**Math 6316, Spring 2016**
**Due 5 April 2016**

**Homework 6 – Trigonometric Polynomials and Derivative Approximation**

**1.** Define the function space

$$L^2(0, 2\pi) = \left\{ f : (0, 2\pi) \to \mathbb{C} : \|f\|_{L^2(0,2\pi)} < \infty \right\},$$

where

$$\|f\|_{L^2(0,2\pi)} = \sqrt{\langle f,\ f \rangle} \qquad \text{and} \qquad \langle f,\ g \rangle = \int_0^{2\pi} f(x)\, \overline{g(x)}\, \mathrm{d}x.$$

Note: in this function space the Cauchy-Schwarz inequality holds, i.e.

$$\langle f,\ g \rangle \le \|f\|_{L^2(0,2\pi)} \|g\|_{L^2(0,2\pi)}.$$

Let $\phi_k(x) = e^{ikx}$.

(a) Prove that $\langle \phi_k,\ \phi_j \rangle = 0$ for all $k \neq j$.

Let $S_N = \text{Span}\,\{\phi_k : |k| \le N\}$, and define the two function $f$ and $f_N$ such that

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k\, \phi_k(x) \quad \text{and}$$

$$f_N(x) = \sum_{k=-N}^{N} \hat{f}_k\, \phi_k(x).$$

(b) Prove that

$$\|f - f_N\|_{L^2(0,2\pi)} = \min_{g \in S_N} \|f - g\|_{L^2(0,2\pi)}.$$

In both parts (a) and (b), show and explain all main steps.

*Hints: (a) follows from the definitions of $\phi_k$ and $\langle f,\ g \rangle$, along with some basic calculus; (b) then follows a similar argument as we used in section 10.1 for orthogonal polynomial spaces.*

**2.** In class we derived an $\mathcal{O}(h^6)$-accurate compact finite difference formula for approximating the first derivative, $u_i \approx f'(x_i)$:

$$\frac{1}{3}u_{i-1} + u_i + \frac{1}{3}u_{i+1} = \frac{7}{9h}(f_{i+1} - f_{i-1}) + \frac{1}{36h}(f_{i+2} - f_{i-2}), \qquad i = 2, \ldots, n-2.$$

Following that approach, derive similar formulas for approximating the second derivative, $u_i \approx f''(x_i)$, based on the values

$$h^2 \left[ \alpha_{-1} u_{i-1} + u_i + \alpha_1 u_{i+1} \right] = \beta_{-1} f_{i-1} + \beta_0 f_i + \beta_1 f_{i+1}, \qquad i = 1, \ldots, n.$$

Show your steps in this derivation. What is the order of accuracy for this approximation?

Under the assumption that $f$ and its derivatives are periodic on the approximation interval, i.e. $u_0 = u_n$ and $f_0 = f_n$, write the full linear system that encodes your compact finite difference formula, both in the interior and those equations that utilize periodicity.

Matlab: create a function of the form

```
function [ddf,x] = compact_fd(f, a, b, n)
```

that implements your approach above. The inputs to this function are: `f` is the function handle or name to be differentiated, `a` and `b` define the approximation interval $[a, b]$, and `n` is the number of unique nodes to use (since $x_0 = x_n$). The outputs `ddf` and `x` are arrays of length `n`, with `ddf(i)` containing the derivative approximation at `x(i)`, for $i = 1, \ldots, n$.

Write a script named `hw6.m` that uses this function to approximate the second derivative of the function $f(x) = e^{\cos(x)}$ over the interval $[0, 4\pi]$, using $n_j = 5 * 2^j$ for $j = 0, \ldots, 8$. For each $n$ output the maximum error in the approximation over the interval, $\|f'' - f''_{\text{approx}}\|_\infty$, and for $j > 0$, compute the convergence rate of your approximations.