HW 5 Nicole Deyerl

① 6-11 - AB and AM based on (6.1) w/ integration over $[t_n, t_{n+1}]$ of a polynomial interpolating $Y'(t) = f(t, Y(t))$

- consider integration over $[t_{n-1}, t_{n+1}]$ giving
$$Y(t_{n+1}) = Y(t_{n-1}) + \int_{t_{n-1}}^{t_{n+1}} f(t, Y(t))\,dt$$
- replace the integral with a constant interpolant giving $\int_{t_{n-1}}^{t_{n+1}} f(t, Y(t))\,dt \approx \int_{t_{n-1}}^{t_{n+1}} f(t_n, Y(t_n))\,dt = 2hf(t_n, Y(t_n))$
- leads to num. method
$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n) \quad n \geq 1$$
- Show that
$$Y(t_{n+1}) - [Y(t_{n-1}) + 2hf(t_n, Y(t_n))] = +\frac{1}{3}h^3 Y'''(t_n) + O(h^4)$$

Taylor expanding $Y(t_{n+1})$ around t_n

$$Y(t_{n+1}) = Y(t_n) + hY'(t_n) + \frac{h^2}{2}Y''(t_n) + \frac{h^3}{6}Y'''(t_n) + O(h^4)$$
$$= Y(t_n) + hf(t_n, Y_n) + \frac{h^2}{2}f'(t_n, Y_n) + \frac{h^3}{6}f''(t_n, Y_n) + O(h^4) \quad \text{by } Y'(t) = f(t, Y(t))$$
$$Y(t_{n-1}) = Y(t_n) - hY'(t_n) + \frac{h^2}{2}Y''(t_n) - \frac{h^3}{6}Y'''(t_n) + O(h^4)$$
$$= Y(t_n) - hf(t_n, Y_n) + \frac{h^2}{2}f'(t_n, Y_n) - \frac{h^3}{6}f''(t_n, Y_n) + O(h^4)$$

Plugging into the method

$$Y(t_{n+1}) - Y(t_{n-1}) - 2hf(t_n, Y_n) =$$
$$= Y(t_n) + hf(t_n, Y_n) + \frac{h^2}{2}f'(t_n, Y_n) + \frac{h^3}{6}f''(t_n, Y_n) - Y(t_n) + hf(t_n, Y_n) - \frac{h^2}{2}f'(t_n, Y_n)$$
$$+ \frac{h^3}{6}f''(t_n, Y_n) + O(h^4) - 2hf(t_n, Y_n)$$

$$= Y(t_n) - Y(t_n) + hf(t_n, Y_n) + hf(t_n, Y_n) - 2hf(t_n, Y_n) + \frac{h^2}{2} f'(t_n, Y_n) - \frac{h^2}{2} f(t_n, Y_n)$$

$$+ \frac{h^3}{6} f''(t_n, Y_n) + \frac{h^3}{6} f''(t_n, Y_n) + O(h^4)$$

$$= 0 + 0 + 0 + \frac{2h^3}{6} f''(t_n, Y_n) + O(h^4)$$

$$= \frac{h^3}{6} f''(t_n, Y_n) + O(h^4)$$

$$= \frac{h^3}{6} Y'''(t_n, Y_n) + O(h^4)$$ by defn $Y'(t) = f(t, Y(t))$

the desired result.

