Feb 11th

Systems of equations, linear systems, and: A quick course in linear algebra (Part I)

(what follows contains material from section 7.1)

A system of differential equations is one where we have a number n of unknown functions $x_1(t), \dots, x_n(t)$ and an equal number of equations for these functions and their first derivatives:

$$(*)\begin{cases} \dot{x}_1(t) = f_1(x_1(t), \dots, x_n(t)) \\ \dot{x}_2(t) = f_2(x_1(t), \dots, x_n(t)) \\ \vdots \\ \dot{x}_n(t) = f_n(x_1(t), \dots, x_n(t)) \end{cases}$$

A solution to this system in some time interval (a,b) consists of n functions $x_1, \dots, x_n$ defined in (a,b) such that those functions and their derivatives satisfy the n equations in $(*)$ for every value t in (a,b).

additionally, for a system like (*), we are often given an "initial condition", this is n values $x_1^{(0)}, \dots, x_n^{(0)}$ meant to be satisfied at some $t = t_0$ in $[a, b]$

$$\begin{cases} x_1(t_0) = x_1^{(0)} \\ \vdots \\ x_n(t_0) = x_n^{(0)} \end{cases}$$

Typically, $a = 0$ and $t_0 = 0$, so the initial condition reflects the state of the system at time $t=0$.

Example

(First, a rather trivial one)

$$\begin{cases} \dot{x}_1 = 2x_1 \\ \dot{x}_2 = tx_2 \end{cases}$$

This system consists of two equations which are independent of each other (the right hand side of the first equation is independent of the second unknown function, and conversely for the second equation).

Then we can solve each of the two equations separately - say, using separation of variables. We obtain the formula:

$$x_1(t) = x_1(0)\, e^{2t}$$
$$x_2(t) = x_2(0)\, e^{-t^2/2}$$

and all such pairs for a given initial condition $(x_1(0), x_2(0))$ form all solutions to the equation.

Example (a less trivial example)

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases}$$

Now this is a less trivial system! This is one we will learn to solve in the coming weeks.

For now, we give two examples of a solution: First, the pair

$$x_1(t) = \cos(t)$$
$$x_2(t) = \sin(t)$$

is a solution, as well as

$$x_1(t) = -\sin(t)$$
$$x_2(t) = \cos(t)$$

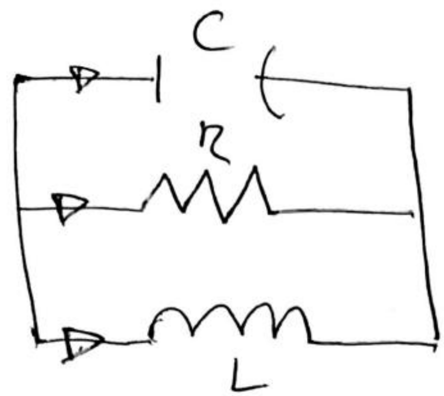
What does the general solution to this equation looks like? We will learn all solutions are given by

$$x_1(t) = C_1 \cos(t) - C_2 \sin(t)$$
$$x_2(t) = C_1 \sin(t) + C_2 \cos(t)$$

where C_1 and C_2 are constants determined from an initial condition..

Example (LRC circuits)

$I(t)$ = current, $V(t)$ = voltage



$$\begin{cases} \dot{I} = \frac{1}{L} V \\ \dot{V} = -\frac{1}{C} I - \frac{1}{RC} V \end{cases}$$

Here L, R, C are constants representing the circuits' inductance, resistance, and capacitance.

Given L, R, C and the initial voltage and current, determine $I(t)$ and $V(t)$ for all later times.

Example (Lotka-Volterra equations for 1920's predator/prey models)

$x_1(t)$ = prey population,
$x_2(t)$ = predator population

Lotka-Volterra equations
$$\begin{cases} \dot{x}_1 = x_1(a - \alpha x_2) \\ \dot{x}_2 = x_2(-c + \gamma x_1) \end{cases}$$

The constants a, c, α, and γ are parameters representing the environment where the predators and prey interact.

