

## HW 5

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- ① 6.11 - AB and AM based on (6.1) w/ integration over  $[t_n, t_{n+1}]$  of a polynomial interpolating

$$Y'(t) = f(t, Y(t))$$

- consider integration over  $[t_{n-1}, t_{n+1}]$  giving

$$Y(t_{n+1}) = Y(t_{n-1}) + \int_{t_{n-1}}^{t_{n+1}} f(t, Y(t)) dt$$

- replace the integral with a constant interpolant

$$\text{giving } \int_{t_{n-1}}^{t_{n+1}} f(t, Y(t)) dt \approx \int_{t_{n-1}}^{t_{n+1}} f(t_n, Y(t_n)) dt = 2h f(t_n, Y(t_n))$$

- leads to num. method

$$y_{n+1} = y_{n-1} + 2h f(t_n, y_n) \quad n \geq 1$$

- Show that

$$Y(t_{n+1}) - [Y(t_{n-1}) + 2h f(t_n, Y(t_n))] = +\frac{1}{3} h^3 Y'''(t_n) + O(h^4)$$

Taylor expanding  $Y(t_{n+1})$  around  $t_n$

$$Y(t_{n+1}) = Y(t_n) + h Y'(t_n) + \frac{h^2}{2} Y''(t_n) + \frac{h^3}{6} Y'''(t_n) + O(h^4)$$

$$= Y(t_n) + h f(t_n, Y_n) + \frac{h^2}{2} f'(t_n, Y_n) + \frac{h^3}{6} f''(t_n, Y_n) + O(h^4) \quad \text{by } Y'(t) = f(t, Y(t))$$

$$Y(t_{n-1}) = Y(t_n) - h Y'(t_n) + \frac{h^2}{2} Y''(t_n) - \frac{h^3}{6} Y'''(t_n) + O(h^4)$$

$$= Y(t_n) - h f(t_n, Y_n) + \frac{h^2}{2} f'(t_n, Y_n) - \frac{h^3}{6} f''(t_n, Y_n) + O(h^4)$$

Plugging into the method

$$Y(t_{n+1}) - Y(t_{n-1}) - 2h f(t_n, Y_n) =$$

$$= Y(t_n) + h f(t_n, Y_n) + \frac{h^2}{2} f'(t_n, Y_n) + \frac{h^3}{6} f''(t_n, Y_n) - Y(t_n) + h f(t_n, Y_n) - \frac{h^2}{2} f'(t_n, Y_n) \\ + \frac{h^3}{6} f''(t_n, Y_n) + O(h^4) - 2h f(t_n, Y_n)$$

$$\begin{aligned}
&= Y(t_n) - Y(t_n) + hf(t_n, Y_n) + hf(t_n, Y_n) - 2hf(t_n, Y_n) + \frac{h^2}{2} f'(t_n, Y_n) - \frac{h^2}{2} f(t_n, Y_n) \\
&\quad + \frac{h^3}{6} f''(t_n, Y_n) + \frac{h^3}{6} f''(t_n, Y_n) + O(h^4) \\
&= 0 + 0 + 0 + \frac{2h^3}{6} f''(t_n, Y_n) + O(h^4) \\
&= \frac{h^3}{6} f''(t_n, Y_n) + O(h^4)
\end{aligned}$$

$$= \frac{h^3}{6} Y'''(t_n, Y_n) + O(h^4) \quad \text{by defn } Y'(t) = f(t, Y(t))$$

The desired result.

