5374 Fall '22

Numerical Linear Algebra Lecture 18

Today: * The Fast Fourier Transform

* Symmetric matrices and orthogonality

Last time we talked about the Discrete Fourier transform D_m, defined by

$$(D_m)_{ij} = \omega_m^{ij}, \quad \text{where } \omega_m = e^{\frac{2\pi i}{m}}, \quad i,j = 1,\dots,m$$

We saw how this unitary matrix is extremely useful when analyzing 1-d functions (after discretizing them to a m-point grid).

Given a vector $\bar{x} \in \mathbb{C}^m$, computing $D_m \bar{x}$ in the usual way takes about $O(m^2)$ FLOP's (as we know for matrix-vector multiplication).

In the mid 20th-century it was observed that if $m = 2^k$ for some $k \in \mathbb{N}$ then the multiplication $D_m \bar{x}$ could be reduced to two multiplications involving $D_{\frac{m}{2}}$, repeating this procedure one can perform the multiplication $D_m \bar{x}$ not in $O(m^2)$ FLOP's but in $O(m \log m)$ FLOP's. This is a big enough difference that for many applications the only practical way of computing $D_m x$ is in this manner. This is known as the Fast Discrete Fourier Transform algorithm or just Fast Fourier Transform (FFT)

The Fast Fourier Transform

Let's see how computing $D_m x$ is the same as computing $D_{\frac{m}{2}} x^{(1)}$, $D_{\frac{m}{2}} x^{(2)}$ for some well chosen vectors $x^{(1)}, x^{(2)} \in \mathbb{C}^{m/2}$.

Then, let $m \in \mathbb{N}$ be even, $m = 2m'$.
Here is the key observation:

$$\omega_{2m'}^2 = \omega_{m'} \qquad (*)$$

(recall $\omega_m = e^{\frac{2\pi\sqrt{-1}}{m}}$ for every $m \in \mathbb{N}$)

Why is $(*)$ true? It's elementary:

$$\omega_{2m'}^2 = \left(e^{\frac{\cancel{2}\pi\sqrt{-1}}{\cancel{2}m'}}\right)^{\cancel{2}} = e^{\frac{2\pi\sqrt{-1}}{m'}} = \omega_{m'}$$

With this in hand, what does the i-th component of $D_m x$ look like?

$$(D_m x)_i = \sum_{j=1}^{m} (D_m)_{ij}\, x_j$$

$$= \sum_{j=1}^{m} (\omega_m^i)^j x_j$$

Observe that

$$\omega_m^j = \omega_{2m'}^j = \begin{cases} \omega_{m'}^{\frac{j}{2}} & \text{if } j \text{ is even} \\ \omega_{m'}^{\frac{j-1}{2}} \cdot \omega_{2m'}^{1} & \text{if } j \text{ is odd} \end{cases}$$

Then

$$(D_m x)_i = \sum_{j \text{ even}} \omega_{m'}^{\frac{ij}{2}} x_j + \omega_{2m'} \sum_{j \text{ odd}} \omega_{m'}^{i\left(\frac{j-1}{2}\right)} x_j$$

(for $i = 1, 2, \dots, m$)

Let's write $j = 2k$ in the first sum, and $j = 2k-1$ in the second sum, where $k = 1, \dots, m'$.

Then

$$(D_m x)_i = \sum_{k=1}^{m'} \omega_{m'}^{ik} x_{2k} + \omega_{2m'} \sum_{k=1}^{m'} \omega_{m'}^{i(k-1)} x_{2k-1}$$

$$= \sum_{k=1}^{m'} \omega_{m'}^{ik} x_{2k} + \omega_{2m'} \sum_{k=1}^{m'} \omega_{m'}^{-i} \omega_{m'}^{ik} x_{2k-1}$$

(since $\omega_{m'} = \omega_{2m'}^2$)

$$= \sum_{k=1}^{m'} \omega_{m'}^{ik} x_{2k} + \omega_{2m'}^{1-2i} \sum_{k=1}^{m'} \omega_{m'}^{ik} x_{2k-1}$$

We see that $D_m x$ is the sum of two vectors that look a lot like $D_{\frac{m}{2}} = D_{m'}$ applied to something else.

