PSet 3

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① 3.4) Convert the system of 2nd order eqns → larger system of first order eqns

$$x''(t) = \frac{-c\,x(t)}{r(t)^3} \qquad y''(t) = \frac{-c\,y(t)}{r(t)^3} \qquad z''(t) = \frac{-c\,z(t)}{r(t)^3}$$

$$c>0, \quad r(t) = [x(t)^2 + y(t)^2 + z(t)^2]^{1/2}$$

have: $x''(t) = f_1(t,x,y,z) \quad y'' = f_2(t,x,y,z) \quad z'' = f_3(t,x,y,z)$

Change of variables:

Let $a_1(t) = x(t) \qquad a_3(t) = y(t) \qquad a_5(t) = z(t)$

$a_2(t) = x'(t) \qquad a_4(t) = y'(t) \qquad a_6(t) = z'(t)$

$$\Rightarrow r(t) = [a_1(t)^2 + a_3(t)^2 + a_5(t)^2]^{1/2}$$

Then our system becomes

$$x''(t) = \frac{-c\,x(t)}{r(t)^3} \Rightarrow a_2'(t) = \frac{-c\,a_1(t)}{r(t)^3} = \frac{-c\,a_1(t)}{[a_1(t)^2 + a_3(t)^2 + a_5(t)^2]^{3/2}}$$

$$= F_1(t, a_1, a_3, a_5)$$

$$y''(t) = \frac{-c\,y(t)}{r(t)^3} \Rightarrow a_4'(t) = \frac{-c\,a_3(t)}{r(t)^3} = \frac{-c\,a_3(t)}{[a_1(t)^2 + a_3(t)^2 + a_5(t)^2]^{3/2}}$$

$$= F_2(t, a_1, a_3, a_5)$$

$$z''(t) = \frac{-c\,z(t)}{r(t)^3} \Rightarrow a_6' = \frac{-c\,a_5(t)}{r(t)^3} = \frac{-c\,a_5(t)}{[a_1(t)^2 + a_3(t)^2 + a_5(t)^2]^{3/2}}$$

$$= F_3(t, a_1, a_3, a_5)$$

$\Rightarrow$ The system becomes

$$\underline{a}'(t) = \begin{vmatrix} a_1'(t) \\ a_2'(t) \\ a_3'(t) \\ a_4'(t) \\ a_5'(t) \\ a_6'(t) \end{vmatrix} = \begin{vmatrix} x'(t) \\ x''(t) \\ y'(t) \\ y''(t) \\ z'(t) \\ z''(t) \end{vmatrix} = \begin{vmatrix} x'(t) \\ f_1(t,x,y,z) \\ y'(t) \\ f_2(t,x,y,z) \\ z'(t) \\ f_3(t,x,y,z) \end{vmatrix} = \begin{vmatrix} a_2(t) \\ F_1(t,a_1,a_3,a_5) \\ a_4(t) \\ F_2(t,a_1,a_3,a_5) \\ a_6(t) \\ F_3(t,a_1,a_3,a_5) \end{vmatrix}$$

$$\Rightarrow \underline{a}'(t) = \begin{vmatrix} a_2(t) \\ F_1(t,a_1,a_3,a_5) \\ a_4(t) \\ F_2(t,a_1,a_3,a_5) \\ a_6(t) \\ F_3(t,a_1,a_3,a_5) \end{vmatrix} \quad \text{with } \underline{a}(t) = \begin{vmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \\ a_5(t) \\ a_6(t) \end{vmatrix}, \; a_0 = \begin{vmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ z_0' \end{vmatrix}$$

② - ODE system

$$y_1' = -3y_1 + y_2 - e^{-2t} \qquad y_1(0) = 2$$

$$y_2' = y_1 - 3y_2 + e^{-t} \qquad y_2(0) = 1$$

$$t \in [0, 3]$$

- analytic soln:

$$y' = \begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix} y + \begin{vmatrix} -e^{-2t} \\ e^{-t} \end{vmatrix} = Ay(t) + G(t)$$

This is a matrix exponential type problem

- eigen value decompose A

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)(-3-\lambda) - 1$$

$$= 9 + 3\lambda + 3\lambda + \lambda^2 - 1$$

$$= \lambda^2 + 6\lambda + 8$$

$$= (\lambda + 4)(\lambda + 2) = 0$$

eigenvalues $\lambda = -4$ or -2

$$\lambda_{-4} : (A - \lambda_{-4} I) = \begin{vmatrix} -3-(-4) & 1 \\ 1 & -3-(-4) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{matrix}$$

$$\Rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = -x_1 \end{matrix}$$

$$\Rightarrow v_{-4} = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

$$\lambda_2 : (A - \lambda_{-2} I) = \begin{vmatrix} -3-(-2) & 1 \\ 1 & -3-(-2) \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \rightarrow \begin{matrix} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{matrix}$$

$$\Rightarrow \begin{matrix} x_2 = x_1 \\ x_1 = x_2 \end{matrix}$$

$$\Rightarrow v_{-2} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$A = V \Lambda V^{-1} = P \Lambda P^T$ where P is the normalization of V from real and symmetric A

$$\Lambda = \begin{vmatrix} \lambda_{-4} & 0 \\ 0 & \lambda_{-2} \end{vmatrix} = \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} \qquad V = \begin{vmatrix} \underline{v}_{-4}, \underline{v}_{-2} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\Rightarrow P = \frac{1}{\sqrt{2}} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$P^T = P$$

substitute $y = P\underline{w}$ into $y' = Ay + g$

$$(P\underline{w})' = A(P\underline{w}) + g(t)$$

$$P^{-1}\left[P\underline{w} = AP\underline{w} + g(t) \right] \quad \text{(on the left)}$$

$$\underline{w} = P^{-1}AP\underline{w} + P^{-1}g(t)$$

$$= P^T A P \underline{w} + P^{-1} g(t)$$

$$= \Lambda \underline{w} + P^{-1} g(t)$$

$$\begin{vmatrix} -e^{-2t} \\ e^{-t} \end{vmatrix}$$

$$\begin{vmatrix} w_1 \\ w_2 \end{vmatrix}' = \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} + \frac{1}{\sqrt{2}} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} -e^{-2t} \\ e^{-t} \end{vmatrix}$$

$$\begin{vmatrix} w_1 \\ w_2 \end{vmatrix}' = \begin{vmatrix} -4w_1 + \frac{1}{\sqrt{2}}(e^{-2t} + e^{-t}) \\ -2w_2 + \frac{1}{\sqrt{2}}(-e^{-2t} + e^{-t}) \end{vmatrix}$$

2 uncoupled, linear homog odes

$$w_1' = -4w_1 + \frac{1}{\sqrt{2}}(e^{-2t} + e^{-t})$$

$$\Rightarrow w_1' + 4w_1 = \frac{1}{\sqrt{2}}(e^{-2t} + e^{-t}) \quad \text{integrating factor } e^{\int 4\,dt} = e^{4t}$$

$(4-2)t = 2t \qquad (4-1)t = 3t$

$$\Rightarrow e^{4t} w_1' + 4e^{4t} w_1 = \frac{1}{\sqrt{2}} e^{4t}(e^{-2t} + e^{-t})$$

$$\Rightarrow \int (e^{4t} w_1)'\,dt = \frac{1}{\sqrt{2}} \int e^{2t} + e^{3t}\,dt$$

$$\Rightarrow \left[e^{4t} w_1 = \frac{1}{2\sqrt{2}} e^{2t} + \frac{1}{3\sqrt{2}} e^{3t} + c\right] e^{-4t}$$

$$\Rightarrow w_1 = \frac{1}{2\sqrt{2}} e^{-2t} + \frac{1}{3\sqrt{2}} e^{-t} + c_1 e^{-4t}$$

$$w_2' = -2w_2 + \frac{1}{\sqrt{2}}(-e^{-2t} + e^{-t})$$

$$\Rightarrow w_2' + 2w_2 = \frac{1}{\sqrt{2}}(-e^{-2t} + e^{-t}) \quad \text{integrating factor} \quad e^{\int 2dt} = e^{2t}$$

$$\Rightarrow e^{2t} w_2' + 2e^{2t} w_2 = \frac{1}{\sqrt{2}} e^{2t}(-e^{-2t} + e^{-t}) \qquad 2t - 2t = 0 \quad 2t - t = t$$

$$\Rightarrow \int (e^{2t} w_2)' \, dt = \frac{1}{\sqrt{2}} \int (-1 + e^{t}) \, dt$$

$$\Rightarrow \left[e^{2t} w_2 = \frac{1}{\sqrt{2}}(e^{t} - t + c)\right] e^{-2t} \qquad -2t + t = -t$$

$$\Rightarrow w_2 = \frac{1}{\sqrt{2}}(e^{-t} - te^{-2t}) + c_2 e^{-2t} \qquad w_1 = \frac{1}{2\sqrt{2}} e^{-2t} + \frac{1}{3\sqrt{2}} e^{-t} + c_1 e^{-4t}$$

