Feb 4, 2020 (Sec. 2.1, 2.2)

More on separation of variables

Another way of seeing separation of variables is that one rewrites the equation (whenever possible)

$$\frac{dy}{dx} = f(x,y)$$

in the form

$$g(y)\frac{dy}{dx} = h(x)$$

for <u>some</u> function g an some funct h. (The book in turn writes this as

$$M(x) + N(y)\frac{dy}{dx} = 0,$$

so that, abusing the notation a bit, one can write the equation in term of differentials as

$$M(x)\,dx + N(y)\,dy = 0$$)

From

$$g(y)\frac{dy}{dx} = h(x)$$

we integrate on both sides with respect to the x variable, to obtain

$$\int g(y)\frac{dy}{dx}\,dx = \int h(x)\,dx + C$$

Now, if we let G and H be two function such that

$$G' = g, \quad H' = h.$$

Then the above can be rewritten as

$$G(y(x)) = H(x) + C.$$

EX| Equation from last class:

$$y' = \frac{x}{1+y}$$

In this case

$$g(y) = 1+y, \quad h(x) = x$$

so

$$G(y) = y + \frac{1}{2}y^2, \quad H(x) = \frac{x^2}{2}.$$

Let's do another example

<u>EX 1.</u>

$$y' = x/y$$

At least for $x, y \neq 0$ the above is equivalent to

$$y \cdot y' = -x$$

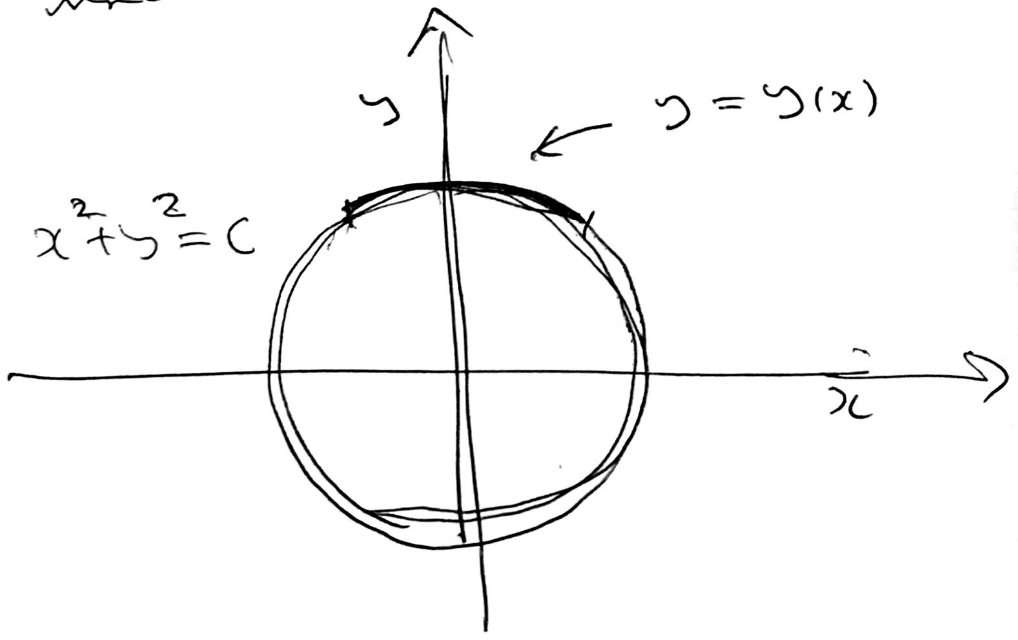
Integrating, this becomes

$$\frac{1}{2} y^2 = -\frac{1}{2} x^2 + C \qquad (*)$$

so

$$\boxed{y(x) = \pm\sqrt{-x^2 + 2C}}$$

One way to see what is going on, if $y(x)$ is a solution, then $(*)$ above says that the graph of $y(x)$ lies inside a circle.



Notice that depending on the size and sign of C, a solution of the equation only exists on a certain interval!

The previous example generalizes to all separable equations. With g, G, h, and H have the same meaning as before, $y(x)$ solving the equation

