

# QUADRATURE FOR PARABOLIC GALERKIN BEM WITH MOVING SURFACES

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**ABSTRACT.** The successful implementation of the Galerkin Boundary Element Method hinges on the accurate and effective quadrature of the influence coefficients. For parabolic boundary integral operators quadrature must be performed in space and time where integrals have singularities when source- and evaluation times coincide. For problems where the surface is fixed, the time integration can be performed analytically, but for moving geometries numerical quadrature in space and time must be used. For this case a set of transformations is derived that render the singular space-time integrals into smooth integrals that can be treated with standard tensor product Gauss quadrature rules. This methodology can be applied to the heat equation and to the transient Stokes flow.

**Keywords.** Boundary element method, Galerkin method, moving geometry time dependent problem, numerical quadrature, error analysis

## 1. INTRODUCTION

The boundary element method is a well known technique to solve parabolic boundary value problems which has generated considerable interest in engineering [8, 11, 21] as well as in the mathematics literature [14, 6, 13, 1]. Time dependence is reflected in the fact that parabolic layer potential operators involve integrals over time in addition to the boundary surface. For instance, for the heat equation, the single layer potential of the density  $g(\cdot)$  on a boundary surface  $\Gamma$  is

$$(1) \quad \mathcal{V}g(\mathbf{x}, t) = \int_0^t \int_{\Gamma} \frac{1}{(4\pi(t-\tau))^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-\tau)}\right) g(\mathbf{y}, \tau) ds(\mathbf{y}) d\tau.$$

To accommodate more general parabolic equations and potentials we assume the kernel has the form

$$(2) \quad G(\mathbf{x}, \mathbf{y}, t, \tau) = \frac{1}{(t-\tau)^{\frac{n}{2}}} K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}),$$

where

$$(3) \quad \boldsymbol{\rho} = \frac{\mathbf{x} - \mathbf{y}}{\sqrt{4(t-\tau)}}.$$

Here,  $n$  is a positive integer. Because of the causality of parabolic boundary operators we have  $G = 0$  whenever  $t < \tau$ .

We make the following following assumptions about  $K$ . Later, we will show that they are satisfied for the potentials of the heat and transient Stokes flow.

**Assumption 1.1.** (1) *The function  $K$  is  $C^\infty$  in all variables.*

(2) *There is an integer  $M$  such that the function*

$$\mu \rightarrow \frac{1}{\mu^M} K\left(\frac{\boldsymbol{\rho}}{\mu}, \mathbf{x}, \mathbf{y}\right)$$

*is  $C^\infty[0, 1]$  for fixed  $|\boldsymbol{\rho}| > 0$ .*

Note that for  $M$  sufficiently large, the kernel  $G$  is singular only if  $(\mathbf{y}, \tau) = (\mathbf{x}, t)$ . There are several options for the discretization of parabolic boundary equations, they are surveyed in [7]. Here we focus on the Galerkin method, because of its effectiveness for non-smooth and time dependent geometries. These advantages come at a price, as the calculation of the influence coefficients of the discretized layer potentials involve complicated and possibly singular space-time integrals. To motivate the ensuing discussion, consider a time-dependent surface  $\Gamma = \Gamma(t)$  which has been subdivided into a conforming triangulation with patches  $\Gamma_p(t)$ ,  $p = 1, \dots, P$ . To keep things manageable we assume that  $\Gamma(t)$  and its triangulation are isomorphic to the initial surface and triangulation. Further, the time interval  $I = [0, T]$  has been subdivided into equal subintervals  $I_i = [ih_t, (i+1)h_t]$ , where  $h_t$  is the time stepsize. The test- and ansatz spaces consist of piecewise polynomials on this space-time subdivision.

To realize the Galerkin discretization integrals of the form

$$(4) \quad I_{p,q}^{i,j} = \int_{I_i} \int_{\Gamma_p(t)} \int_{I_j} \int_{\Gamma_q(\tau)} G(\mathbf{x}, \mathbf{y}, t, \tau) \phi_i^I(\tau) \phi_j^I(t) \phi_k^\Gamma(\mathbf{y}) \phi_l^\Gamma(\mathbf{x}) ds(\mathbf{y}) d\tau ds(\mathbf{x}) dt$$

must be computed, where  $\phi_i^I, \phi_j^I, \phi_k^\Gamma, \phi_l^\Gamma$  are polynomial shape functions.

When the surface is stationary one can rearrange the order of integration such that the time variables appear inside of the space variables. For the heat kernel and the transient Stokeslet these time integrals can be expressed in closed form using incomplete gamma functions [5, 17]. The result is a space-dependent kernel  $G_{ij}(\mathbf{x}, \mathbf{y})$ , that is smooth if the time intervals  $I_i$  and  $I_j$  are well-separated. Otherwise, if  $i = j$  or  $i = j+1$  the functions or derivatives of  $G_{ij}(\cdot)$  are singular for  $\mathbf{x} = \mathbf{y}$ . In these cases, the singularity removing transformations for elliptic boundary operators [9, 19] are applicable to compute the integrals (4) by numerical quadrature, see [15, 16].

Unfortunately, this trick is limited to stationary surfaces whereas in many applications of interest the geometry is time dependent. For instance, melting and solidification problems are governed by the heat equation and the objective is to determine the time dependent interface between the solid- and liquid phase, see e.g. [2, 20]. Also problems governed by Stokes flow often involve moving surfaces, such as in Mems devices [22], micro-swimmers [10] and sedimentation [12].

The technical difficulty of moving geometries is that all variables in (4) must be handled by numerical quadrature. Because of the singularity new transformations in space-time must be derived, this is the goal of this work. One should note here that the papers [4, 3] also deal with singularity removing transformation that can be seen as an extension of the construction in [9, 19] to higher dimensions. However, the mentioned papers address elliptic operators. Because of the space-time scaling expressed by the variable  $\boldsymbol{\rho}$  in the kernel of (2), the parabolic case is different and we did not see an obvious way to apply the earlier work to the case considered here.

The outline of the remainder of this paper is as follows. In section 2 we will state some minimal assumptions on the geometry which ensure that the transformed integrals have indeed smooth integrands. Our construction has three stages, we first apply a transformation for the spatial variable (section 3), followed by a transformation in time (section 4). Since these two transformations still result in singular integrands we will then apply a simple Duffy transform in section 5 that will finally lead to a regular integrand. Note that section 3 follows closely the discussion in [19] where a set of suitable transformations for elliptic boundary integral operators are derived. However, we felt that it was necessary to include this here at the expense of some repetition, in order to set the stage for the new contributions and to improve self consistency and readability of this work. We conclude in section 7 with an illustration of the Galerkin BEM for parabolic moving boundary problems.