Then, $y = P\underline{w}$

$$\Rightarrow y = \frac{1}{\sqrt{2}} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} -w_1 + w_2 \\ w_1 + w_2 \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{e^{-2t}}{4} - \frac{e^{-t}}{6} - \frac{c_1 e^{-4t}}{\sqrt{2}} + e^{-2t}\left(\frac{c_2}{\sqrt{2}} - \frac{t}{2}\right) + \frac{e^{-t}}{2} \\ \frac{e^{-2t}}{4} + \frac{e^{-t}}{6} + \frac{c_1 e^{-4t}}{\sqrt{2}} + e^{-2t}\left(\frac{c_2}{\sqrt{2}} - \frac{t}{2}\right) + \frac{e^{-t}}{2} \end{vmatrix}$$

$$= \begin{vmatrix} e^{-t}\left(\frac{1}{2} - \frac{1}{6}\right) + e^{-2t}\left(\frac{c_2}{\sqrt{2}} - \frac{1}{4} - \frac{t}{2}\right) - \frac{c_1 e^{-4t}}{\sqrt{2}} \\ e^{-t}\left(\frac{1}{2} + \frac{1}{6}\right) + e^{-2t}\left(\frac{c_2}{\sqrt{2}} + \frac{1}{4} - \frac{t}{2}\right) + \frac{c_1 e^{-4t}}{\sqrt{2}} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{3}e^{-t} + e^{-2t}\left(\frac{c_2}{\sqrt{2}} - \frac{1}{4} - \frac{t}{2}\right) - \frac{c_1}{\sqrt{2}} e^{-4t} \\ \frac{2}{3}e^{-t} + e^{-2t}\left(\frac{c_2}{\sqrt{2}} + \frac{1}{4} - \frac{t}{2}\right) + \frac{c_1}{\sqrt{2}} e^{-4t} \end{vmatrix}$$

$$y_1(0) = 2 \Rightarrow \frac{1}{3} + \frac{c_2}{\sqrt{2}} - \frac{1}{4} - 0 - \frac{c_1}{\sqrt{2}} = 2$$

$$\Rightarrow \frac{c_2}{\sqrt{2}} = 2 - \frac{1}{3} + \frac{1}{4} + \frac{c_1}{\sqrt{2}}$$

$$\Rightarrow \left[\frac{c_2}{\sqrt{2}} = \frac{23}{12} + \frac{c_1}{\sqrt{2}}\right]\sqrt{2}$$

$$\Rightarrow c_2 = \frac{23}{12}\sqrt{2} + c_1$$

$$y_2(0) = 1 \Rightarrow \frac{2}{3} + \frac{c_2}{\sqrt{2}} + \frac{1}{4} - \frac{0}{2} + \frac{c_1}{\sqrt{2}} = 1$$

$$\Rightarrow \frac{2}{3} + \frac{1}{\sqrt{2}}\left(\frac{23}{12}\sqrt{2} + c_1\right) + \frac{1}{4} + \frac{c_1}{\sqrt{2}} = 1$$

$$\Rightarrow \frac{2}{3} + \frac{23}{12} + \frac{1}{4} + \frac{2c_1}{\sqrt{2}} = 1$$

$$\Rightarrow \frac{2}{\sqrt{2}} c_1 = 1 - \frac{2}{3} - \frac{23}{12} - \frac{1}{4}$$

$$\Rightarrow \left[\frac{2}{\sqrt{2}} c_1 = \frac{-11}{6}\right]\frac{\sqrt{2}}{2}$$

$$\Rightarrow c_1 = \frac{-11\sqrt{2}}{12} \quad \Rightarrow c_2 = \frac{23}{12}\sqrt{2} - \frac{11\sqrt{2}}{12} = \frac{12}{12}\sqrt{2} = \sqrt{2}$$

$$y = \begin{vmatrix} \frac{1}{3}e^{-t} + e^{-2t}\left(1 - \frac{1}{4} - \frac{t}{2}\right) - -\frac{11}{12} e^{-4t} \\ \frac{2}{3}e^{-t} + e^{-2t}\left(1 + \frac{1}{4} - \frac{t}{2}\right) - \frac{11}{12} e^{-4t} \end{vmatrix}$$

$$y = \begin{vmatrix} \frac{1}{3}e^{-t} + e^{-2t}\left(\frac{3}{4} - \frac{t}{2}\right) + \frac{11}{12} e^{-4t} \\ \frac{2}{3}e^{-t} + e^{-2t}\left(\frac{5}{4} - \frac{t}{2}\right) - \frac{11}{12} e^{-4t} \end{vmatrix}$$

③ single fixed point iteration

$$y_{n+1} = y_n + hf(t_{n+1}, y_n + hf(t_n, y_n))$$

• stability region: Dahlquist test problem $\quad y' = \lambda y \quad y(0) = 1$

$\rightarrow f(t,y) = \lambda y$

$\Rightarrow y_n = y_{n-1} + hf(t_n, y_{n-1} + hf(t_{n-1}, y_{n-1}))$

$\Rightarrow y_n = y_{n-1} + hf(t_n, y_{n-1} + h \cdot \lambda y_{n-1})$ plugging $t_n, y_n \rightarrow f(t,y)$

$= y_{n-1} + hf(t_n, (1+h\lambda) y_{n-1})$

$= y_{n-1} + h \cdot \lambda[(1+h\lambda) y_{n-1}]$ plugging $t_{n+1}, (1+h\lambda)y_n \rightarrow f(t,y)$

$= y_{n-1} + h\lambda y_{n-1} + h^2\lambda^2 y_{n-1}$

$= (h^2\lambda^2 + h\lambda + 1) y_{n-1}$

$= (h^2\lambda^2 + h\lambda + 1)[(h^2\lambda^2 + h\lambda + 1) y_{n-2}]$

$= (h^2\lambda^2 + h\lambda + 1)^2 y_{n-2}$

...

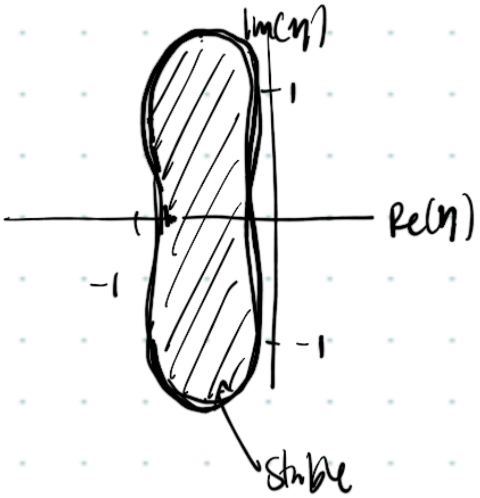
$= (h^2\lambda^2 + h\lambda + 1)^n y_0$

$\Rightarrow y_n = (h^2\lambda^2 + h\lambda + 1)^n$

$\Rightarrow \|y_n\| = \|h^2\lambda^2 + h\lambda + 1\|^n$ $\quad R(h\lambda) = h^2\lambda^2 + h\lambda + 1$ or $R(\eta) = (\eta^2 + \eta + 1)$

$\Rightarrow \|y_n\| \le 1$ iff $\|h^2\lambda^2 + h\lambda + 1\| \le 1$

Stability Region: $\{\eta \in \mathbb{C} \mid \|R(\eta)\| \le 1\}$ plotted using maple



