Today: - Python code and examples
(relevant for next homework)
(check email with link just sent)

- Variation of parameters, some examples

- Start 2nd order linear equations

(Parts of Section 7.9: "Variation of parameters")

Inhomogeneous equations:

"Forcing term"

$$\dot{x} = A(t)x + b(t)$$

so $A(t)$ is a given $n \times n$ matrix
and $b(t)$ is a n column vector.

EX: Solve

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ t \end{pmatrix}$$

with the following initial conditions:

(a) $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(5) $x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(we will learn to solve this in a moment)

Key idea #1

Decomposition of solutions

Every solution to the system

$$\dot{x} = A(t)x + b(t)$$

can be written as the sum of a particular solution $x_p(t)$ and a solution $x_h(t)$ the corresponding homogeneous equation

$$\dot{x} = A(t)x$$

$$x(t) = x_p(t) + x_h(t)$$

Stated differently: once you have found one solution to the inhomogeneous equation, all other solutions can be obtained by adding a solution of the homogeneous equation to your particular solution.

Key idea #2

Variation of Parameters

This is basically a matrix version of integrating factor

Recall integrating factor:

(1-d eqn) $\dot{x} = p(t)\,x + q(t)$

Then $\dot{x} - p(t)x = q(t)$

$$e^{-P(t)}\dot{x} - e^{-P(t)}p(t)x = e^{-P(t)}q$$

where $P(t) = \int_0^t p(s)\,ds$

$$\left(e^{-P(t)}x\right)^{\bullet} = e^{-P(t)}q(t)$$

Integrating, we obtained

$$x(t) = e^{P(t)}x(0) + e^{P(t)}\int_0^t e^{-P(s)}q(s)\,ds$$

Alternative way of thinking about this.

Look for solutions of the form (Ansatz*: $e^{P(t)}c(t)$)

*(german for "just guess")

How should $c(t)$ be in order for $x(t) = e^{P(t)} c(t)$ to be a solution?

Let's compute

$$\dot{x} = \left(e^{P(t)}\right)^{\cdot} c(t) + e^{P(t)} \dot{c}(t)$$

$$= p \boxed{e^{P(t)} c(t)} + e^{P(t)} \dot{c}(t)$$

$$= px + e^{P(t)} \dot{c}$$

want this to equal to

$$\rightarrow = px + q \text{, so}$$

$$q = e^{P(t)} \dot{c}$$

Solving for $c(t)$:

$$\boxed{c(t) = c(0) + \int_0^t e^{-P(s)} q(s)\, ds}$$

Plugging this in our ansatz we get the same formula as before

$$x(t) = e^{P(t)} c(0) + e^{P(t)} \int_0^t e^{-P(s)} q(s) ds$$

Variation of parameters for systems with constant coefficients

$$\dot{x} = Ax + b(t)$$

Then we look for a solution of the form

$$x(t) = e^{tA} \underbrace{c(t)}_{\text{parameter vector}}$$

what should the vector $c(t)$ be like if x is to solve $\dot{x} = Ax + b$?

Let's compute $\dot{x}$!

Using the product rule component by component, you can see that

$$\frac{d}{dt}\left(e^{tA}c(t)\right)$$

$$= \left(\frac{d}{dt}e^{tA}\right)c(t) + e^{tA}\left(\frac{d}{dt}c\right)$$

Well, $\frac{d}{dt}e^{tA} = Ae^{tA}$, so

$$\dot{x} = \frac{d}{dt}\left(e^{tA}c(t)\right)$$

$$= A\underbrace{e^{tA}c(t)}_{=x} + e^{tA}\frac{d}{dt}c$$

so $\dot{x} = Ax + e^{tA}\frac{d}{dt}c$

and we see that c must be chosen so that

$$e^{tA}\frac{d}{dt}c = b$$

$$\frac{d}{dt} c = e^{-tA} b$$

Integrating...

$$c(t) = c(0) + \int_0^t e^{-sA} b(s)\,ds$$

Substituting

$$x(t) = e^{tA}\left(c(0) + \int_0^t e^{-sA} b(s)\,ds\right)$$

is easy to see taking $t=0$, that $c(0) = x(0)$, so the formula for the solution is

$$\boxed{x(t) = e^{tA} x(0) + e^{tA}\int_0^t e^{-sA} b(s)\,ds}$$

(this is known as the Variation of Parameters formula / Duhamel's formula)

Let's put this to use.

