Today: - Variation of parameters (7.9)
- Second order equations (7.1, 3.1)
- Characteristic Polynomial: real and complex roots (3.1 and 3.3)

The general linear inhomogeneous equation

$$\dot{x} = A(t)x + b(t)$$

Last time we learned how to solve this when $A(t)$ is constant ($= A$)

$$\boxed{x(t) = e^{tA}x(0) + e^{tA}\int_0^t e^{-sA}b(s)\,ds}$$

Note: The inverse to any matrix of the form e^A is e^{-A}, that is

$$\left(e^A\right)^{-1} = e^{-A}.$$

So if you ever have to compute the inverse of e^{tA}, it's the same as e^{-tA}. For example

$$e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

$$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}^{-1} = e^{-t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

What to do in general?

Well, if we can solve the homogeneous equation, then we can obtain a similar expression as what we obtained in the simpler case of $A(t)$ constant.

So. Let's suppose that we have $\Psi(t)$, a fundamental matrix for

$$\boxed{\dot{x} = A(t)\,x}$$

In particular, this means: $\Psi(t)$ is invertible for every t and

$$\dot{\Psi}(t) = A(t)\,\Psi(t).$$

So, variation of parameters says that the inhomogeneous problem has a solution given by

$$x(t) = \Psi(t)\,C(t)$$

where $C(t)$ is a n-vector of parameters that depend on t and that must be determined.

How do we choose $c(t)$?

$$\dot{x} = (\Psi(t) c(t))^{\cdot}$$

$$= \dot{\Psi}(t) c(t) + \Psi(t) \dot{c}(t)$$

$(\dot{\Psi} = A(t)\Psi(t))$

$$= A(t) \underbrace{\Psi(t) c(t)}_{= x(t)} + \Psi(t) \dot{c}(t)$$

Then
$$\dot{x} = Ax + \Psi(t) \dot{c}(t)$$

(want) $= Ax + b(t)$

It follows that $c(t)$ must solve

$$\Psi(t) \dot{c}(t) = b(t)$$

or
$$\boxed{\dot{c}(t) = (\Psi(t))^{-1} b(t)}$$

(in the cases where $\Psi(t) = e^{tA}$, this became $\dot{c}(t) = e^{-tA} b(t)$)

Integrating, we obtain

$$c(t) = c(0) + \int_0^t \dot{c}(s)\, ds$$

$$= c(0) + \int_0^t (\Psi(s))^{-1} b(s)\,ds$$

In conclusion

$$\boxed{x(t) = \Psi(t)c(0) + \Psi(t)\int_0^t (\Psi(s))^{-1} b(s)\,ds}$$

and $c(0)$ is computed from the initial condition $x(0)$

(in fact, $c(0) = \Psi(0)^{-1}x(0)$)

<u>Example</u>

$$\dot{x} = \begin{pmatrix} \frac{3t}{2} & t/2 \\ t/2 & 3t/2 \end{pmatrix} x + \begin{pmatrix} t \\ -t \end{pmatrix}$$

$$A(t) = \begin{pmatrix} 3t/2 & t/2 \\ t/2 & 3t/2 \end{pmatrix}, \quad b(t) = \begin{pmatrix} t \\ -t \end{pmatrix}$$

A fundamental matrix for the system is

$$\Psi(t) = \begin{pmatrix} e^{t^2} & -e^{t} \\ e^{t^2} & e^{t} \end{pmatrix}$$

Let's solve the system with initial condition

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\Psi(t)^{-1} = \frac{1}{2}\begin{pmatrix} e^{-t^2} & e^{-t^2} \\ -e^{t} & e^{t} \end{pmatrix}$$

Then
$$\left(\Psi(t)\right)^{-1} b(t) = \begin{pmatrix} 0 \\ e^{-t} t \end{pmatrix}$$

Integrating
$$\int_0^t \left(\Psi(s)\right)^{-1} b(s)\, ds = \int_0^t \begin{pmatrix} 0 \\ e^{-s} s \end{pmatrix} ds = \begin{pmatrix} 0 \\ \int_0^t e^{-s} s\, ds \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 1 - e^{-t} - e^{-t} t \end{pmatrix}$$