② - AM methods are based on

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$

- the integrand f is replaced by a polyn. interpolant through the data $\{(t_{n-p}, f_{n-p}), \dots, (t_n, f_n), (t_{n+1}, f_{n+1})\}$
- Alternatively, integrate over interval $[t_{n-1}, t_{n+1}]$

$$y_{n+1} = y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} f(t, y(t))\,dt$$

- Replace integrand f w/ polyn. interpolant thru points

$$\{(t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})\}$$

- Derive the resulting LMM + plot its stability region
  → is it A-stable?

$$y_{n+1} = y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} f(t, y(t))\,dt \quad \text{thru } (t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})$$

$$= y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} \sum_{j=-1}^{1} f_{n-j}\, l_j(t)\,dt$$ as in class, where $l_j(t)$ is the lagrange interpolating polynomial

$$= y_{n-1} + \sum_{j=-1}^{1} f_{n-j} \cdot \int_{t_{n-1}}^{t_{n+1}} l_j(t)\,dt$$

$$= y_{n-1} + f_{n+1} \int_{t_{n-1}}^{t_{n+1}} l_{-1}(t)\,dt + f_n \int_{t_{n-1}}^{t_{n+1}} l_0(t)\,dt + f_{n-1} \int_{t_{n-1}}^{t_{n+1}} l_1(t)\,dt$$

$$l_{-1}(t) = \frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} = \frac{(t-t_n)(t-t_{n+1})}{-h \cdot -2h} = \frac{1}{2h^2} \underbrace{(t-t_n)(t-t_{n+1})}_{f}$$

$$\int_{t_{n-1}}^{t_{n+1}} l_{-1}(t)\,dt = \frac{1}{2h^2} \int_{t_{n-1}}^{t_{n+1}} (t-t_n)(t-t_{n+1})\,dt = \frac{1}{2h^2}\left[\frac{2h}{6}\left[f(t_{n-1}) + 4f(t_n) + f(t_{n+1})\right]\right]$$

by Simpson's rule perfectly approximating quadratic polynomials

$t_n - h - t_n \quad t_n - h - t_n - h$

$$= \frac{1}{6h}\left[-h \cdot -2h + 4 \cdot 0 + 0\right] = \frac{2h^2}{6h} = \frac{h}{3}$$

$$\ell_0(t) = \frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} = \frac{(t-t_{n-1})(t-t_{n+1})}{h\cdot -h} = \frac{-1}{h^2}\underbrace{(t-t_{n-1})(t-t_{n+1})}_{f}$$

$$\int_{t_{n-1}}^{t_{n+1}} \ell_0(t)\,dt = \frac{-1}{h^2}\int_{t_{n-1}}^{t_{n+1}} (t-t_{n-1})(t-t_{n+1})\,dt = \frac{-1}{h^2}\left[\frac{2h}{6}\left(f(t_{n-1})+4f(t_n)+f(t_{n+1})\right)\right]$$

by Simpson's rule

$$= \frac{-1}{3h}\left[0+4\cdot h\cdot -h+0\right] = \frac{4h}{3}$$

$$\ell_1(t) = \frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)} = \frac{(t-t_{n-1})(t-t_n)}{2h\cdot h} = \frac{1}{2h^2}(t-t_{n-1})(t-t_n)$$

$$\int_{t_{n-1}}^{t_{n+1}} \ell_1(t)\,dt = \frac{1}{2h^2}\int_{t_{n-1}}^{t_{n+1}} (t-t_{n-1})(t-t_n)\,dt = \frac{1}{2h^2}\left[\frac{2h}{6}\left[f(t_{n-1})+4f(t_n)+f(t_{n+1})\right]\right]$$

$$= \frac{1}{6h}\left[0+4\cdot 0+2h\cdot h\right] = \frac{2h}{6} = \frac{h}{3}$$

$$\Rightarrow \quad y_{n+1} = y_{n-1} + \frac{h}{3}f_{n+1} + \frac{4h}{3}f_n + \frac{h}{3}f_{n-1}$$

$$\Rightarrow \quad \boxed{y_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n+1}+4f_n+f_{n-1}\right]}$$

