April 9, 2020

- The exponential of a matrix (7.7)
- Complex valued solutions (end of 7.4, 7.6)
- Method of Variation of Parameters (7.9) ← Today and next class

The exponential

$$e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n} \qquad (x \text{ a real variable})$$

$n! = n(n-1)(n-2)\cdots 1$, $\quad 0! = 1, \; 1! = 1 \ldots$

$$\begin{aligned}(e^{x})' &= \sum_{n=0}^{\infty} \frac{1}{n!}(x^{n})' \\ &= \sum_{n=0}^{\infty} \frac{n}{n!} x^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^{n} = e^{x}\end{aligned}$$

$$\left( \frac{n}{n!} = \frac{\cancel{n}}{\cancel{n}(n-1)\cdots 1} = \frac{1}{(n-1)!}, \; n \neq 0 \right)$$

EXERCISE: Check without using the chain rule that

$$\left(e^{\lambda x}\right)' = \lambda e^{\lambda x}$$

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To compute the exponential of $x$, we required

- Taking powers of $x$
- Adding such powers of $x$ (multiplied by $\frac{1}{n!}$)
- Taking limits

$$e^{x} = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} x^{n}$$

Each of these things can be done with a square matrix $A$:

- Powers of $A$: $A^0 = I, A^1, A^2, \dots$
- We can add them up

$$\sum_{k=0}^{N} \frac{1}{k!} A^k = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots + \frac{1}{k!}A^n$$

- $$\boxed{\lim_{N\to\infty} \sum_{k=0}^{\infty} \frac{1}{k!} A^k}$$

**Definition**: The exponential of a square matrix $A$, denoted $e^A$, is the matrix given by

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

**Properties of $e^A$** (properties of $e^x$, $x$ real: ① $e^x \neq 0$ for all $x$ ② $e^0 = 1$ ③ $(e^x)' = e^x$ ④ $e^{a+b} = e^a e^b$)

① If $A$ and $B$ are matrices which commute (meaning $AB = BA$) then

$$\boxed{e^{A+B} = e^A \cdot e^B = e^B \cdot e^A}$$

(one can check that this is true manipulating the power series defining $e^{A+B}$: $\sum_{k=0}^{\infty} \frac{1}{k!}(A+B)^k$ ← A good exercise!)

② If $A = 0$ $\left(A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & \cdots & & 0 \end{pmatrix}\right)$ then

$$e^A = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}$$

(why? well if $A=0$, then $A^k = 0$ for all powers $k$, so

$$e^A = I + 0 + \frac{1}{2}0 + \frac{1}{6}0 + \cdots$$
$$= I.)$$

③ If $A$ is a diagonal matrix, then so is $e^A$.

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & & \\ \vdots & & \lambda_3 & & 0 \\ & & & \ddots & \\ 0 & \cdots & & 0 & \lambda_n \end{pmatrix}$$

Note $A^k$ is simply the diagonal matrix where the entries of $A$ are raised to the $k$-th power

$$A = \begin{pmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & & \\ \vdots & & \lambda_3^k & & 0 \\ & & & \ddots & \\ 0 & \cdots & & 0 & \lambda_n^k \end{pmatrix}$$

Then,
$$e^A = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_2^k & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdot & & e^{\lambda_n} \end{pmatrix}$$

(4) If $v$ is an eigenvector of the matrix $A$ with eigenvalue $\lambda$, then $v$ is also an eigenvector of $e^A$, but with eigenvalue $e^\lambda$.

(why? If $v$ is an eigenvector, then

$$A^k v = A^{k-1}(Av) = \lambda A^{k-1} v$$
$$= \lambda^2 A^{k-2} v \dots$$
$$= \lambda^k v$$

Then
$$e^A v = \sum_{k=0}^{\infty} \frac{1}{k!} A^k v = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k v$$
$$= \left(\sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k\right) v = e^\lambda v. \;)$$

In particular, if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, then

$$e^{tA} v = e^{\lambda t} v$$

This is interesting because we know that $e^{\lambda t} v$ is a solution of $\dot{x} = Ax$ when $v$ is an eigenvector with eigenvalue $\lambda$.

Is it the case that for any $v$, then
$$e^{tA} v$$
is a solution of $\dot{x} = Ax$?

