Texas State University
MATH 3323: Differential Equations
Instructor: Nestor Guillen

Problem Set 1

(1) Compute each indefinite integral

$$\int \frac{1}{1-x^2}\,dx$$
$$\int \frac{x}{1+x^2}\,dx$$
$$\int \frac{\sin(x)}{5-2\cos(x)}\,dx$$

(2) For each case, find the solution $x(t)$ for the differential equation which has the given value

$$\dot{x}(t) = 2x(t) + e^{2t}\cos(t), \quad x(2) = 1$$
$$\dot{x}(t) = -10x(t) - e^{-10t}t^2, \quad x(0) = 0$$
$$\dot{x}(t) = x(t) + 1, \quad x(10) = -1$$

(3) Compute $\frac{df}{dx}$ for each given f

$$f(x) = x^4 + x^3 + x^2 + x + 1$$
$$f(x) = \cos(3x) + \cos(2x) + \cos(x) + 1$$
$$f(x) = \frac{\sin(x^2)}{2+\cos(x)} - \frac{(\sin(x))^2}{2+\cos(x+2)}$$

(4) Find the value α such that if $x(t)$ solves the initial value problem

$$\dot{x} = -\tfrac{2}{3}x + 1 - \tfrac{1}{2}t, \ x(0) = \alpha$$

then $x(t)$ does not change sign but takes the value $x(t) = 0$ for at least some t.

(5) (BONUS) Let $x(t)$ be a positive function satisfying the **inequality**

$$\dot{x}(t) \leq -\lambda x(t) \ \text{ for all } t$$

for some number $\lambda > 0$. Show that

$$x(t) \leq x(0)e^{-\lambda t} \text{ for } t > 0.$$

Discuss: what is the relationship between the value of λ and the behavior of $x(t)$ as t goes to infinity? (for example, take $x_1(t)$ and $x_2(t)$ functions as above with two different constants λ_1 and λ_2, what can you say about $x_1(t)/x_2(t)$ as $t \to \infty$ if $\lambda_1 > \lambda_2$?).

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Problem Set 2

(1) Write down a explicit formula for the solution $x(t)$ of each initial value problem

$$\text{a) } \dot{x} = (1-2t)x^2, \ \ x(0) = -3$$

$$\text{b) } \dot{x} = \frac{tx}{\sqrt{1+t^2}}, \ \ x(0) = 1$$

$$\text{c) } \dot{x} = 5t^{-1}x, \ \ x(0) = 5$$

(2) Find the function $y(x)$ such that $y(0) = 0$ and which solves the equation

$$y'(x) = \frac{2-e^x}{3+2y(x)}$$

Once you find $y(x)$, find the value of x where it attains its maximum value.

(3) Consider the nonlinear differential equation

$$\dot{x} = \tfrac{1}{3}x(\tfrac{7}{23} - x)$$

Then

(a) Find the general formula for the solution (in terms of the initial value $x(0)$).
(b) When $x(0) = 1$, what happens with $x(t)$ as $t \to \infty$?
(c) Find a solution $x(t)$ of the equation which is constant in time.
(d) Give an example of initial value $x(0)$ so that $\lim_{t\to\infty} x(t) = 0$.

(4) (BONUS) Let $x_1(t)$ and $x_2(t)$ be two solutions to the equation

$$\dot{x} = f(x),$$

where all know about $f(x)$ is there is some $L > 0$ such that

$$f(x) - f(y) \leq L(y-x) \text{ whenever } x < y.$$

Show the following "differential inequality" holds

$$\frac{d}{dt}(x_1 - x_2)^2 \leq -2L(x_1 - x_2)^2.$$

Use this to show the following inequality for solutions

$$|x_1(t) - x_2(t)| \leq e^{-Lt}|x_1(0) - x_2(0)|, \ \ \text{for } t > 0.$$

What do you think is the significance of this inequality? For the sake of concreteness, think for a second $x_1(t)$ and $x_2(t)$ represent the state of some physical system, what does this last inequality say about the behavior of the state of the system as time increases?

Hint: Once you obtain the differential inequality, use problem 5 from Problem Set 1.

