Laplace Transform

Last time we defined for $f:(0,\infty)\to\mathbb{R}$ the function

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Warm up: The Laplace transform for some basic functions

(1) $f(t) = 1$

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}\cdot 1\,dt = \int_0^\infty e^{-st}\,dt$$

$(s>0)$

$$= -\frac{1}{s}e^{-st}\Big|_0^\infty = \lim_{L\to\infty}\left(-\frac{1}{s}e^{-sL} - \left(-\frac{1}{s}\right)\right)$$

$$= 0 + \frac{1}{s} = \frac{1}{s}$$

$$\boxed{\mathcal{L}(1)(s) = \frac{1}{s}} \quad s>0$$

(2) $f(t) = e^t$

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}\cdot e^t\,dt$$

$$= \int_0^\infty e^{-st+t}\,dt = \int_0^\infty e^{(1-s)t}\,dt$$

This integral converges if $s>1$. Then

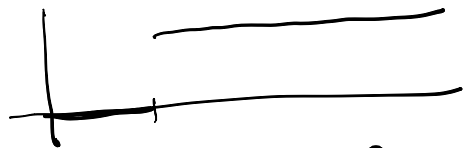
$$= \frac{1}{1-s}e^{(1-s)t}\Big|_0^\infty = -\frac{1}{1-s}e^{(1-s)\cdot 0}$$

$$= \frac{1}{s-1}$$

$$\boxed{\mathcal{L}(e^t)(s) = \frac{1}{s-1}} \quad \text{for } s>1$$

(3) $f(t) = e^{\lambda t}$, $\lambda \in \mathbb{R}$

$$\mathcal{L}(f)(s) = \frac{1}{s-\lambda} \quad \text{for } s > \lambda.$$

(4)  $h(t) = \begin{cases} 1 & \text{if } t > 1 \\ 0 & \text{if } t < 0 \end{cases}$

$$\mathcal{L}(h(t))(s) = \int_0^\infty e^{-st} h(t)\, dt$$

$$= \int_1^\infty e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_1^\infty$$

$$(\text{For } s>0) = 0 - \left(-\frac{1}{s} e^{-s\cdot 1}\right)$$

$$= \frac{e^{-s}}{s}$$

$$\boxed{\mathcal{L}(h(t))(s) = \frac{e^{-s}}{s}} \quad s>0$$

⑤ More generally, consider the indicator function of an interval (a,b)

$$\chi_{(a,b)}(t) := \begin{cases} 1 & \text{if } a<t<b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{L}\left(\chi_{(a,b)}(t)\right)(s) = \int_0^\infty e^{-st}\chi_{(a,b)}(t)\,dt$$

$$= \int_a^b e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_a^b$$

$$= -\frac{1}{s}e^{-bs} + \frac{1}{s}e^{-as}$$

$$= \frac{1}{s}\left(e^{-as} - e^{-bs}\right)$$

$$\boxed{\mathcal{L}\left(\chi_{(a,b)}\right)(s) = \frac{1}{s}\left(e^{-as} - e^{-bs}\right)}$$

(for all $s \in \mathbb{R}$)

Using $\mathcal{L}$ to solve differential equation.

Last time we saw:

If I know $y(t)$ solves
$$ay'' + by' + cy = f$$
Then $Y(s) = \mathcal{L}(y(t))(s)$ is given by
$$(as^2 + bs + c)Y(s) = (as+b)y(0) + ay'(0) + \mathcal{L}(f)(s)$$

or,
$$Y(s) = \frac{(as+b)y(0) + ay'(0) + \mathcal{L}(f)(s)}{as^2 + bs + c}$$

The idea is to use this formula for $Y(s)$ to obtain a formula for $y(t)$.

