Texas State University
MATH 3323: Differential Equations
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Problem Set 8

(1) For each matrix A bellow, compute e^{tA}. In each case you are given a basis of eigenvectors for A that you can use in your computation (the respective eigenvalues are not given, but those you can find easily given the eigenvectors!).

$$\text{(a) } A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\text{(b) } A = \begin{pmatrix} 2 & -1 \\ 10 & 5 \end{pmatrix}, \; v_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\text{(c) } A = \begin{pmatrix} -2 & 8 \\ 2 & -2 \end{pmatrix}, \; v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(2) Find the general solution to $\dot{x} = Ax + b$ for each A and b given bellow (note you are explicitly given the exponential matrix e^{tA} for each case)

$$\text{(a) } A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; b = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}, \; e^{tA} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

$$\text{(b) } A = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -2/3 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ -3t \end{pmatrix}, \; e^{tA} = \begin{pmatrix} e^{(\sqrt{3})t} & 0 \\ 0 & e^{-(2/3)t} \end{pmatrix}$$

$$\text{(c) } A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 5e^{3t} \\ 10e^{3t} \end{pmatrix}, \; e^{tA} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

(3) Find the general solution to $\dot{x} = Ax + b$, where

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

Hint: Remember that the general solution is a sum of the general solution to the homogeneous equation plus a particular solution to the inhomogeneous equation.

(4) (BONUS) Find a fundamental matrix $\Psi(t)$ (for example, e^{tA}) for the system $\dot{x} = Ax$ for each of the following A's.

$$\text{(a) } A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \quad \text{(b) } A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

(5) (BONUS) Consider a linear second order differential equation

$$\ddot{x} + B\dot{x} + Cx = 0,$$

such that $B^2 - 4C < 0$. In this case the roots of the characteristic polynomial are not real numbers.

(a) Show that if the roots are $\lambda = \mu \pm i\omega$ then μ and ω are given by the formulas

$$\mu = -\tfrac{1}{2}B \text{ and } \omega = \tfrac{1}{2}\sqrt{4C - B^4}$$

(b) Write down a general formula for solutions $x(t)$ using μ and ω.

(c) Let $x(t)$ be a real solution of the differential equation, using the formula obtained in part (b) show that the function $x(t)$ will always take both positive and negative values as t varies, except if x is the trivial solution given by $x(t) = 0$ for all t.