- Two fixed point iterations:

$$y_{n+1}^{(1)} = y_n + hf(t_{n+1}, y_{n+1}^{(0)}) = y_n + hf(t_{n+1}, y_n)$$

$$y_{n+1}^{(2)} = y_n + hf(t_{n+1}, y_{n+1}^{(1)})$$

$$y_{n+1}^{(3)} = y_n + hf(t_{n+1}, y_{n+1}^{(2)})$$

$$\rightarrow y_{n+1} = y_n + hf(t_{n+1}, y_n + hf(t_{n+1}, y_n + hf(t_{n+1}, y_n)))$$

∘ Stability Region: Dahlquist test problem $y' = f(t,y) = \lambda y \quad y(0) = 1$

$$\Rightarrow y_n = y_{n-1} + hf(t_{n+1}, y_{n-1} + hf(t_{n+1}, y_{n-1} + hf(t_{n+1}, y_{n-1})))$$

$$= y_{n-1} + hf(t_{n+1}, y_{n-1} + hf(t_{n+1}, y_{n-1} + h\lambda y_{n-1}))$$

$$= y_{n-1} + hf(t_{n+1}, y_{n-1} + h\lambda(y_{n-1} + h\lambda y_{n-1}))$$

$$= y_{n-1} + h\lambda(y_{n-1} + h\lambda(y_{n-1} + h\lambda y_{n-1}))$$

$$= y_{n-1} + h\lambda y_{n-1} + h^2\lambda^2 y_{n-1} + h^3\lambda^3 y_{n-1}$$

$$y_n = (h^3\lambda^3 + h^2\lambda^2 + h\lambda + 1) y_{n-1}$$

$= (h^3\lambda^3 + h^2\lambda^2 + h\lambda + 1)[(h^3\lambda^3 + h^2\lambda^2 + h\lambda + 1) y_{n-2}]$

$= (h^3\lambda^3 + h^2\lambda^2 + h\lambda + 1)^2 y_{n-2}$

$\circ \circ \circ$

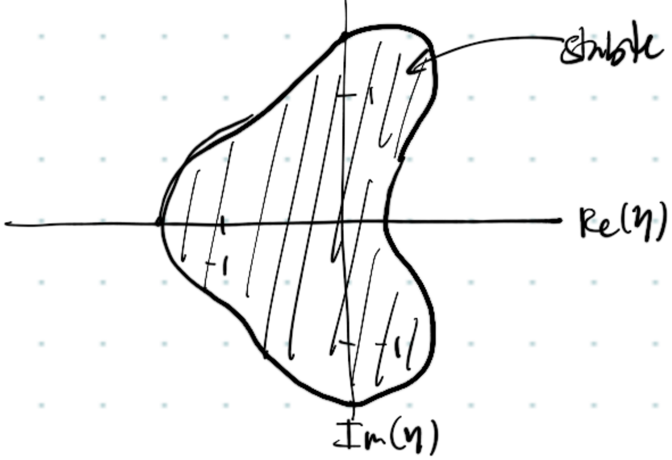
$= (h^3\lambda^3 + h^2\lambda^2 + h\lambda + 1)^n y_0$

$= (h^3\lambda^3 + h^2\lambda^2 + h\lambda + 1)^n$ by $y_0 = y(0) = 1$

$\Rightarrow R(h\lambda) = (h^3\lambda^3 + h^2\lambda^2 + h\lambda + 1)^n$

$\Leftrightarrow R(\eta) = (\eta^3 + \eta^2 + \eta + 1)^n$

Stability Region: $\{\eta \in \mathbb{C} \mid \|R(\eta)\| \leq 1\}$ plotted w/ maple



I disagree with the recommendation of the book. A stiff problem has very negative λ, which will make $h\lambda = \eta$ stretch far into the left half plane of the stability region. Neither of these stability regions cover much of the left half plane, so I dont think it makes much sense to use either of these methods for very stiff problems.

④ -generalized midpoint:

$$y_{n+1} = y_n + hf(t_n + \theta h, (1-\theta)y_n + \theta y_{n+1})$$

• Stability Region → Dahlquist test problem $y' = f(t,y) = h\lambda$ $y(0)=1$

$$\Rightarrow y_n = y_{n-1} + hf(t_{n-1} + \theta h, (1-\theta)y_{n-1} + \theta y_n)$$

$$= y_{n-1} + h\cdot\lambda\cdot((1-\theta)y_{n-1} + \theta y_n)$$

$$= y_{n-1} + h\lambda(1-\theta)y_{n-1} + h\lambda\theta y_n$$

$$y_n = y_{n-1} + h\lambda y_n - h\lambda\theta y_{n-1} + h\lambda\theta y_n$$

$$\Rightarrow y_n - h\lambda\theta y_n = (1 + h\lambda - h\lambda\theta)y_{n-1}$$

$$\Rightarrow (1-h\lambda\theta)y_n = (1+h\lambda - h\lambda\theta)y_{n-1}$$

$$\Rightarrow y_n = \frac{(1+h\lambda - h\lambda\theta)}{(1-h\lambda\theta)} y_{n-1}$$

$$\Rightarrow y_n = \frac{(1+h\lambda-h\lambda\theta)}{(1-\theta h\lambda)} \cdot \left[\frac{(1+h\lambda-h\lambda\theta)}{(1-\theta h\lambda)} y_{n-2}\right]$$

$$= \frac{(1+h\lambda-h\lambda\theta)^2}{(1-h\lambda\theta)^2} y_{n-2}$$

$$\cdots$$

$$= \frac{(1+h\lambda-h\lambda\theta)^n}{(1-h\lambda\theta)^n} y_0$$

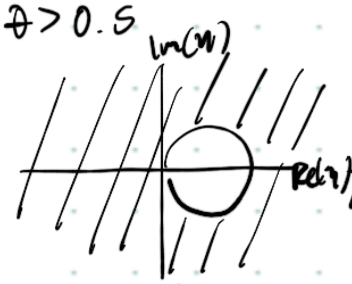
$$= \left[\frac{(1+h\lambda-h\lambda\theta)}{(1-h\lambda\theta)}\right]^n \quad \text{by } y_0 = y(0) = 1$$

$$= \left[\frac{(1+(1-\theta)h\lambda)}{(1-h\lambda\theta)}\right]^n$$

$$R(\eta, \theta) = \left[\frac{1 + (1-\theta)\eta}{1 - h\lambda\theta} \right] = \left[\frac{1 + \eta - \theta\eta}{1 - h\lambda\theta} \right]$$

- Stability Region: $\{ \eta \in \mathbb{C} \mid \| R(\eta) \| \leq 1 \}$

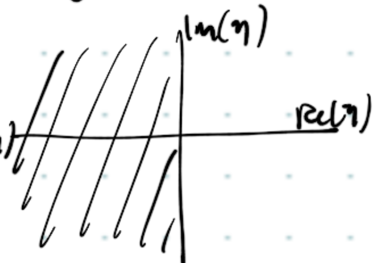
$\theta > 0.5$



the radius of the circle shrinks as θ increases, but regardless the left half plane is stable.

A-stable

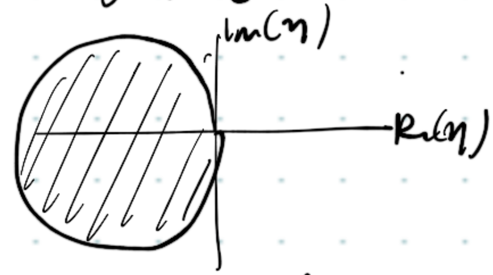
$\theta = 0.5$



left half plane is stable

A-stable

$\theta < 0.5$



The radius of the circle shrinks as θ decreases. when θ is small enough $(\theta < -10)$ the circle disappears, and the whole plane becomes stable