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Problem Set 3

(1) Find the solution to the differential equation taking the given value for a given t

$$\text{a) } \dot{x} = \left(1 + \frac{1}{t}\right) x + t,\ x(\ln(2)) = 1,$$
$$\text{b) } \dot{x} = -\frac{2}{t}x + \frac{1}{t^2}\cos(t)\ \ x(\pi) = 0,$$
$$\text{c) } \dot{x} = \pi\cos(t)x + e^{\pi\sin(t)}\sin(t),\ x(0) = 0.$$

(2) Solve each system of equations below (i.e. if there are solutions find all of them, or else explain why there aren't any solutions)

$$\text{a) } \begin{cases} 9x_1 & -x_2 & = 3 \\ 3x_1 & +x_2 & = 1 \end{cases}$$
$$\text{b) } \begin{cases} x_1 & +x_2 & -x_3 & = 1 \\ 2x_1 & +x_2 & +x_3 & = 1 \\ x_1 & -x_2 & +2x_3 & = 1 \end{cases}$$
$$\text{c) } \begin{cases} x_1 & & -x_3 & = 0 \\ 3x_1 & +x_2 & +x_3 & = 1 \\ -x_1 & +x_2 & +2x_3 & = 2 \end{cases}$$

(3) For each family of vectors, determine wether the vectors are linearly independent or not, and in case they are linearly dependent, find a linear relation between them.

a) $\mathbf{x}_1 = (2, 2, 0),\ \ \mathbf{x}_2 = (0, -2, 2),\ \ \mathbf{x}_3 = (1, 0, 1)$
b) $\mathbf{x}_1 = (2, 1, 0),\ \ \mathbf{x}_2 = (0, 1, 0),\ \ \mathbf{x}_3 = (-1, 2, 0)$
c) $\mathbf{x}_1 = (1, 1, 0, 0),\ \ \mathbf{x}_2 = (0, 1, 1, 0),\ \ \mathbf{x}_3 = (0, 0, 1, 1),\ \ \mathbf{x}_4 = (0, 0, 0, 1)$

(4) Consider the two vector valued functions

$$\mathbf{x}_1(t) = (2e^t, 3) \text{ and } \mathbf{x}_1(t) = (4, 6e^{-t}).$$

For any given fixed value t_0, show that the two dimensional vectors $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$ are linearly dependent. At the same time, show that $\mathbf{x}_1$ and $\mathbf{x}_2$ as functions of t are linearly independent.

(5) (BONUS) Determine a function $c(t)$ so that if we define the function

$$x(t) = e^{5t}c(t)$$

Then x solves

$$\dot{x} = 5x + \sin(t).$$

Discuss any similarities or relationship to one of the methods to solve differential equations discussed in class.

(6) (BONUS) Let f be a real function of a single real variable with the property that for some constant $L > 0$ we have that

$$|f(x) - f(y)| \leq L|x - y| \text{ for all numbers } x, y.$$

Show that given an initial value α, the problem

$$\dot{x} = f(x), \quad x(0) = \alpha$$

cannot have two different solutions.

Hint: Assume $x_1(t)$ and $x_2(t)$ are two solutions, and prove that $(x_1(t) - x_2(t))^2$ satisfies a differential inequality like the one in problem 5 from Problem Set 2.

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Problem Set 4

(1) Consider the real valued functions $x_1(t) = e^{-2t}$ and $x_2(t) = e^{-3t}$, then

(a) By direct computation, check that each function solves

$$\ddot{x} + 5\dot{x} + 6x = 0$$

(b) Find α_1 and α_2 such that the function given by

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

solves

$$\begin{aligned} &\ddot{x} + 5\dot{x} + 6x = 0 \\ &x(0) = 2,\ \dot{x}(0) = 9. \end{aligned}$$

(2) Check whether the following families of functions of t are linearly independent or not

(a) $t^2 + 1,\ 2t,\ 4(t+1)^2$

(b) $\sin(t)\cos(t),\ \sin(2t) + \cos(2t),\ \cos(2t)$

(c) $e^{2t},\ e^{-2t},\ 2e^t$

(d) $2e^t,\ 3\cosh(t),\ 13\sinh(t)$

(e) $\dfrac{1}{t^2-1},\ \dfrac{1}{t+1},\ \dfrac{1}{t-1}$

(3) In each item below, compute the derivative of the given vector-valued function $\mathbf{x}$ (whose components are denoted $x_1(t)$ and $x_2(t)$) and match it to the differential equation it solves from the list on the right column

i) $\mathbf{x}(t) = \begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$

ii) $\mathbf{x}(t) = \begin{pmatrix} e^{2t}(t+1) \\ e^{2t} \end{pmatrix}$

iii) $\mathbf{x}(t) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$

iv) $\mathbf{x}(t) = \begin{pmatrix} t + \frac{1}{2}(e^{2t} - 1) \\ \frac{1}{3}(e^{3t} - 1) \end{pmatrix}$

a) $\dot{x} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} x$

b) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) + \frac{\sqrt{2}}{2} \end{cases}$

c) $\dot{x} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} x + \begin{pmatrix} -2t \\ 1 \end{pmatrix}$

d) $\dot{x} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x$

(4) Consider the matrix-valued functions of t

$$\mathbf{A}(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 2e^{-t} \\ 0 & 2e^{-t} & 2 \end{pmatrix} \quad \mathbf{B}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^t \\ 0 & e^{-t} & -1 \end{pmatrix}$$