Dahlquist test problem $\quad y' = \lambda y \quad y(0)=1$

For a LMM, the stability region boundary is given by

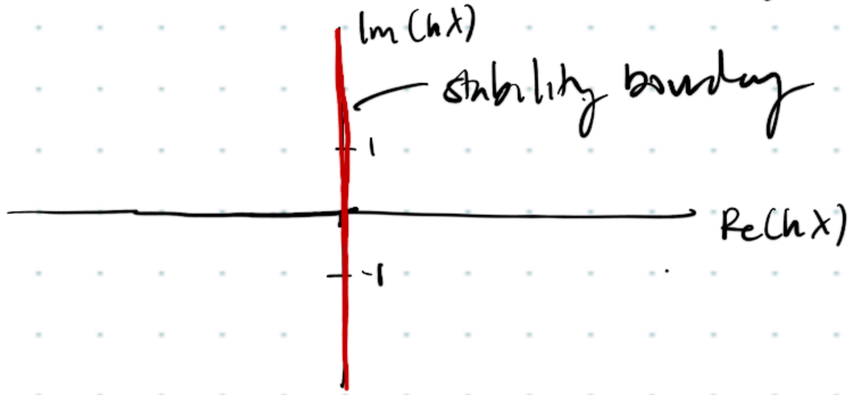
$$h\lambda = \frac{r^{p+1} - \sum_{j=0}^{p} a_j r^{p-j}}{\sum_{j=-1}^{p} b_j r^{p-j}}$$

- for this method, $p=1$ and $a_1=1$, $a_j=0$ for $\forall j\neq 1$

- the constants b_j are given by the integrals of the lagrange interpolating polynomials $\cdot \frac{1}{h}$

$$b_{-1}=\frac{1}{3} \quad b_0=\frac{4}{3} \quad b_1=\frac{1}{3}$$

Plotting using these coefficients w/ Dan Reynolds LMM_stab_eval.m and LMM_stability.m files:



Fill doesn't work, so we can check the sign of r for $h\lambda$ on either side of the boundary.

$$h\lambda = \frac{r^2 - \sum_{j=0}^{1} a_j r^{1-j}}{\sum_{j=-1}^{1} b_j r^{1-j}}$$

$$\Rightarrow h\lambda = \frac{r^2 - a_0 r - a_1}{b_{-1} r^2 + b_0 r + b_1} = \frac{r^2 - 0r - 1}{\frac{1}{3}r^2 + \frac{4}{3}r + \frac{1}{3}}.$$

So, $h\lambda = -1 \Rightarrow -1 = \dfrac{r^2 - 1}{\frac{1}{3}r^2 + \frac{4}{3}r + \frac{1}{3}}$

By wolfram, $r = \underbrace{-\frac{1}{2} - \frac{\sqrt{3}}{2}}_{r_1}, \ \frac{\sqrt{3}}{2} - \frac{1}{2}$

$-1.36 \qquad 0.36$

so, $|r_1| \nleq 1 \Rightarrow h\lambda = -1$ is in an unstable region.

Definitely not A-stable since there are values in the left half plane which make the method unstable.

To finish the stability graph

$h\lambda = 1 \Rightarrow 1 = \dfrac{r^2 - 1}{\frac{1}{3}r^2 + \frac{4}{3}r + \frac{1}{3}}$