② - AM methods are based on

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y(t)) dt$$

- the integrand  $f$  is replaced by a polyn. interpolant through the data  $\{(t_n-p, f_{n-p}), \dots, (t_n, f_n), (t_{n+1}, f_{n+1})\}$
- Alternatively, integrate over interval  $[t_{n-1}, t_{n+1}]$

$$y_{n+1} = y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} f(t, y(t)) dt$$

- Replace integrand  $f$  w/ polyn. interpolant thru points  $\{(t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})\}$

- Derive the resulting LMM + plot its stability region  
→ Is it A-stable?

$$\begin{aligned} y_{n+1} &= y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} f(t, y(t)) dt \quad \text{thru } (t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1}) \\ &= y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} \sum_{j=-1}^1 f_{n+j} l_j(t) dt \quad \text{as in class, where } l_j(t) \text{ is the} \\ &\quad \text{lagrange interpolating polynomial} \\ &= y_{n-1} + \sum_{j=-1}^1 f_{n+j} \cdot \int_{t_{n-1}}^{t_{n+1}} l_j(t) dt \\ &= y_{n-1} + f_{n+1} \int_{t_{n-1}}^{t_{n+1}} l_{-1}(t) dt + f_n \int_{t_{n-1}}^{t_{n+1}} l_0(t) dt + f_{n-1} \int_{t_{n-1}}^{t_{n+1}} l_1(t) dt \end{aligned}$$

$$l_{-1}(t) = \frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} = \frac{(t-t_n)(t-t_{n+1})}{-h \cdot -2h} = \frac{1}{2h^2} \underbrace{\frac{(t-t_n)(t-t_{n+1})}{f}}_f$$

$$\begin{aligned} \int_{t_{n-1}}^{t_{n+1}} l_{-1}(t) dt &= \frac{1}{2h^2} \int_{t_{n-1}}^{t_{n+1}} (t-t_n)(t-t_{n+1}) dt = \frac{1}{2h^2} \left[ \frac{2h}{6} [f(t_{n-1}) + 4f(t_n) + f(t_{n+1})] \right] \\ &\quad \begin{matrix} t_n-h-t_n & t_n-h-t_n-h \end{matrix} \quad \begin{matrix} \text{by Simpson's rule perfectly approximat} \\ \text{ing quadratic polynomials} \end{matrix} \\ &= \frac{1}{6h} [-h \cdot -2h + 4 \cdot 0 + 0] = \frac{2h^2}{6h} = \frac{h}{3} \end{aligned}$$

$$l_0(t) = \frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} = \frac{(t-t_{n-1})(t-t_{n+1})}{h \cdot -h} = \frac{-1}{h^2} \underbrace{(t-t_{n-1})(t-t_{n+1})}_f$$

$$\int_{t_{n-1}}^{t_{n+1}} l_0(t) dt = -\frac{1}{h^2} \int_{t_{n-1}}^{t_{n+1}} (t-t_{n-1})(t-t_{n+1}) dt = -\frac{1}{h^2} \left[ \frac{2h}{6} [f(t_{n-1}) + 4f(t_n) + f(t_{n+1})] \right]$$

by Simpson's rule

$$= -\frac{1}{3h} [0 + 4 \cdot h \cdot h + 0] = \frac{4h}{3}$$

$$l_1(t) = \frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)} = \frac{(t-t_{n-1})(t-t_n)}{2h \cdot h} = \frac{1}{2h^2} (t-t_{n-1})(t-t_n)$$

$$\int_{t_{n-1}}^{t_{n+1}} l_1(t) dt = \frac{1}{2h^2} \int_{t_{n-1}}^{t_{n+1}} (t-t_{n-1})(t-t_n) dt = \frac{1}{2h^2} \left[ \frac{2h}{6} [f(t_{n-1}) + 4f(t_n) + f(t_{n+1})] \right]$$

$$= \frac{1}{6h} [0 + 4 \cdot 0 + 2h \cdot h] = \frac{2h}{6} = \frac{h}{3}$$

$$\Rightarrow y_{n+1} = y_{n-1} + \frac{h}{3} f_{n+1} + \frac{4h}{3} f_n + \frac{h}{3} f_{n-1}$$

$$\Rightarrow \boxed{y_{n+1} = y_{n-1} + \frac{h}{3} [f_{n+1} + 4f_n + f_{n-1}]}$$