$\Rightarrow$ Not A-stable for $0.5 < \theta < -10$

- generalized trapezoid

$$y_{n+1} = y_n + (1-\theta) h f(t_n, y_n) + \theta h f(t_{n+1}, y_{n+1})$$

- Stability Region $\rightarrow$ Dahlquist test problem

$$\Rightarrow y_n = y_{n-1} + (1-\theta) h f(t_{n-1}, y_{n-1}) + \theta h f(t_n, y_n)$$

$$= y_{n-1} + (1-\theta) h \cdot \lambda \cdot y_{n-1} + \theta h \cdot \lambda \cdot y_n$$

$$y_n = (1 + (1-\theta) h\lambda) y_{n-1} + \theta h \lambda y_n$$

$$\Rightarrow y_n - \theta h \lambda y_n = (1 + (1-\theta) h\lambda) y_{n-1}$$

$$\Rightarrow (1 - \theta h \lambda) y_n = (1 + (1-\theta) h\lambda) y_{n-1}$$

$$\Rightarrow y_n = \frac{(1 + (1-\theta) h\lambda)}{(1 - \theta h\lambda)} y_{n-1}$$

$$= \frac{(1 + (1-\theta) h\lambda)}{(1 - \theta h\lambda)} \left[\frac{(1 + (1-\theta) h\lambda)}{(1 - \theta h\lambda)} y_{n-2} \right]$$

$$= \left[\frac{(1+(1-\theta)h\lambda)}{(1-\theta h\lambda)} \right]^2 y_{n-2}$$

$$\cdots$$

$$= \left[\frac{(1+(1-\theta)h\lambda)}{(1-\theta h\lambda)} \right]^n y_0$$

$$= \left[\frac{(1+(1-\theta)h\lambda)}{(1-\theta h\lambda)} \right]^n \quad \text{by } y_0 = y(0) = 1$$

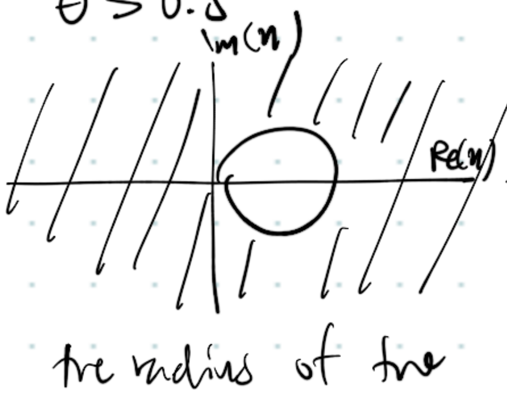
$$R(\eta, \theta) = \left[\frac{(1+(1-\theta)\eta)}{(1-\theta\eta)} \right] = \left[\frac{1+\eta-\theta\eta}{1-\theta\eta} \right]$$

∘ Stability Region : $\{\eta \in \mathbb{C} \mid \|R(\eta)\| \le 1\}$

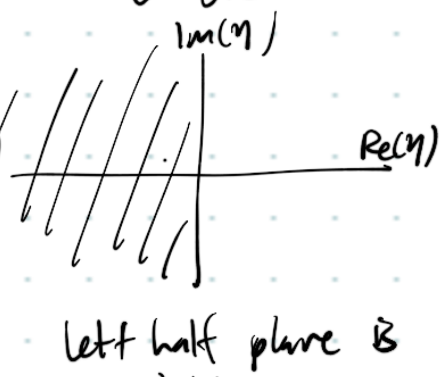
Same as above. Same residual → same graphs

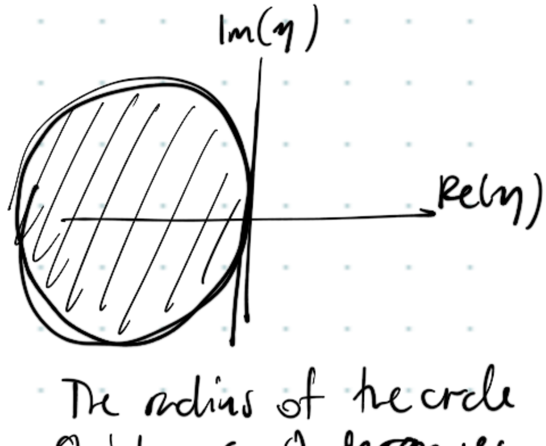
The method is A-stable for $\theta \ge 0.5$, and $\theta < -10$ by the same reasoning as General midpoint

$\theta > 0.5$



$\theta = 0.5$





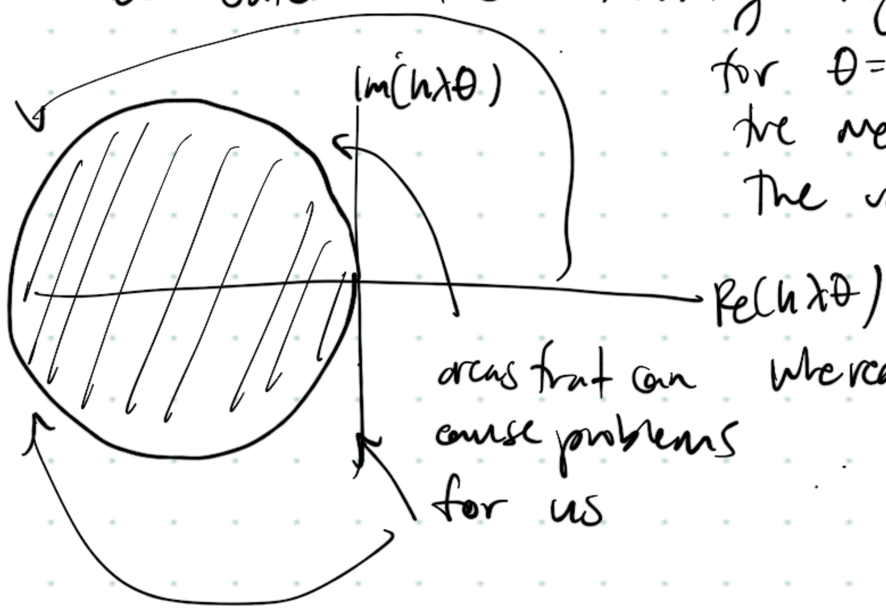
the radius of the circle shrinks w/ increasing θ, but either way the left half plane is stable for $\forall \theta > 0.5$
→ A-stable

left half plane is stable.
→ A-stable

The radius of the circle shrinks as θ decreases, until the circle disappears and the whole plane becomes stable (this happens for $\theta < -10$)
→ not A-stable for $0.5 < \theta < -10$

5. For $\theta \geq 0.5$, the method is stable and convergent, which is supported by the graphs of our stability region.

For $\theta = 0.45$, the method is no longer A-stable, but a large portion of the left half plane is stable. However, for a stiff enough problem, $\theta h \lambda$ can be outside the stability region. This explains why



for $\theta = 0.45$, $\lambda = -20{,}000$, the method appears unstable. The very negative λ value has pushed us out of the stability region. Whereas for the tests with $\theta \geq 0.5$, $h\lambda\theta$ was always in the stability region, since the whole left half plane was stable.

$restart;$
$with(plots);$

[*animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *densityplot*, *display*, *dualaxisplot*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *intersectplot*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *rootlocus*, *semilogplot*, *setcolors*, *setoptions*, *setoptions3d*, *shadebetween*, *spacecurve*, *sparsematrixplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*] **(1)**

$OneStep := y\_new - y\_old - h \cdot f(x\_new, y\_old + h \cdot f(x\_old, y\_old)) = 0;$

$$y\_new - y\_old - h\, f(x\_new, y\_old + h\, f(x\_old, y\_old)) = 0 \quad \textbf{(2)}$$

$f := (x, y) \rightarrow \mathrm{lambda} * y;$

$$(x, y) \rightarrow \lambda\, y \quad \textbf{(3)}$$

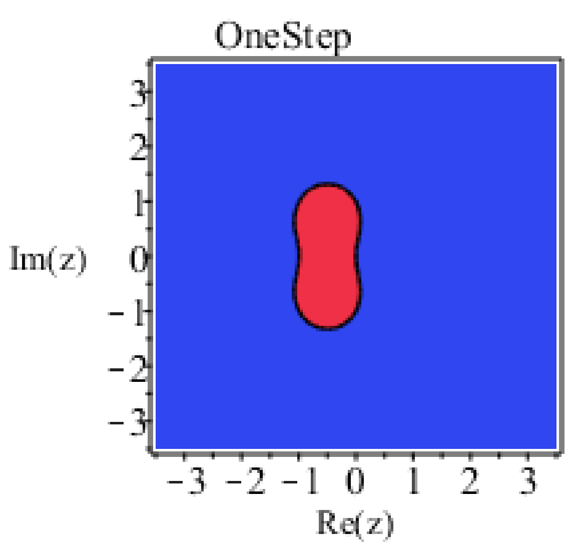
$OneStep := factor(isolate(OneStep, y\_new) / y\_old);$

$$\frac{y\_new}{y\_old} = h^2 \lambda^2 + h\,\lambda + 1 \quad \textbf{(4)}$$

$stencils := [OneStep];$
$stencil\_names := [\text{"OneStep"}];$

$$\left[\frac{y\_new}{y\_old} = h^2 \lambda^2 + h\,\lambda + 1\right]$$
$$[\text{"OneStep"}] \quad \textbf{(5)}$$

```
for i from 1 to 1 do:
 contourplot(simplify(subs(lambda=z/h, z=x+I*y, abs(rhs(stencils[i])))), x=-3.5..3.5, y=-3.5..3.5, contours=[1],
 grid=[50, 50], axes=boxed, filled=true, title=stencil_names[i],
labels=["Re(z)", "Im(z)"]);
 od;
```



