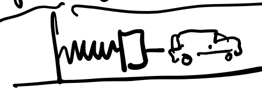
Laplace transform (Sections 6.1, 6.2, 6.3)

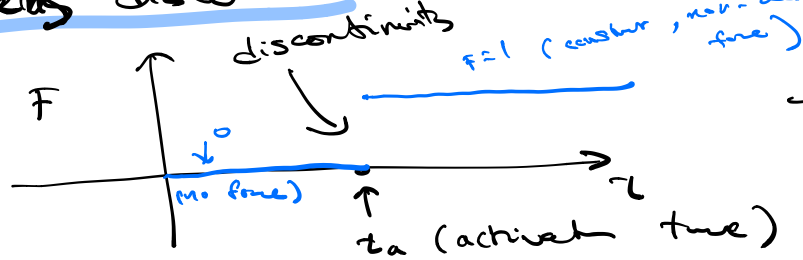
Motivation: ① Consider the spring system with an external force:



(at time t_a the car goes from not moving to having constant acceleration)

$$\ddot{x} = -kx + F(t)$$

where the force represents a kind of mechanical switch reflected in the force being discontinuous



What's the best method to analyze linear differential equations with discontinuous forcing terms?

② The unreasonable effectiveness of exponentials

In the previous example consider instead a force given by an exponential

$$\ddot{x} = -kx + e^{-\lambda t}$$

Using undetermined coefficients it then makes sense to look for a solution of the form

$$x(t) = c\,e^{-\lambda t}$$

let's plug this in and determine the value of c that makes this a solution.

$$\dot{x} = -\lambda c e^{-\lambda t}, \quad \ddot{x} = \lambda^2 c e^{-\lambda t}$$

$$\ddot{x} + kx = \lambda^2 c e^{-\lambda t} + k c e^{-\lambda t}$$

$$= \underbrace{(\lambda^2+k)c}_{=1} e^{-\lambda t} \overset{\text{goal}}{=} e^{-\lambda t}$$

Then,

$$\boxed{c = \frac{1}{\lambda^2+k}}$$

We observe that when it comes to exponentials, the operation of differentiation and the solution of differential equations becomes simple.

This fact behind the power of certain integral transformation in solving differential equations

General recipe:

1. Take an equation, say
$$\ddot{x} = -kx + \sigma\dot{x}$$

2. Use an integral transformation

on $x(t)$; that is compute a new function from t, like

$$X(s) = L(x(t))(s) = \underbrace{\int_a^b x(t)K(s,t)\,dt}_{\text{"integral transform"}}$$

new variable

③ Depending on the nature of the linear transformation, the the transformed function will have some special property* that allows to get a formula for $X(s)$

(*this will be connected to the equation solved by x)

④ Having determined a formula for $X(s)$ reverse the integral transformation to obtain a formula for $x(t)$.

The most popular integral transformations are:

(1) The Fourier transform: if $x(t)$ is defined for all t, then it's Fourier transform is the function

$$\hat{x}(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-2\pi i s t}\, dt$$

(2) The Laplace transform: if $x(t)$ is a function of $t > 0$, then its Laplace transform is

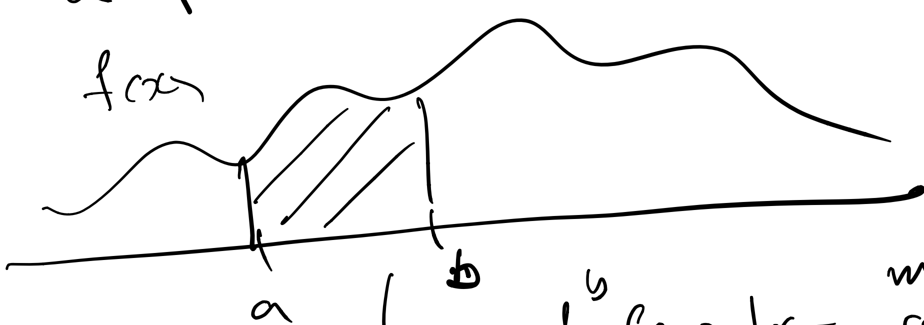
$$L(x)(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

new variable

Think of this expression as a limit of linear combination of exponentials, with the coefficient of e^{-ts} given by $x(t)$.

Some history: (Pierre Simon Laplace) ~ late 18th cent to early 19th century

If $f(x)$ is a mass distribution (or a probability distribution) then



$\int_a^b f(x)\,dx =$ mass contained in (a,b)

or $\int_a^b f(x)\,dx =$ probability that a random variable given by f takes value in (a,b).

