A little review

1. We did a lot of linear algebra
2. We learned about eigenvalues, eigenvectors, and determinants of matrices
3. We learned how these are used to solve systems

Last few classes

(1) We started studying linear systems such as

$$\dot{x} = Ax$$

a n-vector

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

a $n \times n$ matrix ("the coefficient matrix")

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

(2) We learned about eigenvectors and eigenvalues of a square matrix A:

$v \neq 0$ is an **eigenvector** of A with **eigenvalue** λ if

$$Av = \lambda v$$

③ Key fact:

If v is an eigenvector of A with eigenvalue λ, then the function

$$x(t) = e^{\lambda t} v$$

is a solution of

$$\dot{x} = Ax$$

This means that solving linear systems can be reduced to finding eigenvectors.

④ Linear superposition:

If $v_1, \dots, v_k$ are eigenvectors of A, with eigval $\lambda_1, \dots, \lambda_k$ then for any combination of numbers $c_1, c_2, \dots, c_k$ the function

$$\boxed{x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_k e^{\lambda_k t} v_k}$$

is a solution of

$$\dot{x} = Ax$$

Solving $\dot{x} = Ax \leftarrow$ (A is $n \times n$)
(Step by Step)

① Write down the characteristic polynomial of A:

$$P(\lambda) = \det(A - \lambda I)$$

and find its roots: $\lambda_1, \lambda_2, \dots, \lambda_n$

② For each eigenvalue λ_k, find an eigenvector v_k.
In general (for instance, if there are no repeated roots) the eigenvectors $v_1, v_2, \dots, v_n$ will form a basis of the space.

③ Every solution to $\dot{x} = Ax$ can be obtained from

$$\boxed{x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n}$$

for the right combination of $c_1, \dots, c_n$.

The general solution to $\dot{x} = Ax$

④ If we are dealing with a given initial condition x_0 then we must find $c_1, c_2, \dots, c_n$ so that

$$x_0 = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

Then,

$$\boxed{x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n}$$

solves $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$

Example $\quad \dot{x} = \begin{pmatrix} 10 & 4 \\ 0 & 5 \end{pmatrix} x$

① Find $P(\lambda)$ $\quad (P_A(\lambda))$

$$P_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{pmatrix} 10 & 4 \\ 0 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \det\left(\begin{pmatrix} 10-\lambda & 4 \\ 0 & 5-\lambda \end{pmatrix} \right)$$

$$= (10-\lambda)(5-\lambda) - 4 \cdot 0$$

so $\boxed{P_A(\lambda) = (10-\lambda)(5-\lambda)}$

~~Roots~~: Eigenvalues: $\lambda_1 = 10, \; \lambda_2 = 5$

② Find eigenvectors :

Let's look for v_1 such that

$$\boxed{A v_1 = 10 v_1} \qquad \left(v_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}\right)$$

$$(A - 10 I) v_1 = 0$$

$$\begin{pmatrix} 0 & 4 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0\cdot v_{11} + 4\cdot v_{12} = 0 \rightarrow v_{12}=0$$

$$0\cdot v_{11} - 5\, v_{12} = 0$$

v_{11} is free for us to choose, so let's take $v_{11}=1$. This gives

$$\boxed{v_1 = \begin{pmatrix}1\\0\end{pmatrix}}$$

Let's look for v_2 such that

$$\boxed{Av_2 = 5v_2}$$

$$\begin{pmatrix}10-5 & 4\\ 0 & 5-5\end{pmatrix}\begin{pmatrix}v_{21}\\v_{22}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}5 & 4\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_{21}\\v_{22}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$5v_{21} + 4\, v_{22} = 0$$

no information from this equation $\rightarrow$ $0\cdot v_{21} + 0\, v_{22} = 0$

$$v_{22} = -\frac{5}{4} v_{21}$$

and one v_{21} is chosen (freely) v_{22} is determined. Let's take $v_{21}=1$, in which case $v_{22}=-5/4$, so

$$\boxed{v_2 = \begin{pmatrix}1\\-5/4\end{pmatrix}}$$

(alternatively: $v_{21}'=4$, then $v_{22}'=-5$, and

$$v_2' = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad (v_2' = 4v_2))$$

(3) The general solution to $\dot{x} = Ax$ is

$$\boxed{x(t) = c_1 e^{10t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ -5/4 \end{pmatrix}}$$

<u>Example</u> (continued)

Same equation as before, but find solution with $x(0) = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$.

