April 2nd, 2020

Relevant book sections: 7.4, 7.5, (maybe) 7.6

1. Fundamental family of solutions
2. Fundamental matrices
3. Wronskian of a family of solutions

For today, a lot of our discussion extends to linear systems with time-dependent coefficients

$$\dot{x} = A(t)\, x$$

(entries of the matrix are allowed to change with t)

BUT we won't deal just yet with examples of such time-dependent systems

1. Fundamental Family of Solutions

(book calls them "fundamental set of solutions)

Consider a linear system

$$\dot{x} = A(t)\, x \qquad (1)$$

We will say that a family of solutions
$$x_1(t), x_2(t), \dots, x_n(t)$$
is a "Fundamental Family of the equation (1)" if any solution $x(t)$ of (1) can be expressed as a linear combination
$$x(t) = c_1 x_1(t) + \dots + c_n x_n(t)$$
and the coefficients $c_1, \dots, c_n$ are unique.

EXAMPLE: Recall from last class:
$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$$
We found the solutions
$$x_1(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_2(t) = e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
These solutions are such that $x_1(t)$ and $x_2(t)$ are linearly independent, and any solution to the equation is a combination
$$c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Consider a new family in this example, $y_1(t)$ and $y_2(t)$ given by

$$y_1(t) = x_1(t) + x_2(t)$$
$$y_2(t) = x_1(t) - x_2(t)$$

Claim: y_1, y_2 is another fundamental family for this equation

EXAMPLE: Consider

$$\dot{x} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x$$

Let us find solutions to this system

(1) Find the characteristic polynomial

$$A = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}, \quad P_A(\lambda) = \det \begin{pmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{pmatrix}$$

$$P_A(\lambda) = (-3-\lambda)(-2-\lambda) - (\sqrt{2})^2$$
$$= (-3-\lambda)(-2-\lambda) - 2$$

$$p_A(\lambda) = (\lambda+1)(\lambda+4)$$

Eigenvalues : $\lambda_1 = -1$, $\lambda_2 = -4$

$$\left((3+\lambda)(2+\lambda) - 2 = 6 + 5\lambda + \lambda^2 - 2 \right.$$
$$= \lambda^2 + 5\lambda + 4$$
$$\left. = (\lambda+1)(\lambda+4) \right)$$

② Eigenvectors

v_1 : $Av_1 = -1 v_1$

$$(A - (-1)I) v_1 = 0$$

$$\begin{pmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2v_{11} + \sqrt{2}\, v_{12} = 0$$
$$\sqrt{2}\, v_{11} - 1\, v_{12} = 0$$

$\Leftrightarrow$ $v_{12} = \sqrt{2}\, v_{11}$

Choose $v_{11} = 1$, so

$$v_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

(another eigenvector $\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{pmatrix}$)

v_2 : $Av_2 = -4v_2$

$$(A - (-4)I) v_2 = 0$$

$$\begin{pmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_{21} + \sqrt{2}\, v_{22} = 0$$
$$\sqrt{2}\, v_{21} + 2\, v_{22} = 0$$

$$v_{22} = -\frac{\sqrt{2}}{2} v_{21}$$

Choose $v_{21} = 1$,

$$v_2 = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

③ $x_1(t) = e^{\lambda_1 t} v_1$
$x_2(t) = e^{\lambda_2 t} v_2 \quad \dots$
$x_n(t) = e^{\lambda_n t} v_n$

Fundamental Family

From a pair of eigenvalue λ and corresponding eigenvector v we build a special solution

$$(Av = \lambda v)$$

$$v \quad \lambda$$

$$x(t) = e^{\lambda t} v$$

For a 2-d system, once you find 2 linearly independent solutions you have a fundamental system of solutions. Which in this case is:

$$x_1(t) = e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad x_2(t) = e^{-4t}\begin{pmatrix} 1 \\ -\sqrt{2}/2 \end{pmatrix}$$

$$\left(\text{or } \overset{\text{alternative}}{e^{-t}}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{pmatrix}\right)$$

Then any solution to the system can be written as

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$
$$= c_1 e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

EXAMPLE (continued)

