5374 Fall '22

Numerical Linear Algebra — Lecture 13

Pset 6 $\qquad f(t) = \|A_\lambda x_t\|^2 , \qquad A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$

$f(t)$ has critical points at

$$\pm \frac{1}{2}\arctan\left(\frac{2}{\lambda}\right), \quad \pm\left(-\frac{1}{2}\arctan\left(\frac{2}{\lambda}\right) + \frac{\pi}{2}\right)$$

$$\left(-\frac{1}{2}\arctan\left(\frac{2}{\lambda}\right) + k\frac{\pi}{2}, \quad k = 1, 2, 3, 4 \right)$$

$t \in [0, 2\pi)$

$$\begin{aligned} f(t) &= \left\| \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \right\|^2 \\ &= (\cos(t) + \lambda\sin(t))^2 + (\sin(t))^2 \\ &= \cancel{(\cos(t))^2} + 2\lambda\sin(t)\cos(t) + \cancel{(\sin(t))^2} + \lambda^2(\sin(t))^2 \\ &= 1 + \lambda\sin(2t) + \lambda^2(\sin(t))^2 \end{aligned}$$

$$\cos(2t) = (\cos(t))^2 - (\sin(t))^2 = 1 - 2(\sin(t))^2$$

$$(\sin(t))^2 = \frac{1 - \cos(2t)}{2}$$

$$f(t) = 1 + \lambda\sin(2t) + \lambda^2\frac{1}{2}(1 - \cos(2t))$$

$$\begin{aligned} f'(t) &= 2\lambda\cos(2t) + \lambda^2\frac{1}{\cancel{2}}(\cancel{2}\sin(2t)) \\ &= \lambda(2\cos(2t) + \lambda\sin(2t)) \end{aligned}$$

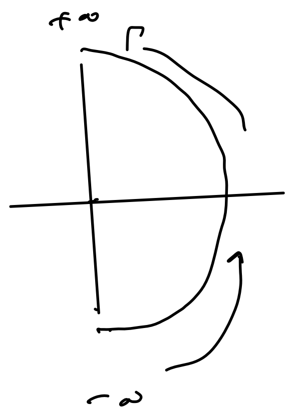
$f'(t) = 0 \iff 2\cos(2t) + \lambda \sin(2t) = 0$

(say, $\cos(2t) \neq 0$

$2 + \lambda \tan(2t) = 0$

$\tan(2t) = -\frac{2}{\lambda}$)

$2t = \arctan\left(-\frac{2}{\lambda}\right) = -\arctan\left(\frac{2}{\lambda}\right)$



$t = -\frac{1}{2}\arctan\left(\frac{2}{\lambda}\right)$

$\exists! \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ s.t.

$\alpha = -\arctan\left(\frac{2}{\lambda}\right)$

How many t's are there in $[0, 2\pi)$ s.t.

$\alpha = 2t \mod \pi \quad (\alpha = 2t + \pi)$

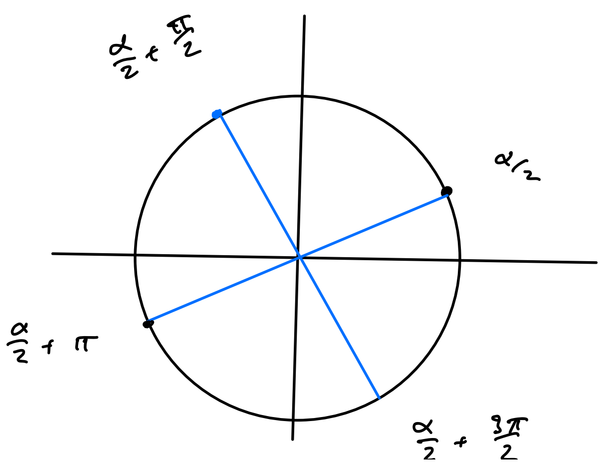
$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \qquad -\frac{\pi}{4} \leq \frac{\alpha}{2} \leq \frac{\pi}{4}$

