Reminders: - Problem Set 10 is posted
(Problem Set 11 will be due on the last day of class)

- Exam 2 will take place this Thursday in the following manner
  - At 6pm on Thursday you will receive the exam via email, you must submit the answer by 6pm on Friday.
  - If you foresee this presents scheduling problems email me immediately.

Today
- Wronskian (3.2, 3.6)
- Mechanical and electrical vibrations (3.7)
- Undetermined coefficients (3.5) and Variation of parameters (3.6).

Wronskian

Consider the equation

$$\boxed{a(t)\ddot{u} + b(t)\dot{u} + c(t)u = 0 \quad (*)}$$

Suppose you have two solutions $u_1(t)$ and

$u_2(t)$ and you want to solve the initial value problem for (*) with

$$u(0) = u_0 \ , \ \dot{u}(0) = \dot{u}_0$$

($u_0, \dot{u}_0$ some given numbers)

We look for a solution of the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t)$$

c_1, c_2 numbers to be determined

The numbers c_1, c_2 are given by the system of algebraic equations

$$c_1 u_1(0) + c_2 u_2(0) = u_0$$
$$c_1 \dot{u}_1(0) + c_2 \dot{u}_2(0) = \dot{u}_0$$

There is one condition on the coefficients $u_1(0), u_2(0), \dot{u}_1(0), \dot{u}_2(0)$ that guarantees this always has one, and exactly one solution c_1, c_2.

$$\left(\begin{array}{l} a c_1 + b c_2 = u_0 \\ c c_1 + d c_2 = \dot{u}_0 \end{array} \right)$$

The condition is

$$\det \begin{pmatrix} u_1(0) & u_2(0) \\ \dot{u}_1(0) & \dot{u}_2(0) \end{pmatrix} \neq 0$$

The Wronskian of two solutions u_1 and u_2 is the function of t given by

$$W(u_1,u_2;t) = \det\begin{pmatrix} u_1(t) & u_2(t) \\ \dot{u}_1(t) & \dot{u}_2(t) \end{pmatrix}$$

$$= u_1(t)\dot{u}_2(t) - \dot{u}_1(t)u_2(t)$$

Exercise (See Problem set 10)

We have seen that for a 2-d system $\dot{x} = Ax$ and two fundamental solutions x_1, x_2, that their Wronskian $\left(= \det\left(\left(x_1(t)\ x_2(t)\right)\right)\right)$ solves the equation

$$\dot{W} = tr(A)\,W$$

where for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$tr(A) = a_{11} + a_{22}.$$

Examples: For each equation bellow let's find a set of fundamental of solutions and compute their Wronskian.

1) $$\ddot{u} + 4u = 0$$

Characteristic polynomial : $\lambda^2 + 4$

Roots are $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$\Rightarrow$ e^{2it} and e^{-2it} are solutions; their real and imaginary parts are then also solutions, so we get

$$\boxed{u_1(t) = \cos(2t) \quad , \quad u_2(t) = \sin(2t)}$$

Their Wronskian is

$$W(u_1, u_2; t) = u_1 \dot{u}_2 - \dot{u}_1 u_2$$
$$= \cos(2t)\, 2\cos(2t) - (-2\sin(2t))\sin(2t)$$
$$= 2\cos(2t)^2 + 2\sin(2t)^2$$
$$= 2 \neq 0 .$$

$$\boxed{W(u_1, u_2; t) = 2}$$

2) $$\ddot{u} + 2\dot{u} + 4u = 0$$

Characteristic polynomial $\lambda^2 + 2\lambda + 4$

Roots (use quadratic formula) $\lambda_{\pm} = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 4}}{2}$

$$= -1 \pm \tfrac{1}{2}\sqrt{-12}$$
$$= -1 \pm \tfrac{1}{2}\sqrt{4 \cdot 3 \cdot (-1)}$$
$$= -1 \pm \sqrt{3}\, i$$

real part (for -1) imaginary part (for $\sqrt{3}$)

Then we have

$$\boxed{\begin{aligned} u_1(t) &= e^{-t}\cos(\sqrt{3}t) \\ u_2(t) &= e^{-t}\sin(\sqrt{3}t) \end{aligned}}$$

Wronskian:

$$\dot{u}_1(t) = -e^{-t}\cos(\sqrt{3}t) - \sqrt{3}\,e^{-t}\sin(\sqrt{3}t)$$

$$\dot{u}_2(t) = -e^{-t}\sin(\sqrt{3}t) + \sqrt{3}\,e^{-t}\cos(\sqrt{3}t)$$

(after some cancellations)

$$u_1\dot{u}_2 - \dot{u}_1 u_2 \overset{\downarrow}{=} \sqrt{3}\,e^{-2t}\cos(\sqrt{3}t)^2 + \sqrt{3}\,e^{-2t}\sin(\sqrt{3}t)^2$$

$$= \sqrt{3}\,e^{-2t}$$

$$\boxed{W(u_1, u_2; t) = \sqrt{3}\,e^{-2t}}$$

3) $\ddot{u} + 5\dot{u} + 4u = 0$

$\lambda^2 + 5\lambda + 4 = 0$, Roots: $\dfrac{-5 \pm \sqrt{25-16}}{2}$

$\dfrac{-5 \pm 3}{2} = \begin{cases} -1 \\ -4 \end{cases}$

(Note: both eigenvalues are negative!)