$TwoStep := y\_new - y\_old - h \cdot f(x\_new, y\_old + h \cdot f(x\_old, y\_old + h \cdot f(x\_old, y\_old))) =$

0;

$$y\_new - y\_old - h\,\lambda\,\big(y\_old + h\,\lambda\,(h\,\lambda\,y\_old + y\_old)\big) = 0 \tag{6}$$

$f := (x, y) \rightarrow \text{lambda} * y;$

$$(x, y) \rightarrow \lambda\,y \tag{7}$$

$TwoStep := factor(isolate(TwoStep, y\_new)\,/\,y\_old);$

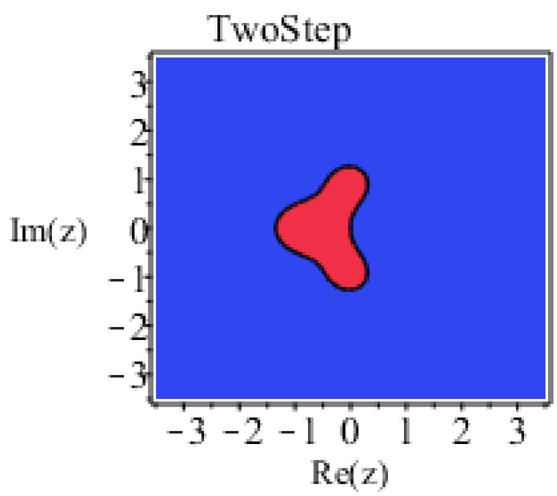
$$\frac{y\_new}{y\_old} = (h\,\lambda + 1)\,\big(h^2\,\lambda^2 + 1\big) \tag{8}$$

$stencils := [TwoStep];$
$stencil\_names := [\text{"TwoStep"}];$

$$\left[\frac{y\_new}{y\_old} = (h\,\lambda + 1)\,\big(h^2\,\lambda^2 + 1\big)\right]$$

$$[\text{"TwoStep"}] \tag{9}$$

```
for i from 1 to 1 do:
 contourplot(simplify(subs(lambda = z/h, z = x + I*y, abs(rhs(stencils
[i])))), x =-3.5 ..3.5, y =-3.5 ..3.5, contours = [1],
 grid = [50, 50], axes = boxed, filled = true, title = stencil_names[i],
labels = ["Re(z)", "Im(z)"]);
 od;
```



$restart;$
$with(plots);$

$$[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra\_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] \quad (1)$$

$GenMidm20 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (-20), (1 - (-20)) \cdot y\_old + (-20) \cdot y\_new) = 0;$
$GenMidm10 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (-10), (1 - (-10)) \cdot y\_old + (-10) \cdot y\_new) = 0;$
$GenMidm1 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (-1), (1 - (-1)) \cdot y\_old + (-1) \cdot y\_new) = 0;$
$GenMidm5 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (-.5), (1 - (-.5)) \cdot y\_old + (-.5) \cdot y\_new) = 0;$
$GenMid0 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (0), (1 - (0)) \cdot y\_old + (0) \cdot y\_new) = 0;$
$GenMid45 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (.45), (1 - (.45)) \cdot y\_old + (.45) \cdot y\_new) = 0;$
$GenMid5 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (.5), (1 - (.5)) \cdot y\_old + (.5) \cdot y\_new) = 0;$
$GenMid1 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (1), (1 - (1)) \cdot y\_old + (1) \cdot y\_new) = 0;$
$GenMid10 := y\_new - y\_old - h \cdot f(x\_old + h \cdot (10), (1 - (10)) \cdot y\_old + (10) \cdot y\_new) = 0;$

$$y\_new - y\_old - h\,\lambda\,(21\,y\_old - 20\,y\_new) = 0$$
$$y\_new - y\_old - h\,\lambda\,(11\,y\_old - 10\,y\_new) = 0$$
$$y\_new - y\_old - h\,\lambda\,(2\,y\_old - y\_new) = 0$$
$$y\_new - y\_old - h\,\lambda\,(1.5\,y\_old - 0.5\,y\_new) = 0$$
$$-h\,\lambda\,y\_old + y\_new - y\_old = 0$$
$$y\_new - y\_old - h\,\lambda\,(0.55\,y\_old + 0.45\,y\_new) = 0$$
$$y\_new - y\_old - h\,\lambda\,(0.5\,y\_old + 0.5\,y\_new) = 0$$
$$-h\,\lambda\,y\_new + y\_new - y\_old = 0$$
$$y\_new - y\_old - h\,\lambda\,(-9\,y\_old + 10\,y\_new) = 0 \quad (2)$$

$f := (x, y) \rightarrow \text{lambda} * y;$

$$(x, y) \rightarrow \lambda\,y \quad (3)$$

$GenMidm20 := factor(isolate(GenMidm20, y\_new) / y\_old);$
$GenMidm10 := factor(isolate(GenMidm10, y\_new) / y\_old);$
$GenMidm1 := factor(isolate(GenMidm1, y\_new) / y\_old);$

```
GenMidm5 := factor(isolate(GenMidm5, y_new) / y_old) ;
GenMid0 := factor(isolate(GenMid0, y_new) / y_old) ;
GenMid45 := factor(isolate(GenMid45, y_new) / y_old) ;
GenMid5 := factor(isolate(GenMid5, y_new) / y_old) ;
GenMid1 := factor(isolate(GenMid1, y_new) / y_old) ;
GenMid10 := factor(isolate(GenMid10, y_new) / y_old) ;
```

$$\frac{y\_new}{y\_old} = \frac{21\,h\,\lambda + 1}{20\,h\,\lambda + 1}$$

$$\frac{y\_new}{y\_old} = \frac{11\,h\,\lambda + 1}{10\,h\,\lambda + 1}$$

$$\frac{y\_new}{y\_old} = \frac{2\,h\,\lambda + 1}{h\,\lambda + 1}$$

$$\frac{y\_new}{y\_old} = \frac{1. + 1.500000000\,h\,\lambda}{1. + 0.5000000000\,h\,\lambda}$$

$$\frac{y\_new}{y\_old} = h\,\lambda + 1$$

$$\frac{y\_new}{y\_old} = -\frac{1. + 0.5500000000\,h\,\lambda}{-1. + 0.4500000000\,h\,\lambda}$$

$$\frac{y\_new}{y\_old} = -\frac{1. + 0.5000000000\,h\,\lambda}{-1. + 0.5000000000\,h\,\lambda}$$