<u>Example</u> (Second order equations as first order systems)

Consider the second order equation

$$\ddot{x} - 2\dot{x} + 2x = 0 .$$

Let's see this can be recast as a first order system of equations: If $x(t)$ solves this equation, then the functions

$$\begin{cases} x_1(t) = x(t) \\ x_2(t) = \dot{x}(t) \end{cases}$$

Solve the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 + 2x_2 \end{cases}$$

and viceversa, any solution $(x_1(t), x_2(t))$ to this system is such that $x_1(t)$ solves the second order equation above.

Example (Newtonian gravity)

In this example

$x(t), y(t) =$ coordinates for a planet going around the sun, which is located at the center of the coordinate system.

Newton's equations

$$\begin{cases} \ddot{x} = -\dfrac{Gm_s}{(x^2+y^2)^{3/2}}\,x \\[2ex] \ddot{y} = -\dfrac{Gm_s}{(x^2+y^2)^{3/2}}\,y \end{cases}$$

← This is a second order, two-dimensional system

Having seen examples of systems, our focus for the next few classes will be to learn some basics of linear algebra of direct relevance to differential equations and systems of differential equations.

A rapid course on linear algebra (I)

<u>Vector spaces</u>: These are mathematical objects sharing many properties of the Cartesian plane and space (vectors made out of two (x,y) and three (x,y,z) coordinates) in that elements of the space can be added together and can be multiplied by numbers.

To illustrate the variety of examples of vector spaces, let us go over several examples

Example 1 :

V = the Cartesian plane (denoted $\mathbb{R}^2$)

elements denoted as $x = (x_1, x_2)$, $y = (y_1, y_2)$, etc.

Operation

$$x + y = (x_1 + y_1, x_2 + y_2)$$
$$\lambda x = (\lambda x_1, \lambda x_2)$$
(for λ a number)

For example : $x = (1, 7)$, $y = (\pi, 9)$

$x + y = (1 + \pi, 16)$, $3x = (3, 21)$
$x - y = (1 - \pi, -2)$, $\frac{2}{\pi} y = \left(2, \frac{18}{\pi}\right)$

Example 2

V = the 3-d Cartesian space (denoted by $\mathbb{R}^3$)

As before, but with 3 coordinates (x_1, x_2, x_3)

For example, take $x = (1, 0, 1)$ and $y = (0, 2, 3)$

Then

$$x - \tfrac{1}{2} y = \left(1, -1, \tfrac{1}{2}\right)$$
$$2y = (0, 4, 6)$$

Example 3: The set of all polynomials

Think of polynomials as functions, you can add them and multiply them by numbers, resulting in new polynomials, the resulting vector space is often denoted as $\mathbb{R}[x]$.

For instance, take

$$P_1(x) = 1 + x + x^2$$
$$P_2(x) = 35 + 137x^{10}$$
$$P_3(x) = 1 + x^2$$

Then

$$P_1 + P_2 = 2 + x + 2x^2$$
$$P_1 - P_2 = x$$
$$2P_2 + P_3 = 71 + x^2 + 274x^{10}$$

Example 4: The set of all polynomials of degree at most 2

The degree of a polynomial is the highest power of x that appears in it. If two polynomials have degree ≤ 2, then so will any sum of them. For instance

$$P_1(x) = 2x + 1, \qquad P_3(x) = x.$$
$$P_2(x) = x^2 - 1,$$

Observe that to specify a polynomial (1) of degree at most 2, we only need 3 numbers: the coefficients of $1 (= x^0)$, x^1, and x^2.

Each polynomial of degree at most 2 is a combination of these three:

$$P_0(x) = 1$$

$$P_2(x) = x$$

$$P_3(x) = x^2.$$

Feb 13th

A quick course in linear algebra (Part II)

More examples of vector spaces

Example 5 : V = the set of all real valued, continuous functions in the interval $[0,1]$.

