5374 Fall '22

Numerical Linear Algebra Lecture 17

Question (Pset 3 Prob # 3)

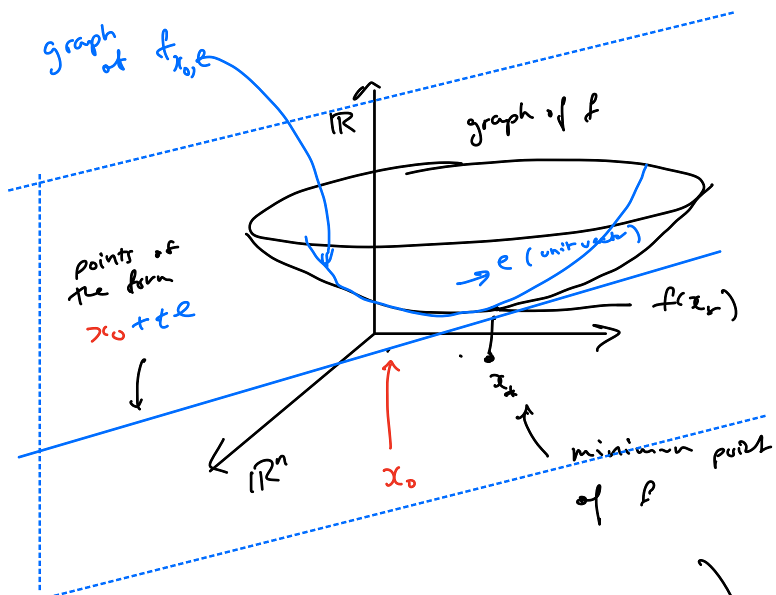
$$((I-2P)u, (I-2P)v) = (u,v) \quad \forall\, u,v$$

$$\Updownarrow$$

$$\|(I-2P)u\|_2 = \|u\|_2 \quad \forall\, u$$

Pset 3 Prob # 2

$$f(x) = \|Ax-b\|_2^2$$



$$f_{x_0,e}(t) := f(x_0+te) = \|A(x_0+te)-b\|_2^2$$

Today: * More on orthogonal projections
* Discrete Fourier transform
(and fast Fourier transform algorithm)

* Scipy. fft.

* Interpolation / regression with the Fourier transform.

Remark (1) In the space $C([0,1], \mathbb{C})$ $\left(L^2(0,1) \supset \right)$ we can consider the family of functions:

$$e_k(x) = e^{2\pi i k x}, \quad k \in \mathbb{R}$$

If $k \in \mathbb{Z}$ and $k \neq 0$, then

$$\int_0^1 e_k(x)\,dx = \int_0^1 e^{2\pi i k x}\,dx$$

$$= \frac{1}{2\pi i k} e^{2\pi i k x} \Big|_{x=0}^{x=1} \quad (k \neq 0)$$

$$= \frac{1}{2\pi i k}\left(\underbrace{e^{2\pi i k}}_{=1 \text{ since } k \in \mathbb{Z}} - 1 \right) = 0$$

In particular, if $k_1, k_2 \in \mathbb{Z}$ and $k_1 \neq k_2$

$$\int_0^1 e^{2\pi i(k_1-k_2)x}\,dx = 0$$

$$\Rightarrow \int_0^1 e^{2\pi i k_1 x}\cdot e^{-2\pi i k_2 x}\,dx = 0$$

$$(e_{k_1}, e_{k_2}) = 0$$

The family of functions $\{e_k\}_{k\in\mathbb{Z}}$ form an orthonormal basis of $L^2(0,1)$. This means the following:

① For any $f \in L^2(0,1)$ (or just $f \in C([0,1];\mathbb{C})$) there exists numbers $\{c_n\}_{n\in\mathbb{Z}}$ such that $c_n \in \mathbb{C}\ \forall n$ and such that if we define:

$$f_N(x) := \sum_{n=-N}^{N} c_n e_n(x)$$

Then

$$\lim_{N\to\infty} \int_0^1 |f(x) - f_N(x)|^2\,dx = 0$$

② The sequence $\{c_n\}$ is uniquely determined for each f by

$$c_n := (f, e_n) = \int_0^1 f(x)\,\overline{e_n(x)}\,dx$$

and

$$\int_0^1 |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |c_n|^2 \quad \text{(Parseval's formula)}$$

The map $f \longmapsto c_n = (f, e_n)$ defines a linear transformation from

$$L^2(0,1) \quad \text{to} \quad \ell^2(\mathbb{Z})$$

$$\left(\begin{array}{c} f : [0,1] \to \mathbb{C} \\ \int_0^1 |f(x)|^2 dx < \infty \end{array} \right) \qquad \left(\begin{array}{c} c : \mathbb{Z} \to \mathbb{C} \\ \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty \end{array} \right)$$

This is an isometry between Hilbert spaces (first pointed out by Von Neumann), and at least in parts of math is called the Fourier transform.

(2) The Fourier transform, as understood by most people deals with functions in $L^2(\mathbb{R})$

If $f : \mathbb{R} \to \mathbb{C}$ and f is sufficiently nice, we define a new function $\hat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y} dx$$

This is what is most often called the Fourier transform. Most importantly, we have again a Parseval formula

$$\int_{\mathbb{R}} |f(x)|^2\,dx = \int_{\mathbb{R}} |\hat{f}(y)|^2\,dy$$

The Discrete Fourier transform

Following the example from the end of last class, for each $m \in \mathbb{N}$, we define a $m \times m$ matrix as follows:

Let $\omega_m := e^{\frac{2\pi i}{m}}$ (this is called a primitive m-root of unity)

Then define D_m as follows:

$$D_m = \frac{1}{\sqrt{m}} \begin{array}{c} \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \omega_m & \omega_m^2 & \cdots & \omega_m^{m-1} \\ \vdots & \omega_m^2 & \omega_m^4 & \cdots & \omega_m^{2(m-1)} \\ (\omega_m^k)^0 & (\omega_m^k)^1 & (\omega_m^k)^2 & \cdots & (\omega_m^k)^{m-1} \\ \vdots & \vdots & & & \\ 1 & \omega_m^{m-1} & & & (\omega_m^{m-1})^{m-1} \end{pmatrix} \\ \text{— } m \text{ columns —} \end{array} \begin{array}{c} | \\ m \text{ rows} \\ | \end{array}$$

$$(D_m)_{ij} = \frac{1}{\sqrt{m}} (\omega_m^i)^j$$

From the computation we did at the end of last class, we have that

$$D_m D_m^* = I$$

i.e. $$D_m^{-1} = \overline{D_m^t}$$

This D_m matrix is called the Discrete Fourier Transform

(Since $(D_m)_{ij} = (D_m)_{ji}$, it follows that

$$D_m^{-1} = \overline{D_m}$$

D_m^{-1} is called the Inverse Discrete Fourier Transform)

The fact that D_m is unitary (i.e. that $D_m D_m^* = I$) is the discrete analogue of Parseval's identity:

$$\forall\ z \in \mathbb{C}^m: \qquad \| D_m z \|_2 = \| z \|_2$$

What is the Fourier transform good for?
It is a more efficient way of representing/approximating smooth "functions" (vectors in $\mathbb{C}^m$) in contrast to the canonical basis.