Roughly speaking, since
$$\mathcal{L}(y)(s) = \frac{(as+b)y(0) + ay'(0) + \mathcal{L}(f)(s)}{as^2 + bs + c}$$

we would like to say
$$y(t) = \text{"}\mathcal{L}^{-1}\text{"}\left(\frac{(as+b)y(0) + ay'(0) + \mathcal{L}(f)(s)}{as^2+bs+c}\right)$$

To do this we need to think about:

(1) Does the operator "$\mathcal{L}^{-1}$" makes sense? Answer: Yes.

(2) Is there some formula, perhaps involving an integral such that from it we can compute $\mathcal{L}^{-1}(Y(s))$? Answer: Yes!

(3) Is the formula simple enough to be practical? Answer: NO!

However, we can do the following: if we know enough Laplace transforms, then we can try to express $Y(s)$ as a combination of those known transforms which may allow us to effectively find $\mathcal{L}^{-1}(Y(s))$.

More concretely: Suppose we can write

$$Y(s) = Y_1(s) + Y_2(s) + \cdots + Y_k(s)$$

where the $Y_1, Y_2, \ldots, Y_k$ are known functions for which we know functions $y_1, y_2, \ldots, y_k$ such that

$$\mathcal{L}(y_1) = Y_1, \quad \mathcal{L}(y_2) = Y_2, \ldots$$

$$\ldots \mathcal{L}(y_k) = Y_k$$

Then
$$Y(s) = \mathcal{L}(y_1) + \mathcal{L}(y_2) + \cdots + \mathcal{L}(y_k)$$
$$= \mathcal{L}(y_1 + y_2 + \cdots + y_k)$$

and then we would say

$$\mathcal{L}^{-1}\big(Y(s)\big) = y_1(t) + y_2(t) + \cdots + y_k(t).$$

Example (1st order equation).

Let y solve
$$y' + y = 2 + e^{2t}, \quad y(0) = 0$$
what is $y(t)$? Well, let's write

$$Y(s) = \mathcal{L}(y)$$

(since $y(0) = 0$) ↓

$$sY + Y = \mathcal{L}(2 + e^{2t})$$

$$\downarrow \qquad = \mathcal{L}(2) + \mathcal{L}(e^{2t})$$

$$(s+1)Y(s) = 2\mathcal{L}(1) + \mathcal{L}(e^{2t})$$

$$Y(s) = \frac{2\mathcal{L}(1)(s)}{s+1} + \frac{\mathcal{L}(e^{2t})(s)}{s+1}$$

$\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$, so

$$\boxed{Y(s) = \frac{2}{s(s+1)} + \frac{1}{(s-2)(s+1)}}$$

The right hand side (using partial fraction) can be written as a sum of functions of the ~~form~~ $\frac{A}{s-a}$, $A, a \in \mathbb{R}$

What do we know about such functions?

$$\mathcal{L}(Ae^{at})(s) = \frac{A}{s-a}$$

Apply partial fractions

$$Y(s) = \frac{2}{s} - \frac{1}{s+1} - \frac{1}{s+2}$$

$$= \mathcal{L}(2 \cdot e^{0 \cdot t}) - \mathcal{L}(e^{-t}) - \mathcal{L}(e^{-2t})$$

$$= \mathcal{L}(2 - e^{-t} - e^{-2t})$$

$$\Longrightarrow \boxed{y(t) = 2 - e^{-t} - e^{-2t}}$$

This solves the equation with $y(0) = 0$

<u>EX.</u> Use Laplace transform to solve

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 0, \; y'(0) = 1 \end{cases}$$

Let $Y(s) = \mathcal{L}(y)$, then $(as+b)y(0) - by'(0)$

$$(s^2 - s - 2)Y(s) = 1$$

$$Y(s) = \frac{1}{s^2 - s - 2}$$

Roots of $s^2 - s - 2$ are $2, -1$, so

$$s^2 - s - 2 = (s+1)(s-2)$$

Applying partial fractions we have

$$Y(s) = -\frac{1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$

$$\mathcal{L}(-\tfrac{1}{3}e^{-t}) \qquad \mathcal{L}(\tfrac{1}{3}e^{2t})$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}}$$

$$\left(y(0) = -\frac{1}{3} + \frac{1}{3} = 0, \quad y'(0) = \frac{1}{3} + \frac{2}{3} = 1 \right)$$