One is interested in probability in calculating the "moments of f"

"1st moment" $\int_{-\infty}^{+\infty} f(x)|x|\,dx \longrightarrow$ Average value or center of mass $\int_{-\infty}^{+\infty} f(x)\,x\,dx$

"2nd moment" $\int_{-\infty}^{+\infty} f(x)\,x^2\,dx \longrightarrow$ Variance of a distribution $\int_{-\infty}^{+\infty} f(x)\,x^2\,dx - \left(\int_{-\infty}^{+\infty} f(x)\,x\,dx\right)^2$

3rd moment:

$$\int_{-\infty}^{+\infty} f(x)\,|x|^3\,dx$$

$$\vdots$$

$$\int_{-\infty}^{+\infty} f(x)\,|x|^k\,dx \quad \cdots$$

Laplace made the following observation: If I am given $f(x)$, and consider the function

$$E(s) = \int_{-\infty}^{+\infty} e^{-sx} f(x)\,dx$$

Then E has the following property:

$$E'(0) = -\int_{-\infty}^{+\infty} f(x)\,x\,dx$$

$$E''(0) = \int_{-\infty}^{+\infty} f(x)\,x^2\,dx$$

$$\vdots$$

$$E^{(k)}(0) = (-1)^k \int_{-\infty}^{+\infty} f(x)\,x^k\,dx$$

Then Laplace started experimenting with the transform

$$\boxed{L(f)(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

and realized it reduced many differential equations to algebraic equations.

(A quick review of improper integrals:

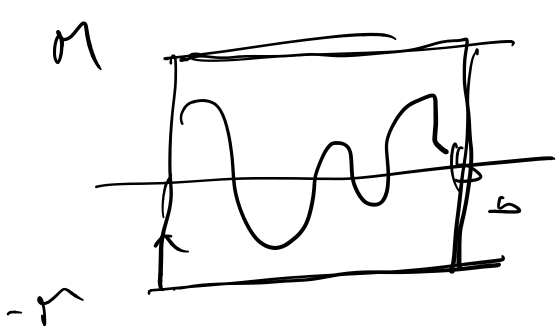
Regular integral:

$$\int_a^b f(x)\, dx$$

① the integration limits are finite

② The integrand $f(x)$ is bounded which mean

$-M \le f(x) \le M$ in the interval (a,b) for some large M



An improper integral is one where the integration interval is not finite or the integrand takes arbitrarily large values.

How do we deal with improper integrals? They are limits of proper integrals:

E.G. $$\int_{-\infty}^{+\infty} f(x)\,dx := \lim_{L\to\infty} \int_{-L}^{L} f(x)\,dx$$

Similarly we handle $\int_{a}^{\infty} f(x)\,dx$ or $\int_{-\infty}^{0} f(x)\,dx$.

If f is not bounded near b, then
$$\int_a^b f(x)\,dx = \lim_{L\to b^-} \int_a^L f(x)\,dx$$

EX. $$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{L\to 0^+} \int_L^1 \frac{1}{\sqrt{x}}\,dx = \lim_{L\to 0^+} \int_L^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{L\to 0^+} \frac{1}{\frac{1}{2}} x^{\frac{1}{2}} \Big|_L^1 = \lim_{L\to 0^+} 2x^{1/2}\Big|_L^1$$

$$= \lim_{L\to 0^+} 2 - 2L^{1/2} = 2$$

Thus $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

EX

$$\int_0^1 \frac{1}{x}\,dx \quad (= +\infty)$$

$$= \lim_{L\to 0^+} \int_0^1 \frac{1}{x}\,dx$$

$$= \lim_{L\to 0^+} \Big(\underbrace{-\ln(L)}_{\to +\infty}\Big).$$

EX

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{L\to\infty} \int_1^L \frac{1}{x^2}\,dx$$

$$= \lim_{L\to\infty} -\frac{1}{x}\Big|_1^L$$

$$= \lim_{L\to\infty} \left(-\frac{1}{L}+1\right)$$

$$= 1.$$

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = -0-\left(-\frac{1}{1}\right)$$

$$= 1.$$

EX

$$\int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty$$

$$\left(= \lim_{L\to\infty} -e^{-t}\Big|_0^L\right)$$

$$= -0 + e^0 = 1$$

EX

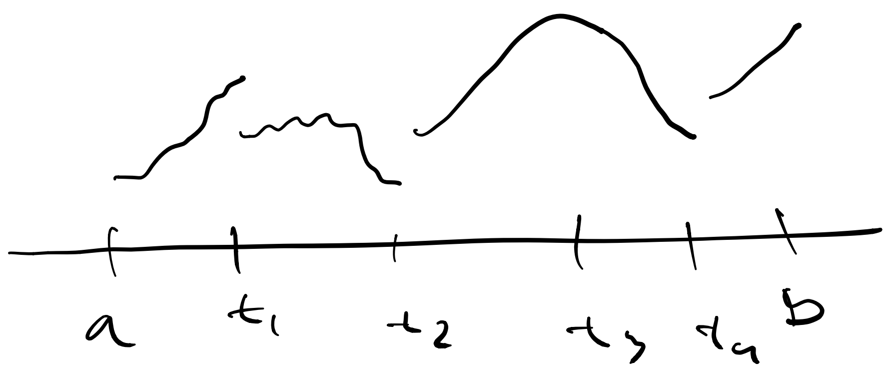
$$\int_0^\infty e^{-2t}\,dt = \frac{1}{2}$$

EX Let $s>0$, then

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

Laplace transform: it's formal definition and its use in solving differential equations

A function $f(t)$ is said to be piecewise continuous in an interval $[a,b]$ if there is a finite list of values in the $t_1, t_2, \dots, t_N$ in this interval such that f is continuous at every t in $[a,b]$ except perhaps at $t_1, \dots, t_n$.