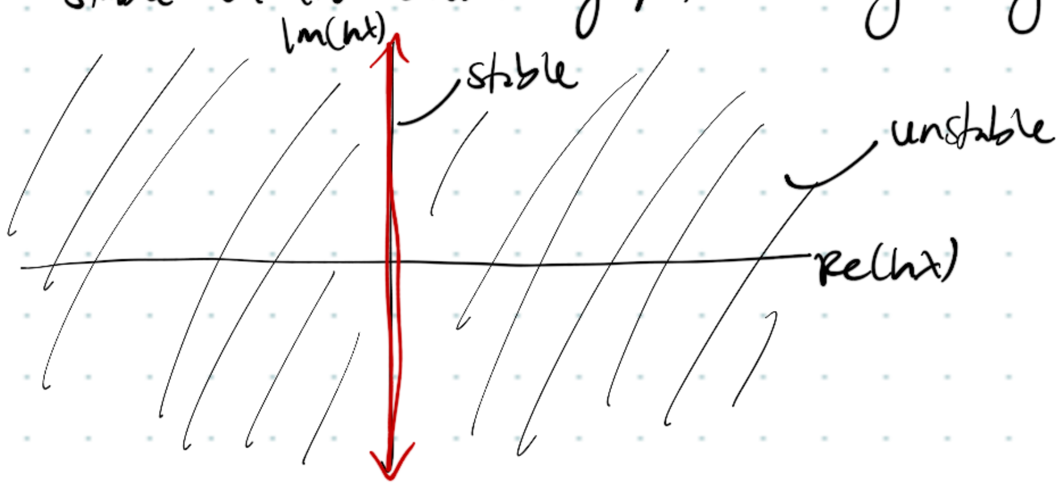
by wolfram

$\Rightarrow r = 1 - \sqrt{3}, \; 1 + \sqrt{3}$

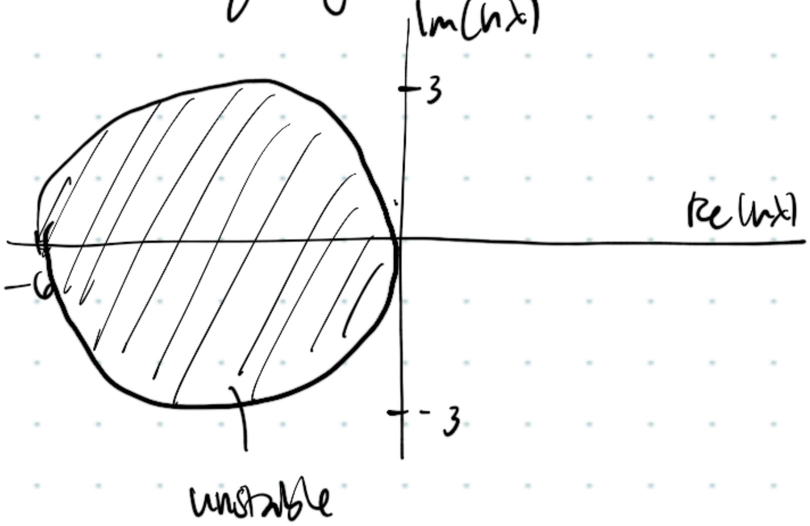
$\approx -0.73 \qquad \approx 2.73$

$|r_2| \nleq 1 \Rightarrow h\lambda = 1$ is an unstable value.

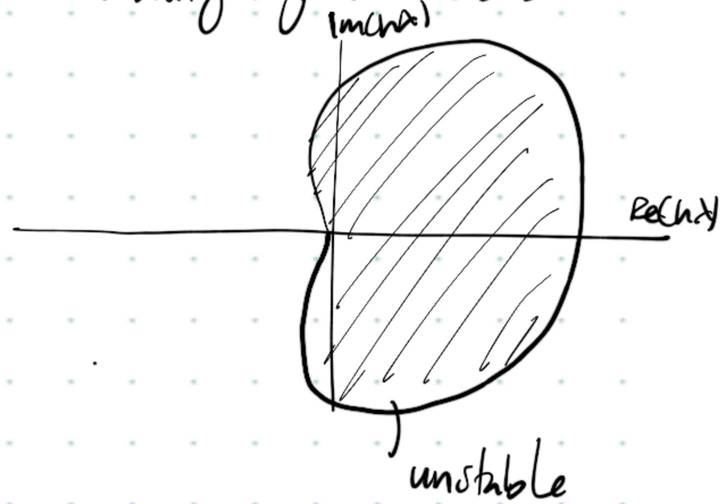
So, we can conclude that this method is only stable on its boundary, the imaginary axis



③ Stability region for AM2



Stability Region for BDF3



From our output, we can see that AM2 produces results that are sometimes stable and $O(h^3)$, and sometimes not stable. In contrast, BDF3 produces results that are $O(h^3)$ (sometimes actually better). Looking at the linear stability regions, we can explain why we are seeing these results. BDF3 is not a-stable, but none of the instability region touches the real (negative) axis, so since our stiffness parameter λ is real (and h is real) we don't see any effects of instability in these examples and has order 3 convergence for the most part. In contrast, AM2 sees instability for negative real values of $h\lambda$. So some combinations of h and λ can pull $h\lambda$ into the unstable region.