<u>Example</u> (continued)

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ t \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad b(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

① Compute e^{tA}.

$$e^{tA} = \begin{pmatrix} \frac{1}{2}\left(e^{-3t} + e^{-t}\right) & \frac{1}{2}\left(-e^{-3t} + e^{-t}\right) \\ \frac{1}{2}\left(-e^{-3t} + e^{-t}\right) & \frac{1}{2}\left(e^{-3t} + e^{-t}\right) \end{pmatrix}$$

② Compute $e^{-sA} b(s)$

$$e^{-sA} b(s) = \begin{pmatrix} \frac{s}{2}\left(-e^{3s} + e^{s}\right) \\ \frac{s}{2}\left(e^{3s} + e^{s}\right) \end{pmatrix}$$

(3) Integrate $\int_0^t e^{-sA} b(s)\, ds$

$$\int_0^t e^{-sA} b(s)\, ds = \begin{pmatrix} \int_0^t \frac{s}{2}\left(-e^{3s} + e^{s}\right) ds \\ \int_0^t \frac{s}{2}\left(e^{3s} + e^{s}\right) ds \end{pmatrix}$$

$$\int_0^t s e^{3s}\, ds = \frac{1}{3} s e^{3s} \Big|_0^t - \frac{1}{3}\int_0^t e^{3s}\, ds$$
$$= \frac{1}{3} t e^{3t} - \frac{1}{3}\int_0^t e^{3s}\, ds$$
$$= \frac{1}{3} t e^{3t} - \frac{1}{9}\left(e^{3t} - 1\right)$$

$$\int_0^t s e^{s}\, ds = s e^{s} \Big|_0^t - \int_0^t e^{s}\, ds$$
$$= t e^{t} - \left(e^{t} - 1\right)$$

So,

$$\int_0^t \frac{s}{2}\left(-e^{3s} + e^{s}\right) ds = -\frac{1}{2}\int_0^t s e^{3s}\, ds + \frac{1}{2}\int_0^t s e^{s}\, ds$$

$$= -\frac{1}{6} t e^{3t} + \frac{1}{18}\left(e^{3t} - 1\right) + \frac{1}{2} t e^{t} - \frac{1}{2}\left(e^{t} - 1\right)$$

and

$$\int_0^t \frac{s}{2}\left(e^{3s}+e^{s}\right)ds = \frac{1}{2}\int_0^t s e^{3s}\,ds + \frac{1}{2}\int_0^t s\, e^{s}\,ds$$

$$= \frac{1}{6}te^{3t} - \frac{1}{18}\left(e^{3t}-1\right) + \frac{1}{2}te^{t} - \frac{1}{2}\left(e^{t}-1\right).$$

$$\int_0^t e^{-sA}b(s)\,ds = \begin{pmatrix} -\frac{1}{6}te^{3t} + \frac{1}{18}(e^{3t}-1) + \frac{1}{2}te^{t} - \frac{1}{2}(e^{t}-1) \\ \frac{1}{6}te^{3t} - \frac{1}{18}(e^{3t}-1) + \frac{1}{2}te^{t} - \frac{1}{2}(e^{t}-1) \end{pmatrix}$$

