Today: - Fundamental Matrices (what are they for?) (7.7)
- The exponential of a matrix.
- Complex-valued solutions, complex eigenvalues and complex eigenvectors (7.6)

What we will see today is that the system of equations

$$\boxed{\dot{x} = Ax}$$

where A is a time-independent $n \times n$ matrix. Then solutions are given by:

$$x(t) = \underbrace{e^{tA}}_{\text{a } n\times n \text{ matrix (a fundamental matrix)}} \underbrace{x(0)}_{\text{initial condition}}$$

In this sense, linear systems are very similar to the 1d exponential differential equation. This will be useful insight since it will allow us to extrapolate the integrating

factor method but for systems
("variation of parameters")

Last time we said that
a Fundamental Matrix of a
system $\dot{x} = A(t)x$
is a matrix function whose columns
form a fundamental family of
solutions to the equation:

$$\Psi(t) = \begin{pmatrix} x_1(t) & \Bigg| & \cdots & \Bigg| & x_n(t) \end{pmatrix}$$

columns

$x_1(t), x_2(t), \ldots, x_n(t)$ are
solutions, and form a fundamental system of solutions

What is this for?

EX 1

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$$

Two solutions to this system are:

$$x_1(t) = \begin{pmatrix} \frac{1}{2}(e^{3t} + e^{-t}) \\ e^{3t} - e^{-t} \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} \frac{1}{4}(e^{3t} - e^{-t}) \\ \frac{1}{2}(e^{3t} + e^{-t}) \end{pmatrix}$$

Note:

$$x_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

They are lin. ind. $\Rightarrow$ x_1, x_2 are a fundamental family of solutions

$\Rightarrow$

$$\Psi(t) = \begin{pmatrix} \frac{1}{2}(e^{3t} + e^{-t}) & \frac{1}{4}(e^{3t} - e^{-t}) \\ e^{3t} - e^{-t} & \frac{1}{2}(e^{3t} + e^{-t}) \end{pmatrix}$$

Also, note: $\Psi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Given any vector x_0, the function

$$x(t) = \Psi(t)\, x_0$$

is a solution of the system

$$\dot{x} = Ax .$$

In particular, if you want to solve

$$\begin{cases} \dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x \\ x(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{cases}$$

All you do is look for x_0 such that

$$x(t) = \Psi(t)\, x_0$$

has initial value $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Take $t=0$

$$x(0) = \Psi(0)\, x_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

in our case $\Psi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so

$$x_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

gives the solution.

Let's see what that looks like

$$x(t) = \begin{pmatrix} \frac{1}{2}(e^{3t}+e^{-t}) & \frac{1}{4}(e^{3t}-e^{-t}) \\ e^{3t}-e^{-t} & \frac{1}{2}(e^{3t}+e^{-t}) \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t}+e^{-t}+\frac{1}{2}(e^{3t}-e^{-t}) \\ 2(e^{3t}-e^{-t}) + e^{3t}+e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}e^{3t}+\frac{1}{2}e^{-t} \\ 3e^{3t}-e^{-t} \end{pmatrix}.$$

Is easy to check $x(t)$ solves the equation, but also $x(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

So: once we have a fundamental matrix $\Phi(t)$ for a system $\dot{x} = A(t)x$, we can easily compute the solution for any initial value problem: all we do is choose a vector

$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ such that

Given

$$\boxed{\Psi(0) C = x(0)}$$

unknown

Given

and in that case

$$x(t) = \Psi(t) C$$

will be our solution.

