Today: - The method of undetermined coefficients
- Equation with periodic forcing

Warm up: Consider the differential equation

$$\ddot{u} = f(t)$$

for various function $f(t)$, and let's find -by guessing- particular solutions.

EX: $\ddot{u} = 1$, $u_1(t) = \frac{1}{2}t^2$

EX: $\ddot{u} = t$, $u_2(t) = \frac{1}{6}t^3$

EX: $\ddot{u} = t^2$, $u_3(t) = \frac{1}{12}t^4$

EX: Using the above, find a special solution to

$$\ddot{u} = 2t^2 + t - 1$$

Solution: $u = 2u_3 + u_2 - u_1$
$= \frac{1}{6}t^4 + \frac{1}{6}t^3 - \frac{1}{2}t^2$

Idea: If p is a polynomial, then $\ddot{p}$ will also be a polynom (two degrees down) so the equation $\ddot{u} = q(t)$ (where q is a polynomial) should have as a special solution a polynomial of degree equal to the degree of $f + 2$.

The same idea works for 2nd order differential equation with constant coefficients

<u>EX</u>. Let's find a special solution of

$$\ddot{u} - u = t + 1$$

Let's look for a solution among polynomials of degree 3. That means a function of the form

$$\boxed{u(t) = c_3 t^3 + c_2 t^2 + c_1 t + c_0 \cdot 1}$$

Then

$$\dot{u} = 3c_3 t^2 + 2c_2 t + c_1$$

$$\ddot{u} = 6c_3 t + 2c_2$$

Combining these,

$$\ddot{u} - u = 6c_3 t + 2c_2 - c_3 t^3 - c_2 t^2 - c_1 t - c_0$$

$$= -c_3 t^3 - c_2 t^2 + (6c_3 - c_1) t + 2c_2 - c_0$$

$$= t + 1$$

Equating the coefficients on either side, we get

$$c_3 = 0$$

$$c_2 = 0$$

$$6c_3 - c_1 = 1 \rightarrow c_1 = -1, \; c_2 = 0$$

$$2c_2 - c_0 = 1 \rightarrow c_0 = -1, \; c_3 = 0$$

The solution is

$$\boxed{u(t) = -t - 1}$$

This idea works with forcing terms of other kinds, such as:

① Trigonometric functions
$\cos(\alpha t)$, $\sin(\alpha t)$

② Exponentials
$e^{\lambda_1 t}$, $e^{\lambda_2 t}$, ... $\quad (e^{\lambda t} p(t))$

③ Polynomials times exponentials or polynomials times trig functions
$\left(e^{\lambda t}\cos(\alpha t),\ e^{\lambda t}\sin(\alpha t) \right)$

EX Find a function solving
$$\ddot{u} - u = \cos(2t)$$

Guess: There should be a solution of the form
$$\boxed{u(t) = A\cos(2t) + B\sin(2t)}$$

Let's compute the various derivatives

$$\dot{u} = -2A\sin(2t) + 2B\cos(2t)$$
$$\ddot{u} = -4A\cos(2t) - 4B\sin(2t)$$

Then

$$\ddot{u} - u = -4A\cos(2t) - 4B\sin(2t) - A\cos(2t) - B\sin(2t)$$

$$= -5A\cos(2t) - 5B\sin(2t)$$

To solve $\ddot{u} - u = \cos(2t)$, so A, B must be such that

$$-5A\cos(2t) - 5B\sin(2t) = \cos(2t) \Rightarrow A = -\frac{1}{5}, B = 0$$

So the particular solution is

$$\boxed{u(t) = -\frac{1}{5}\cos(2t)}$$

<u>EX</u> Consider the equation

$$\ddot{u} + 2\dot{u} + 5u = e^{-t} + 2e^{-2t}$$

Find a particular solution to this differential equation, where this solution has the form

$$\boxed{u(t) = Ae^{-t} + Be^{-2t}}$$

<u>Solution:</u>

$$u(t) = Ae^{-t} + Be^{-2t}$$

Derivatives:

$$\dot{u} = -Ae^{-t} - 2Be^{-2t}$$

$$\ddot{u} = Ae^{-t} + 4Be^{-2t}$$

Then

$$\begin{aligned}\ddot{u} + 2\dot{u} + 5u &= Ae^{-t} + 4Be^{-2t} + 2(-Ae^{-t} - 2Be^{-2t}) \\ &\quad + 5(Ae^{-t} + Be^{-2t}) \\ &= (A - 2A + 5A)e^{-t} + (4B - 4B + 5B)e^{-2t} \\ &= 4Ae^{-t} + 5Be^{-2t}\end{aligned}$$

Since we want

$$\ddot{u} + 2u + 5u = e^{-t} + 2e^{-2t}$$

We need $4A = 1$, $5B = 2$

so $A = \frac{1}{4}$, $B = \frac{2}{5}$, and the special solution is

$$\boxed{u(t) = \frac{1}{4}e^{-t} + \frac{2}{5}e^{-2t}}$$

Differential Operators

A linear transformation of differentiable functions is called a differential operator if it involves the operator $\frac{d}{dt}$ (or $(\dot{\ })$)