$$\frac{y\_new}{y\_old} = -\frac{1}{h\,\lambda - 1}$$

$$\frac{y\_new}{y\_old} = \frac{9\,h\,\lambda - 1}{10\,h\,\lambda - 1} \tag{4}$$

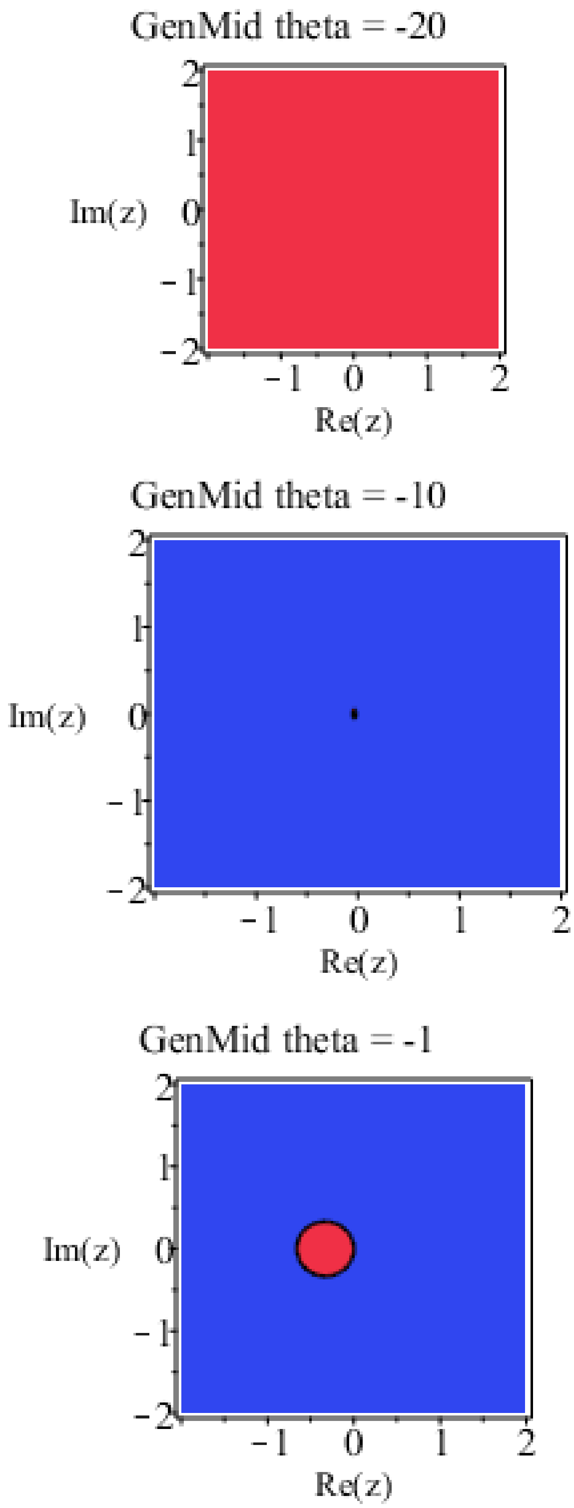
```
stencils := [GenMidm20, GenMidm10, GenMidm1, GenMidm5, GenMid0, GenMid45, GenMid5,
    GenMid1, GenMid10];
stencil_names := ["GenMid theta = -20", "GenMid theta = -10", "GenMid theta = -1",
    "GenMid theta = -.5", "GenMid theta = 0", "GenMid theta = .45", "GenMid theta = .5",
    "GenMid theta = 1", "GenMid theta = 10"];
```

$$\left[\frac{y\_new}{y\_old} = \frac{21\,h\,\lambda + 1}{20\,h\,\lambda + 1}, \frac{y\_new}{y\_old} = \frac{11\,h\,\lambda + 1}{10\,h\,\lambda + 1}, \frac{y\_new}{y\_old} = \frac{2\,h\,\lambda + 1}{h\,\lambda + 1}, \frac{y\_new}{y\_old} = \frac{1. + 1.500000000\,h\,\lambda}{1. + 0.5000000000\,h\,\lambda}, \frac{y\_new}{y\_old} = h\,\lambda + 1, \frac{y\_new}{y\_old} = -\frac{1. + 0.5500000000\,h\,\lambda}{-1. + 0.4500000000\,h\,\lambda}, \frac{y\_new}{y\_old} = -\frac{1. + 0.5000000000\,h\,\lambda}{-1. + 0.5000000000\,h\,\lambda}, \frac{y\_new}{y\_old} = -\frac{1}{h\,\lambda - 1}, \frac{y\_new}{y\_old} = \frac{9\,h\,\lambda - 1}{10\,h\,\lambda - 1}\right]$$

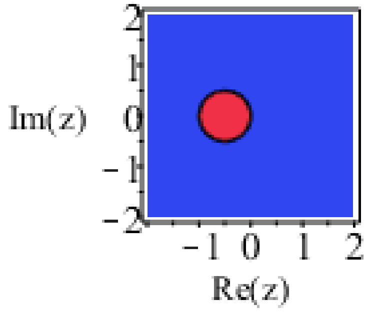
["GenMid theta = -20", "GenMid theta = -10", "GenMid theta = -1", "GenMid theta = -.5", "GenMid theta = 0", "GenMid theta = .45", "GenMid theta = .5", "GenMid theta = 1", "GenMid theta = 10"] **(5)**

for *i* **from** 1 **to** 9 **do**:

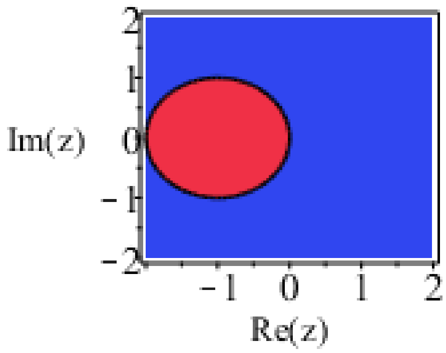
```
contourplot(simplify(subs(lambda = z/h, z = x + I* y, abs(rhs(stencils
[i])))), x =-2 ..2, y =-2 ..2, contours = [1],
grid = [50, 50], axes = boxed, filled = true, title = stencil_names[i],
labels = ["Re(z)", "Im(z)"]);
od;
```