Dahlquist test problem  $y' = \lambda y$   $y(0) = 1$

For a LMM, the stability region boundary is given by

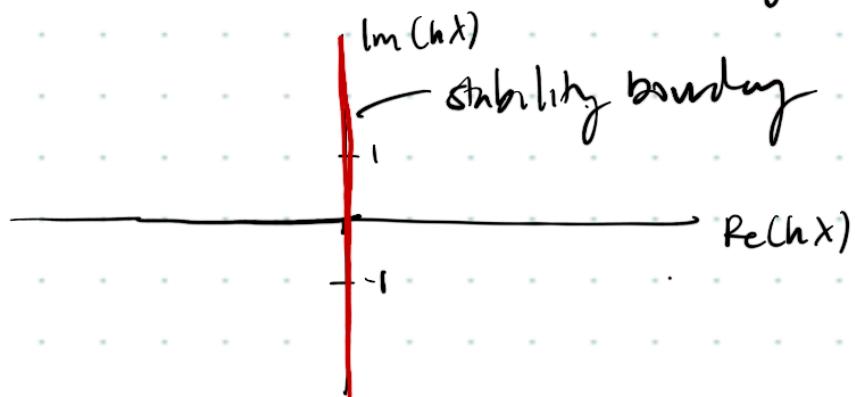
$$h\lambda = \frac{r^{p+1} - \sum_{j=0}^p a_j r^{p-j}}{\sum_{j=-1}^p b_j r^{p-j}}$$

- for this method,  $p=1$  and  $a_1=1, a_j=0$  for  $\forall j \neq 1$

- the constants  $b_j$  are given by the integrals of the Lagrange interpolating polynomials

$$b_{-1} = \frac{1}{3}, \quad b_0 = \frac{4}{3}, \quad b_1 = \frac{1}{3}$$

Plotting using these coefficients w/ Dan Reynolds  
 LMM-stab-erval.m and LMM-stability.m files:



Fill doesn't work, so we can check the sign of  $r$  for  $hx$  on either side of the boundary.

$$h\lambda = \frac{r^2 - \sum_{j=0}^1 a_j r^{1-j}}{\sum_{j=-1}^1 b_j r^{1-j}}$$

$$\Rightarrow h\lambda = \frac{r^2 - a_0 r - a_1}{b_1 r^2 + b_0 r + b_1} = \frac{r^2 - 0r - 1}{\frac{1}{3} r^2 + \frac{4}{3} r + \frac{1}{3}}$$

$$\text{so, } h\lambda = -1 \Rightarrow -1 = \frac{r^2 - 1}{\frac{1}{3} r^2 + \frac{4}{3} r + \frac{1}{3}}$$

$$\text{By wolfram, } r = \frac{-\frac{1}{2} - \frac{\sqrt{3}}{2}}{s_1}, \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$-1.36 \quad 0.36$$

so,  $|r_1| \neq 1 \Rightarrow h\lambda = -1$  is in an unstable region.

Definitely not A-stable since there are values in the lefthalf plane which make the method unstable.

To finish the stability graphs

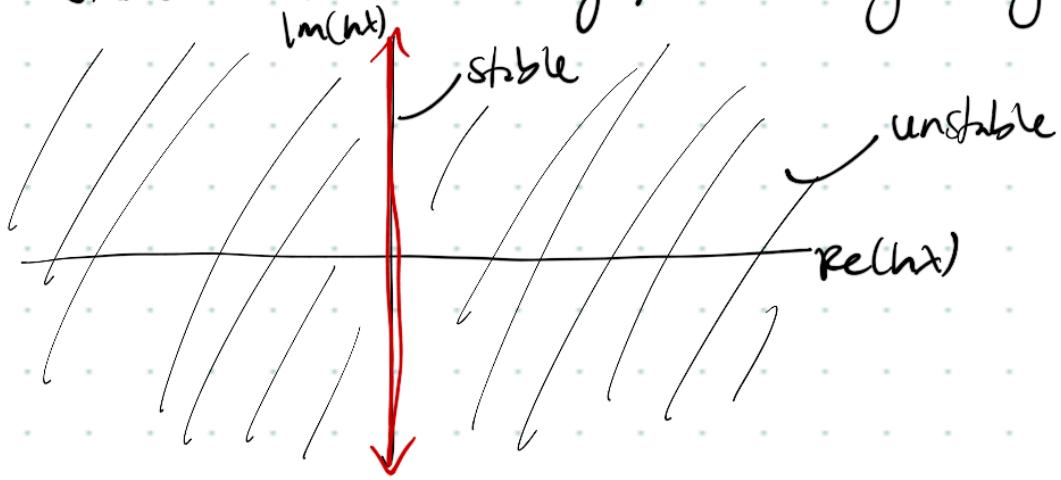
$$h\lambda = 1 \Rightarrow 1 = \frac{r^2 - 1}{\frac{1}{3}r^2 + \frac{4}{3}r + \frac{1}{3}}$$