For example:

$$Lu = 5\frac{d^2}{dt^2}u - 3\frac{d}{dt}u + u$$

$$(= 5\ddot{u} - 3\dot{u} + u)$$

Another example: (variable coefficients)

$$Lu = -\frac{d}{dt}u + \sin(t)\,u$$

What's key is that

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2$$

This means that if you want to solve the differential equation

$$\boxed{a\ddot{u} + b\dot{u} + cu = f}$$

Then you may think about it a solving the linear equation

$$\boxed{Lu = f}$$

Looks a lot like $Ax = b$

in the space of function, L denoting the operator

$$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c$$

This has a number of advantages, for instance. If you know a solution to each of the problem

$$Lu_1 = f_1 \quad , \quad Lu_2 = f_2$$

Then one solution to $Lu = f_1 + f_2$, is

$$u = u_1 + u_2 .$$

Exponentials and differential operators with constant coefficients

Note: For any λ,

$$u(t) = e^{\lambda t}$$

is an "eigenvector" (or rather, eigenfunction) of the operators

$$\frac{d}{dt}, \left(\frac{d}{dt}\right)^2, \ldots$$

$$\left(\frac{d}{dt} u = \lambda u \, , \, \left(\frac{d}{dt}\right)^2 u = \lambda^2 u \, , \, \ldots \right)$$

So if $L = a\left(\frac{d}{dt}\right)^2 + b\frac{d}{dt} + c$, then

$$L(e^{\lambda t}) = (a\lambda^2 + b\lambda + c)e^{\lambda t}$$

Undetermined coefficients and general solutions

When solving

$$a\ddot{u} + b\dot{u} + cu = f$$

via undetermined coefficients, you first apply the method to obtain one particular solution $u_p(t)$, and then you must solve the homogeneous equation

$$a\ddot{u} + b\dot{u} + cu = 0 \rightarrow \begin{matrix}\text{Fundamental solutions:}\\ u_1(t), u_2(t)\end{matrix}$$

and add the general solution of this to your particular solution in order to get the general solution to the inhomogeneous problem:

$$\boxed{u(t) = c_1 u_1(t) + c_2 u_2(t) + u_p(t)}$$

Undetermined Coefficients + Periodic Forcing

We are going to consider equations of the form

$$a\ddot{u} + b\dot{u} + cu = f(t)$$

where f is a periodic function (i.e. a trigonometric function)

Ex. $\ddot{u} + 2\dot{u} + 2u = \cos(2t)$

Undetermined Coefficients

$$u_p(t) = A\cos(2t) + B\sin(2t)$$
$$\dot{u}_p(t) = -2A\sin(2t) - 2B\cos(2t)$$
$$\ddot{u}_p(t) = -4u_p(t)$$

Then

$$\ddot{u}_p + 2\dot{u}_p + 2u_p = -4u_p(t) + 2\dot{u}_p + 2u_p$$
$$= -2u_p(t) + 2\dot{u}_p$$
$$= -2\left(A\cos(2t) + B\sin(2t)\right) + 2\left(-2A\sin(2t) - 2B\cos(2t)\right)$$
$$= (-2A - 4B)\cos(2t) + (-4A - 2B)\sin(2t)$$

So we get $\begin{cases} -2A - 4B = 1 \\ -4A - 2B = 0 \end{cases}$

$B = -2A$, $\quad -2A - 4(-2A) = 1$

$6A = 1$, $A = 1/6$

$B = -1/3$

Solution is

$$\boxed{u_p(t) = \frac{1}{6}\cos(2t) - \frac{1}{3}\sin(2t)}$$

The general solution has the form
$$u(t) = c_1 u_1(t) + C_2 u_2(t) + \frac{1}{6}\cos(2t) - \frac{1}{3}\sin(2t)$$

what is u_1, u_2 in this case?
(i.e.. studying $\ddot{u} + 2\dot{u} + 2u = 0$)

characteristic equation is
$$\lambda^2 + 2\lambda + 2$$

Roots :
$$\frac{-2 \pm \sqrt{4 - 4\cdot 2}}{2} = -1 \pm \frac{1}{2}\sqrt{-4}$$
$$= -1 \pm i$$

Then
$$u_1(t) = e^{-t}\cos(t)$$
$$u_2(t) = e^{-t}\sin(t)$$
and the general solution to the inhomogeneous problem is

$$\boxed{u(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) + \frac{1}{6}\cos(2t) - \frac{1}{3}\sin(2t)}$$

Consider the system

$$a\ddot{u} + b\dot{u} + cu = f \quad (*)$$

and suppose this is a system with damping (that is the roots of the corresponding characteristic equation have negative real part).

Suppose also f is a periodic function.

Then, solution to $(*)$ are decomposed as the sum of two functions

$$u(t) = u_T(t) + u_s(t)$$

transient part of the solution

steady-state part of the solution