This ~~set~~ vector space is so important that it has its own symbol: $C([0,1])$.

Here are some elements in this space:
(we are using x to denote the variable $x \in [0,1]$)

$f_1(x) = x^2$, $f_3(x) = \sin(x)$, $f_5(x) = \cos(x)$,
$f_2(x) = x+1$, $f_4(x) = 2^x$, $f_6(x) = \frac{1}{x-2}$,

$f_7(x) = x$,
$f_8(x) = (x+1)^2$,

An example of a function not in this space:

$$f(x) = \begin{cases} 1 & \text{if } x \le 1/2 \\ -1 & \text{if } x > 1/2 \end{cases}$$

(this function is not contin)

some linear combination of these functions.

(*) $f_1 + f_2 + f_7 = x^2 + x + 1 + x = x^2 + 2x + 1$
$$= (x+1)^2$$

so $f_1 + f_2 + f_7$ is the same as f_8.

$$f_8 = f_1 + f_2 + f_7$$

(we say f_8 is a linear combination of f_1, f_2, and f_7).

(*) The function $f_6(x)$, on the other hand, cannot be obtained by taking linear combinations of the other functions.

(*) ~~The functions~~ Another combination:

$$2 f_3 - f_5 = 2\sin(x) - \cos(x).$$

(*) It is not possible to obtain ~~so~~ f_3 or f_5 as a linear combination of the other functions in the list!

Example 6 (Solution to linear systems of algebraic equations).

Consider the following system of linear algebraic equations:

$$(*)\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ 3x_1 + 0 \cdot x_2 + 2x_3 = 0 \end{cases}$$

$V_1 = \left(-\frac{2}{3}, \frac{1}{3}, 1\right)$ and $V_2 = (2, 1, 3)$

are two solution to this system.

Check: any linear combination of these two vectors (for example, $V_1 + V_2$, or $V_1 - V_2$, or $2V_1 - V_2$) give all vectors which solve the system $(*)$ as well.

In fact: the set of triplets (x_1, x_2, x_3) solving $(*)$ with the usual operations in $\mathbb{R}^3$ form a vector space! It will either be a line, or a plane going through $(0,0,0)$

Example 7: The set of functions solving a given second order differential equation.

Solutions of the differential equation

$$\ddot{x} = 0$$

include functions such as

$$x_1(t) = 0$$
$$x_2(t) = 1$$
$$x_3(t) = 2t$$
$$x_4(t) = t$$

Any sum or linear combination of these functions will result in a function which also has $\ddot{x} = 0$. In fact, <u>all solutions to $\ddot{x} = 0$</u> can be expressed as

$$x(t) = c_1 t + c_2$$

which means the set of solutions is the same as the set of all polynomials of degree at most 1 — that is, a vector space.

Example 8 : The set of all functions $x(t)$ solving

$$(*) \quad \ddot{x} - \lambda^2 x = 0,$$

where $\lambda \in \mathbb{R}$ is fixed.

Later we will learn how to solve such equations systematically! For now, note that any function that solves

$$\dot{x} = \lambda x$$

or

$$\dot{x} = -\lambda x$$

will automatically solve $\ddot{x} = \lambda^2 x$.

Thus, we see we have the solutions

$$x_1(t) = e^{\lambda t}$$

$$x_2(t) = e^{-\lambda t}.$$

Now, you may have guessed it by now: solutions to $(*)$ form a vector space of functions, so automatically we have that given any two numbers C_1 and C_2, the function

$$x(t) = c_1 e^{\lambda t} + c_1 e^{-\lambda t}$$

is a solution to $(*)$, in fact, all solutions will be of this kind.

Example 9: Solutions to

$$\ddot{x} - 3\dot{x} + 2x = 0$$

also form a vector space! Later we will learn how to solve this and other linear second order equations.

Example 10: Vector valued- (or pairs of) continuous functions.