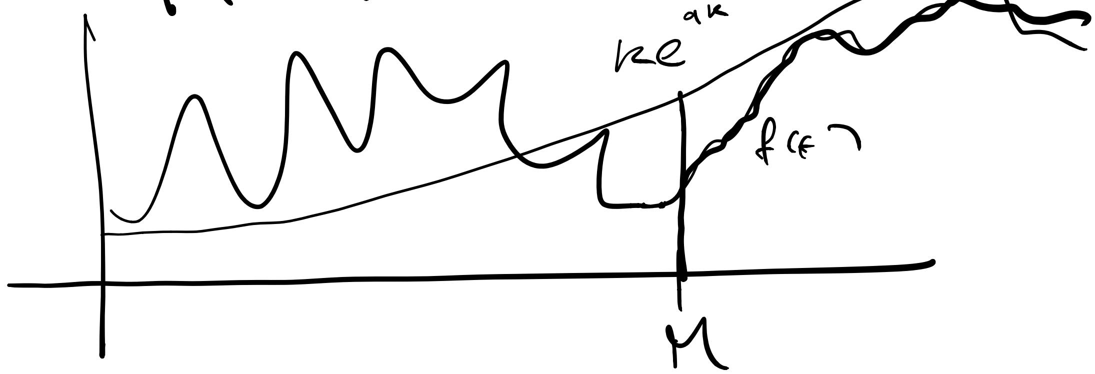


(Typical graph of a piecewise continuous function)

f is said to be piecewise continuous in $(0,\infty)$ if it is piecewise continuous in every interval $(0,b)$ for $b>0$.

A function will be said to be of exponential order if there are numbers a, $K>0$, $M>0$ such that

$$|f(t)| \leq Ke^{at} \quad \text{when } t \geq M$$



Let f be a piecewise continuous function in $[0,\infty)$ with exponential order, then we define its Laplace transform by

$$L(f)(s) = \int_0^\infty f(t)e^{-st}\,dt$$

defined for whichever values of s the integral becomes convergent.

<u>Note</u>: If $|f(t)| \leq Ke^{at}$ for large t, then $L(f)(s)$ is defined

for every $s > a$.

<u>Example</u> (why $\mathcal{L}(f)$ is useful when studying differential equations)

Suppose $y(t)$ is a solution of the differential equation

$$\boxed{ay'' + by' + cy = 0}$$

Then, the function

$$Y(s) = \mathcal{L}(y)(s)$$

solves for every s

$$\boxed{(as^2 + bs + c)Y(s) = (as+b)y(0) + ay'(0)}$$

(we will explain this formula in a second.)

Solving for $Y(s)$ we have

$$\boxed{Y(s) = \frac{(as+b)y(0) + a\,y'(0)}{as^2+bs+c}}$$

Now the question is, from this formula for $Y(s)$, can we deduce a formula for $y(t)$?

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why this connection between $y(t)$ and $Y(s)$?

<u>Lemma</u>: If $f(t)$ is such that f' is piecewise continuous and

$$|f(t)| \leq K e^{at} \quad \text{for } t \geq M$$

Then $\mathcal{L}(f')(s)$ is well defined for $s > a$ and

$$\boxed{\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)}$$

Why is this true? Integrate by parts!

Fix $L>0$, then

$$\int_0^L e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^L - \int_0^L (e^{-st})' f(t)\,dt$$

$$= e^{-sL} f(L) - f(0) - \int_0^L (-se^{-st}) f(t)\,dt$$

Then

$$\int_0^L e^{-st} f'(t)\,dt = s\int_0^L e^{-st} f(t)\,dt - f(0) + e^{-sL} f(L)$$

Taking $L \to \infty$, as long as $s>a$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

We could now repeat this procedure.
If f'' is piecewise contin etc
then

$$\mathcal{L}(f'') = \mathcal{L}((f')')$$
$$= s\mathcal{L}(f') - f'(0)$$
$$= s(s\mathcal{L}(f) - f(0)) - f'(0)$$

So

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s f(0) - f'(0)$$

<u>Exercise</u>: Using the formulas for $\mathcal{L}(f')$ and $\mathcal{L}(f'')$ check that if y solves

$$ay'' + by' + cy = 0$$

Then $Y(s)$ solves

$$(as^2 + bs + c)Y(s) = (as+b)y(0) + ay'(0)$$

<u>Exercise</u>:

Compute the Laplace transform of the following functions

- $f(t) = 1$
- $f(t) = e^{\lambda t}$
- $f(t) = \begin{cases} 1 & \text{in } (a,b) \\ 0 & \text{outside } (a,b) \end{cases}$
- $f(t) = t$.
- $f(t) = \cos(kt)$