## 2. THE SETTING

We consider the boundary surface of a time-dependent finite domain  $\Gamma(t) = \partial\Omega(t)$  in three dimensions which is subdivided into a triangulation

$$(5) \quad \Gamma(t) = \bigcup_{p=1}^P \Gamma_p(t),$$

where each  $\Gamma_p(t)$  is a parametric image of the unit triangle,

$$\sigma^{(2)} = \{\hat{\mathbf{x}} \in \mathbb{R}^2 : 0 \leq \hat{x}_2 \leq \hat{x}_1 \leq 1\},$$

i.e.,

$$\Gamma_p(t) = \left\{ \mathbf{x}_p(\hat{\mathbf{x}}, t), \hat{\mathbf{x}} \in \sigma^{(2)} \right\}.$$

The parametric images of the points  $(0,0)$ ,  $(0,1)$  and  $(1,1)$  are the vertices of the triangulation and the parametric images of the line segments  $(\xi, 0)$ ,  $(1, \xi)$  and  $(\xi, \xi)$ ,  $\xi \in [0, 1]$  are the edges.

When considering a pair of patches  $\Gamma_p(t)$  and  $\Gamma_q(t)$  with a common vertex, then without loss of generality the common vertex can be assumed to be the parametric image of the origin  $(0,0) \in \sigma^{(2)}$ . Otherwise, we can take the affine transformations  $S_p$  and  $S_q$  of  $\sigma^{(2)}$  that map the common vertex into the origin and consider the parameterizations  $\mathbf{x}_p \circ S_p$  and  $\mathbf{x}_q \circ S_q$ . With such a transformation it is also possible to parameterize a common edge by the line segment  $(\xi, 0)$ ,  $\xi \in [0, 1]$ .

We make the following assumptions on the triangulation and the parameterizations.

### Assumption 2.1.

- (1) *The triangulation is time conforming in the sense that for all  $t \in I$  the intersection of two different triangular patches  $\Gamma_p(t) \cap \Gamma_q(t)$  is either empty, a common vertex or a common edge. Moreover, if two patches intersect for some  $t \in I$ , they intersect for all  $t \in I$  in the same vertex or edge. The vertices and edges may, of course, move in time.*
- (2)  *$\mathbf{x}_p \in C^\infty(\sigma^{(2)} \times I)$  and there are constants  $C, c > 0$  such that*

$$\begin{aligned} |\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_p(\hat{\mathbf{y}}, t)| &\geq ch_s |\hat{\mathbf{x}} - \hat{\mathbf{y}}|, \\ |\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_p(\hat{\mathbf{y}}, \tau)| &\leq C(h_s |\hat{\mathbf{x}} - \hat{\mathbf{y}}| + |t - \tau|). \end{aligned}$$

Here  $h_s$  is the maximal diameter of  $\Gamma_p(t)$ .

- (3) If  $\Gamma_p(t)$  and  $\Gamma_q(t)$  intersect in a common vertex or edge, then they satisfy the cone condition stated below in definition 2.2.
- (4) If  $\Gamma_p(t)$  and  $\Gamma_q(t)$  intersect in a common edge, then  $\mathbf{x}_p(\xi, 0, t) = \mathbf{x}_q(\xi, 0, t)$ ,  $\xi \in [0, 1]$ .

**Definition 2.2.**

- (1) If  $\Gamma_p(t)$  and  $\Gamma_q(t)$  have one common vertex  $\mathbf{v}_0$ , then they satisfy the cone condition if there is a  $\gamma \in (0, 1)$  such that for all  $\mathbf{x} \in \Gamma_p$  and  $\mathbf{y} \in \Gamma_q$  and  $t \in I$

$$(\mathbf{x} - \mathbf{v}_0) \cdot (\mathbf{y} - \mathbf{v}_0) \leq (1 - \gamma)|\mathbf{x} - \mathbf{v}_0||\mathbf{y} - \mathbf{v}_0|$$

holds.

- (2) If  $\Gamma_p$  and  $\Gamma_q$  have a common edge then for  $\mathbf{x} = \mathbf{x}_p(\hat{\mathbf{x}}, t) \in \Gamma_p$  the point  $\mathbf{v}_x = \mathbf{x}_p(x_1, 0, t)$  is the projection of  $\mathbf{x}$  on the common edge. Further, the two panels satisfy the cone condition if there is a  $\gamma > 0$  such that for all  $\mathbf{x} \in \Gamma_p$ ,  $\mathbf{y} \in \Gamma_q$  and  $t \in I$

$$(\mathbf{x} - \mathbf{v}_x) \cdot (\mathbf{y} - \mathbf{v}_x) \leq (1 - \gamma)|\mathbf{x} - \mathbf{v}_x||\mathbf{y} - \mathbf{v}_x|$$

holds.

The first cone condition states that for every  $\mathbf{x} \in \Gamma_p$  there is a cone with vertex  $\mathbf{v}_0$  and axis in direction of  $\mathbf{x} - \mathbf{v}_0$  such that  $\Gamma_q$  intersects with this cone only in  $\mathbf{v}_0$ . Similarly, the second condition states that there is a cone with vertex  $\mathbf{v}_x$  and axis in direction of  $\mathbf{x} - \mathbf{v}_x$  such that  $\Gamma_q$  intersects with this cone only in  $\mathbf{v}_x$ . Since  $p$  and  $q$  are interchangeable, a similar condition holds for  $\mathbf{y}$ ,  $\mathbf{v}_y$  and  $\Gamma_p$ .

If  $\mathbf{x} \in \Gamma_p$ ,  $\mathbf{y} \in \Gamma_q$  and  $\mathbf{v}$  is either  $\mathbf{v}_0$  or  $\mathbf{v}_x$  or  $\mathbf{v}_y$  in either cone condition, then the computation

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^2 &= |\mathbf{x} - \mathbf{v}|^2 + |\mathbf{y} - \mathbf{v}|^2 - 2(\mathbf{x} - \mathbf{v}) \cdot (\mathbf{y} - \mathbf{v}) \\ &\geq |\mathbf{x} - \mathbf{v}|^2 + |\mathbf{y} - \mathbf{v}|^2 - 2(1 - \gamma)|\mathbf{x} - \mathbf{v}||\mathbf{y} - \mathbf{v}| \\ &\geq |\mathbf{x} - \mathbf{v}|^2 + |\mathbf{y} - \mathbf{v}|^2 - (1 - \gamma)(|\mathbf{x} - \mathbf{v}|^2 + |\mathbf{y} - \mathbf{v}|^2) \\ &= \gamma(|\mathbf{x} - \mathbf{v}|^2 + |\mathbf{y} - \mathbf{v}|^2) \end{aligned}$$

implies the inequality

$$(6) \quad |\mathbf{x} - \mathbf{y}| \geq \alpha(|\mathbf{x} - \mathbf{v}| + |\mathbf{y} - \mathbf{v}|)$$

with  $\alpha = \sqrt{\gamma/2}$ .

