Numerical Zinear Algebra

Jecture 7

Today:

A Algorithms in scientific computing

of Forward ever, backward ever, and condition numbers.

of Some more linear algebra review

Reprences: Trefetten and Bau, lecture 12 Solomon, Chapter 2 section 2.2.

Note: Bring laptop on Thursday

More on the condition number of a matrix

Last time we introduced the condition number of a nxn matrial A

cond (A) =) ||A|| ||A'|| if A is invertible

If A is murtible, be R' then let 7 be the unique solution of

AZ = 6

We saw lest time that if we chose a matrix A and a weeken is and define (for & small)

$$A_{\varepsilon} = A + \varepsilon A$$

The norm for matrice, here is

 $b_{\varepsilon} = b + \varepsilon b$

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Then

$$\frac{|12-2\varepsilon l|_2}{|12|_2} \leq \operatorname{cond}(A) \left(\frac{|1A-A\varepsilon l|}{|1A|} + \frac{|1b-b\varepsilon l|_2}{|1b|_2} \right) + O(\varepsilon^2)$$

Today, as a kind of wormp, let us show how this inequality can be improved if $\dot{A} = 0$; so $A_{\Sigma} = A$ for all E.

In this case we have Z and ZE which some, respectively

Without knowing 2, and knowing 2E, I would like to estimate how for away is

Ze from Z.

We cannot compute 112-2 E112, four we can compute

Conversely == A'b, ZE= A'bE,
one home

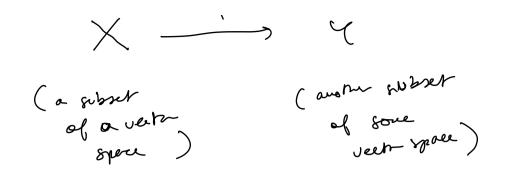
Moreover,

Since b = A2, $||b||_2 \le ||A|| ||2||_2$, implying that $\frac{1}{||2||_2} \le \frac{||A||}{||b||_2}$

$$\frac{||2-2\varepsilon||_2}{||2||_2} \leq ||A|| ||A^{-1}|| \frac{||b-b\varepsilon||_2}{||b||_2}$$
i.e.
$$\frac{||2-2\varepsilon||_2}{||2||_2} \leq cond (A) \frac{||b-b\varepsilon||_2}{||b||_2}$$
This is called the (relative)
$$\frac{||a||_2}{||a||_2} = \frac{||a||_2}{||a||_2} = \frac{||a||_2}{||$$

A framework for mathematical problems and algorithms the solve trem

We will think of a mathematical problem simply as a function



Every $x \in X$ will represent a particular combination of parameters for the problem, and fext will represent the solution of that problem for the instance x.

Examples : (1) Consider the problem of computing the positive square root of a real positive number.

$$X = (0, \infty)$$

$$Y = (0, \infty)$$
and
$$f(x) = + \sqrt{x}$$

2 Commide the problem of, given A invertible (NKN) and belton, to find 2 solving

A == 6

$$X = \frac{1}{2} (A,b) \left(A \text{ is anx} \text{ invertible notra}, b \in \mathbb{R}^n \right)$$

$$C \mathbb{R}^{n^2+n}$$

$$Y = \mathbb{R}^n$$

$$f(A,b) = A^{-1}b$$

(3) Least square problems: Given A, a morn notice, and b, a vector is 12m, find ZEM? minimizing the norm

$$X = \mathbb{R}^{mn+m}$$
, $Y = \mathbb{R}^n$

Eigenvalue problems: Given a symmetric vxv motrix A, compute it's n eigenvalue, $\lambda_1, \lambda_2, ..., \lambda_n$ st in a non-electeory order

An algorithm is a more inprecise notion, it is a lot like a mathematical problem

4: 7-34

except the function of comes with a set of instructions that take x as input and produces text in a prite number of steps.

To distinguish them from nathemetical problems, we will denote algorithms with a A symbol, ic. $\hat{f}: X \longrightarrow Y$.

Our overarching goal is, given a natheratical problem $f: X \rightarrow Y$, the design or arrive at an algorithm $f: X \rightarrow Y$. Not that serves as a good approximation for f.

How de me quantity how "good" un

algorithm 13?

Backward error, forward error, and bookward stability

(In the following discussion I will use 11.11 to refer to some enspectful near in the space ()

Fix a mothematical problem f, and an algorithm f.

J. Forward error

11 fra) - fran 1

This is the error we most come about, but one we cannot compute directly since we lack knowledge of fix!

There is also the Relative forward error

11 for >11

2. Backward error

Here, given x, we look arong all \widehat{x} in X, and then to find one such that $\widehat{f}(x) = f(\widehat{x})$

i.e. $\hat{f}(x)$ may not be the exact solution to problem x, but it is the exact solution of c (hopefulls) close problem, $\hat{\chi}$.

That is, comide

inf d 11x-211 over 2 e X st. f(x)=f(x))

(1 ms myt be empty)

Suppose for a second that $f: X \rightarrow Y$ is Lipschitz continuous with norm L, 10.

11 for - for 11 = L 11x-211 + x, y = X

If so, then repetition $|f(x) - \hat{f}(x)| = ||f(x) - \hat{f}(\hat{x})||$ $\leq L(|x - \hat{x}||,$

forward 3 am bowled by the bookand