For instance:
$$x(t) = (\cos(t), \sin(t))$$
$$y(t) = (t^2 - t, e^{-t})$$

this space consists of functions taking values in the Cartesian plane and whose components/coordinates are continuous functions. If the variable t is confined to $[0,1]$, then we may think of this vector space as two copies of $C([0,1])$.

Example 11: Solution to a system of linear differential equation also form a vector space.

Consider the system:

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1$$

If $x(t) = (x_1(t), x_2(t))$ and $y(t) = (y_1(t), y_2(t))$ both solve this system, then any combination

$$c_1 x(t) + c_2 y(t)$$

will also solve this system, because:

$$c_1 \dot{x}_1 + c_2 \dot{y}_1 = -c_1 x_2 - c_2 y_2$$

$$= -(c_1 x_2 + c_2 y_2)$$

$$c_1 \dot{x}_2 + c_2 \dot{y}_2 = (c_1 x_1 + c_2 y_1)$$

It is easy to check that

$$x(t) = (\cos(t), \sin(t)), \quad y(t) = (-\sin(t), \cos(t))$$

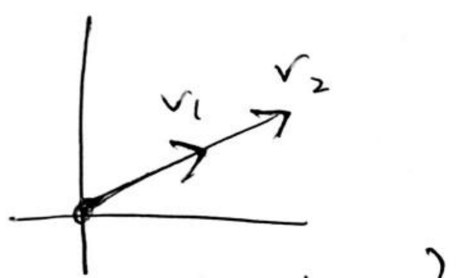
are two solutions to this system. Thus,

$$(c_1 \cos(t) - c_2 \sin(t), c_1 \sin(t) + c_2 \cos(t))$$

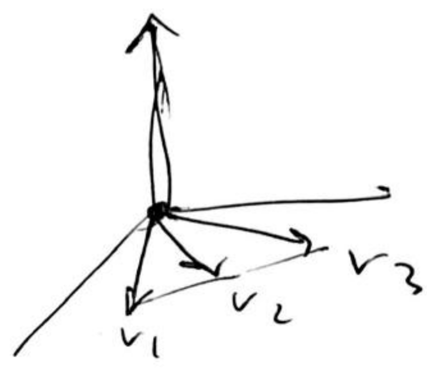
is also a solution, for any choice of c_1 and c_2.

Linear relations between vectors

Are two vectors parallel?.
Are three vector coplanar (with the origin)



$V_2 = \lambda_1 V_1$?



$V_3 = \lambda_1 V_1 + \lambda_2 V_2$?

The notion of linear dependence extends the idea of vectors being parallel or co-planar. We explain it bellow.

Given a vector space V, a "family" in V is a numbered list of vectors in V: $v_1, v_2, \dots, v_k$ for some $k \in \mathbb{N}$.

We will say a family of vectors $v_1, v_2, \dots, v_k$ is linearly dependent if there is some combination of numbers $c_1, c_2, \dots, c_k$, where at least one of them is not zero, and such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0,$$

(at least)

Being linearly dependent means that one vector in the family is redundant, in that it can be ~~expr~~ obtained as a construct of the others. Say, if $c_1 \neq 0$ ~~&~~ above,

then

$$v_1 = -\frac{c_2}{c_1} v_2 - \frac{c_3}{c_1} v_3 - \cdots - \frac{c_k}{c_1} v_k$$

<u>Example</u>: In $\mathbb{R}^3$, consider the vectors

$$v_1 = (2,1,1), \quad v_2 = (0,1,2), \quad v_3 = (4,0,-2)$$

Are they coplanar? (i.e. linearly dependent?)

Look: $v_3 = 2v_1 - 2v_2$. So the answer is these vectors are linearly dependent.

<u>Example</u>: (to discuss next time)

In $C([0,1])$, are the functions

$$x_1(t) = 1$$
$$x_2(t) = t$$
$$x_3(t) = t^2$$

linearly dependent? i.e. can we find c_1, c_2, c_3 not all zero such that $c_1 + c_2 t + c_3 t^2 = 0$?

How can we check this?