There are many practical situations where assumption 2.1 is satisfied. To illustrate these ideas, consider a time conforming triangulation where all patches are flat. If the vertices of  $\Gamma_p$  are  $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$  then an obvious parameterization is

$$\mathbf{x}_p(\hat{\mathbf{x}}, t) = \mathbf{v}_0(t) + \hat{x}_1 \mathbf{a}_1(t) + \hat{x}_2 \mathbf{a}_2(t)$$

where  $\mathbf{a}_1(t) = \mathbf{v}_1(t) - \mathbf{v}_0(t)$ ,  $\mathbf{a}_2(t) = \mathbf{v}_2(t) - \mathbf{v}_1(t)$  are two of the three edges.

If all interior angles are positive, then condition (2.1) is obviously satisfied. Further, if the angle between adjacent panels remains positive for all times, then the cone conditions are also satisfied. Note that for a flat triangle, the vector  $\mathbf{x} - \mathbf{v}_x$  is parallel to  $\mathbf{a}_2$ , see Figure 1.

Below, we will frequently use the following fact which follows from standard calculus of multivariate functions.

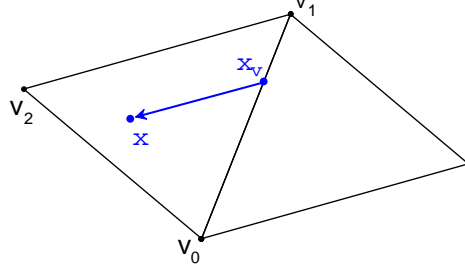


FIGURE 1. A pair of triangles with a common edge and the projection  $\mathbf{x}_v$  of a point  $\mathbf{x}$ .

**Lemma 2.3.** *If  $D \subset \mathbb{R}^d$  is convex and  $\mathbf{f}$  is a smooth and possibly vector valued function in  $D$ , then there are smooth functions  $\mathbf{a}_k$  such that for  $\mathbf{z}, \mathbf{w} \in D$*

$$\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{w}) = \sum_{k=1}^d \mathbf{a}_k(\mathbf{z}, \mathbf{w})(z_k - w_k)$$

holds.

We make the following observations about patches with nonempty intersections in a time dependent triangulation that satisfies assumption 2.1.

**Self Panel,**  $p = q$ . The assumption states that

$$(7) \quad |\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_p(\hat{\mathbf{y}}, t)| \geq ch_s |\hat{\mathbf{x}} - \hat{\mathbf{y}}|.$$

Further, by setting  $\mathbf{z} = (\hat{\mathbf{x}}, t)$ ,  $\mathbf{w} = (\hat{\mathbf{y}}, \tau)$  and  $\mathbf{f} = \mathbf{x}_p$  it follows from Lemma 2.3 that there are smooth functions  $\mathbf{a}_1 = \mathbf{a}_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}, t, \tau)$ ,  $\mathbf{a}_2 = \mathbf{a}_2(\hat{\mathbf{x}}, \hat{\mathbf{y}}, t, \tau)$  and  $\mathbf{v} = \mathbf{v}(\hat{\mathbf{x}}, \hat{\mathbf{y}}, t, \tau)$  such that

$$(8) \quad \mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_p(\hat{\mathbf{y}}, \tau) = \mathbf{a}_1(\hat{x}_1 - \hat{y}_1) + \mathbf{a}_2(\hat{x}_2 - \hat{y}_2) + \mathbf{v}(t - \tau)$$

To make our notations less cluttered we omit the dependence of the vectors  $\mathbf{a}_1$ ,  $\mathbf{a}_2$  and  $\mathbf{v}$  on the variables  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ ,  $t$  and  $\tau$ .

**Common Edge.** If  $\Gamma_p$  and  $\Gamma_q$  have a common edge then it follows from applying (6) two times, that

$$\begin{aligned} & |\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_q(\hat{\mathbf{y}}, t)| \\ & \geq \alpha (|\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_p(x_1, 0, t)| + |\mathbf{x}_p(x_1, 0, t) - \mathbf{x}_q(\hat{\mathbf{y}}, t)|) \\ & \geq \alpha^2 (|\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_p(x_1, 0, t)| + |\mathbf{x}_p(x_1, 0, t) - \mathbf{x}_q(y_1, 0, t)| + |\mathbf{x}_q(y_1, 0, t) - \mathbf{x}_q(\hat{\mathbf{y}}, t)|) \end{aligned}$$

and thus it follows from (7) that there is a constant  $c > 0$  such that

$$(9) \quad |\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_q(\hat{\mathbf{y}}, t)| \geq ch_s (|\hat{x}_1 - \hat{y}_1| + \hat{x}_2 + \hat{y}_2).$$

If, in Lemma 2.3, we let  $\mathbf{z} = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, t, \tau)$ ,  $\mathbf{w} = (\hat{y}_1, 0, \hat{y}_1, 0, \tau, \tau)$  and  $\mathbf{f}(\mathbf{z}) = \mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_q(\hat{\mathbf{y}}, \tau)$  then  $\mathbf{f}(\mathbf{w}) = \mathbf{x}_p(\hat{y}_1, 0, \tau) - \mathbf{x}_q(\hat{y}_1, 0, \tau) = \mathbf{0}$  and thus there are smooth functions  $\mathbf{a}_1, \dots, \mathbf{a}_3$  and  $\mathbf{v}$ , such that

$$(10) \quad \mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_q(\hat{\mathbf{y}}, \tau) = \mathbf{a}_1(\hat{x}_1 - \hat{y}_1) + \mathbf{a}_2 \hat{x}_2 + \mathbf{a}_3 \hat{y}_2 + \mathbf{v}(t - \tau).$$

**Common Vertex.** If  $\Gamma_p$  and  $\Gamma_q$  have exactly one common vertex then

$$(11) \quad |\mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_q(\hat{\mathbf{y}}, t)| \geq ch_s (\hat{x}_1 + \hat{x}_2 + \hat{y}_1 + \hat{y}_2).$$

which can be derived from the cone condition using similar arguments as before.

If we let  $\mathbf{z} = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, t, \tau)$ ,  $\mathbf{w} = (\mathbf{0}, \mathbf{0}, \tau, \tau)$  and  $\mathbf{f}(\mathbf{z}) = \mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_q(\hat{\mathbf{y}}, \tau)$  then  $\mathbf{f}(\mathbf{w}) = \mathbf{x}_p(\mathbf{0}, \tau) - \mathbf{x}_q(\mathbf{0}, \tau) = \mathbf{0}$  and thus it follows from Lemma 2.3 that there are smooth functions  $\mathbf{a}_1, \dots, \mathbf{a}_4$  and  $\mathbf{v}$ , such that

$$(12) \quad \mathbf{x}_p(\hat{\mathbf{x}}, t) - \mathbf{x}_q(\hat{\mathbf{y}}, \tau) = \mathbf{a}_1 \hat{x}_1 + \mathbf{a}_2 \hat{x}_2 + \mathbf{a}_3 \hat{y}_1 + \mathbf{a}_4 \hat{y}_2 + (t - \tau) \mathbf{v}.$$

### 3. SPATIAL TRANSFORMATIONS

It will become apparent in Section 5 below that space and time can be treated separately when designing singularity removing transformations. In this section we begin with suitable transformations for the spatial variables  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  in (4). They are designed to handle singularities of the integrand when  $\mathbf{x} = \mathbf{y}$ , which can happen when  $\Gamma_p$  and  $\Gamma_q$  are either the same patch, or patches with a common edge or vertex. This parallels the construction for elliptic integral operators. We recall this here for completeness, following the discussion in [19] closely.