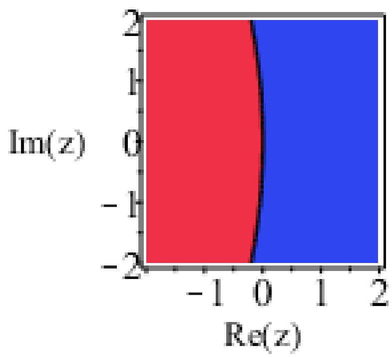
GenMid theta = -.5



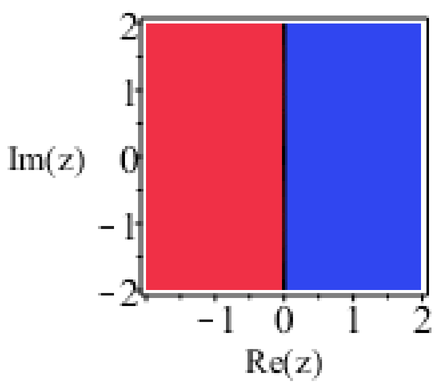
GenMid theta = 0

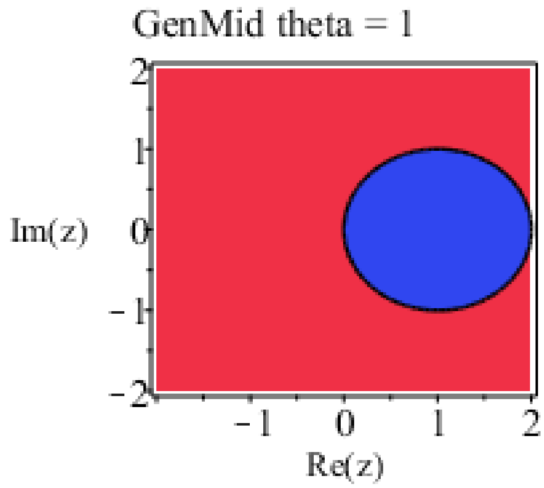


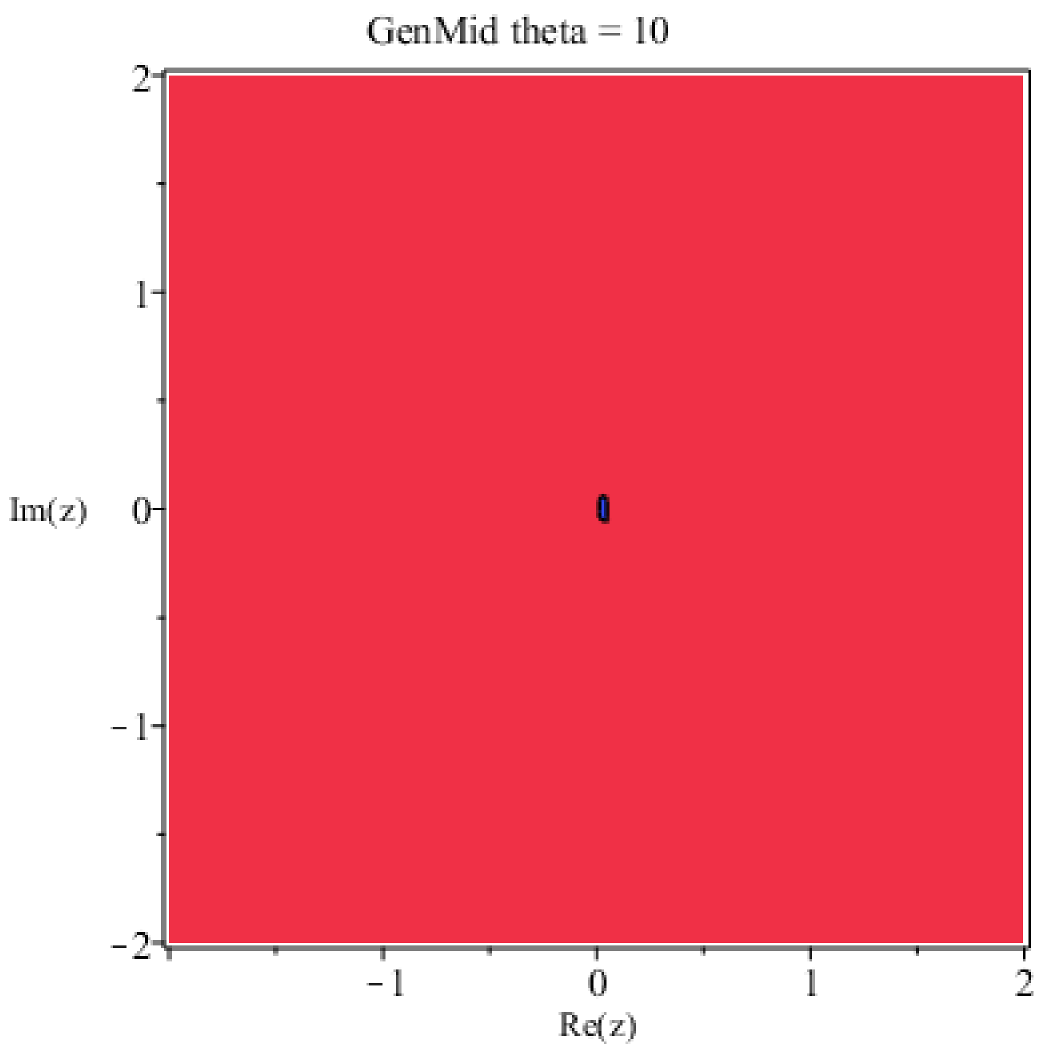
GenMid theta = .45



GenMid theta = .5







$GenTrapm20 := y\_new - y\_old - (1 - (-20)) \cdot h \cdot f(x\_old, y\_old) + (-20) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrapm10 := y\_new - y\_old - (1 - (-10)) \cdot h \cdot f(x\_old, y\_old) + (-10) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrapm1 := y\_new - y\_old - (1 - (-1)) \cdot h \cdot f(x\_old, y\_old) + (-1) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrapm5 := y\_new - y\_old - (1 - (-.5)) \cdot h \cdot f(x\_old, y\_old) + (-.5) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrap0 := y\_new - y\_old - (1 - (0)) \cdot h \cdot f(x\_old, y\_old) + (0) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrap45 := y\_new - y\_old - (1 - (.45)) \cdot h \cdot f(x\_old, y\_old) + (.45) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrap5 := y\_new - y\_old - (1 - (.5)) \cdot h \cdot f(x\_old, y\_old) + (.5) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrap1 := y\_new - y\_old - (1 - (1)) \cdot h \cdot f(x\_old, y\_old) + (1) \cdot h \cdot f(x\_new, y\_new) = 0;$

$GenTrap10 := y\_new - y\_old - (1 - (10)) \cdot h \cdot f(x\_old, y\_old) + (10) \cdot h \cdot f(x\_new, y\_new) = 0;$

$$-20\,h\,\lambda\,y\_new - 21\,h\,\lambda\,y\_old + y\_new - y\_old = 0$$

$$-10\,h\,\lambda\,y\_new - 11\,h\,\lambda\,y\_old + y\_new - y\_old = 0$$

$$-h\,\lambda\,y\_new - 2\,h\,\lambda\,y\_old + y\_new - y\_old = 0$$

$$y\_new - y\_old - 1.5\,h\,\lambda\,y\_old - 0.5\,h\,\lambda\,y\_new = 0$$

$$-h\,\lambda\,y\_old + y\_new - y\_old = 0$$

$$y\_new - y\_old - 0.55\,h\,\lambda\,y\_old + 0.45\,h\,\lambda\,y\_new = 0$$

$$y\_new - y\_old - 0.5\,h\,\lambda\,y\_old + 0.5\,h\,\lambda\,y\_new = 0$$

$$h\,\lambda\,y\_new + y\_new - y\_old = 0$$

$$10\,h\,\lambda\,y\_new + 9\,h\,\lambda\,y\_old + y\_new - y\_old = 0 \quad \textbf{(6)}$$

$f := (x, y) \rightarrow \text{lambda} * y;$

$$(x, y) \rightarrow \lambda\,y \quad \textbf{(7)}$$

$GenTrapm20 := factor(isolate(GenMidm20, y\_new) / y\_old);$
$GenTrapm10 := factor(isolate(GenMidm10, y\_new) / y\_old);$
$GenTrapm1 := factor(isolate(GenMidm1, y\_new) / y\_old);$
$GenTrapm5 := factor(isolate(GenMidm5, y\_new) / y\_old);$
$GenTrap0 := factor(isolate(GenMid0, y\_new) / y\_old);$
$GenTrap45 := factor(isolate(GenMid45, y\_new) / y\_old);$
$GenTrap5 := factor(isolate(GenMid5, y\_new) / y\_old);$
$GenTrap1 := factor(isolate(GenMid1, y\_new) / y\_old);$
$GenTrap10 := factor(isolate(GenMid10, y\_new) / y\_old);$

$$\frac{y\_new}{y\_old} = \frac{21\,h\,\lambda + 1}{20\,h\,\lambda + 1}$$

$$\frac{y\_new}{y\_old} = \frac{11\,h\,\lambda + 1}{10\,h\,\lambda + 1}$$

$$\frac{y\_new}{y\_old} = \frac{2\,h\,\lambda + 1}{h\,\lambda + 1}$$

$$\frac{y\_new}{y\_old} = \frac{1. + 1.500000000\,h\,\lambda}{1. + 0.5000000000\,h\,\lambda}$$

$$\frac{y\_new}{y\_old} = h\lambda + 1$$

$$\frac{y\_new}{y\_old} = -\frac{1.\ (1. + 0.5500000000\ h\lambda)}{-1. + 0.4500000000\ h\lambda}$$

$$\frac{y\_new}{y\_old} = -\frac{1.\ (1. + 0.5000000000\ h\lambda)}{-1. + 0.5000000000\ h\lambda}$$

$$\frac{y\_new}{y\_old} = -\frac{1}{h\lambda - 1}$$

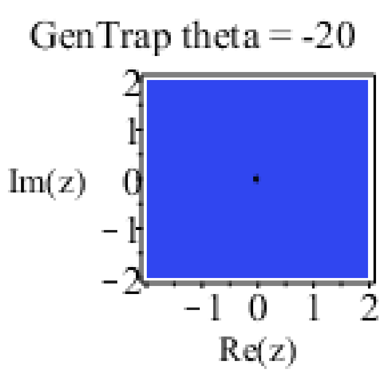
$$\frac{y\_new}{y\_old} = \frac{9\ h\lambda - 1}{10\ h\lambda - 1} \quad \textbf{(8)}$$

stencils := [*GenTrapm10*, *GenTrapm1*, *GenTrapm5*, *GenTrap0*, *GenTrap45*, *GenTrap5*, *GenTrap1*, *GenTrap10*];
stencil_names := ["GenTrap theta = -20", "GenTrap theta = -10", "GenTrap theta = -1", "GenTrap theta = -.5", "GenTrap theta = 0", "GenTrap theta = .45", "GenTrap theta = .5", "GenTrap theta = 1", "GenTrap theta = 10"];