Then
$$\Psi(t) \int_0^t \left(\Psi(s)\right)^{-1} b(s)\, ds$$
$$= \left((e^{t} - 1) - t \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

From the variation of parameters formula we arrive at

$$x(t) = \Psi(t)\, c(0) + \left((e^{t} - 1) - t \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

we want $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boxed{x(t) = \begin{pmatrix} e^{t^2} - ((e^{t} - 1) - t) \\ e^{t^2} + ((e^{t} - 1) - t) \end{pmatrix}}$$

Second Order (Linear) Equations

(Today 3.1, 3.3 from the book)

Example (See also Example in Sec 7.1)

Consider the equation for a scalar function $u(t)$

$$\ddot{u} + k^2 u = 0 \qquad (k \text{ is a parameter})$$

This equation is equivalent to a 2×2 system.

If u solves

$$\begin{cases} \ddot{u} + k^2 u = 0 \\ u(0) = u_0 \\ \dot{u}(0) = \dot{u}_0 \end{cases}$$

Then, $x(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ solves

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix} x$$

$$x(0) = \begin{pmatrix} u_0 \\ \dot{u}_0 \end{pmatrix}$$

(and viceversa)

Why? Set $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

where $x_1(t) = u(t)$, $x_2(t) = \dot{u}(t)$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{u} \\ (\dot{u})^{\cdot} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \ddot{u} \end{pmatrix}$$

from the equation $\ddot{u} = -k^2 u$

so $$\dot{x} = \begin{pmatrix} \dot{u} \\ -k^2 u \end{pmatrix} = \begin{pmatrix} x_2 \\ -k^2 x_1 \end{pmatrix}$$
which corresponds to the matrix
$$A = \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix}$$

In general, a second order equation
$$P(t)\,\ddot{u} + Q(t)\,\dot{u} + R(t)\,u = 0$$
is equivalent to a 2-dim system
$$\dot{x} = A(t)\,x \quad (*)$$
where $$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{R(t)}{P(t)} & -\frac{Q(t)}{P(t)} \end{pmatrix}$$

Then, to solve the second order equation, we solve the system $(*)$ and then the first component of the solution vector, will give us the solution to the original second order equation

<u>Remark</u> : The class of 2-dim linear systems arising from 2nd order equations has special properties and further methods are available to solve them. Still, from the connection with systems explained above we get the following important fact :

$\boxed{\begin{array}{l}\text{If } u_1(t) \text{ and } u_2(t) \text{ are two linearly independent solutions to the second order equation} \\ \qquad P(t)\,\ddot{u} + Q(t)\,\dot{u} + R(t)\,u = 0 \\ \text{Then any solution to the equation can be written as} \\ \qquad A u_1(t) + B u_2(t) \\ \text{for the suitable constants } A, B.\end{array}}$

The moral of the story: to solve a second order equation we must find two special solutions which are linearly independent.

Constant Coefficient Second Order Equations

Let us focus on equations of the form

$$a\ddot{u} + b\dot{u} + cu = 0$$

$$(a, b, c \text{ constants})$$

What happens if $a=0$? We get a 1st order equation which we know how to solve.

If $a \neq 0$, the equation is effectively the same if we divide everything by a:

$$\ddot{u} + \underbrace{\left(\frac{b}{a}\right)}_{b}\dot{u} + \underbrace{\left(\frac{c}{a}\right)}_{c}u = 0$$

Then we really just have to study

$$\boxed{\ddot{u} + b\dot{u} + cu = 0}$$

In the space of functions of the variable t we have the linear transformation given by

$$Lu(t) = \ddot{u}(t) + b\dot{u}(t) + cu(t)$$

Then solution to the second order equation are simply the solutions to

$$Lu = 0 \quad (0 = \text{"the 0 function"})$$

How does L transform special class of functions?

EX : If $u(t) = 1$, then

$$Lu = c.$$

EX : Consider $Lu = \ddot{u}$. Then

$$L\cos(t) = -\cos(t)$$

so -1 is an eigenvalue of the operator L in this case, and $u = \cos(t)$ the corresponding eigenvector.