**Self Panel,  $p = q$ .** Since the integrand is singular when  $\hat{\mathbf{x}} = \hat{\mathbf{y}}$ , we introduce the transformation

$$(13) \quad \begin{aligned} z_1 &= \hat{y}_1 - \hat{x}_1 & \hat{x}_1 &= w_1 \\ z_2 &= \hat{y}_2 - \hat{x}_2 & \hat{x}_2 &= w_2 \\ w_1 &= \hat{x}_1 & \hat{y}_1 &= z_1 + w_1 \\ w_2 &= \hat{x}_2 & \hat{y}_2 &= z_2 + w_2 \end{aligned}$$

For  $\hat{\mathbf{x}}, \hat{\mathbf{y}}$  in  $\sigma^{(2)}$  it is easy to see that the  $\mathbf{z} = (z_1, z_2)$ -variables must be in the union of six triangles

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in D_1 \cup \dots \cup D_6,$$

see, Figure 2. Each triangle  $D_k$  can be mapped on the unit square with the Duffy

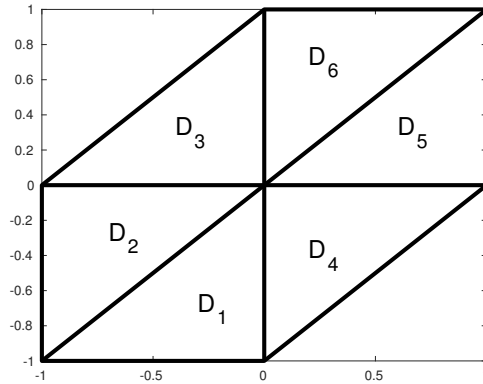


FIGURE 2. Domain of  $z$ -integration for the self panel case

transform. Thus

$$(14) \quad \mathbf{z} = \xi \mathbf{r}_k(\eta_1), \quad \mathbf{z} \in D_k, \quad \xi, \eta_1 \in [0, 1]^2,$$

for  $k \in \{1, \dots, 6\}$ . The domain of integration of the  $\mathbf{w} = (w_1, w_2)$ -variables depends on  $\mathbf{z}$ . From (13) it follows that

$$W_k(\mathbf{z}) = \sigma^{(2)} \cap \left( -\mathbf{z} + \sigma^{(2)} \right)$$

Since  $W_k(\mathbf{z})$  is the intersection of  $\sigma^{(2)}$  with a shifted version of  $\sigma^{(2)}$  it is an axiparallel isosceles right triangle with sidelength  $1 - \xi$ . Thus the transformation that maps the unit square to  $W_k(\mathbf{z})$  is

$$\mathbf{w} = \mathbf{v}_k + (1 - \xi) \begin{bmatrix} \eta_2 \\ \eta_2 \eta_3 \end{bmatrix}, \quad (\eta_2, \eta_3) \in [0, 1]^2.$$

where  $\mathbf{v}_k$  is the lower left corner. For the six subdomains we thus obtain

$$(15) \quad \begin{aligned} k = 1 : \quad \mathbf{r}_1 &= \begin{bmatrix} -\eta_1 \\ -1 \end{bmatrix} & \mathbf{v}_1 &= \begin{bmatrix} \xi(1 - \eta_1) \\ 0 \end{bmatrix} \\ k = 2 : \quad \mathbf{r}_2 &= \begin{bmatrix} -1 \\ -\eta_1 \end{bmatrix} & \mathbf{v}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ k = 3 : \quad \mathbf{r}_3 &= \begin{bmatrix} -\eta_1 \\ 1 - \eta_1 \end{bmatrix} & \mathbf{v}_3 &= \begin{bmatrix} \xi(1 - \eta_1) \\ \xi(1 - \eta_1) \end{bmatrix} \\ k = 4 : \quad \mathbf{r}_4 &= \begin{bmatrix} 1 - \eta_1 \\ -\eta_1 \end{bmatrix} & \mathbf{v}_4 &= \begin{bmatrix} \xi \\ 0 \end{bmatrix} \\ k = 5 : \quad \mathbf{r}_5 &= \begin{bmatrix} 1 \\ \eta_1 \end{bmatrix} & \mathbf{v}_5 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ k = 6 : \quad \mathbf{r}_6 &= \begin{bmatrix} \eta_1 \\ 1 \end{bmatrix} & \mathbf{v}_6 &= \begin{bmatrix} \xi \\ \xi \end{bmatrix} \end{aligned}$$

In all cases the Jacobian of the transformation is  $J = \xi(1 - \xi)^2 \eta_2$ .

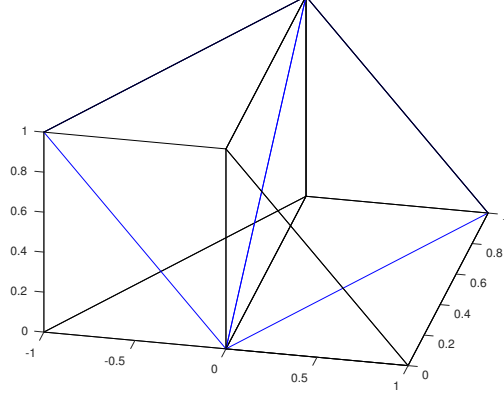
**Common Edge.** In this case we introduce the transformation

$$(16) \quad \begin{aligned} z_1 &= \hat{x}_1 - \hat{y}_1 & \hat{x}_1 &= z_1 + w_1 \\ z_2 &= \hat{x}_2 & \hat{x}_2 &= z_2 \\ z_3 &= \hat{y}_2 & \hat{y}_1 &= w_1 \\ w_1 &= \hat{y}_1 & \hat{y}_2 &= z_3 \end{aligned}$$

For  $\hat{\mathbf{x}}, \hat{\mathbf{y}}$  in  $\sigma^{(2)}$  it is easy to see that the  $\mathbf{z} = (z_1, z_2, z_3)$ -variables must be in the union of four polyhedra  $D_1 \cup D_2 \cup D_3 \cup D_4$ , which have one vertex in the origin and are characterized by the opposite face see, Figure 3.