$u_1(t) = e^{-t}$ $\quad u_2(t) = e^{-4t}$

$\dot{u}_1(t) = -e^{-t}$ $\quad \dot{u}_2(t) = -4e^{-4t}$

$$W(t) = e^{-t}(-4e^{-4t}) - (-e^{-t})e^{-4t}$$

$$= e^{-5t}(-4+1)$$

$$= -3e^{-5t}$$

4) $\ddot{u} + 3\dot{u} + 4u = 0$

$\lambda^2 + 3\lambda + 4 = 0$

Roots $\dfrac{-3 \pm \sqrt{9-16}}{2} = \dfrac{-3 \pm \sqrt{-7}}{2}$

$= -\dfrac{3}{2} \pm i\sqrt{7}$

real part imaginary part

$u_1(t) = e^{-\frac{3}{2}t}\cos(\sqrt{7}\,t)$

$u_2(t) = e^{-\frac{3}{2}t}\sin(\sqrt{7}\,t)$

$W(t) =$ (exercise!)

A general look at $a\ddot{u} + b\dot{u} + cu = 0$

The characteristic equation for this equation is

$$a\lambda^2 + b\lambda + c = 0$$

The roots are

$$\lambda_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Note that the sign of

$$b^2-4ac$$

determines if the roots are real or not.

If $b=0$, and $a,c>0$ then the roots are purely imaginary

$$\pm i\sqrt{\frac{c}{a}}$$

and solutions are

$$\cos\left(\sqrt{\frac{c}{a}}\,t\right), \quad \sin\left(\sqrt{\frac{c}{a}}\,t\right)$$

If $b>0$, and $a,c>0$, then if b is small we will get $b^2-4ac<0$ and the roots will be

>0

$$\lambda_1 = -\frac{b}{2a} + i\,\frac{\sqrt{4ac-b^2}}{2a}$$

$$\lambda_2 = -\frac{b}{2a} - i\,\frac{\sqrt{4ac-b^2}}{2a}$$

If we write

$$\mu = -\frac{b}{2a}$$

$$\omega = \frac{\sqrt{4ac-b^2}}{2a}$$

the solution becomes

$$u_1(t) = e^{\mu t}\cos(\omega t), \quad u_2(t) = e^{\mu t}\sin(\omega t)$$

so there are functions oscillating but also decaying exponentially.

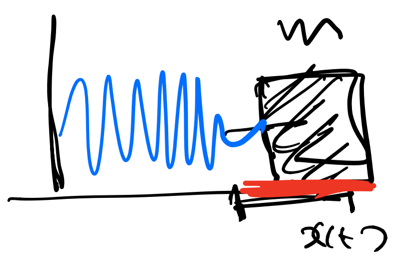
Lastly, if $b>0$ and sufficiently large then $b^2-4ac>0$ and the roots will be real $\Rightarrow$ no oscillation.

Damping force and restorative force

$$m\ddot{x} = \overbrace{-kx}^{\text{restorative force}} \qquad (k>0)$$

(Hooke's law)

$$m\ddot{x} = -kx - \underbrace{\gamma\dot{x}}_{\text{damping/friction force}}$$



The resulting equation has the form

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

(where $m,\gamma,k>0$)

In summary

- No damping ($b=0$) : perfectly oscillatory solution
- Some damping ($b>0$, small) : oscillation + exponential decay
- More damping ($b>0$, large) : no more oscillation, purely exponential behavior

One more term...

$$m\ddot{x} = -kx - \gamma\dot{x} + \underbrace{f(t)}_{\text{External Force}}$$

This leads to considering inhomogeneous equations

$$\boxed{a\ddot{u} + b\dot{u} + cu = f(t)}$$

Inhomogeneous Equations

Two methods

1) Variation of parameters

2) Undetermined coefficients

Var. of parameters: Two ways of thinking about it:

① The book's (Section 3.6)

② Reduction to 2-d systems (Var of parameters in Chapter 7)

I'll discuss ②.

When dealing with

$$a\ddot{u} + b\dot{u} + cu = f(t)$$

one approach is to consider the corresponding 2-d system

(Recall $x(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ $\leftarrow x_1(t)$, $\uparrow x_2(t)$

$$\ddot{u} = -\frac{c}{a}u - \frac{b}{a}\dot{u} + \frac{1}{a}f$$)

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{a} f(t) \end{pmatrix}$$

to this system we can apply variation of parameters, get a solution $x(t)$, of which the first component will be <u>the solution</u> to the 2nd order equation.

We get the following: if $u_1(t)$, $u_2(t)$ are a fundamental system of solutions to

$$a\ddot{u} + b\dot{u} + cu = 0$$

Then a particular solution to

$$a\ddot{u} + b\dot{u} + cu = f$$

is given by

$u_p(t)$

$$= \left(- \int_0^t \frac{u_2(s) f(s)}{W(u_1, u_2; s)} \, ds \right) u_1(t)$$

$$+ \left(\int_0^t \frac{u_1(s) f(s)}{W(u_1, u_2; s)} \, ds \right) u_2(t)$$

The general solution is then simply

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + u_p(t)$$