Find c_1, c_2 so that

$$\begin{pmatrix} 6 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -5/4 \end{pmatrix}$$

$$\left(\begin{array}{l} 6 = c_1 + c_2 \\ 9 = 0c_1 - \frac{5}{4}\cdot c_2 \end{array}\right. \rightarrow c_2 = -\frac{4}{5}9 = -\frac{36}{5}$$

$$c_1 = 6 - c_2 = 6 + \frac{36}{5} = \frac{66}{5}$$

Solution: $$\boxed{x(t) = \frac{66}{5} e^{10t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{36}{5} e^{5t}\begin{pmatrix} 1 \\ -5/4 \end{pmatrix}}$$

(Alternative way of writing it:

$$\boxed{x(t) = \begin{pmatrix} \frac{66}{5} e^{10t} - \frac{36}{5} e^{5t} \\ 9 e^{5t} \end{pmatrix}}$$)

(check answer: $x(0) = \begin{pmatrix} \frac{66}{5}\cdot 1 - \frac{36}{5}\cdot 1 \\ 9\cdot 1 \end{pmatrix} = \begin{pmatrix} \frac{30}{5} \\ 9 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$)

$\det\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = ac$

Example: Find the general solution to

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 20 \\ 1 & 16 & 8 \end{pmatrix} x$$

and then find the solution starting from (i) $x(0) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and

(ii) $x(0) = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$

① Find $P_A(\lambda)$

$$P_A(\lambda) = \det(A - \lambda I) =$$

$$= \det\begin{pmatrix} -1-\lambda & 0 & 0 \\ 0 & 4-\lambda & 20 \\ 1 & 16 & 8-\lambda \end{pmatrix}$$

$$= (-1-\lambda)\det\begin{pmatrix} 4-\lambda & 20 \\ 16 & 8-\lambda \end{pmatrix}$$

$$+ \cancel{0 \cdot \det\begin{pmatrix} 20 & 0 \\ 8-\lambda & 1 \end{pmatrix}}$$

$$+ \cancel{0 \det\begin{pmatrix} 0 & 4-\lambda \\ 1 & 16 \end{pmatrix}}$$

$$= (-1-\lambda)\left((4-\lambda)(8-\lambda) - 20\cdot 16\right)$$

$$= -(\lambda+1)(32 - 12\lambda + \lambda^2 - 320)$$

$$= -(\lambda+1)(\lambda^2 - 12\lambda - 288)$$

Eigenvalues: $\lambda_1 = -1$

$\lambda_2 = -12$

$\lambda_3 = 24$

($-12, 24$ are the roots of $\lambda^2 - 12\lambda - 288$)

(2) Find eigenvectors

For $\boxed{Av_1 = -1v_1}$, $\boxed{v_1 = \begin{pmatrix} 55 \\ -4 \\ 1 \end{pmatrix}}$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 20 \\ 1 & 16 & 9 \end{pmatrix}}_{A-(-1)I} v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $Av_2 = -12v_2$,

$$\boxed{v_2 = \begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix}}$$

For $Av_3 = 24 v_3$

$$\boxed{v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

3 The general solution is

$$x(t) = c_1 e^{-t}\begin{pmatrix} 55 \\ -4 \\ 1 \end{pmatrix} + c_2 e^{-12t}\begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix} + c_3 e^{24t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

4 Initial conditions

(i) $x(0) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

Solve:

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = c_1\begin{pmatrix} 55 \\ -4 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$1 = 55c_1 + 0 + 0 \rightarrow \boxed{c_1 = \frac{1}{55}}$$

$$0 = -4c_1 - 5c_2 + c_3$$

$$3 = c_1 + 4c_2 + c_3$$

Substituting,

$$\begin{cases} -5c_2 + c_3 = -\frac{4}{55} \\ 4c_2 + c_3 = -\frac{3}{55} \end{cases}$$

$$-9c_2 = -\frac{4}{55} + \frac{3}{55} = -\frac{1}{55}$$

$$\boxed{c_2 = \frac{1}{9 \cdot 55}}$$

$$\boxed{= \frac{1}{495}}$$

$$c_3 = -\frac{3}{55} - 4c_2 = -\frac{3}{55} - \frac{4}{495}$$

$$= -\frac{27}{495} - \frac{4}{495} = \boxed{-\frac{31}{495}}$$

Solution for $x(0) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$,

$$\boxed{x(t) = \frac{1}{55} e^{t} \begin{pmatrix} 55 \\ -4 \\ 1 \end{pmatrix} + \frac{1}{495} e^{-12t} \begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix} - \frac{31}{495} e^{24t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

Example $\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$

① $P_A(\lambda)$,

$$P_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

Eigenvalues : $\lambda_1 = 3, \lambda_2 = -1$

② Eigenvectors

$A v_1 = 3 v_1$

$$\underbrace{\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}}_{(A-3I)} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2v_{11} + v_{12} = 0$$
$$4v_{11} - 2v_{12} = 0$$

One solution : $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$A v_2 = -v_2$

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}}_{(A-(-1)I)} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

One solution $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(3) General solution:

$$\boxed{x(t) = c_1 e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-2\end{pmatrix}}$$