Thus we write

$$(17) \quad \mathbf{z} = \xi \mathbf{r}_k(\eta_1, \eta_2), \quad \mathbf{z} \in D_k, \quad \xi, \eta_1, \eta_2 \in [0, 1]^3,$$

FIGURE 3. Domain of  $z$ -integration for the case of a common edge

for  $k \in \{1, \dots, 4\}$ , where  $\mathbf{r}_k(\eta_1, \eta_2)$  parameterizes the opposite face. Moreover, it follows from (16) that  $w_1$  must be in the intersection  $w_1 \in [-z_1, 1 - z_1] \cap [0, 1]$ .  
(18)

$$\begin{aligned}
 k = 1 : \quad \mathbf{r}_1 &= \begin{bmatrix} -\eta_1 \\ 1 - \eta_1 \\ \eta_2 \end{bmatrix} & w_1 &= \xi + (1 - \xi)\eta_3 & J_1 &= (1 - \xi)\xi^2 \\
 k = 2 : \quad \mathbf{r}_2 &= \begin{bmatrix} -\eta_1\eta_2 \\ (1 - \eta_1)\eta_2 \\ 1 \end{bmatrix} & w_1 &= \xi + (1 - \xi)\eta_3 & J_2 &= (1 - \xi)\xi^2\eta_2 \\
 k = 3 : \quad \mathbf{r}_3 &= \begin{bmatrix} 1 - \eta_1 \\ \eta_2 \\ \eta_1 \end{bmatrix} & w_1 &= (1 - \eta_1)\xi + (1 - \xi)\eta_3 & J_3 &= (1 - \xi)\xi^2 \\
 k = 4 : \quad \mathbf{r}_4 &= \begin{bmatrix} \eta_1\eta_2 \\ 1 \\ (1 - \eta_1)\eta_2 \end{bmatrix} & w_1 &= \xi(1 - \eta_1\eta_2) + (1 - \xi)\eta_3 & J_4 &= (1 - \xi)\xi^2\eta_2
 \end{aligned}$$

**Common Vertex.** In this case the transformation is simply

$$\begin{aligned}
 z_1 &= \hat{x}_1 \\
 z_2 &= \hat{x}_2 \\
 z_3 &= \hat{y}_2 \\
 z_4 &= \hat{y}_1
 \end{aligned}$$

The  $\mathbf{z}$ -domain can be split into two regions  $\sigma^{(2)} \otimes \sigma^{(2)} = D_1 \cup D_2$ , where

$$D_1 = \left\{ \mathbf{z} : \begin{array}{l} 0 \leq z_2 \leq z_1 \leq 1 \\ 0 \leq z_4 \leq z_3 \leq 1 \\ z_3 \leq z_1 \end{array} \right\}, \text{ and } D_2 = \left\{ \mathbf{z} : \begin{array}{l} 0 \leq z_2 \leq z_1 \leq 1 \\ 0 \leq z_4 \leq z_3 \leq 1 \\ z_1 \leq z_3 \end{array} \right\}.$$

Thus

$$(19) \quad \mathbf{z} = \xi \mathbf{r}_k(\eta_1, \eta_2 \eta_3), \quad \mathbf{z} \in D_k, \quad \xi, \eta_1, \eta_2 \in [0, 1]^3,$$



where

$$(20) \quad \mathbf{r}_1 = \begin{bmatrix} 1 \\ \eta_1 \\ \eta_2 \\ \eta_2\eta_3 \end{bmatrix} \text{ and } \mathbf{r}_2 = \begin{bmatrix} \eta_2 \\ \eta_2\eta_3 \\ 1 \\ \eta_1 \end{bmatrix}.$$

In both cases the Jacobian is  $J = \xi^3\eta_2$ .

#### 4. TEMPORAL TRANSFORMATIONS

We start with a transformation for a singular time integral

$$I_0 = \int_{t_i}^{t_{i+1}} \int_{t_i}^t \frac{f(t, \tau)}{(t - \tau)^{\frac{n}{2}}} d\tau dt$$

where  $f$  is a smooth function. The transformation

$$(21) \quad \begin{aligned} \zeta &= \sqrt{(t - \tau)/h_t} & t &= h_t(\zeta^2 + \omega) + t_i, \\ \omega &= (\tau - t_i)/h_t & \tau &= h_t\omega + t_i, \end{aligned}$$

which has Jacobian  $J = 2h_t^2\zeta$ . This transformation maps the triangular  $(t, \tau)$ -integration domain to  $\{(\zeta, \omega) : 0 \leq \omega \leq 1 - \zeta^2, 0 \leq \zeta \leq 1\}$ . In the second step we transform the  $\omega$ -variable

$$\omega = (1 - \zeta^2)\eta_4$$

which gives a unit square in the  $(\zeta, \eta_4)$ -plane. Thus

$$I_0 = 2h_t^{2-\frac{n}{2}} \int_0^1 \int_0^1 \zeta^{1-n} \tilde{f}(\zeta, \eta_4) d\zeta d\eta_4$$

where  $\tilde{f}$  is a smooth function that incorporates the Jacobian from the  $\eta_4$ -transformation. The integrand is smooth as long as  $n \leq 1$ .

In the case  $i = j + 1$  the integral the singularity occurs only at one point  $t = \tau = t_i$ , thus we split the integral into two triangular regions

$$I_1 = \int_{t_i}^{t_{i+1}} \int_{t_{i-1}}^{t_i} \frac{f(t, \tau)}{(t - \tau)^{\frac{n}{2}}} d\tau dt = \int_{t_i}^{t_{i+1}} \int_{t_{i-1}}^{t-h_t} \frac{f(t, \tau)}{(t - \tau)^{\frac{n}{2}}} d\tau dt + \int_{t_i}^{t_{i+1}} \int_{t-h}^{t_i} \frac{f(t, \tau)}{(t - \tau)^{\frac{n}{2}}} d\tau dt$$

where the first integral can be treated by standard means and the second integral (denoted by  $I_1^+$ ) contains the singularity. To derive the transformation for the second integral consider the transformation

$$(22) \quad \begin{aligned} \zeta &= \sqrt{(t - \tau)/h_t} & t &= h_t(\zeta^2 - \omega) + t_i, \\ \omega &= (t_i - \tau)/h_t & \tau &= t_i - h_t\omega, \end{aligned}$$

which has Jacobian  $J = 2h_t^2\zeta$ . This transformation maps the triangle in the  $(t, \tau)$  plane to the domain  $\{(\zeta, \omega) : 0 \leq \omega \leq \zeta^2, 0 \leq \zeta \leq 1\}$ . In the second step we transform the  $\omega$ -variable

$$\omega = \zeta^2\eta_4$$

which gives a unit square in the  $(\zeta, \eta_4)$ -plane. Thus

$$I_1^+ = h_t^{2-\frac{n}{2}} \int_0^1 \int_0^1 \zeta^{3-n} \tilde{f}(\zeta, \eta_4) d\zeta d\eta_4$$

has a smooth integrand as long as  $n \leq 3$ .

## 5. COMBINING THE SPACE AND TIME TRANSFORMATIONS

We now turn to the type of integrals that have to be computed in the implementation of the Galerkin boundary element method of parabolic problems. In the parametrization of section 2 the integral (4) appears as

$$(23) \quad I_{p,q}^{i,j} = \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \int_{\sigma^{(2)}} \int_{\sigma^{(2)}} \frac{1}{(t-\tau)^{\frac{n}{2}}} K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}) \hat{\Phi}(\hat{\mathbf{x}}, \hat{\mathbf{y}}, t, \tau) d^2 \hat{\mathbf{x}} d^2 \hat{\mathbf{y}} d\tau dt.$$

where  $\boldsymbol{\rho}$ ,  $\mathbf{x}$  and  $\mathbf{y}$  are now considered functions of  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  and the smooth function  $\hat{\Phi}$  incorporates all shape functions and Jacobians of surface parameterizations.