Let's find $y_1(t)$ and $y_2(t)$ each a solution to

$$\dot{x} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x$$

where $y_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solution for y_1: We know y_1 can be written as

$$y_1(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

at $t=0$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$1 = c_1 + c_2$$
$$0 = \cancel{\sqrt{2}}c_1 - \cancel{\frac{\sqrt{2}}{2}}c_2 \rightarrow c_2 = 2c_1$$

$$1 = c_1 + 2c_1 = 3c_1$$
$$\Rightarrow c_1 = \frac{1}{3}$$
$$\Rightarrow c_2 = \frac{2}{3}$$

Then,

$$\boxed{y_1(t) = \frac{1}{3}e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + \frac{2}{3}e^{-4t}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}}$$

Solution for y_2

$$y_2(t) = c_1 e^{-t}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix} + c_2 e^{-4t}\begin{pmatrix}1\\ -\frac{\sqrt{2}}{2}\end{pmatrix}$$

At $t=0$ this becomes

$$\begin{pmatrix}0\\1\end{pmatrix} = c_1\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix} + c_2\begin{pmatrix}1\\ -\frac{\sqrt{2}}{2}\end{pmatrix}$$

or,

$$0 = c_1 + c_2 \longrightarrow c_2 = -c_1$$

$$1 = \sqrt{2}c_1 - \frac{\sqrt{2}}{2}c_2$$

$$1 = \sqrt{2}c_1 + \frac{\sqrt{2}}{2}c_1$$

$$1 = \frac{3}{2}\sqrt{2}c_1$$

$$\Rightarrow c_1 = \frac{2}{3\sqrt{2}}, \quad c_2 = -\frac{2}{3\sqrt{2}}$$

Then

$$\boxed{y_2(t) = \frac{2}{3\sqrt{2}}e^{-t}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix} - \frac{2}{3\sqrt{2}}e^{-4t}\begin{pmatrix}1\\ -\frac{\sqrt{2}}{2}\end{pmatrix}}$$

The functions y_1, y_2 form another fundamental family of solutions for

$$\dot{x} = \begin{pmatrix}-3 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}x$$

How do I check that a family of solutions to $\dot{x} = A(t)x$ is fundamental?

<u>Theorem</u>: If $x_1(t), \dots, x_k(t)$ are solutions to a linear system

$$\dot{x} = A(t)x$$

and at $t=0$ the vectors

$$x_1(0), \dots, x_k(0)$$

are linearly independent, then the solutions $x_1(t), \dots, x_k(t)$ are linearly independent.

Moreover, if $k=n$ (n is the dimension of the system) then the family of solutions is fundamental.

This theorem is closely related to Picard's existence and uniqueness theorem.

(Different Picard)

So, if you have n solutions
$$x_1(t), \dots, x_n(t)$$
to a n-dimensional system
$$\dot{x} = A(t)x$$
then if
$$x_1(0), \dots, x_n(0)$$
are linearly independent, then the family $x_1(t), \dots, x_n(t)$ is fundamental.

(Check this is so in all previous examples)

Fundamental Matrices

Given a system (of dimension n)
$$\dot{x} = A(t)x$$
then a $n \times n$ matrix function $U(t)$ will be called a Fundamental Matrix if the n columns of the system of the matrix $U(t)$ form a fundamental system of solutions:
$$U(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{pmatrix}$$
...

fundamental family
of solutions

<u>EXAMPLE</u>: Consider the system

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$$

Let $x_1(t)$ solve this system with initial condition $x_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let $x_2(t)$ solve it with $x_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$x_1(t) = \begin{pmatrix} \frac{1}{2}(e^{3t} + e^{-t}) \\ e^{3t} - e^{-t} \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} \frac{1}{4}(e^{3t} - e^{-t}) \\ \frac{1}{2}(e^{3t} + e^{-t}) \end{pmatrix}$$

These form a fundamental family of solutions and thus

$$U(t) = \begin{pmatrix} \frac{1}{2}(e^{3t} + e^{-t}) & \frac{1}{4}(e^{3t} - e^{-t}) \\ e^{3t} - e^{-t} & \frac{1}{2}(e^{3t} + e^{-t}) \end{pmatrix}$$

is a fundamental matrix.

Another fundamental matrix would come from

$$e^{3t}\begin{pmatrix}1\\2\end{pmatrix} \qquad e^{-t}\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\upsilon_2(t) = \begin{pmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{pmatrix}$$

Exercise:

Compute $\dot{\upsilon}_1$ and $\dot{\upsilon}_2$ and note that

$$\dot{\upsilon}_1 = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \upsilon_1$$

and

$$\dot{\upsilon}_2 = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \upsilon_2$$