by wolfram

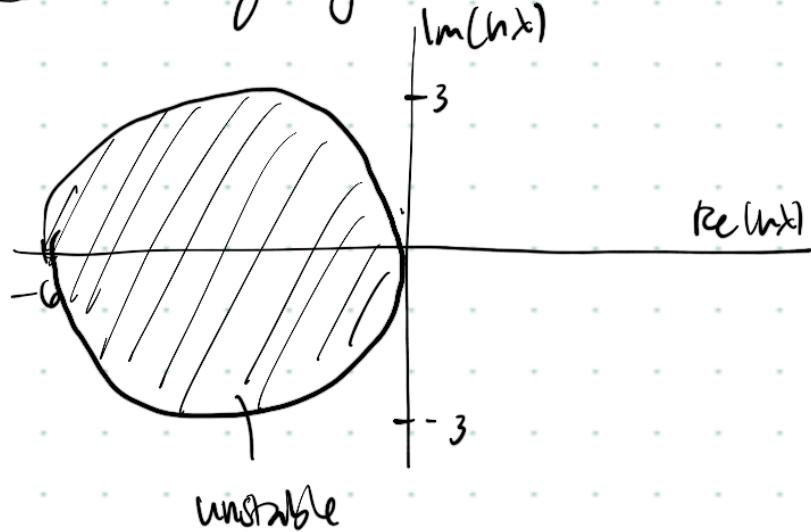
$$\Rightarrow r = \begin{matrix} 1 - \sqrt{3} \\ \text{ss} \\ -0.73 \end{matrix}, \begin{matrix} 1 + \sqrt{3} \\ \text{ss} \\ 2.73 \end{matrix}$$

$|r_2| \neq 1 \Rightarrow h\lambda = 1$  is an unstable value.

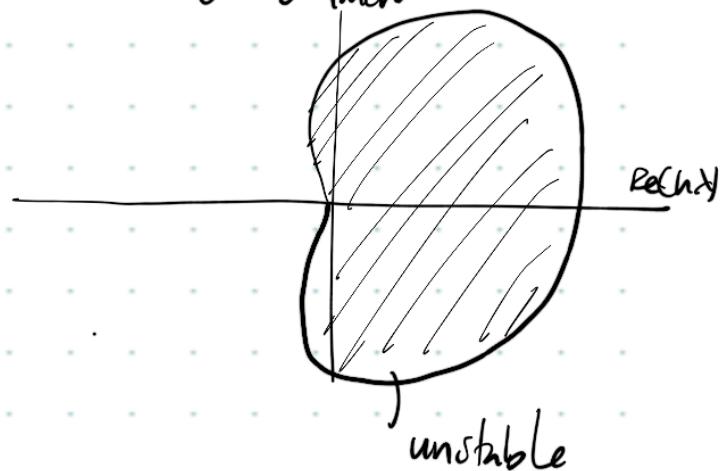
so, we can conclude that this method is only stable on its boundary, the imaginary axis



③ Stability region for AM2



Stability Region for BDF3



From our output, we can see that AM2 produces results that are sometimes stable and  $O(h^3)$ , and sometimes not stable. In contrast, BDF3 produces results that are  $O(h^3)$  (sometimes actually better). Looking at the linear stability regions, we can explain why we are seeing these results. BDF3 is not a-stable, but none of the instability region touches the real (negative) axis, so since our stiffness parameter  $\lambda$  is real (and  $h$  is real) we don't see any effects of instability in these examples. In contrast, AM2 sees instability for negative real values of  $h\lambda$ . So some combinations of  $h$  and  $\lambda$  can pull  $h\lambda$  into the unstable region. (when  $h$