This integral is singular only if  $i = j$  or  $i = j + 1$  and if the panels are identical or have a common edge or vertex. Together with the temporal transforms (21) or (22) we obtain from (23) either six, four or two integrals of the type

$$(24) \quad I = \int_0^1 \int_0^1 \int_{[0,1]^4} \frac{\xi^m}{\zeta^{n-1}} K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}) \tilde{\Phi}(\xi, \zeta, \boldsymbol{\eta}) d^4 \boldsymbol{\eta} d\xi d\zeta,$$

where  $\xi^m$ ,  $m \in \{1, 2, 3\}$  comes from the Jacobian in the spatial transformations,  $\tilde{\Phi}$  is the function  $\Phi$  in the transformed coordinates and also contains Jacobians and  $\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}$  are functions of the new variables.

Integral (24) is still singular. To regularize it split the unit square in the  $(\xi, \zeta)$ -plane into two triangles and apply the Duffy transforms

$$(25) \quad \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \mu \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = \lambda \begin{bmatrix} \mu \\ 1 \end{bmatrix}.$$

The Jacobians contributes an extra factor of  $\lambda$ . This results in two integrals  $I = I_a + I_b$ , where

$$(26) \quad I_a = \int_0^1 \int_0^1 \int_{[0,1]^4} \frac{\lambda^{2+m-n}}{\mu^{n-1}} K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}) \Phi(\lambda, \mu, \boldsymbol{\eta}) d^4 \boldsymbol{\eta} d\lambda d\mu,$$

$$(27) \quad I_b = \int_0^1 \int_0^1 \int_{[0,1]^4} \lambda^{2+m-n} \mu^m K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}) \Phi(\lambda, \mu, \boldsymbol{\eta}) d^4 \boldsymbol{\eta} d\lambda d\mu.$$

In the remainder of this section we show that  $I_a$  and  $I_b$  are regular integrals and therefore standard rules can be used for their quadrature. In all singular cases, it follows from (8), (10), (12), and the discussion in Section 3 that

$$(28) \quad \mathbf{x} - \mathbf{y} = \xi A \mathbf{r} + (t - \tau) \mathbf{v}.$$

where  $A = A(\xi, \boldsymbol{\eta}, t, \tau) \in \mathbb{R}^{3 \times 2}$  is a matrix, and  $\mathbf{r} = \mathbf{r}(\boldsymbol{\eta})$  is the vector defined in (14), (17) and (19), respectively.

**Lemma 5.1.** *There are constants  $c > 0$ ,  $\tau_0 > 0$  such that for  $(t, \tau) \in \sigma^{(2)}$  with  $\tau \geq t - \tau_0$  the estimate*

$$(29) \quad |A\mathbf{r}| \geq \frac{c}{2}h_s$$

*holds.*

*Proof.* First consider the case  $t = \tau$ . It follows from (7), (9), (11) and going through the transformations in section 3 that

$$(30) \quad |\mathbf{x} - \mathbf{y}| \geq ch_s|\mathbf{z}| = ch_s\xi|\mathbf{r}|$$

Recall that in all singular cases (15) (18) and (20)  $\mathbf{r} = \mathbf{r}(\boldsymbol{\eta})$  parameterizes a side of the  $\mathbf{z}$ -domain that is opposite to the origin. Hence the vector  $\mathbf{r}$  is always non-zero, in fact, it can be found that  $|\mathbf{r}| \geq 1/\sqrt{2}$ . Moreover, because of  $t = \tau$  equation (28) simplifies to

$$(31) \quad \mathbf{x} - \mathbf{y} = \xi A\mathbf{r}.$$

Combining (30) and (31) gives

$$(32) \quad |A\mathbf{r}| \geq \frac{c}{\sqrt{2}}h_s$$

which holds for  $A = A(\xi, \boldsymbol{\eta}, t, t)$ . By continuity of  $A$ , we can conclude that for some  $\tau_0 > 0$  and a smaller constant on the right hand side, the above inequality also holds for  $\tau$  when  $t - \tau_0 \leq \tau \leq t$ . This is the assertion.  $\square$

**Theorem 5.2.** *If  $2 + m - n \geq 0$  and  $n \leq M + 1$  then the integrands of  $I_a$  and  $I_b$  are  $C^\infty$  in all variables.*

*Proof.* Since the transformations  $(\lambda, \mu, \boldsymbol{\eta}) \rightarrow (\mathbf{x}, \mathbf{y}, t, \tau)$  are all  $C^\infty$  the regularity of  $I_b$  follows from the smoothness of  $K$  and  $\Phi$ .

Because of the negative powers of  $\mu$  in  $I_a$  the smoothness of the transformations only implies smoothness of the integrand only for  $\mu \in (0, 1]$ . For  $\mu \rightarrow 0$  we have from lemma 29 and  $\zeta/\xi = \mu$  that

$$|\boldsymbol{\rho}| = \frac{|\mathbf{x} - \mathbf{y}|}{\sqrt{4(t - \tau)}} = \frac{\xi}{\zeta} \frac{|A\mathbf{r}|}{\sqrt{4h_t}} \geq \frac{c}{4} \frac{h_s}{\sqrt{h_t}} \frac{1}{\mu}.$$

Therefore assumption 1.1 implies that  $K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y})$  is smooth for the given range of  $n$  and the proof is complete.  $\square$

## 6. EXAMPLES OF KERNELS

In this section we verify that the assumptions of 1.1 and theorem 5.2 are indeed satisfied for the typical layer potentials of the Heat equation and transient Stokes flow.

**Single Layer Heat Kernel.** For the single layer operator of the heat equation the integrand of (23)

$$K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}) = \exp\left(-|\boldsymbol{\rho}|^2\right)$$

with  $n = 3$ . The assumption 1.1 are satisfied for any positive integer  $M$  and thus by 5.2 the transformations discussed in the previous sections will lead to a regular integral.

