Name:

Math 534
Spring 2015

Final

There are three parts, you only need to do the indicated portion of problems from each one. Part I is mostly computational, but it does not hurt to state what method or theorem you are using. The exam, even if written, is a conversation, so write as much as you feel is necessaray! you may use an informal tone, but aim to be clear and concise.

...May the fourth be with you!

PART I (40 points): Solve FOUR of the SIX problems below.

(1) Let D be the disc of radius 1 centered at $(0,0)$. Find a formula for the solution of

$$\begin{cases} \Delta u &= f \text{ in } D, \\ u &= 1 \text{ on } \partial D. \end{cases}$$

in the case where $f(x) = x_2$. *Hint: Use separation of variables!.*

(2) Let D be the disc of radius 1 centered at $(0,0)$. Find an explicit formula for the solution of the boundary value problem

$$\begin{cases} \Delta u &= 0 \text{ in } D, \\ u &= x_1^2 \text{ on } \partial D. \end{cases}$$

Hint: Remember the identity $\cos(\theta)^2 - \sin(\theta)^2 = \cos(2\theta)$, *equivalently,* $\cos(\theta)^2 = (\cos(2\theta)+1)/2$.

(3) Find an explicit formula for the solution of the following non-homogeneous problem

$$\begin{cases} \partial_t u &= \partial_{xx} u + \sin(x)\cos(x) \text{ for } x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \cos(x)^2 - \sin(x)^2 \text{ for } x \in \mathbb{R}, \end{cases}$$

(4) Use Duhamel's principle to find a explicit formula for the initial value problem

$$\begin{cases} \partial_t u + 5\partial_x u &= e^{-t}\sin(x) \text{ for } x \in \mathbb{R},\ t > 0, \\ u(x,0) &= (1-x^2)_+ \text{ for } x \in \mathbb{R}, \end{cases}$$

(5) Use the method of characteristics to find a formula for the solution to

$$\begin{cases} \partial_t u + u\partial_x u &= 0 \text{ for } x \in \mathbb{R},\ t > 0, \\ u(x,0) &= x \text{ for } x \in \mathbb{R}, \end{cases}$$

Hint: Note that for this particular problem the characteristics do not intercept!

(6) Find the wave function $\psi(x,t)$ solving the Schrödinger equation

$$i\partial_t \psi + \partial_{xx}\psi = x^2\psi, \quad x \in \mathbb{R}, t > 0$$

and such that $\psi(x,0) = (1+x+x^2)e^{-x^2/2}$.

Hint: Express the initial condition in terms of Hermite functions ψ_k. *Recall that this is a family of functions given by the equations* $\partial_{xx}\psi_k - x^2\psi_k = -(2k+1)\psi_k$, $k = 0,1,2\ldots$

Solutions to Problems in Part I.

Solutions to Problems in Part I (continued).

PART II (30 points): Solve TWO of the FOUR problems below.

(1) Find an explicit formula for the solution of the following initial value problem with homogeneous Neumann boundary conditions

$$\begin{cases} \partial_t u &= \partial_{xx} u + tx \text{ for } 0 < x < \pi, \ t > 0, \\ u(x,0) &= 1 \text{ for } 0 < x < \pi, \\ \partial_x u(0,t) &= \partial_x u(\pi,t) = 0 \text{ for } t > 0. \end{cases}$$

(2) Let D as usual denote the disc of radius 1 with center at $(0,0)$. Use separation of variables to determine the set of numbers $k \in \mathbb{R}$'s for which the eigenvalue problem

$$\begin{cases} \Delta u &= ku \text{ in } D \\ u &= 0 \text{ on } \partial D. \end{cases}$$

admits a nontrivial solution (i.e. different from $u \equiv 0$). **Bonus (5 points):** How does the answer change if the disc has radius $r > 0$ not necessarily equal to 1?.

(3) For some number $D > 0$ and numbers $b, c \in \mathbb{R}$ let us consider the initial value problem

$$\begin{cases} \partial_t u &= D\partial_{xx} u + b\partial_x u + cu \text{ for } x \in \mathbb{R}, \ t > 0, \\ u(x,0) &= u_0(x) \text{ for } x \in \mathbb{R}. \end{cases}$$

(a) Suppose that $u_0 \equiv 0$, show that then $u(x,t) \equiv 0$. Conclude that for any initial data u_0 there is at most one solution to the above initial value problem. *Hint: Maximum principle?.*
(b) Do a change of the "dependent" variable $u(x,t)$ into a new variable $v(x,t)$, in such a way that if u solves the above equation then v solves $\partial_t v = \partial_{xx} v$ for $x \in \mathbb{R}$, $t > 0$.
(c) Use b) to discover an integral formula for the solution $u(x,t)$ that uses $u_0(x)$, D, b, c.
(d) Using a-c above explain why for a given initial u_0 there is always one, and only one solution $u(x,t)$ to the above problem.

