1/30/2020 [1]

Integration factor / Separation of variables

The overall idea:

From an equation

$$\frac{d}{dt} x = f(x(t), t)$$

($f(x,t)$ representing some combination of x' and t)

we perform a series of algebraic manipulations until we obtain an equivalent expression, of the form

$$\frac{d}{dt}\Big(G(x(t),t)\Big) = h(t)$$

Then, we can integrate, and conclude that

$$G(x(t),\cancel{t}) = \int h(t)\,dt + C$$

and, inverting G (if possible), obtain a formula

$$x(t) = G^{-1}\left(\int h(t)\,dt + C, t\right)$$

Integrating factors:

$$\frac{d}{dt} x = p(t)\, x + g(t)$$

rewrite as

$$(*) \quad \frac{dx}{dt} - p(t)\, x(t) = g(t)$$

Find a function, $I(t)$ (the integrating factor) with the property that

$$\frac{dI}{dt} = -p(t)\, I$$

Then multiply $(*)$ by $I(t)$,

$$I \frac{dx}{dt} - pIx = g$$

since $\frac{dI}{dt} = -pI$, this is the same as

$$I \left(\frac{dx}{dt}\right) + \left(\frac{dI}{dt}\right) x = g$$

or,

$$\frac{d}{dt}(Ix) = g$$

so,

$$I(t)\, x(t) = \int g(t)\, dt + C$$

and

$$x(t) = \frac{1}{I(t)} \int g(t)\, dt + \frac{C}{I(t)}$$

EX 1 Find all solutions to

$$\frac{dx}{dt} = -3x + e^{-3t}t$$

rearrange:

$$\frac{dx}{dt} \boxed{+3}x = e^{-3t}t$$

we seek $I(t)$ such that

$$\frac{dI}{dt} = 3I.$$

We learned about these functions before!

$$I(t) = Ce^{3t}, \quad C \in \mathbb{R}$$

is a possibility, so we use this with $C=1$ and obtain $I(t) = e^{3t}$. Multiplying the equation, it follows that

$$e^{3t}\frac{dx}{dt} + 3e^{3t}x = e^{3t}\cdot e^{-3t}t$$

Since $e^{3t}\cdot e^{-3t} = e^{3t-3t} = e^{0} = 1$, this simplifies to

$$\frac{d}{dt}\left(e^{3t}x\right) = t$$

This means that $e^{3t}x(t) = \frac{1}{2}t^2 + C$

or,

$$\boxed{x(t) = \frac{1}{2}t^2 e^{-3t} + Ce^{3t}}$$

Let's do another example

EX (let's use notation $y(x)$ this time)

$$y' = -y + \sin(x).$$

Rearrange $y' + y = \sin(x)$.

We seek a function $I(x)$ such that

$$I'(x) = I(x)$$

(we know! $I(x) = e^x$)

so

$$e^x y' + e^x y = e^x \sin(x)$$

This becomes

$$(e^x y)' = e^x \sin(x)$$

or, integrating

you can add this after integrating if you prefer

$$e^x y = \int e^x \sin(x)\,dx + C$$

Now

$$\int e^x \sin(x)\,dx \overset{+C}{=} \frac{1}{2} e^x \left(-\cos(x) + \sin(x)\right) + C$$

Check:

$$\left(\begin{array}{l} \frac{1}{2} e^x \left(\sin(x) + \cos(x) \right) \\ \frac{1}{2} e^x \left(-\cos(x) + \sin(x) \right) \end{array} \right)$$

Then

$$e^x y = \frac{1}{2} e^x \left(-\cos(x) + \sin(x)\right) + C$$

This means

$$\boxed{y = \frac{1}{2}\left(-\cos(x) + \sin(x)\right) + Ce^{-x}}$$

Now, a somewhat ~~an entirely~~ different method that is applicable in different circumstances: Separation of variables.

It works whenever our equation

$$\frac{dx}{dt} = f(x(t), t)$$

is such that the right hand side consists of a product of a function of x and a function of t, like this:

$$\frac{dx}{dt} = g(x(t))\, h(t)$$

EX Find all solution to

$$\frac{dx}{dt} = \frac{3}{t} x$$

The idea: write this as $\frac{d}{dt}\left(G(x(t), t)\right) = h(t)$.

Assume $x \neq 0$, we write

$$\frac{1}{x}\frac{dx}{dt} = \frac{3}{t}$$

Now, $\frac{1}{x}\frac{dx}{dt}$ is simply the ~~integral~~ derivative of $\ln(x(t))$

so

the equation is equivalent to

$$\frac{d}{dt} \ln(|x(t)|) = \frac{3}{t}$$

Integrating, we obtain

$$\ln(|x(t)|) = 3 \ln(t) + c$$

or

$$x(t) = \pm e^{3\ln(t) + c}$$

$$= \left(\pm e^{c}\right) e^{3\ln(t)}$$

Since $e^{3\ln(t)} = t^3$, and calling $c_1 = \pm e^{c}$, we have

$$\boxed{x(t) = c_1 t^3}$$

EX Find all solutions: $y' = \dfrac{x}{1+y}$

This can be rewritten as

$$(1+y)\, y' = x.$$

or for $(1+y)y'$, what is it a derivative of?

Method 1: Guess it

$$(1+y)y' = y' + yy' = (y)' + \left(\tfrac{1}{2}y^2\right)' = \left(y + \tfrac{1}{2}y^2\right)'$$

Method 2: Integrate and change of variable

$$u = y(x)$$

$$du = y'(x)\,dx$$

$$\int (1+y)y'\,dx = \int (1+u)\,du = u + \tfrac{1}{2}u^2 + c = y(x) + \tfrac{1}{2}y(x)^2 + c$$

In any case, we conclude that

$$\left(y(x) + \tfrac{1}{2}y(x)^2\right)' = x = \left(\tfrac{1}{2}x^2\right)'$$

or

$$\boxed{y(x) + \tfrac{1}{2}y(x)^2 = \tfrac{1}{2}x^2 + C}$$

$$\tfrac{1}{2}y(x)^2 + y(x) - \tfrac{1}{2}x^2 - C = 0$$

Formula for $y(x)$? Not so easy this time: (using the quadratic formula:)

$$y(x) = \frac{-1 \pm \sqrt{1 - 4\cdot\tfrac{1}{2}\left(-\tfrac{1}{2}x^2 - C\right)}}{2\,\tfrac{1}{2}} = -1 \pm \sqrt{1 - 2\left(-\tfrac{1}{2}x^2 - C\right)}$$

$$y(x) = -1 \pm \sqrt{1 + x^2 + 2C}$$

Integrating factor vs. Separation of variables

Equation of the form

$$\dot{x} = p(t)\, x + q(t)$$

can always be analyzed via <u>integrating factor</u>. What's important is that the R.H.S. of the equation be a linear (or rather, affine) function of x. You will always obtain an explicit form of x.

Equation of the form

$$\dot{x} = g(x(t))\, h(t)$$

can always be analyzed via separation of variables. You may sometimes only get an implicit formula.