**Double Layer Heat Kernel.** For the double layer operator we have to consider the quantity  $(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y$ . From standard calculus of surface parameterizations, it can be concluded that there is a matrix  $B = B(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \tau)$ , a vector  $\mathbf{b} = \mathbf{b}(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \tau)$  and a scalar  $\beta = \beta(\hat{\mathbf{y}}, t, \tau)$  such that for  $\mathbf{x} \in \Gamma_p$  and  $\mathbf{y} \in \Gamma_q$

$$(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y = \begin{cases} (\hat{\mathbf{x}} - \hat{\mathbf{y}})^T B(\hat{\mathbf{x}} - \hat{\mathbf{y}}) + \beta(t - \tau) & \text{if } \Gamma_p = \Gamma_q, \\ \mathbf{b}^T(\hat{\mathbf{x}} - \hat{\mathbf{y}}) + \beta(t - \tau) & \Gamma_p, \Gamma_q \text{ have a common vertex or edge.} \end{cases}$$

With the transformations of section 3 and 4 it follows that there are scalar functions  $\tilde{b} = \tilde{b}(\xi, \zeta, \boldsymbol{\eta})$  and  $\tilde{\beta} = \tilde{\beta}(\xi, \zeta, \boldsymbol{\eta})$  such that

$$\frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y}{2(t - \tau)} = \begin{cases} \left(\frac{\xi}{\zeta}\right)^2 \tilde{b} + \tilde{\beta} & \text{if } \Gamma_p = \Gamma_q, \\ \frac{\xi}{\zeta^2} \tilde{b} + \tilde{\beta} & \Gamma_p, \Gamma_q \text{ have a common vertex or edge.} \end{cases}$$

The double layer kernel of the heat equation is the product of this term with the single layer kernel. The effect of the additional factor is that the leading term in integral (24) is  $(\xi/\zeta)^3$ , because  $n = 3$  and  $m = 1$  in the self case and  $m \geq 2$  in the case of adjacent panels. Thus the Duffy transform (25) will result in a regular integrals.

**Stokeslet.** We use the form of the time dependent Stokeslet which can be found in [5]

$$K_{kl}(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}) = \frac{1}{s^2} \left[ \delta_{kl} \psi_1(|\boldsymbol{\rho}|) + \rho_k \rho_l \psi_2(|\boldsymbol{\rho}|) \right]$$

where  $k, l \in \{1, 2, 3\}$  and the functions  $\psi_1, \psi_2$  are given by

$$\begin{aligned} \psi_1(\rho) &= \frac{1}{\pi^2} \frac{1}{\rho^3} \left( \gamma\left(\frac{3}{2}, \rho^2\right) - \gamma\left(\frac{5}{2}, \rho^2\right) \right), \\ \psi_2(\rho) &= \frac{1}{\pi^2} \frac{1}{\rho^5} \gamma\left(\frac{5}{2}, \rho^2\right). \end{aligned}$$

Here  $\gamma(\alpha, z)$  is the incomplete gamma function

$$\gamma(\alpha, z) = \int_0^z \zeta^{\alpha-1} e^{-\zeta} d\zeta, \quad \alpha > 0,$$

see e.g. [18]. For our purposes it is only important that  $\gamma(\alpha, z) = z^\alpha \gamma^*(\alpha, z)$  where  $\gamma^*$  is an entire function and that  $\gamma(\alpha, z) \rightarrow \Gamma(\alpha)$  as  $z \rightarrow \infty$  on the real axis. Thus the  $\psi$ -functions are smooth and the Stokeslet satisfies assumption 1.1 with  $n = M = 3$ . Therefore the transformations in the previous sections will result in regular integrals.

**Stresslet.** We now turn to the stresslet of the transient Stokes flow. It is given by

$$(33) \quad T_{ijk}(\boldsymbol{\rho}) = \frac{1}{s^2} \left[ (\delta_{jk} \rho_i + \delta_{ik} \rho_j) \psi_3(|\boldsymbol{\rho}|) + \delta_{ij} \rho_k \psi_4(|\boldsymbol{\rho}|) + \rho_i \rho_j \rho_k \psi_5(|\boldsymbol{\rho}|) \right]$$

where the fuctions  $\psi_3, \psi_4$  and  $\psi_5$  are defined by

$$(34) \quad \begin{aligned} \psi_3(\rho) &= \frac{1}{\pi^{\frac{3}{2}}} \frac{1}{\rho^5} (3\gamma(\frac{5}{2}, \rho^2) - 2\gamma(\frac{7}{2}, \rho^2)), \\ \psi_4(\rho) &= -\frac{2}{\pi^{\frac{3}{2}}} \frac{1}{\rho^5} \gamma(\frac{5}{2}, \rho^2), \\ \psi_5(\rho) &= \frac{4}{\pi^{\frac{3}{2}}} \frac{1}{\rho^7} \gamma(\frac{7}{2}, \rho^2), \end{aligned}$$

see [5]. The double layer kernel is

$$(35) \quad K_{ik}(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y}) = T_{ijk}(\mathbf{r}, s) n_j(\mathbf{y}) + \frac{1}{4\pi} \frac{r_i}{|\mathbf{r}|^3} n_k(\mathbf{y}) \delta(s),$$

where we use the Einstein summation convention. The delta function implies that the double layer potential involves an additional surface integral.

The Stresslet satisfies assumption 1.1 with  $n = 4$  and  $M = 4$ . Hence theorem 5.2 can only be applied for  $m \geq 2$ . In the self panel case, i.e.,  $m = 1$ , the integrals  $I_a$  and  $I_b$  in (26) and (27) are still singular. Likewise, for the surface integrals due to the second term in (35) the spatial transformations of section 3 will result in smooth integrands only when  $m \geq 2$ .

That the self panel case creates difficulties should be expected, because it is known that the singularity in the double layer kernel must be understood as Cauchy principal values, see [5]. In this paper it is shown that for a stationary surface the Galerkin influence coefficients become weakly singular after time integration. The reason is that the time integrated stresslet cancels the second term in (35). This follows easily from the fact that the antiderivatives of the incomplete gamma functions are

$$\partial_t^{-1} \gamma(\alpha, \rho^2) = \frac{|\mathbf{r}|^2}{4} \Gamma(\alpha - 1) + t \tilde{\gamma}(\alpha, \rho^2),$$

where the time dependence is in  $\rho$  and

$$\tilde{\gamma}(\alpha, \rho^2) = \gamma(\alpha, \rho^2) - \rho^2 \gamma(\alpha - 1, \rho^2).$$

Then the time integrated stresslet is

$$(36) \quad \int_0^s T_{ijk}(\mathbf{r}, s') ds' = -\frac{1}{4\pi} \delta_{ij} \frac{r_k}{|\mathbf{r}|^3} + \frac{3}{4\pi} \frac{r_i r_j r_k}{|\mathbf{r}|^5} + \frac{1}{s} T_{ijk}^w(\mathbf{r}, s),$$

where

$$T_{ijk}^w(\mathbf{r}, s) = \left[ (\delta_{jk} \rho_i + \delta_{ik} \rho_j) \tilde{\psi}_3(|\boldsymbol{\rho}|) + \delta_{ij} \rho_k \tilde{\psi}_4(|\boldsymbol{\rho}|) + \rho_i \rho_j \rho_k \tilde{\psi}_5(|\boldsymbol{\rho}|) \right].$$

The fuctions  $\tilde{\psi}_3, \tilde{\psi}_4$  and  $\tilde{\psi}_5$  are defined just as in (34) but with  $\gamma(\alpha, \rho^2)$  replaced by  $\tilde{\gamma}(\alpha, \rho^2)$ . The first term on the right hand side in (36) cancels the second term in (35). The other term in (36) is the stresslet of the steady Stokes and is weakly singular in space. The third term,  $T_{ijk}^w(\mathbf{r}, s)/s$ , is weakly singular in space-time and satisfies the assumptions of theorem 5.2 with  $n = 2$  and  $M = 3$ .