$$g(y)\, y' = h(x)$$

is the same as

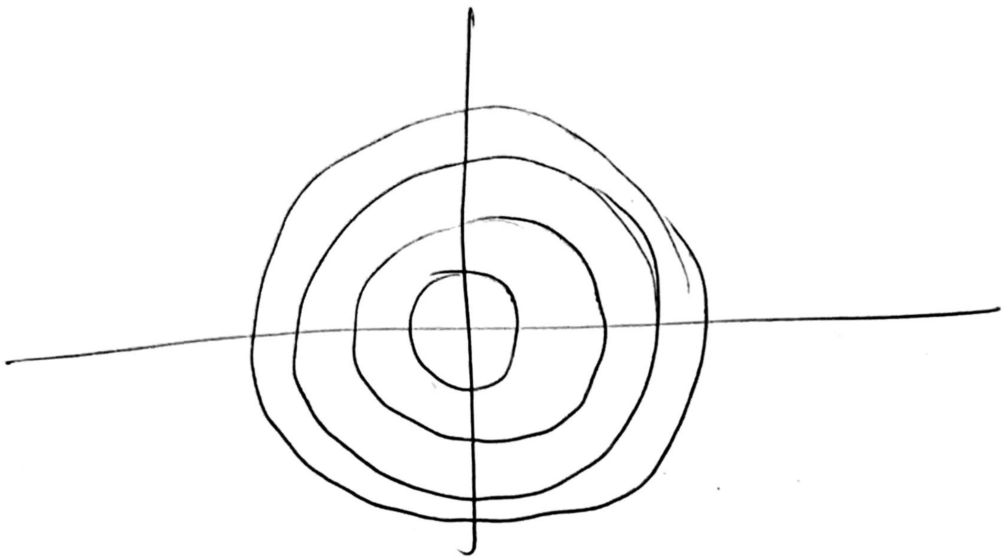
$$G(y(x)) = H(x) + C$$

for some C independent of x, or what is the same, the graph of $y(x)$ lies on a level set curve of the function $F(x,y) = G(y) - H(x)$.

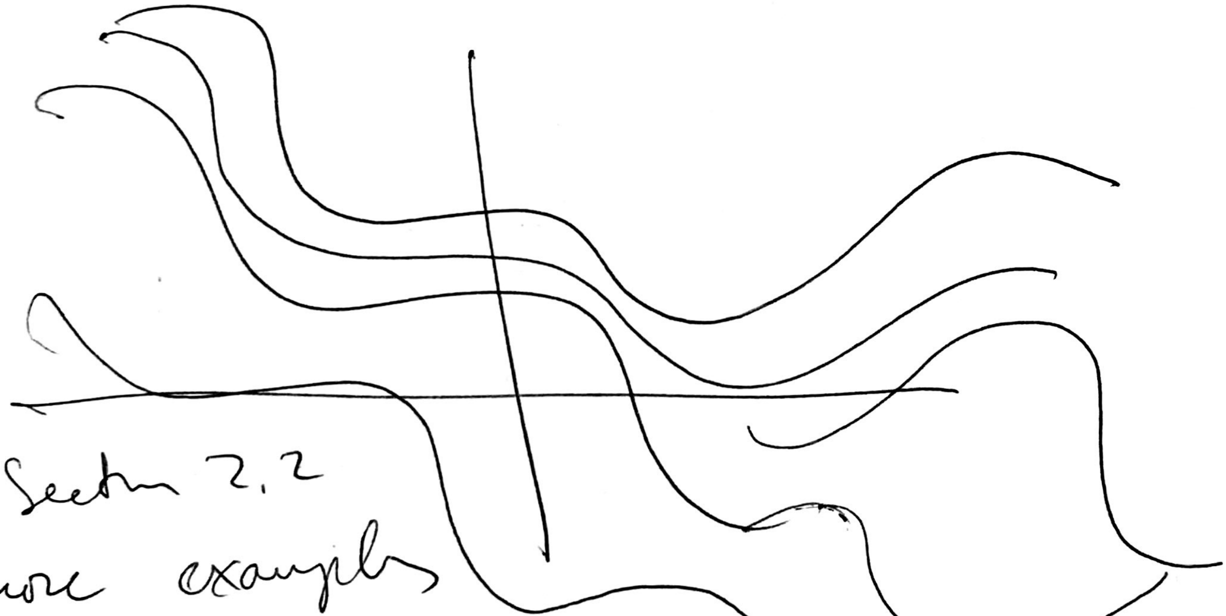
In the previous example,

$$F(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

so the level sets are all circles centered at $(0,0)$



In general, we will have
$F(x,y) = G(y) - H(x) = c$



See Section 2.2
for more examples

EX | Let's try to solve this equation:

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

What do we get?

(Answer: $\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$, or

$$\boxed{y^3 = \sqrt[3]{x^3+3C}}$$)

EX | $y' = 1+e^y$

In this case $h(x) = 1$, $g(y) = (1+e^y)^{-1}$

so

$$\int (1+e^y)^{-1} dy = x + C$$

After some work, we see that

$$\int \frac{1}{1+e^y} dy = y - \ln(e^y+1) + C$$

so solution satisfies the implicit equation

$$\boxed{y(x) - \ln\left(e^{y(x)}+1\right) = x + C}$$

It is not clear in this case that one can solve for $y(x)$ and obtain an explicit formula.

Now, you are equipped to study the most general type of (autonomous*) first order equation

$$y' = f(y)$$

(* means that the f does not depend on x.)

Feb 6th, 2020

Back to <u>first order linear equations</u>
(i.e. integrating factor)

Let us revisit integrating factor now that we know how to solve equations that are separable.