EX : If $u(t) = e^{\lambda t}$, then

$$\dot{u} = \lambda u$$

$$\ddot{u} = \lambda^2 u.$$

This means that if $Lu = \ddot{u} + b\dot{u} + cu$ for arbitrary u, then

$$L(e^{\lambda t}) = (\lambda^2 + b\lambda + c\lambda)e^{\lambda t}$$

Given a constant coefficient second order equation

$$a\ddot{u} + b\dot{u} + cu = 0$$

the characteristic polynomial of the equation is

$$p(\lambda) = a\lambda^2 + b\lambda + c.$$

Then if λ_0 is a root of the characteristic polynomial, then $e^{\lambda_0 t}$ solves the equation.

If λ_1, λ_2 are the two (different) roots of the characteristic polynomial then

$$u_1(t) = e^{\lambda_1 t}$$
$$u_2(t) = e^{\lambda_2 t}$$

are two linearly independent solution of the second order equation and thus any solution can be written as

$$u(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

The A and B can be determined from the initial conditions via

$$u(0) = A + B$$
$$\dot{u}(0) = \lambda_1 A + \lambda_2 B$$

The algorithm to solving

$$a\ddot{u} + b\dot{u} + cu = 0$$

(0) (Divide by a, so $a=1$ without loss of generality)

(1) Find roots λ_1, λ_2 of

$$a\lambda^2 + b\lambda + c = 0$$

(2) If roots are different, then the general solution is

$$u(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

(repeated root case to be covered on another day)

(3) If we have an Initial Value Problem (IVP) when we are asked to find a solution with given initial vals

$$u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0$$

Then we determine A, B by solving

$$A + B = u_0$$

$$\lambda_1 A + \lambda_2 B = \dot{u}_0.$$

Ex. Find $u(t)$ solving

$$\begin{cases} \ddot{u} - 2\dot{u} - 24u = 0 \\ u(0) = 0, \quad \dot{u}(0) = 1 \end{cases}$$

$$p(\lambda) = \lambda^2 - 2\lambda - 24, \quad \lambda_1 = -4, \lambda_2 = 6$$

General solution is

$$u(t) = A e^{-4t} + B e^{6t}$$

We want $u(0) = A + B = 0 \Rightarrow B = -A$

$$\dot{u}(0) = -4A + 6B = 1$$

$$-4A - 6A = 1$$

$$A = -\frac{1}{10}, \quad B = \frac{1}{10}.$$

Solution is $\boxed{u(t) = -\frac{1}{10} e^{-4t} + \frac{1}{10} e^{6t}}$

The case of complex roots

When the roots of the characteristic equation are not Real numbers, then they will always have the form

$$\lambda_1 = \mu + i\omega$$

$$\lambda_2 = \mu - i\omega$$

(μ, ω real numbers, $\omega \neq 0$)

With this notation

$$e^{\lambda_1 t} = e^{\mu t + i t\omega}$$
$$= e^{\mu t}(\cos(t\omega) + i\sin(t\omega))$$

$$e^{\lambda_2 t} = e^{\mu t}(\cos(t\omega) - i\sin(t\omega))$$

The real and imaginary parts of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are (up to a sign) given by

$$e^{\mu t}\cos(t\omega), \quad e^{\mu t}\sin(t\omega)$$

Because $Le^{\lambda_1 t} = 0$, and the coefficients in L are real, this means that the real and imaginary parts of $e^{\lambda_1 t}$ (and $e^{\lambda_2 t}$) both solve the differential equation.

(note we get no new solutions from λ_2, so it is enough for complex roots to use one)

Therefore the function

$$u_1(t) = e^{\mu t}\cos(\omega t)$$

$$u_2(t) = e^{\mu t}\sin(\omega t)$$

are two (linearly independent) solutions to the differential equation, so the general solution is

$$u(t) = A e^{\mu t}\cos(\omega t) + B e^{\mu t}\sin(\omega t)$$

__Remark__: (See Pset 9, question 4)

Any function as given above can be rewritten as

shifted cosine function

exponential function

$$R e^{\mu t}\cos(\omega t - \delta)$$