Consider now a typical integral for the double layer potential, that is, an integral of the form (4) where the function  $G$  is now  $K_{ik}$  of (35). To simplify notations,

we set  $d = 0$ . We split this integral into two pieces

$$I_{p,p}^w = \int_0^{h_t} \int_0^t \int_\sigma \int_\sigma K_{ik}(\mathbf{x} - \mathbf{y}, t - \tau) \phi(\hat{\mathbf{x}}, t, \hat{\mathbf{y}}, \tau) - K_{ik}(\mathbf{x} - \mathbf{y}', t - \tau) \phi(\hat{\mathbf{x}}, t, \hat{\mathbf{y}}, t) d\hat{\mathbf{y}} d\hat{\mathbf{x}} d\tau dt,$$

and

$$I_{p,p}^s = \int_0^{h_t} \int_\sigma \int_\sigma \int_0^t K_{ik}(\mathbf{x} - \mathbf{y}', t - \tau) d\tau \phi(\hat{\mathbf{x}}, t, \hat{\mathbf{y}}, t) d\hat{\mathbf{y}} d\hat{\mathbf{x}} dt.$$

where  $\mathbf{x} = \mathbf{x}_p(\hat{\mathbf{x}}, t)$ ,  $\mathbf{y} = \mathbf{x}_p(\hat{\mathbf{y}}, \tau)$  and  $\mathbf{y}' = \mathbf{x}_p(\hat{\mathbf{y}}, t)$ , that is the primed quantities are evaluated at  $t$  instead of  $\tau$ .

For the first integral we have from Lemma 2.3 and (7) that

$$\rho - \rho' = \mathbf{b}(\hat{\mathbf{x}}, \hat{\mathbf{y}}, t, \tau) (\hat{\mathbf{x}} - \hat{\mathbf{y}}) \sqrt{t - \tau},$$

where  $b$  is a smooth function. Since the functions in (34) are analytic, the factor  $\sqrt{t - \tau}$  contributes an additional factor of  $\zeta$  in (24). Thus we get  $n = 3$  in theorem 5.2 and therefore  $I_{p,p}^w$  has a smooth integrand in the transformed coordinates.

In integral  $I_{p,p}^s$  the  $\tau$  integration gives the time integrated stresslet. Since the  $r_k/|\mathbf{r}|$  part cancels, it remains to compute a surface integral for the stresslet of the steady Stokes kernel and an integral with  $T_{ijk}^w(\mathbf{r}, t)$ . The former is smooth in the transformed spatial coordinates. For the latter the spatial transformations and  $t = h_t \zeta^2$  result in an integral of the form

$$I_{p,p}^w = \int_0^1 \int_0^1 \int_{[0,1]^3} \tilde{T}_{ijk}^w \left( \frac{\xi}{\zeta} A \mathbf{r}, \xi, \zeta, \boldsymbol{\eta} \right) d^3 \boldsymbol{\eta} d\xi d\zeta.$$

Proceeding as before, with the Duffy transform (25) leads to an integral with a smooth integrand.

## 7. A NUMERICAL EXAMPLE

We have implemented the quadrature method to solve the two-dimensional heat equation. We consider a time dependent curve that is broken up into segments  $\Gamma_p(t)$ , which are parameterized as  $\mathbf{x} = \mathbf{x}_p(\hat{x}, t)$  where  $\hat{x} \in [0, 1]$ . Singularities occur when two segments are either identical or have a common end point. These can be treated by transforms similar to those described in Section 3. Since this is a straight forward application of this methodology we omit the details here, we only mention that after the time transforms of section 4 one obtains integrals of the form (24) with the only difference that the variable  $\boldsymbol{\eta}$  has now two instead of four components. Then the remainder of section 5 implies that we obtain smooth integrands.

For a moving geometry, the Green's representation formula is

$$\begin{aligned} \int_0^t \int_{\Gamma(t)} G(x, y, t, \tau) \frac{\partial u}{\partial n}(y, \tau) ds(y) d\tau = \\ \frac{1}{2} u(x, t) + \int_0^t \int_{\Gamma(t)} \left( \frac{\partial G}{\partial n}(x, y, t, \tau) - G(x, y, t, \tau) v_n(y, \tau) \right) u(y, \tau) ds(y) d\tau. \end{aligned}$$

Here,  $v_n$  is the normal velocity which is an additional term for time dependent surfaces. We solve the Dirichlet problem, where the ansatz and test space consist

of the piecewise constants in space and time. Furthermore,  $u$  is replaced by an interpolation of the given Dirichlet condition, that is constant in space and linear in time. We have tested two geometies, a unit sqare that rotatates in the time interval  $[0, 1]$  once about its center and a translating unit circle whose center has time dependent coordinats  $x_0(t) = 2 \cos(2\pi t) - 2$  and  $y_0(t) = 2 \sin(2\pi t)$ . We use tensor-product Gauss-Legendre quadrature to compute the transformed integrals. Thus the error of the solution depends on the meshwidth in space and time as well as the quadrature order. Tables 1 and 2 show the errors for mesh refinements and a fixed quadrature order of six.

$N_x \backslash N_t$	20	80	320	1280
10	0.57371	0.53451	0.55592	0.56598
20	0.48216	0.31812	0.31938	0.32498
40	0.53286	0.18653	0.16466	0.16906
80	16.7322	0.20098	0.08928	0.08544
160	****	0.44416	0.10549	0.04709

TABLE 1.  $L_2$  errors of the Galerkin solution. Rotating Square

$N_x \backslash N_t$	20	80	320	1280
10	0.97917	0.78284	0.70532	0.62064
20	0.91415	0.34885	0.34608	0.30676
40	0.97190	0.47609	0.10426	0.15189
80	13.9612	0.59329	0.17262	0.04972
160	****	0.51388	0.24019	0.07202

TABLE 2.  $L_2$  errors of the Galerkin solution. Translating Circle

We see that the theoretical  $O(\sqrt{h_t} + h_s)$  behaviour is well reproduced near the diagonals of the table. It is also apparent that errors in the lower left corners of the table become larger. This corresponds to small mesh ratios  $h_s/\sqrt{h_t}$ . The growth of error is not from discretization but from the quadrature error. This can be explained with equation (32). There it can be inferred that if the mesh ratio is decreased the function  $\mu \rightarrow K(\boldsymbol{\rho}, \mathbf{x}, \mathbf{y})$  becomes closer to a singular function, hence the quadrature error deteriorates. To illustrate this we compute the first column of table 1 for different quadrature orders. The errors shown in table ?? make clear that the overall error is indeed more sensitive on the quadrature error when  $h_s$  is small.

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