⑤ $e^{tA}$ as a matrix function has the following property

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$$

( compare with $(e^{\lambda t})' = \lambda e^{\lambda t}$ )

( why is this true?

$$\frac{d}{dt} e^{tA} = \lim_{h \to 0} \frac{1}{h} \left( e^{(t+h)A} - e^{tA} \right)$$

Note : $e^{(t+h)A} = e^{tA + hA} = e^{hA} \cdot e^{tA}$

$$e^{(t+h)A} - e^{tA} = e^{hA} e^{tA} - e^{tA}$$

$$= \left( e^{hA} - I \right) e^{tA}$$

Observe : $e^{hA} = I + hA + h^2 \frac{A^2}{2} + h^3 \frac{A^3}{6} + \cdots$

so $e^{hA} - I = hA + h^2 \frac{A^2}{2} + h^3 \frac{A^3}{6} + \cdots$

$$\frac{1}{h}\left(e^{hA} - I\right) = A + \underbrace{h \frac{A^2}{2} + h^2 \frac{A^3}{6} + \cdots}_{\text{this sum} \to 0 \text{ when } h \to 0}$$

So

$$\lim_{h \to 0} \frac{1}{h}\left(e^{(t+h)A} - e^{tA}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(e^{hA} - I\right) e^{tA}$$

$$= A e^{tA} \quad )$$

⑥ If $x_0$ is any initial vector,

then $\boxed{x(t) = e^{tA} x_0}$ solves

$$\boxed{\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}}$$

(why! $\dot{x}(t) = \frac{d}{dt}\left(e^{tA} x_0\right)$

$= \left(\frac{d}{dt}(e^{tA})\right)x_0 + e^{tA} \cancel{\frac{d}{dt} x_0}^{=0}$

(by property 5) $= A e^{tA} x_0 = A\left(e^{tA} x_0\right)$

$= A\, x(t)$.

$x(0) = e^{0 \cdot A} x_0$ (by property 2)

$= e^{0} x_0 = I x_0 = x_0$ )

Then is clear that solving a system $\dot{x} = Ax$ is the same as finding an expression for $e^{tA}$.

Remark: In general, $\frac{d}{dt} e^{\int_0^t A(s)ds} \neq A(t) e^{\int_0^t A(s)ds}$

because $e^{A+B} = e^A \cdot e^B$ only when $AB = BA$, which is hardly true for matrices in general. This is why the matrix exponential is of little help for system with time-dependent coefficients

$$\dot{x} = A(t)\,x$$

except when $A(t)$ is such that

$$A(t)\,A(s) = A(s)\,A(t)$$

for all $t$ and $s$)

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How do we actually find the exponential? We already know!

<u>Observation</u>: If $\Psi(t)$ is a fundamental matrix for the system

$$\dot{x} = A x$$

then

$$\Psi(t)\,\Psi(0)^{-1} = e^{tA}$$

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So, to compute $e^{tA}$, we

1. Compute the characteristic polynomial of $A$
2. Find eigenvalues and (hopefully) $n$ linearly independent eigenvectors $v_1, \ldots, v_n$

(3) We form the fundamental matrix

$$\Psi(t) = \left( e^{\lambda_1 t} v_1 \,\middle|\, e^{\lambda_2 t} v_2 \,\middle|\, \cdots \,\middle|\, e^{\lambda_n t} v_n \right)$$

and then compute $\Psi(0)^{-1}$ and then

$$e^{tA} = \Psi(t)\,\Psi(0)^{-1}$$

<u>EX</u> Recall our "favorite" equation

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

We found in previous lecture that a fundamental matrix is given by

$$\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

Then

$$e^{t\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}} = \Psi(t)\,\Psi(0)^{-1}$$

$$= \begin{pmatrix} \frac{1}{2}(e^{3t} + e^{-t}) & \frac{1}{4}(e^{3t} - e^{-t}) \\ e^{3t} - e^{-t} & \frac{1}{2}(e^{3t} - e^{-t}) \end{pmatrix}$$

## Complex Valued Solutions

(end of Section 7.4, and all of Section 7.6)

A vector function $x(t)$ will be called real valued if all the components are real valued, and will be called complex valued function, and will be called complex valued if all the components are complex.