Then, compute the following expressions

a) $\mathbf{A} - 2\mathbf{B}$	**d)** $\int_0^1 \mathbf{B}(t)\, dt$
b) $\mathbf{AB}$	**e)** $\frac{d}{dt}(\mathbf{AB})$
c) $\frac{d}{dt}\mathbf{A}$	**f)** $(\frac{d}{dt}\mathbf{A})\mathbf{B} + \mathbf{A}(\frac{d}{dt}\mathbf{B})$

(5) (BONUS) Consider the function

$$x(t) = e^{2t}$$

Using the chain rule, show that given any numbers A, B, and C, then for all t we have the formula

$$A\ddot{x}(t) + B\dot{x}(t) + Cx(t) = (A4 + B2 + C)x$$

Determine similar formulas for $A\ddot{x}(t) + B\dot{x}(t) + Cx(t)$ for the following functions

$$e^t,\ e^{-t},\ e^{3t},\ e^{-3t}.$$

Based on your findings, try to find a (non-zero) function $x(t)$ solving the equation

$$\ddot{x} - 5\dot{x} - 24x = 0.$$

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Problem Set 5

(1) For each matrix, find all the eigenvalues, and provide an eigenvector for each eigenvalue

$$\text{a)} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \qquad \text{b)} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{c)} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

(2) For each part of this problem you are given a two dimensional differential equation and two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of said equation, in each case check that the given solutions are linearly independent at $t = 0$, and use them to find a solution $\mathbf{x}$ to the differential equation with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$\text{a) } \dot{\mathbf{x}} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}_1(t) = \begin{pmatrix} \cos(4t) \\ \sin(4t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} \sin(4t) \\ -\cos(4t) \end{pmatrix}.$$
$$\text{b) } \dot{\mathbf{x}} = \begin{pmatrix} 10 & 0 \\ 0 & -10 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}_1(t) = \begin{pmatrix} e^{10t} \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 0 \\ e^{-10t} \end{pmatrix}.$$
$$\text{c) } \dot{\mathbf{x}} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}_1(t) = \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 3e^{3t} \\ 0 \end{pmatrix}.$$

(3) Compute the following matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Conclude that if $ad - bc \neq 0$ then the matrix on the left is invertible and provide a formula for its inverse –compare this formula with your answer to problem 2 a) in Problem Set #3.

(4) (BONUS) Let x be a real number, find the limit of the sequence

$$\lim_{n \to \infty} \frac{x^n}{n!}$$

(remember that $n!$ denotes the product $n(n-1)(n-2)\dots 1$). Based on your answer, determine the set of values of x for which the following series converges and is finite

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

(5) (BONUS) Given differentiable functions $a_{11}(t), a_{12}(t), a_{21}(t)$, and $a_{22}(t)$, compute the derivative of

$$\det \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$$

and show the result coincides with the sum

$$\det \begin{pmatrix} \dot{a}_{11}(t) & a_{12}(t) \\ \dot{a}_{21}(t) & a_{22}(t) \end{pmatrix} + \det \begin{pmatrix} a_{11}(t) & \dot{a}_{12}(t) \\ a_{21}(t) & \dot{a}_{22}(t) \end{pmatrix}$$

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Problem Set 6

(1) Find the solution to $\dot{x} = Ax + b(t)$ with $x(0) = (0,0,0)$, for each A and $b(t)$ bellow

$$\text{a) } A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad b(t) = \begin{pmatrix} 1 \\ e^{-2t} \\ 7e^{-2t} \end{pmatrix}$$

$$\text{a) } A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad b(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{a) } A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 1 & -4 \end{pmatrix}, \quad b(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(2) Consider the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Let P denote the matrix $P = A - 3\mathrm{I}$ (I denotes the identity matrix). Then

(a) Show by direct computation that P^2 is the zero matrix – i.e. all its entries are zero.
(b) Compute a formula for e^{tP} (since $(tP)^2 = 0$, the series simplifies considerably!)
(c) Use this to compute a simple formula for e^{tA}.