④ - Derive a LMM of the form

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + h[b_{-1} f(t_{n+1}, y_{n+1}) + b_0 f(t_n, y_n) + b_1 f(t_{n-1}, y_{n-1})]$$

$\Rightarrow b_2 = 0$

- all coeffs non zero
- at least 3rd order accurate, consistent, stable and convergent
- Plot the linear stability region for your method

- A LMM is <u>consistent</u> if $\tau(h) = \|\tau_n(y)\|_\infty \to 0$ as $h \to 0$ over $t \in [t_p, T_f]$
- This happens if the LMM has rate of convergence/global error $\tau(h) = O(h^m)$ with $m \geq 1$.
- Since we want an $O(h^3)$ method, if we can find this method it will be <u>convergent and consistent</u>.
- We can guarantee an $O(h^m)$ method with $m=3$ by placing the following constraints of the coefficients a_i and b_i.

($p=2$ by problem given)

$$\sum_{j=0}^{2} a_j = 1 \qquad \sum_{j=-1}^{2} b_j - \sum_{j=0}^{2} j \cdot a_j = 1 \qquad (b_2 = 0)$$

and $\displaystyle\sum_{j=0}^{2} (-j)^i a_j + i \sum_{j=-1}^{2} (-j)^{i-1} b_j = 1$ for $i = 0:3$ $(b_2 = 0)$

$\Rightarrow$ The a_j and b_j's need to satisfy eqns

$a_0 + a_1 + a_2 = 1$ (1)

$b_{-1} + b_0 + b_1 - 0 \cdot a_0 - a_1 - 2a_2 = 1$

$\Rightarrow b_{-1} + b_0 + b_1 - a_1 - 2a_2 = 1 \quad (2)$

$i=0$: same as (1)

$i=1$: same as (2)

$i=2$: $(-0)^2 a_0 + (-1)^2 a_1 + (-2)^2 a_2 + 2\left[(1)^1 b_{-1} + (0)^1 b_0 + (-1)^1 b_1\right] = 1$

$\Rightarrow a_1 + 4a_2 + 2b_{-1} - 2b_1 = 1 \quad (3)$

$i=3$: $(-0)^3 a_0 + (-1)^3 a_1 + (-2)^3 a_2 + 3\left[(1)^2 b_{-1} + (0)^2 b_0 + (-1)^2 b_1\right] = 1$

$\Rightarrow -a_1 - 8a_2 + 3b_{-1} + 3b_1 = 1 \quad (4)$

So, now we have a linear system:

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & 1 & 1 \\ 0 & 1 & 4 & 2 & 0 & -2 \\ 0 & -1 & -8 & 3 & 0 & 3 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ b_{-1} \\ b_0 \\ b_1 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$$

using math.odu.edu linear system solver

$\rightarrow a_0 = \frac{-23}{22} b_0 - \frac{8}{11} b_1 + \frac{18}{11}$

$a_1 = \frac{14}{11} b_0 + \frac{4}{11} b_1 - \frac{9}{11}$

$a_2 = -\frac{5}{22} b_0 + \frac{4}{11} b_1 + \frac{2}{11}$

$b_{-1} = -\frac{2}{11} b_0 + \frac{1}{11} b_1 + \frac{6}{11}$

$b_0 = \alpha$

$b_1 = \beta$ free variables

we also need the a_i's to satisfy the root condition to guarantee <u>stability</u>