④ - Derive a LMM of the form

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + h [b_{-1} f(t_{n+1}, y_{n+1}) + b_0 f(t_n, y_n) + b_1 f(t_{n-1}, y_{n-1})] \Rightarrow b_2 = 0$$

- all coeffs non zero
- at least 3<sup>rd</sup> order accurate, consistent, stable and convergent
- Plot the linear stability region for your method

- A LMM is consistent if  $\tau(h) = \| \tau_h(y) \|_\infty \rightarrow 0$  as  $h \rightarrow 0$  over  $t \in [t_p, T_f]$

- This happens if the LMM has rate of convergence / global error  $\tau(h) = O(h^m)$  with  $m \geq 1$ .

- Since we want an  $O(h^3)$  method, if we can find this method it will be convergent and consistent.

- We can guarantee an  $O(h^m)$  method with  $m=3$  by placing the following constraints of the coefficients  $a_i$  and  $b_i$ :

( $p=2$  by problem given)

$$\sum_{j=0}^2 a_j = 1 \quad \sum_{j=-1}^2 b_j - \sum_{j=0}^2 j \cdot a_j = 1 \quad (b_2 = 0)$$

$$\text{and } \sum_{j=0}^2 (-j)^i a_j + i \sum_{j=-1}^2 (-j)^{i-1} b_j = 1 \quad \text{for } i=0:3 \quad (b_2 = 0)$$

$\Rightarrow$  The  $a_j$  and  $b_j$ 's need to satisfy eqns

$$a_0 + a_1 + a_2 = 1 \quad (1)$$

$$b_{-1} + b_0 + b_1 - 0 \cdot a_0 - a_1 - 2a_2 = 1$$

$$\Rightarrow b_{-1} + b_0 + b_1 - a_1 - 2a_2 = 1 \quad (2)$$

$i=0$ : same as (1)

$i=1$ : same as (2)

$$i=2: (-0)^2 a_0 + (-1)^2 a_1 + (-2)^2 a_2 + 2[(-1)^1 b_{-1} + (0)^1 b_0 + (-1)^1 b_1] = 1$$

$$\Rightarrow a_1 + 4a_2 + 2b_{-1} - 2b_1 = 1 \quad (3)$$

$$i=3: (-0)^3 a_0 + (-1)^3 a_1 + (-2)^3 a_2 + 3[(1)^2 b_{-1} + (0)^2 b_0 + (-1)^2 b_1] = 1$$

$$\Rightarrow -a_1 - 8a_2 + 3b_{-1} + 3b_1 = 1 \quad (4)$$

So, now we have a linear system:

$$\left| \begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & 1 & 1 \\ 0 & 1 & 4 & 2 & 0 & -2 \\ 0 & -1 & -8 & 3 & 0 & 3 \end{array} \right| \left| \begin{array}{c} a_0 \\ a_1 \\ a_2 \\ b_{-1} \\ b_0 \\ b_1 \end{array} \right| = \left| \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right|$$

using math.odu.edu linear system solver

$$\rightarrow a_0 = \frac{-23}{22}b_0 - \frac{8}{11}b_1 + \frac{18}{11}$$

$$a_1 = \frac{14}{11}b_0 + \frac{4}{11}b_1 - \frac{9}{11}$$

$$a_2 = -\frac{5}{22}b_0 + \frac{4}{11}b_1 + \frac{2}{11}$$

$$b_{-1} = -\frac{2}{11}b_0 + \frac{1}{11}b_1 + \frac{6}{11}$$

$$b_0 = \alpha$$

$$b_1 = \beta \quad \text{free variables}$$

we also need the  $a_i$ 's to satisfy the root condition to guarantee stability

$$p(r) = r^{2+1} - \sum_{j=0}^2 a_j r^{2-j}$$

$$= r^3 - a_0 r^2 - a_1 r - a_2$$

$$= r^3 - \left( \frac{-23}{22} b_0 - \frac{8}{11} b_1 + \frac{18}{11} \right) r^2 - \left( \frac{14}{11} b_0 + \frac{4}{11} b_1 - \frac{9}{11} \right) r \\ - \left( -\frac{5}{22} b_0 + \frac{4}{11} b_1 + \frac{2}{11} \right)$$

$$= 0 \quad \text{with roots } |r_j| \leq 1, \text{ and } |r_j| = 1 \Rightarrow p(r_j) \neq 0$$