__EX__. Use Laplace's transform to solve

$$\begin{cases} y'' + 2y' - 3y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

Characteristic Polynomial is $s^2 + 2s - 3$ and the roots are $-3, 1$. So

$$s^2 + 2s - 3 = (s+3)(s-1)$$

So if $Y = \mathcal{L}(y)$, then

$$(s^2 + 2s - 3)Y = s + 2 \qquad \left(\begin{matrix} y(0) = 1 \\ y'(0) = 0 \end{matrix} \right)$$

$$Y(s) = \frac{s+2}{s^2+2s-3} = \frac{s+2}{(s+3)(s-1)}$$

Partial fractions yield

$$Y(s) = \frac{1}{4}\frac{1}{s+3} + \frac{3}{4}\frac{1}{s-1}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\mathcal{L}\left(\tfrac{1}{4}e^{-3t}\right) \qquad \mathcal{L}\left(\tfrac{3}{4}e^{t}\right)$$

$$\Rightarrow \boxed{y(t) = \frac{1}{4}e^{-3t} + \frac{3}{4}e^{t}}$$

Check: $y(0) = 1$, $y'(0) = 0$ and y solves $y'' + 2y' - 3y = 0$.

Let's compute more Laplace transforms.

$$\mathcal{L}(\cos(t))(s) = \int_0^\infty e^{-st}\cos(t)\,dt$$

(for $s > 0$)
$$= -\frac{1}{s}e^{-st}\cos(t)\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}(-\sin(t))\,dt$$

$$= \frac{1}{s} - \frac{1}{s}\int_0^\infty e^{-st}\sin(t)\,dt$$

$$\left(= \frac{1}{s} - \frac{1}{s}\mathcal{L}(\sin(t)) \right)$$

$$\int_0^\infty e^{-st}\sin(t)\,dt = -\frac{1}{s}e^{-st}\sin(t)\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\cos(t)\,dt$$

$$(s>0) \qquad = 0 + \int_0^\infty \frac{1}{s}e^{-st}\cos(t)\,dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\cos(t)\,dt$$

$$\mathcal{L}(\cos(t))(s) = \frac{1}{s} - \frac{1}{s^2}\mathcal{L}(\cos(t))(s)$$

$$\left(1+\frac{1}{s^2}\right)\mathcal{L}(\cos(t))(s) = \frac{1}{s}$$

$$\mathcal{L}(\cos(t))(s) = \frac{1}{s}\,\frac{1}{1+\frac{1}{s^2}}$$

$$= \frac{1}{s+\frac{1}{s}} = \frac{s}{s^2+1}$$

$$\boxed{\mathcal{L}(\cos(t))(s) = \frac{s}{s^2+1}}$$

$$\boxed{\mathcal{L}(\sin(t))(s) = \frac{1}{s^2+1}}$$

Exercise: Compute

$\mathcal{L}(\cos(\alpha t))$, $\mathcal{L}(\sin(\alpha t))$

and

$\mathcal{L}\left(e^{\mu t}\cos(\alpha t)\right), \mathcal{L}\left(e^{\mu t}\sin(\alpha t)\right)$

EX. Solve (using Laplace transform)

$$\begin{cases} y'' + 2y' + 6y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

$Y(s) = \mathcal{L}(y)$, then

$$(s^2 + 2s + 6)Y = 1$$

$$Y(s) = \frac{1}{s^2 + 2s + 6} \quad \left(\simeq \frac{1}{(s+d)^2 + e} \right)$$

Note: $4 - 4 \cdot 6 < 0$ so the roots are not real.
$(b^2 - 4ac)$

Q: Can we write $s^2 + 2s + 6$ as $(s+d)^2 + e$? $(d, e > 0)$

(this way it looks more like $s^2 + 1$

Next class: More on complex roots, Laplace trans for equations with discont. forcing term.