$$\rho(r) = r^{2+1} - \sum_{j=0}^{2} a_j r^{2-j}$$

$$= r^3 - a_0 r^2 - a_1 r - a_2$$

$$= r^3 - \left(\frac{-23}{22} b_0 - \frac{8}{11} b_1 + \frac{18}{11}\right) r^2 - \left(\frac{14}{11} b_0 + \frac{4}{11} b_1 - \frac{9}{11}\right) r$$
$$- \left(-\frac{5}{22} b_0 + \frac{4}{11} b_1 + \frac{2}{11}\right)$$

$= 0$ with roots $|r_j| \le 1$, and $|r_j| = 1 \Rightarrow \rho'(r_j) \ne 0$

Guess $b_0 = \frac{1}{11}$ $b_1 = \frac{2}{11}$

$\rightarrow a_0 = \frac{31}{22}$ $a_1 = \frac{-7}{11}$ $a_2 = \frac{5}{22}$ $b_{-1} = \frac{6}{11}$

$\Rightarrow r = 1, \frac{1}{44}(9 - i\sqrt{359}), \frac{1}{44}(9 + i\sqrt{359})$ via wolfram

dont think roots can be imaginary.

Guess $b_0 = \frac{-1}{11}$ $b_1 = \frac{-2}{11}$

$\Rightarrow a_0 = \frac{41}{22}$ $a_1 = -1$ $a_2 = \frac{3}{22}$ $b_{-1} = \frac{6}{11}$

$\Rightarrow r = 1, \frac{1}{44}(19 - \sqrt{97}), \frac{1}{44}(19 + \sqrt{97})$

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check $r = 1$:

$$\rho'(r) = 3r^2 - \frac{41}{11} r + 1$$

$$\rho'(1) = \frac{3}{11} \ne 0$$

The root condition is satisfied!

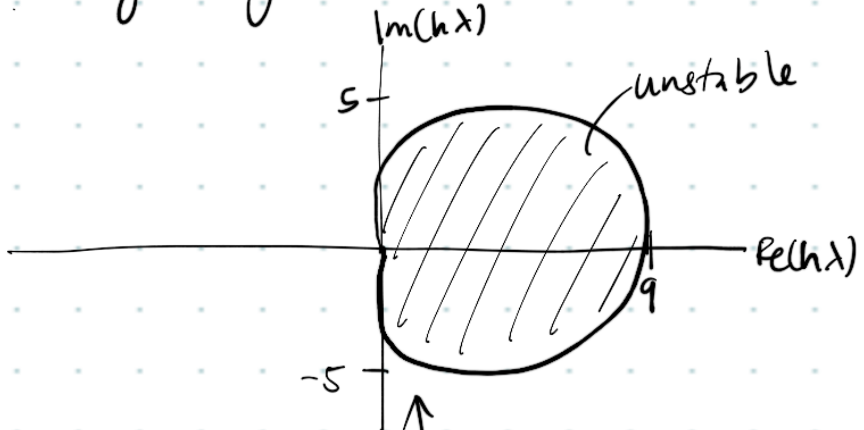
So, this method is consistent, convergent and $O(h^3)$ by satisfying the 4 eqns we used to build the linear system, and it is stable by satisfying the root condition.

$$y_{n+1} = \frac{41}{22} y_n - y_{n-1} + \frac{3}{22} y_{n-2} + h\Big[\frac{6}{11} f(t_{n+1}, y_{n+1}) - \frac{1}{11} f(t_n, y_n) - \frac{2}{11} f(t_{n-1}, y_{n-1})\Big]$$

Stability Region (via LMM_stab_eval.m)

using BDF_a = $[\frac{41}{22}, -1, \frac{3}{22}]$

BDF_b = $[\frac{6}{11}, \frac{-1}{11}, \frac{-2}{11}]$



zoom in