Guess  $b_0 = \frac{1}{11}, b_1 = \frac{2}{11}$

$$\Rightarrow a_0 = \frac{31}{22}, a_1 = -\frac{7}{11}, a_2 = \frac{5}{22}, b_{-1} = \frac{6}{11}$$

$\Rightarrow r = 1, \frac{1}{4} \left( 9 - i\sqrt{359} \right), \frac{1}{4} \left( 9 + i\sqrt{359} \right)$  via wolfram  
don't think roots can be imaginary.

Guess  $b_0 = -\frac{1}{11}, b_1 = -\frac{2}{11}$

$$\Rightarrow a_0 = \frac{41}{22}, a_1 = -1, a_2 = \frac{3}{22}, b_{-1} = \frac{6}{11}$$

$$\Rightarrow r = 1, \frac{1}{4} \left( 19 - \sqrt{97} \right), \frac{1}{4} \left( 19 + \sqrt{97} \right) \\ .2079 \quad .656$$

check  $r=1$ :

$$p'(r) = 3r^2 - \frac{41}{11}r + 1$$

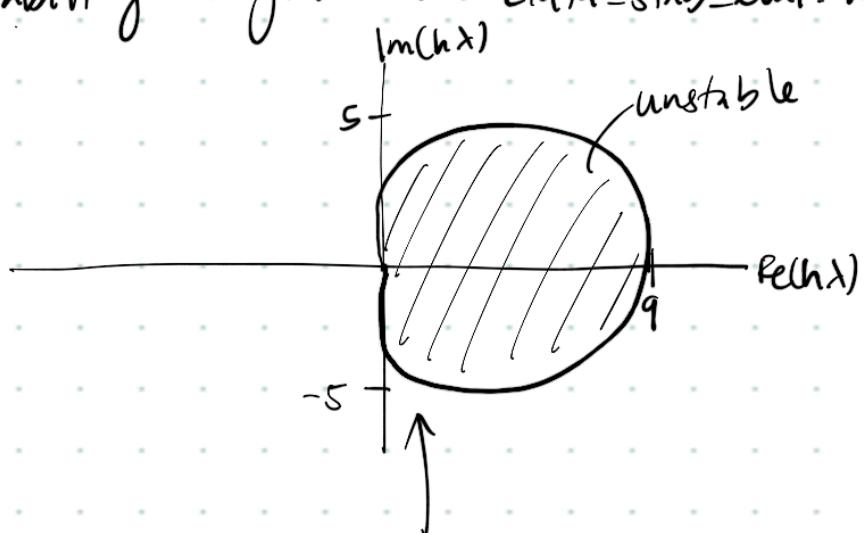
$$p'(1) = \frac{3}{11} \neq 0$$

The root condition is satisfied!

So, this method is consistent, convergent and  $O(h^3)$  by satisfying the 4 eqns we used to build the linear system, and it is stable by satisfying the root condition.

$$y_{n+1} = \frac{41}{22}y_n - y_{n-1} + \frac{3}{22}y_{n-2} + h\left[\frac{6}{11}f(t_{n+1}, y_{n+1}) - \frac{1}{11}f(t_n, y_n) - \frac{2}{11}f(t_{n-1}, y_{n-1})\right]$$

Stability Region (via LMM-stab-oval.m)



$$\text{now } BDF\_a = \left[ \frac{41}{22}, -1, \frac{3}{22} \right]$$

$$BDF\_b = \left[ \frac{6}{11}, -\frac{1}{11}, -\frac{2}{11} \right]$$