$t_1 = \frac{\alpha}{2}$

$t_2 = \frac{\alpha}{2} + \frac{\pi}{2}$

$t_3 = \frac{\alpha}{2} + \pi$

$t_4 = \frac{\alpha}{2} + \frac{3\pi}{2}$



We go back to

$$f(t) = 1 + \lambda \sin(2t) + \lambda^2 (\sin(t))^2$$

$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$$

$$\frac{1}{(\tan\theta)^2} + 1 = \frac{1}{(\sin\theta)^2}$$

$$(\sin\theta)^2 = \frac{1}{\frac{1}{(\tan\theta)^2} + 1}$$

$$\theta = 2t = \alpha = \arctan\left(-\frac{2}{\lambda}\right) \quad \left(t_1 = \frac{\alpha}{2}\right)$$

$$(\tan(\theta))^2 = \frac{4}{\lambda^2} \quad (\sin(2t_1))^2 = \frac{1}{\frac{1}{4/\lambda^2} + 1}$$

$$\Longrightarrow \quad (\sin(2t_1))^2 = \frac{1}{\frac{\lambda^2}{4} + 1} = \frac{4}{\lambda^2 + 4}$$

Since $0 < \alpha < \frac{\pi}{2}$, $\sin(\alpha) > 0$, so

$$\sin(2t_1) = \frac{2}{\sqrt{\lambda^2 + 4}}$$

Now we compute $(\sin(t_1))^2 = \dfrac{1 - (\cos(2t))}{2}$

$$= \frac{1 - \sqrt{1 - (\sin(2t_1))^2}}{2}$$

$$\left(\sin\left(\arctan\left(-\frac{2}{x}\right)\right)\right)^2 = \frac{1-\sqrt{1-\frac{1}{\frac{x^2}{4}+1}}}{2}$$

Today: Orthogonality and the QR decomposition

Last time: Q is orthogonal if

$$Q^{-1} = Q^t$$

If we write $Q = \left(q_1 \middle| \cdots \middle| q_n \right)$

$$Q^t Q = \begin{pmatrix} q_1^t \\ \hline q_2^t \\ \hline \vdots \\ \hline q_n^t \end{pmatrix} \left(q_1 \middle| q_2 \middle| \cdots \middle| q_n \right)$$

$$= \begin{pmatrix} q_1^t q_1 & q_1^t q_2 & \cdots & q_1^t q_n \\ q_2^t q_1 & q_2^t q_2 & \cdots & q_2^t q_n \\ \vdots & & & \\ q_n^t q_1 & q_n^t q_2 & \cdots & q_n^t q_n \end{pmatrix}$$

If $Q^t Q = I$ this means that

$$(q_i, q_j) = q_i^t q_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So Q is orthogonal if and only if the columns of Q form an orthonormal basis.

Alternatively, if $q_1, \dots, q_n$ is an orthonormal basis, then $\forall\ x \in \mathbb{R}^n$,

$$
\begin{aligned}
x &= (q_1, x) q_1 + \cdots + (q_n, x) q_n \\
&= \left(q_1 \middle| \cdots \middle| q_n \right) \begin{pmatrix} (q_1, x) \\ \vdots \\ (q_n, x) \end{pmatrix} \\
&= Q \begin{pmatrix} (q_1, x) \\ \vdots \\ (q_n, x) \end{pmatrix}
\end{aligned}
$$

But $\begin{pmatrix} (q_1, x) \\ \vdots \\ (q_n, x) \end{pmatrix} = Q^t x$

so, we have shown

$$x = Q Q^t x \qquad \forall\ x \in \mathbb{R}^n$$

i.e.

$$Q Q^t = I.$$

The following are equivalent:

- Q is orthogonal $\iff$ Q^t is orthogonal
- The columns of Q form an orthonormal basis
- The rows of Q form an orthonormal basis
- $\|Qx\|_2 = \|x\|_2 \quad \forall\, x \in \mathbb{R}^n$

$$\Big(\quad (Qx, Qx) = (x, x) \quad \forall \quad x \in \mathbb{R}^n$$

$$\iff \quad (Q^t Q x, x) = (x, x) \quad \forall\, x \in \mathbb{R}^n$$

(check why this is so) $$\iff \quad Q^t Q = I \quad \Big)$$

Remark : Recall that

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Then:

The matrix $Q := e^A$ is orthogonal if and only if $A^t = -A$.