For the equation

$$\dot{x} = p(t) x + g(t)$$

we use integrating factor:
we rearrange the equation as

$$\dot{x} - p(t) x = g(t)$$

then look for a function $I(t)$ such that

$$I \dot{x} - p I x = I \dot{x} + \dot{I} x \left(= (Ix)^{\cdot} \right)$$

by the product rule

From this last equation we see that

$$\dot{I} = -p(t) I.$$

This is a separable equation! We solve it:

we have

$$\frac{\dot{I}}{I} = -p(t)$$

which means that $\frac{d}{dt}\left(\ln(I)\right) = -p(t)$,

so

$$\ln(I) = -\int p(t)\,dt + C$$

or

$$\boxed{I(t) = e^{-\int p(t)dt + C}}$$

Then, since for the purposes of the integrating factor it doesn't matter what C we use, choose C so that the function $P(t) = -\int p(t)\,dt + C$ is $= 0$ at $t=0$ (this is just a personal choice)

then

$$I(t) = e^{-P(t)}$$

and

$$\left(e^{-P(t)}x\right)^{\cdot} = e^{-P(t)}g(t)$$

Integrating, we get

$$e^{-P(t)}x = \int e^{-P(t)}g(t)\,dt + C$$

or

$$x(t) = e^{P(t)} C + e^{P(t)} \int e^{-P(t)} g(t)\,dt$$

EX:

$$\dot{x} = tx + t$$

$$\dot{x} - tx = 1 \quad , \quad I\dot{x} - tIx = It$$

Int. factor equation:

$$\dot{I} = -tI$$

A solution: $I(t) = e^{-\frac{1}{2}t^2}$, so $\left(e^{-\frac{1}{2}t^2} x\right)^{\cdot} = e^{-\frac{1}{2}t^2} t$

It follows that

$$e^{-\frac{1}{2}t^2} x(t) = \int e^{-\frac{1}{2}t^2} t\,dt + C$$

or

$$x(t) = e^{\frac{1}{2}t^2} \int e^{-\frac{1}{2}t^2} t\,dt + e^{\frac{1}{2}t^2} C$$

Note (by a substitution)

$$\int e^{-\frac{1}{2}t^2} t\,dt = -e^{-\frac{1}{2}t^2} + C$$

so,

$$x(t) = -1 + e^{\frac{1}{2}t^2} C$$

A note on ~~for~~ notation and integration (definite integrals vs indefinite integrals)

When we have

$$\frac{d}{dt}\left(I(t)x(t)\right) = I(t)g(t)$$

we could also do a definite integral between two times $t=t_1$ and $t=t_0$, resulting in

$$\int_{t_0}^{t_1} \frac{d}{dt}\left(I(t)x(t)\right)dt = \int_{t_0}^{t_1} I(t)g(t)\,dt$$

The left hand side is of course equal to

$$I(t_1)x(t_1) - I(t_0)x(t_0)$$

so we arrive at

$$I(t_1)x(t_1) = I(t_0)x(t_0) + \int_{t_0}^{t_1} I(t)g(t)\,dt$$

or, use $I(t) = e^{-P(t)}$,

$$x(t_1) = e^{P(t_1)-P(t_0)}x(t_0) + e^{P(t_1)}\int_{t_0}^{t_1} e^{-P(t)}g(t)\,dt$$

Linear superposition

We have learned that the solutions to the equation

$$\dot{x} = p(t)\, x + q(t) \qquad (*)$$

are all given by

$$x(t) = e^{P(t)} x(0) + e^{P(t)} \int_0^t e^{-P(s)} q(s)\, ds$$

where $P'(t) = p(t)$ and $P(0) = 0$. This expression has two parts worth analyzing in parts

$$x(t) = \underbrace{e^{P(t)} x(0)}_{\substack{\text{(homogeneous part)} \\ x_h(t)}} + \underbrace{e^{P(t)} \int_0^t e^{-P(s)} q(s)\, ds}_{\substack{\text{(inhomogeneous part)} \\ x_i(t)}}$$

Note that $x_h(t)$ solves:

$$\begin{cases} \dot{x}_h = p(t)\, x \\ x_h(0) = x(0) \end{cases}$$

while $x_i(t)$ solves

$$\begin{cases} \dot{x}_i = p(t)\, x + q(t) \\ x_i(0) = 0 \end{cases}$$

The first of these equation is called "the homogeneous equation", and it corresponds to (*) without the "forcing" term $g(t)$.

This homogeneous equation has the following important property: if $x_1(t)$ and $x_2(t)$ are two solutions, then

$$x(t) = x_1(t) + x_2(t)$$

is again a solution of the equation, moreover, if you take any two constants α_1 and α_2, then

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

is also a solution of the same equation.

Expression like $x_1(t) + x_2(t)$ or $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ are called <u>linear combinations</u> of the functions $x_1(t)$ and $x_2(t)$.

Thus the equation $\dot{x} = p(t)x$ satisfies the <u>linear superposition principle</u>: if x_1 and x_2 are two solutions, then any linear combination of x_1 and x_2 is a solution to the same equation

Linear superposition and the equation that ~~so~~ have it will be our chief concern for ~~...~~ a good part of the semester.

Let's see some examples of other equations that have it.

<u>EX</u>| The harmonic oscillator. That is the second order equation

$$\frac{d^2}{dt^2}x = -k^2 x$$

where k is a given physical constant.

Two solution one:

$$x_1(t) = \cos(kt)$$

and

$$x_2(t) = \sin(kt).$$

It is easy to see that then any combination $x(t) = \alpha_1 \cos(kt) + \alpha_2 \sin(kt)$ is again a solution.