First, look at $i = 1, \dots, m'$ only. Define

$$x^{(1)} = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \\ \vdots \\ x_{2m'} \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2m'-1} \end{pmatrix}$$

If we take only the first m' components of $D_m x$ $(\in \mathbb{C}^{2m'})$, let's denote this $P D_m x$. We see that

$$P D_m x = \underbrace{D_{m'} x^{(1)}}_{\left(\sum_{k=1}^{m'} \omega_{m'}^{ik} x_{2k} \right)_{i=1,\dots,m'}} + \underbrace{\Lambda_m D_{m'} x^{(2)}}_{\left(\omega_{2m'}^{1-2i} \sum_{k=1}^{m'} \omega_{m'}^{ik} x_{2k-1} \right)_{i=1,\dots,m'}}$$

where

$$\Lambda_m = \begin{pmatrix} \omega_{2m'}^{1-2} & 0 & 0 & \cdots & 0 \\ 0 & \omega_{2m'}^{1-4} & & & \\ & & \omega_{2m'}^{1-6} & & \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & \omega_{2m'}^{1-2m'} \end{pmatrix}$$

(Counting operations if we multiply the usual way

$$\begin{array}{cc} & \begin{array}{cc} m' & m' \end{array} \\ \begin{array}{c} m' \\ m' \end{array} & \left(\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right) \end{array} \left(\begin{array}{c} \quad \\ \hline \quad \end{array}\right)\begin{array}{c} m' \\ m' \end{array} = 4 \times \left(m' \left(\quad\right)\left(\quad\right) m' \right)$$

Notice that computing $PD_m x$ amounts to computing two multiplications of $D_{\frac{m}{2}}$ and a multiplication by a $\frac{m}{2} \times \frac{m}{2}$ diagonal matrix (this takes $O(\frac{m}{2})$ FLOP's). What about the other m' components of $D_m x$?

Well, if $i > m'$, then $i = m' + \ell$, $\ell = 1, \dots, m'$.

Then

$$\omega_{m'}^{ik} = \omega_{m'}^{m'k + \ell k} = \overbrace{\cancel{\left(\omega_{m'}^{m'}\right)^{k}}}^{=1} \omega_{m'}^{\ell k}$$
$$= \omega_{m'}^{\ell k}$$

and likewise

$$\omega_{m'}^{1-2i} = \omega_{m'}^{1-2(m'+\ell)} = \overbrace{\cancel{\omega_{m'}^{-2m'}}}^{=1} \omega_{m'}^{1-2\ell}$$
$$= \omega_{m'}^{1-2\ell}$$

so $(D_m x)_i =$

$$= \sum_{k=1}^{m'} \omega_{m'}^{\ell k} x_k + \omega_{m'}^{1-2\ell} \sum_{k=1}^{m'} \omega_{m'}^{\ell k} x_{2k-1}$$

This shows the remaining m' coordinates of $D_m x$ are the same as the first m'.

The above shows how computing D_m is reduced to two computation of $D_{m/2}$, a multiplication by a diagonal $m/2 \times m/2$ matrix, and some post processing.

If $m = 2^k$ for some $k \in \mathbb{N}$, we can iterate this k times, Reducing the computation of D_{2^k} to the computation of 2^k D_2's, plus 2^k diagonal matrix multiplication.

The 2^k multiplication by D_2's takes $O(2^k) = O(m)$ FLOP's, while the 2^k diagonal matrix multiplication take more time since it involves matrices of size 2^ℓ for $\ell = 1, \ldots, k$. This takes $O(k 2^k)$ FLOP's, that is $O(m \log m)$ FLOP's.