④ Compute $e^{tA}\int_0^t e^{-sA}b(s)\,ds$

$$\begin{pmatrix} \frac{1}{2}(e^{-3t}+e^{-t}) & \frac{1}{2}(-e^{-3t}+e^{-t}) \\ \frac{1}{2}(-e^{-3t}+e^{-t}) & \frac{1}{2}(e^{-3t}+e^{-t}) \end{pmatrix} \begin{pmatrix} -\frac{1}{6}te^{3t} + \frac{1}{18}(e^{3t}-1) + \frac{1}{2}te^{t} - \frac{1}{2}(e^{t}-1) \\ \frac{1}{6}te^{3t} - \frac{1}{18}(e^{3t}-1) + \frac{1}{2}te^{t} - \frac{1}{2}(e^{t}-1) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(e^{-3t}+e^{-t})\left(-\frac{1}{6}te^{3t}+\frac{1}{18}(e^{3t}-1)+\frac{1}{2}te^{t}-\frac{1}{2}(e^{t}-1)\right)+\frac{1}{2}(-e^{-3t}+e^{-t})\left(\frac{1}{6}te^{3t}-\frac{1}{18}(e^{3t}-1)+\frac{1}{2}te^{t}-\frac{1}{2}(e^{t}-1)\right) \\ \frac{1}{2}(-e^{-3t}+e^{-t})\left(-\frac{1}{6}te^{3t}+\frac{1}{18}(e^{3t}-1)+\frac{1}{2}te^{t}-\frac{1}{2}(e^{t}-1)\right)+\frac{1}{2}(e^{-3t}+e^{-t})\left(\frac{1}{6}te^{3t}-\frac{1}{18}(e^{3t}-1)+\frac{1}{2}te^{t}-\frac{1}{2}(e^{t}-1)\right) \end{pmatrix}$$

Since $x(0) = \begin{pmatrix}0\\0\end{pmatrix}$, $e^{tA}x(0) = \begin{pmatrix}0\\0\end{pmatrix}$, so the vector function above is the solution to

$$\text{(a)} \quad \begin{cases} \dot{x} = Ax + \begin{pmatrix}0\\t\end{pmatrix} \\ x(0) = \begin{pmatrix}0\\0\end{pmatrix} \end{cases}$$

For (b), all we do is add to the previous solution the function

$$e^{tA}\begin{pmatrix}1\\-1\end{pmatrix}.$$

EX

$$\dot{x} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}x + \begin{pmatrix}0\\1\end{pmatrix}$$

(a) $x(0) = \begin{pmatrix}1\\0\end{pmatrix}$ ($\leftarrow$ This is a fixed point)

(b) $x(0) = \begin{pmatrix}0\\1\end{pmatrix}$

① Compute e^{tA}.

$$e^{tA} = \begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}$$

② Compute $e^{-sA}b(s)$

$$e^{-sA}b(s) = \begin{pmatrix}\sin(-s)\\ \cos(-s)\end{pmatrix} = \begin{pmatrix}-\sin(s)\\ \cos(s)\end{pmatrix}$$

(3) Integrate

$$\int_0^t e^{-sA} b(s)\,ds = \begin{pmatrix} -\int_0^t \sin(s)\,ds \\ \int_0^t \cos(s)\,ds \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t) - 1 \\ \sin(t) \end{pmatrix}$$

(4) Compute $e^{tA}x(0)$ and $\underline{e^{tA}\int_0^t e^{-sA} b(s)\,ds}$

$$e^{tA}\int_0^t e^{-sA} b(s)\,ds = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \cos(t) - 1 \\ \sin(t) \end{pmatrix}$$

$$= (\cos(t) - 1)\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + \sin(t)\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \cos(t) \\ \sin(t) \end{pmatrix}$$

The solution in both cases is

$$x(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} x(0) + \begin{pmatrix} 1 - \cos(t) \\ \sin(t) \end{pmatrix}$$

(a) $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\boxed{x(t) = \begin{pmatrix} \cancel{\cos(t)} \\ \cancel{-\sin(t)} \end{pmatrix} + \begin{pmatrix} 1 - \cancel{\cos(t)} \\ \cancel{\sin(t)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \leftarrow \text{(indeed, a fixed point)}$$

(b) $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{x(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} + \begin{pmatrix} 1 - \cos(t) \\ \sin(t) \end{pmatrix} = \begin{pmatrix} \sin(t) - \cos(t) + 1 \\ \sin(t) + \cos(t) \end{pmatrix}}$$