(4) Let $u(x,t)$ be a solution to the porous medium equation,

$$\partial_t u = \partial_{xx}(u^2) \text{ if } x \in \mathbb{R}, t > 0$$
$$u(x,0) = u_0(x)$$

Assume that $u_0(x) \geq 0$ everywhere, $u_0(x) = 0$ if $|x| \geq 1$ and $\max u_0(x) \leq 1$.
(a) Given an explicit exampple of an initial condition u_0 satisfying the above conditions (other than $u_0 \equiv 0$), and write down a formula for the solution.
(b) Find some function $R(t) > 0$ which is growing with t such that for any u_0 satisfying the above conditions we have,

$$u(x,t) = 0 \text{ if } |x| > R(t)$$

(c) Could the phenomenon in b) still happen if we had the heat equation instead?.

Solutions to Problems in Part II.

Solutions to Problems in Part II (continued).

PART III (20 points): Solve TWO of the FOUR problems below.

(1) Let u, v be harmonic functions in some ball $B_R \subset \mathbb{R}^d$ which are continuous up to its boundary.
   (a) Suppose $u \geq v$ everywhere in B_R, what happens if $u = v$ at some x_0 in the interior of B_R?
   (b) Suppose again that $u \geq v$ everywhere in B_R, but now that $u(x_0) \leq v(x_0) + \varepsilon$ at some point $x_0 \in B_{R/2}$ for some $\varepsilon > 0$. Show that there is a constant $C > 0$ such that

$$\max_{B_{R/2}} |u(x) - v(x)| \leq C\varepsilon$$

   Note: You do not need to write down the constant explicitly!.
   (c) **Bonus(5 points):** Suppose you have a sequence of harmonic functions $u_n(x)$ in B_R, and another function $v(x)$. Suppose that for any fixed point x in the interior of B_R we have

$$\lim_{n\to\infty} u_n(x) = v(x)$$

   Show that for every $r < R$ we have $\lim\limits_{n\to\infty} \max\limits_{B_r} |u_n(x) - v(x)| = 0$ and that $v(x)$ is harmonic!

(2) Consider the function

$$H(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

   (a) Check that $\partial_t H = \partial_{xx} H$ for $t > 0$.
   (b) Given an arbitrary function $f(x,t)$, define the function

$$u(x,t) = \int_0^t (H(\cdot,s) * f(\cdot,t-s))(x)\, ds := \int_0^t \int_{\mathbb{R}} H(x-y,t-s) f(y,s) dy\, ds.$$

   Show that u solves the nonhomogeneous initial value problem

$$\begin{aligned} \partial_t u &= \partial_{xx} u + f(x,t), \quad && \text{for } \mathbb{R}, t > 0 \\ u(x,0) &= 0 && \text{for } x \in \mathbb{R} \end{aligned}$$

(3) Consider a harmonic function $u(x)$ defined in the square $[-1,1] \times [-1,1]$. Suppose that along the sequence of points $x_k = (1/k, 1/k)$ we have

$$u(x_k) = 0 \quad \text{for each } k.$$

Show that then u must be zero everywhere.

(4) To any function $\psi : \mathbb{R}^3 \to \mathbb{C}$, we assign the following "energy"

$$E(\psi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \psi|^2 + |x|^{-1} |\psi|^2\, dx$$

Prove:
   (a) For any two functions ψ and ϕ: $E(\psi + \phi) = E(\psi) + E(\phi) + \int_{\mathbb{R}^3} \nabla\psi \cdot \nabla\phi\, dx + \int_{\mathbb{R}^3} |x|^{-1} \psi\phi\, dx$
   (b) Suppose that among all functions ψ that go to zero at infinity there is one ψ_0 that minimizes E: $E(\psi_0) \leq E(\psi)$ for any other ψ. Show that ψ_0 solves the equation

$$-\Delta\psi_0 + |x|^{-1}\psi_0 = 0 \text{ in } \mathbb{R}^3$$

   (c) **Bonus(5 points):** See how the conclusion in b) changes if ψ_0 was a minimizer only among those functions vanishing at infinity and such that

$$\int_{\mathbb{R}^3} |\psi|^2\, dx = 1$$

Solutions to Problems in Part III.

Solutions to Problems in Part III (continued).