We take the inverse matrix to $\Psi(0)$, that is the matrix (denoted by $\Psi(0)^{-1}$) such that

$$\Psi(0)^{-1} \Psi(0) = I$$

with this,

$$\Psi(0) c = x(0)$$

becomes

$$\cancel{\Psi(0)^{-1}} \cancel{\Psi(0)} c = \Psi(0)^{-1} x(0)$$

$$\boxed{c = \Psi(0)^{-1} x(0)}$$

We are going to make use for 2x2 systems of the following formula:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\det(A) = ad - bc$ is not zero, then

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(see problem set 8 where this formula is studied more carefully)

So when we have any fundamental matrix $\Psi(t)$, the solution to

$$\begin{cases} \dot{x} = A(t)x \\ x(0) = x_0 \end{cases}$$

Is given by

$$x(t) = \Psi(t)\left(\Psi(0)^{-1}x_0\right)$$

Alternatively

$$x(t) = \left(\underbrace{\Psi(t)\Psi(0)^{-1}}\right)x_0$$

The matrix
$$\tilde{\Psi}(t) = \Psi(t)\Psi(0)^{-1}$$
is again a fundamental matrix,
and $\tilde{\Psi}(0) = I$.

<u>EX</u> | Last class we studied the equation:
$$\dot{x} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}$$
and we found the fundamental solutions
$$x_1(t) = \begin{pmatrix} e^{-t} \\ e^{-t}\sqrt{2} \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} e^{-4t} \\ -e^{-4t}\frac{\sqrt{2}}{2} \end{pmatrix}$$
Let's form $\Psi(t)$, the fundamental matrix formed by these solutions, and let's find
$$\Psi(0) \quad \text{and} \quad \Psi(0)^{-1}$$

We have:

$$\Psi(t) = \begin{pmatrix} e^{-t} & e^{-4t} \\ e^{-t}\sqrt{2} & -e^{-4t}\frac{\sqrt{2}}{2} \end{pmatrix}$$

Then, $\Psi(0) = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

$$\det(\Psi(0)) = -\frac{\sqrt{2}}{2} - \sqrt{2} = -\frac{3\sqrt{2}}{2}$$

so, $$\Psi(0)^{-1} = -\frac{2}{3\sqrt{2}} \begin{pmatrix} -\frac{\sqrt{2}}{2} & -1 \\ -\sqrt{2} & 1 \end{pmatrix}$$

$$= \frac{2}{3\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{2} & 1 \\ \sqrt{2} & -1 \end{pmatrix}$$

Now, let us find

$$\boxed{\tilde{\Psi}(t) = \Psi(t)\,\Psi(0)^{-1}}$$

$$\tilde{\Psi}(t) = \frac{2}{3\sqrt{2}} \begin{pmatrix} e^{-t} & e^{-4t} \\ e^{-t}\sqrt{2} & -e^{-4t}\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 1 \\ \sqrt{2} & -1 \end{pmatrix}$$

$$= \frac{2}{3\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{2}e^{-t} + \sqrt{2}e^{-4t} & e^{-t} - e^{-4t} \\ e^{-t} - e^{-4t} & e^{-t}\sqrt{2} + e^{4t}\frac{\sqrt{2}}{2} \end{pmatrix}$$

Check that the columns of $\Psi(0)$ are nothing but the solutions $y_1(t)$ and $y_2(t)$ we found for this example in the previous class.

$$y_1(t) = \frac{1}{3}e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + \frac{2}{3}e^{-4t}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$y_2(t) = \frac{2}{3\sqrt{2}}e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} - \frac{2}{3\sqrt{2}}e^{-4t}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

(7.4) <u>Wronskian</u>

(Won't be so useful right now, but it will pay off when we study 2nd order equations)

Given a family of n solutions to a n-dimensional system

$$\dot{x} = A(t)\,x,$$

$$x_1(t),\ x_2(t),\ \ldots,\ x_n(t).$$

Then, the Wronskian of this family of solutions is a real valued function defined as the determinant of the matrix formed by the solutions

$$\det\left(\left(x_1(t) \,\middle|\, x_2(t) \,\middle|\, \cdots \,\middle|\, x_n(t)\right)\right)$$

This is denoted by $W(t)$, or sometimes

$$W[x_1, \ldots, x_n](t).$$

<u>Theorem</u>: In this situation $W(t)$ solves the differential equation

$$\frac{dW}{dt} = \left(A_{11}(t) + A_{22}(t) + \cdots + A_{nn}(t)\right)W$$

also written as

$$\frac{dW}{dt} = \mathrm{tr}(A(t))\,W$$

("tr" denotes the trace of the matrix, which is simply the sum of its diagonal elements)