Texas State University
MATH 3323: Differential Equations
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Problem Set 5

(1) For each matrix, find all the eigenvalues, and provide an eigenvector for each eigenvalue

$$\text{a) } \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \qquad \text{b) } \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{c) } \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

(2) For each part of this problem you are given a two dimensional differential equation and two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of said equation, in each case check that the given solutions are linearly independent at $t = 0$, and use them to find a solution $\mathbf{x}$ to the differential equation with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$\text{a) } \dot{\mathbf{x}} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}_1(t) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix}.$$

$$\text{b) } \dot{\mathbf{x}} = \begin{pmatrix} 10 & 0 \\ 0 & -10 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}_1(t) = \begin{pmatrix} e^{10t} \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 0 \\ e^{-10t} \end{pmatrix}.$$

$$\text{c) } \dot{\mathbf{x}} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}_1(t) = \begin{pmatrix} 2e^{3t} \\ 1e^{3t} \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 3e^{3t} \\ 0 \end{pmatrix}.$$

(3) Compute the following matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Conclude that if $ad - bc \neq 0$ then the matrix on the left is invertible and provide a formula for its inverse –compare this formula with your answer to problem 2 a) in Problem Set #3.

(4) (BONUS) Let x be a real number, find the limit of the sequence

$$\lim_{n\to\infty} \frac{x^n}{n!}$$

(remember that $n!$ denotes the product $n(n-1)(n-2)\dots 1$). Based on your answer, determine the set of values of x for which the following series converges and is finite

$$\sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

(5) (BONUS) Given differentiable functions $a_{11}(t), a_{12}(t), a_{21}(t)$, and $a_{22}(t)$, compute the derivative of

$$\det \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$$

and show the result coincides with the sum

$$\det \begin{pmatrix} \dot{a}_{11}(t) & a_{12}(t) \\ \dot{a}_{21}(t) & a_{22}(t) \end{pmatrix} + \det \begin{pmatrix} a_{11}(t) & \dot{a}_{12}(t) \\ a_{21}(t) & \dot{a}_{22}(t) \end{pmatrix}$$