(3) For each item below, find the solution of the given initial value problem and describe its behavior as $t \to \infty$ (does it converge to a definite limit, does it oscillate indefinitely, or does it grow to $\pm\infty$?)

a) $y'' + y' - 2y = 0,\ y(0) = 1, y'(0) = 1$
b) $y'' + 4y' + 3y = 0,\ y(0) = 2, y'(0) = -1$
c) $y'' + 3y' = 0,\ y(0) = -2, y'(0) = 3$
d) $4y'' - y = 0,\ y(-2) = 1, y'(-2) = -1$

(4) Find the second order linear differential equation whose general solution $x(t)$ is

$$x(t) = c_1 e^{2t} + c_2 e^{-3t}$$

(5) (BONUS) For each of the equations in problem 1 an problem 3 write down the corresponding characteristic polynomial and find all its roots.

(6) Solve the following linear systems

a)

b)

c)

d)

(7) Recall that given a square matrix A, the exponential e^A is defined by the series

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Then, do the following

(a) Compute e^{tA}, in each of the following cases

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hint: for the third matrix note that A^n is the identity matrix if n is divisible by 4, so the powers of A^n cycle between 4 different matrices. Break the series for e^{tA} in two, where one contains all the even powers of A and the other all the odd powers of A, compare this with the power series for $\sin(t)$ and $\cos(t)$:

$$\sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

(b) Show that for any A that $e^A e^{-A} = \mathrm{I}$ and thus e^A is always invertible.
Hint: Multiply the series for e^A and for e^{-A} and use the identity

$$\sum_{n=0}^{m} \frac{(-1)^n}{n!(m-n)!} = 0$$

which is valid for every $m > 1$ (compare this with problem 5).

(8) Using a computer (see class' website) plot several solutions to each of the systems below, compute the eigenvalues for A in each case and discuss what (if any) relationship is there between the eigenvalues of A and the behavior of the solutions. Can you explain your observations in terms of the matrix exponential?

(9) (BONUS) Suppose A and B are two square matrices which commute, that is

$$AB = BA$$

Show then that the usual formula from the standard exponential holds, namely

$$e^{A+B} = e^A e^B = e^B e^A$$

(10) Compute the Wronskian for the following pair of functions

(11) For each of the second order linear equations below, find the longets interval where the IVP is certain to have a unique twice differentiable solution

(12) For each second order equation below, find the Wronskian of two solutions of the given equation without solving the equation

(13) Problem 4

(14) (BONUS) Consider a two dimensional nonlinear system given by

$$\dot{x} = y$$
$$\dot{y} = -H'(x)$$

If $(x(t), y(t))$ is any solution to this, show that there is some constant c such that

$$\frac{1}{2}y(t)^2 + H(x(t)) = c \text{ for all } t.$$

Apply this to the following problem: you are given a pendulum which initially at position ****?, what is the angular velocity of the pendulum when the pendulum is perfectly vertical?

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Problem Set 7

(last problem set before second midterm: this covers variation of parameters and oscillations and Duhamel's formula)

(1) Problem 1
(2) Problem 2
(3) Problem 3
(4) Problem 4

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Problem Set 8

(1) Problem 1
(2) Problem 2
(3) Problem 3
(4) Problem 4
(5) (BONUS)

We consider a function $L(x,p)$, called the Lagrangian.

$$\mathcal{J}(x) = \int_0^1 L(x,\dot{x})\;dt$$

$$\int_0^1 \frac{\partial L}{\partial x}(x,\dot{x})\psi + \frac{\partial L}{\partial p}(x,\dot{x})\dot{\psi}\;dt = 0$$

$$\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial p}(x,\dot{x}) = 0.$$

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Problem Set 9

(1) Problem 1
(2) Problem 2
(3) Problem 3
(4) Problem 4
(5) (BONUS) Uniqueness for nonlinear problems in arbitrary dimension via a Gronwall-type lemma.
(6) (BONUS) The Brachistocrone problem (uses a bonus problem from the previous problem set)

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Problem Set 10

(1) (a problem about Euler's method)
(2) (a problem about autonomous equations)
(3) (a problem about exact differential equations)
(4) Problem 4
(5) (BONUS) (perhaps a numerical experiment?)

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Problem Set 11

(1) Problem 1 (problems about series)
(2) Problem 2
(3) Problem 3
(4) Problem 4

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Problem Set 12

(last problem set before third midterm)

(1) Problem 1
(2) Problem 2
(3) Problem 3
(4) Problem 4

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Problem Set 13

(1) Problem 1
(2) Problem 2
(3) Problem 3
(4) Problem 4

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Problem Set 14

(1) Problem 1
(2) Problem 2
(3) Problem 3
(4) Problem 4
