `lambda`, `mu`, `eta`, `theta`, `ga` and `gb` are the scalar inputs that define the boundary conditions,
- `a` and `b` are the scalars that define the problem interval, (a,b),
- `n` is the spatial mesh size to use, that defines the finite-difference mesh $\{x_j\}_{j=0}^n$ where $x_j = a + jh$ with $h = \frac{(b-a)}{n}$.

The outputs are the approximation of the solution, $\{u_j\}_{j=0}^n$, and the spatial mesh nodes $\{x_j\}_{j=0}^n$. Ensure that these outputs not only have the same length, but also the same shape.

Write a script named `hw8.m` that uses this function to approximate solutions to the boundary value problem

$$
\begin{aligned}
-\left[(1+x^2)\,u'(x)\right]' + (2+2x^2)\,u(x) &= f(x), \qquad x \in (0, 2\pi),\\
2u(0) + u'(0) &= -2,\\
u(2\pi) - u'(2\pi) &= 5,
\end{aligned}
$$

where $f(x) = 4x\,(\sin(2x) + 2\cos(2x)) + 6(1+x^2)\,(\cos(2x) - 2\sin(2x))$.

The true solution to this problem is $u(x) = \cos(2x) - 2\sin(2x)$.

Approximate solutions to this problem using the mesh sizes $n_k = 10 * 2^k$ for $k = 1, \ldots, 8$. For each n_k:

- Print to the screen the maximum error in the solution, $\|u_h - u\|_\infty$, and for $k > 1$, compute and print to the screen the convergence rate of your approximations.
- Create a separate `figure()` window containing a plot of the analytical solution and approximate solution overlaid on one another. The plots must use distinct colors, and each must be properly annotated with legend, axis labels and title indicating the relevant "n" value used in the calculation.

When your `hw8.m` script completes, 8 separate figure windows must be open with the displayed plots, 8 errors must be printed to output, and 7 convergence rate estimates must be printed to output.