Math 6316, Spring 2016
Due 12 February 2016

Homework 2 – Interpolation Part I

1. Write two Matlab functions:

```
function p = Newton_interp(x, y, z)
function p = Hermite_interp(x, y, dy, z)
```

The first function, `Newton_interp`, should take as inputs:

- `x` – the interpolation nodes, $\{x_k\}_{k=0}^{n}$, stored as a 1D vector of length $n+1$,
- `y` – the function values $\{f(x_k)\}_{k=0}^{n}$, stored as a 1D vector of length $n+1$,
- `z` – the points at which to evaluate the interpolant, stored as a 1D vector of length m,

and should return as output a 1D array `p`, containing the Newton interpolating polynomial $\pi_n f$, that interpolates the data $\{(x_k, f(x_k))\}_{k=0}^{n}$, evaluated at each point in the array `z`, as described in section 8.2 of the book.

The second function, `Hermite_interp`, should take as inputs:

- `x` – the interpolation nodes, $\{x_k\}_{k=0}^{\nu}$, stored as a 1D vector of length $\nu+1$,
- `y` – the function values $\{f(x_k)\}_{k=0}^{\nu}$, stored as a 1D vector of length $\nu+1$,
- `dy` – the derivative function values $\{f'(x_k)\}_{k=0}^{\nu}$, stored as a 1D vector of length $\nu+1$,
- `z` – the points at which to evaluate the interpolant, stored as a 1D vector of length m,

and should return as output a 1D array `p`, containing the Hermite interpolating polynomial $H_{2\nu+1}$, that interpolates the data $\{(x_k, f^{(j)}(x_k))\}$ for $k = 0, \ldots, n$ and $j = 0, 1$, evaluated at each point in the array `z`, as described in section 8.5 of the book.

Test both of the above functions using evenly-spaced data points on the function $f(x) = \operatorname{atan}(2x^2)$, over the interval $[-3, 3]$. Evaluate these functions using 201 evenly-spaced data points `z` in this same interval. To provide fair comparisons, for each $n \in \{5, 11, 21, 41\}$, both methods should construct a polynomial of degree n, i.e. for the Hermite interpolant you should use $\nu + 1 = \frac{n+1}{2}$ so that it uses a total of $n+1$ data points in `y` and `dy` combined.

For each n value, create 2 plots:

- a "normal" plot showing $f(x)$, $\pi_n f(x)$ and $H_n(x)$, overlaid on one another,
- a "semilogy" plot showing $|f(x) - \pi_n f(x)|$ and $|f(x) - H_n(x)|$, overlaid on one another.

All plots should be labeled appropriately (x/y axis labels, titles, legends), and all curves should use different line styles and/or colors.

2. In this problem we'll prove a theorem on the interpolation error from Hermite interpolation using the data $\{(x_k, f^{(j)}(x_k))\}$ for $k = 0, \ldots, n$ and $j = 0, 1$, in a similar manner as done in class for the polynomial interpolation error bound. Here's the theorem:

Hermite Interpolation Error. *Let $\{x_k\}_{k=0}^n$ be distinct nodes in D, and let $x \in D$, where D is the domain of a function $f \in C^{2n+2}(D)$. Let $I_x \subset D$ be the smallest interval containing x and $\{x_k\}_{k=0}^n$. Let $H_{2n+1} \in \mathbb{P}_{2n+1}$ be the Hermite interpolating polynomial satisfying*

$$H_{2n+1}^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, \ldots, n, \quad j = 0, 1.$$

Then the interpolation error at the point x is given by

$$E_{2n+1}(x) = f(x) - H_{2n+1}(x) = \frac{f^{(2n+2)}(\eta)}{(2n+2)!}\,\Omega_{2n+2}(x),$$

where $\eta \in I_x$ and

$$\Omega_{2n+2} = \prod_{k=0}^{n} (x - x_k)^2.$$

Prove this with the following steps; in each step you may assume that you have proven all preceding steps correctly (whether you actually have or not):

(a) Prove that the result is trivially true if $x = x_k$ for any $k = 0, \ldots, n$ (case 1). For case 2 (the rest of this problem), assume that $x \neq x_k$ for all $k = 0, \ldots, n$.

(b) Let $t \in I_x$ be arbitrary, and define

$$G(t) = E_{2n+1}(t) - \Omega_{2n+2}(t)\,\frac{E_{2n+1}(x)}{\Omega_{2n+2}(x)}$$

Prove that $G \in C^{2n+2}(I_x)$, and that G has $(n+2)$ roots in I_x.

(c) Prove that G' has at least $(2n+2)$ different roots. *Hint: $(n+1)$ of these result from step (b), and another $(n+1)$ of these result from how $G(t)$ was constructed. Clearly explain both sets of these, along with why each set must be distinct, ensuring at least $(2n+2)$ roots.*

(d) Prove that $G^{(2n+2)}$ has at least one root, $\eta \in I_x$.

(e) Using the equation $G^{(2n+2)}(\eta) = 0$, prove the final result.