If $x(t)$ is complex valued, we can write it as

$$x(t) = x_1(t) + i\, x_2(t)$$

where $x_1(t)$ and $x_2(t)$ are real valued vector functions.

**EX**

$$x(t) = \begin{pmatrix} \cos(t) + i\sin(t) \\ \sin(t) - i\cos(t) \end{pmatrix}$$

Then

$$x(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + \begin{pmatrix} i\sin(t) \\ -i\cos(t) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + i \begin{pmatrix} \sin(t) \\ -\cos(t) \end{pmatrix}$$

**EX (continued)**

Let's check that $x(t)$ from the previous example is a complex valued solution to the system

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x$$

On one hand

$$\dot{x}(t) = \begin{pmatrix} \frac{d}{dt}(\cos(t) + i\sin(t)) \\ \frac{d}{dt}(\sin(t) - i\cos(t)) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(t) + i\cos(t) \\ \cos(t) + i\sin(t) \end{pmatrix}$$

on the other hand

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x = \begin{pmatrix} -(\sin(t) - i\cos(t)) \\ \cos(t) + i\sin(t) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(t) + i\cos(t) \\ \cos(t) + i\sin(t) \end{pmatrix}$$

so we note that in this case

$$\dot{x} = Ax$$

It is also easy to check that the following two functions are also solutions to the system

$$\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sin(t) \\ -\cos(t) \end{pmatrix}$$

but! these are also the real and imaginary parts of $x(t)$.

Thus we conclude with the following

Observation: A complex valued function $x(t)$ is a solution to a differential equation
$$\dot{x} = Ax$$
where A has only real entries if and only if the real and imaginary parts of $x(t)$ are also solutions.

(symbolically: $x(t) = x_1(t) + i x_2(t)$, $x_1$ and $x_2$ real valued solves $\dot{x} = Ax$ if and only if $\dot{x}_1 = Ax_1$ and $\dot{x}_2 = Ax_2$)

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How does this help us? Suppose you have the system from before
$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x$$
The characteristic polynomial in this case is
$$p(\lambda) = \lambda^2 + 1$$
and the roots are $\pm\sqrt{-1}$, i.e. $\pm i$.

This means that a complex valued solution to this equation would like
$$x(t) = e^{it} v \quad ?$$
for some complex eigenvector $v$.

## Complex exponential

Let $z = \alpha + i\beta$ ($\alpha, \beta$ real numbers)

Then we define

$$\boxed{e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k}$$

Properties:

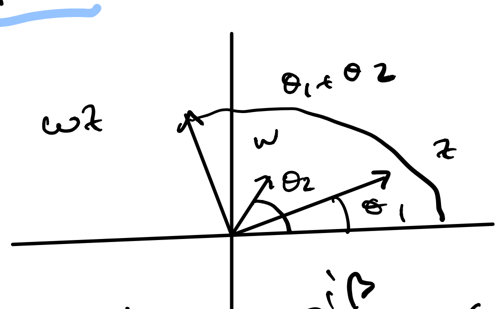
① $e^{z+w} = e^z \cdot e^w$ for all complex numbers $z$ and $w$

② if $z$ is real, then $e^z$ is the usual exponential

③ $e^{i\beta} = \cos(\beta) + i\sin(\beta)$, for real $\beta$.

(why? remember complex multiplication involves amplitudes or rotation

$|wz| = |w|\,|z|$




The reason why $e^{i\beta} = \cos(\beta) + i\sin(\beta)$ involves the power series of $e^x$, $\cos(x)$, $\sin(x)$ and we will discuss that some other time)

EX (let's go back to the system)

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x$$

The eigenvals are $i$ and $-i$, and an eigenvector for $i$ is

$$V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

so a solution is

$$x_1(t) = e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t) + i\sin(t) \\ (\cos(t) + i\sin(t))\,i \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t) + i\sin(t) \\ -\sin(t) + i\cos(t) \end{pmatrix}$$

so the complex valued solution we saw earlier came from a complex eigenvector

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- Tomorrow: I will post a video where I go over more examples
- Tuesday: we will learn to solve systems like

$$\dot{x} = A(t)x + b(t)$$