EX: If $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $e^{At} = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Numerically, solving

$$Ax = b \qquad (*)$$

when A is orthogonal is rather trivial:

$$x = A^{-1}b$$
$$= A^t b$$

This produces an answer to $(*)$ in $O(n^2)$ FLOP's

Why? Is simply a matrix-vector multiplication since A^{-1} is so easily available:

To compute $A^t b$ is the same as computing n inner products (one between b and each row of A) and each inner product $= 2n-1$ FLOP's per inner product, for a total of $2n^2 - n^2$ FLOP's.

For A not orthogonal but invertible, can we somehow reduce $Ax=b$ to the case of an orthogonal A?

The answer is yes, and it involves the QR decomposition of a matrix.

The QR decomposition.

Let A be a $n \times n$ matrix, and invertible. Then there is a pair of matrices Q and R such that

- Q is orthogonal
- R is upper triangular
- $A = QR$

Moreover, there is exactly one such pair with R whose diagonal entries are all > 0.

As it turns out, finding Q and R given A is relatively straightforward and it involves the Gram-Schmidt process; let's review the process:

Gram-Schmidt

Input: A basis $a_1, \dots, a_n$

Output: An orthonormal basis $q_1, \dots, q_n$ such that for $1 \le k \le n$

$$\text{span}(a_1, \dots, a_k) = \text{span}(q_1, \dots, q_k)$$

Step 1.

$$\tilde{q}_1 := a_1$$

$$q_1 := \frac{\tilde{q}_1}{\|\tilde{q}_1\|_2}$$

Step 2

$$\tilde{q}_2 := a_2 - (a_2, q_1)\, q_1$$

$$q_2 := \frac{\tilde{q}_2}{\|\tilde{q}_2\|_2}$$

⋮

Step K

$$\tilde{q}_k = a_k - \sum_{j=1}^{k-1} (a_k, q_j)\, q_j$$

$$q_k = \frac{\tilde{q}_k}{\|\tilde{q}_k\|_2}$$

So by the time we are done we have completed $O(n^3)$ FLOP's and obtain vectors $q_1, \dots, q_n$ such that

$$\|\tilde{q}_1\|\, q_1 = a_1$$

$$\|\tilde{q}_2\|\, q_2 = a_2 - (a_2, q_1)\, q_1$$

$$\vdots$$

$$\|\tilde{q}_k\|\, q_k = a_k - (a_k, q_1)\, q_1 - \cdots - (a_k, q_{k-1})\, q_{k-1}$$

Rearranging,

$$
(*) \quad \left\{ \begin{aligned}
a_1 &= \underbrace{\|\tilde{q}_1\|_2}_{r_{11}} q_1 \\
a_2 &= \underbrace{(a_2, q_1)}_{r_{12}} q_1 + \underbrace{\|\tilde{q}_2\|_2}_{r_{22}} q_2 \\
a_3 &= \\
&\vdots \\
a_k &= (a_k, q_1) q_1 + \cdots + (a_k, q_{k-1}) q_{k-1} + \|\tilde{q}_k\| q_k
\end{aligned} \right.
$$

Define

$$
r_{ij} = \begin{cases} (a_i, q_j) & \text{if} \quad j < i \\ \|\tilde{q}_j\| & \text{if} \quad j = i \\ 0 & \text{if} \quad j > i \end{cases}
$$

$(*)$ Can be written in matrix notation as

$$
\left(a_1 \,\middle|\, \cdots \,\middle|\, a_n \right) = \left(q_1 \; \cdots \; q_n \right) \begin{pmatrix} r_{11} & r_{12} & & \\ 0 & r_{22} & \ddots & \cdots \\ \vdots & \vdots & & \\ 0 & 0 & & \end{pmatrix}
$$