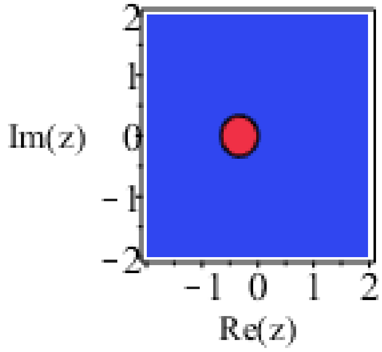
$$\left[\frac{y\_new}{y\_old} = \frac{11\ h\lambda + 1}{10\ h\lambda + 1}, \frac{y\_new}{y\_old} = \frac{2\ h\lambda + 1}{h\lambda + 1}, \frac{y\_new}{y\_old} = \frac{1. + 1.500000000\ h\lambda}{1. + 0.5000000000\ h\lambda}, \frac{y\_new}{y\_old} = h\lambda + 1, \frac{y\_new}{y\_old} = -\frac{1.\ (1. + 0.5500000000\ h\lambda)}{-1. + 0.4500000000\ h\lambda}, \frac{y\_new}{y\_old} = -\frac{1.\ (1. + 0.5000000000\ h\lambda)}{-1. + 0.5000000000\ h\lambda}, \frac{y\_new}{y\_old} = -\frac{1}{h\lambda - 1}, \frac{y\_new}{y\_old} = \frac{9\ h\lambda - 1}{10\ h\lambda - 1}\right]$$

["GenTrap theta = -20", "GenTrap theta = -10", "GenTrap theta = -1", "GenTrap theta = -.5", "GenTrap theta = 0", "GenTrap theta = .45", "GenTrap theta = .5", "GenTrap theta = 1", "GenTrap theta = 10"] **(9)**

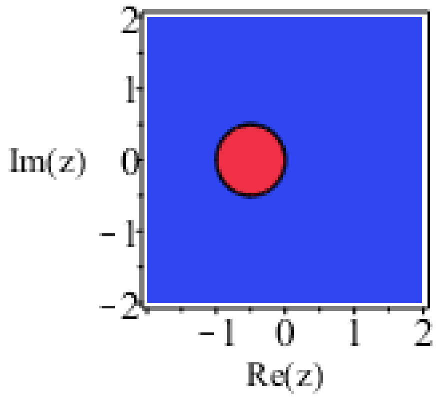
```
for i from 1 to 9 do:
 contourplot(simplify(subs(lambda = z/h, z = x + I*y, abs(rhs(stencils
[i])))), x = -2..2, y = -2..2, contours = [1],
 grid = [50, 50], axes = boxed, filled = true, title = stencil_names[i],
labels = ["Re(z)", "Im(z)"]);
 od;
```



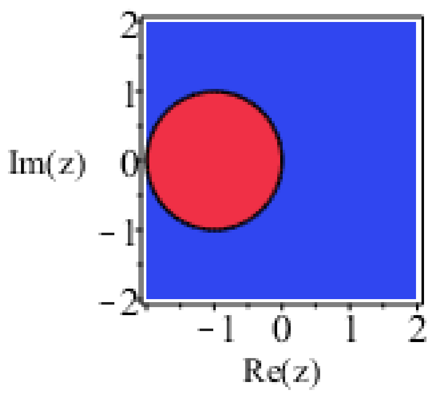
GenTrap theta = -10



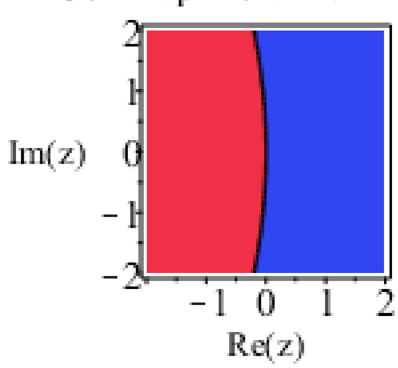
GenTrap theta = -1



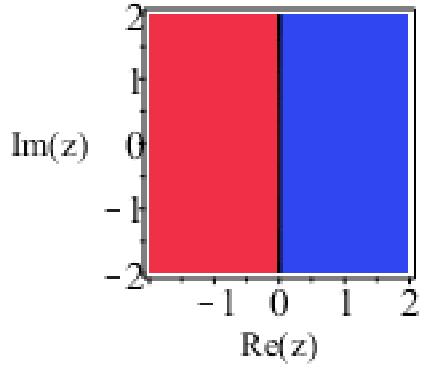
GenTrap theta = -.5



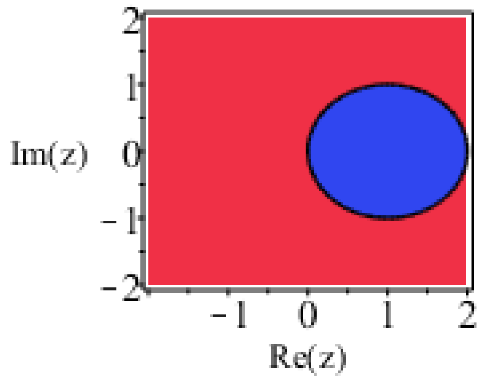
GenTrap theta = 0

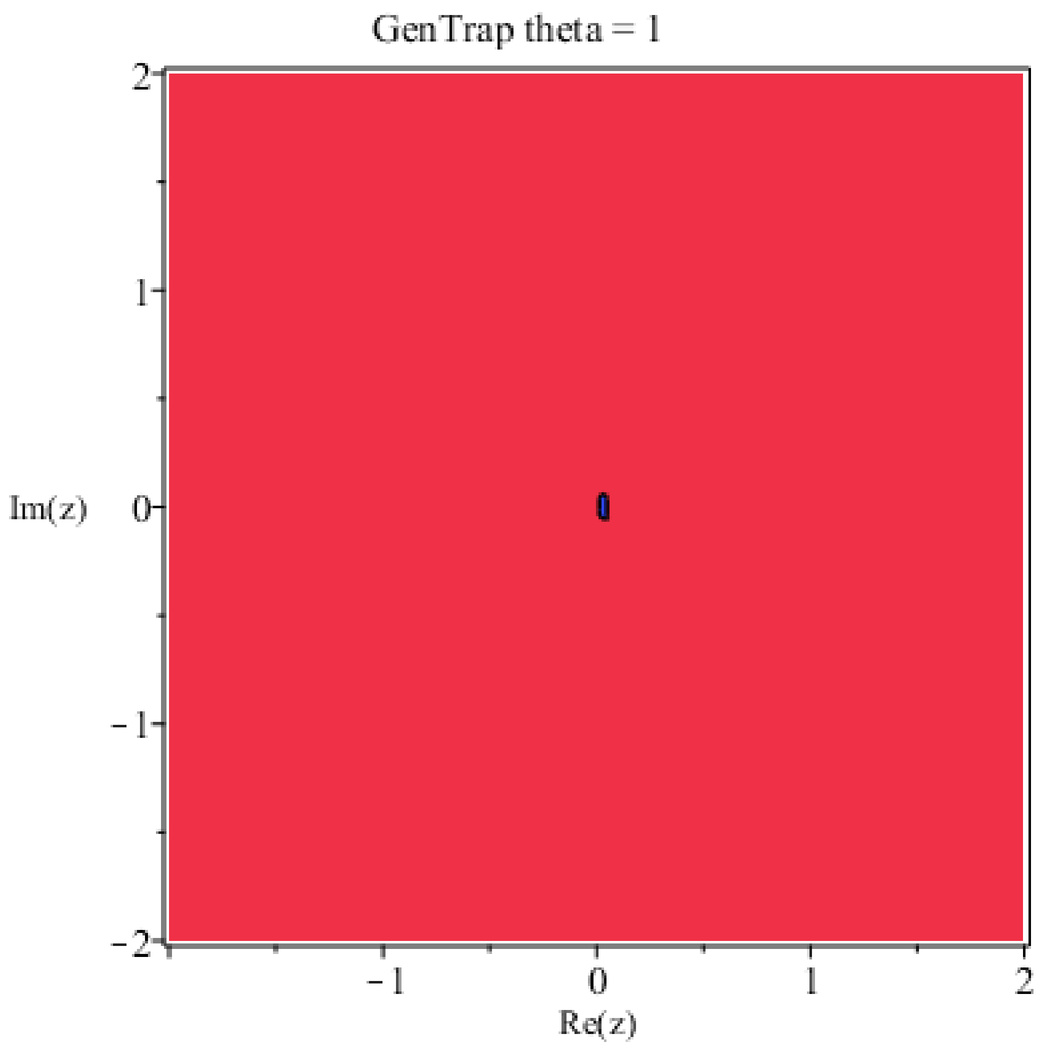


GenTrap theta = .45



GenTrap theta = .5





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