5374 Fall '22

Numerical Linear Algebra Lecture 11

Conjugation of matrices

Suppose A is a $m \times n$ matrix, this is the same as having a linear transformation

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Fix new bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, and let

M = $m \times m$ matrix

N = $n \times n$ matrix

such that the columns of M and N represent respectively the new bases in $\mathbb{R}^m$ and $\mathbb{R}^n$

We have a transform $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$L(x) = Ax \qquad (x \in \mathbb{R}^n)$$

what does L look like with respect to the new bases?

Let $x \in \mathbb{R}^n$, Then x can be written

as $\quad x = N x'$ for some $x' \in \mathbb{R}^n$.
What this means is that a vector represented by x in the canonical basis is represented by x' in the new basis in $\mathbb{R}^n$.

Likewise, if $y \in \mathbb{R}^m$ there is a $y' \in \mathbb{R}^m$ such that $\quad y = M y'$, and y' represents the coordinates of the vector y in the new basis in $\mathbb{R}^m$.

Now, take $x \in \mathbb{R}^n$, and look at $L(x)$,

$$Lx = Ax$$

In the new basis, $x = N x'$, so

$$L(x) = ANx'$$

$L(x)$ is an element of $\mathbb{R}^m$, so it has an expression in the new basis, meaning there is a $y' \in \mathbb{R}^m$ such that

$$L(x) = M y'$$

so we have that x' and y' are related by

$$My' = ANx'$$

i.e

$$y' = (M^{-1}AN)x'$$

(This corresponds to $y = Ax$ in the canonical basis)

In other words, the matrix of L in the new basis is

$$M^{-1}AN.$$

This is called a conjugation, and it can be visualized as follows

$$\begin{tikzcd}
x \quad \mathbb{R}^n \arrow[rr, "L(x)=Ax"] \arrow[d, leftrightarrow, "\substack{N^{-1}x \leftarrow x \\ x' \to Nx'}"'] & & \mathbb{R}^m \quad y \arrow[d, leftrightarrow, "\substack{M^{-1}y \leftarrow y \\ y' \to My'}"] \\
x' \quad \mathbb{R}^n \arrow[rr, "x' \to (M^{-1}AN)x'"'] & & \mathbb{R}^m \quad y'
\end{tikzcd}$$

It is particularly interesting to look at such conjugation when $m = n$ and $M = N$.

If A and A' are two matrices such that

$$A' = M^{-1} A M$$

for some $n \times n$ matrix M, we say A and A' are conjugate matrices

Exercise : Suppose A, A', and M are as above. Suppose also that M preserves the Euclidean norm, i.e. $\|Ax\|_2 = \|x\|_2$ for all x. Show that

$$\|A\| = \|A'\|$$

$$\|A^{-1}\| = \|(A')^{-1}\|$$

In particular, $\operatorname{cond}(A) = \operatorname{cond}(A')$

Inner products and orthogonality

Given a vector space X, an inner product is a real valued function

$$X \times X \longrightarrow \mathbb{R} \quad (\text{or } \mathbb{C})$$

which is denoted (x, y) given $x, y \in X$ and which satisfies the following:

- Bilinearity

$$(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$$

$$\forall \ \alpha_1, \alpha_2 \in \mathbb{R}, \ x_1, x_2, y \in X$$

$$(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 (x, y_1) + \beta_2 (x, y_2)$$

$$\forall \ \beta_1, \beta_2 \in \mathbb{R}, \ x, y_1, y_2 \in X$$

- Symmetry: $\quad (x, y) = (y, x) \quad \forall \ x, y \in X$

- Positivity: $\quad (x, x) \geq 0 \quad \forall \ x \in X$

and

$$(x, x) = 0 \iff x = 0.$$

When a vector space has an inner product we can define a norm in it, via

$$\|x\| = \sqrt{(x, x)}$$

If the space is complete with respect to this norm then $(X, (\cdot, \cdot))$ is called a (real) Hilbert space.

There is an analogue of an inner product for complex vector spaces called a Hermitian form

(named after Hermite)

Hermitian forms

A map $X \times X \longrightarrow \mathbb{C}$

Denoted $x, y \longrightarrow \langle x, y \rangle$

- Linearity and sesqui-linearity

$$\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$$

$$\forall \ \alpha_1, \alpha_2 \in \mathbb{C}$$
$$x_1, x_2, y \in X$$

$$\langle x, \beta_1 y_1 + \beta_2 y_2 \rangle = \bar{\beta}_1 \langle x, y_1 \rangle + \bar{\beta}_2 \langle x, y_2 \rangle$$

$$\forall \ \beta_1, \beta_2 \in \mathbb{C}$$
$$x, y_1, y_2 \in X$$

- "Symmetry" $\quad \overline{\langle x, y \rangle} = \langle y, x \rangle$

- Positivity $\quad \langle x, x \rangle \geq 0 \quad \forall \ x \in X$

$$= 0 \iff x = 0$$

In this case we can define a norm in X

$$\|x\| := \sqrt{\langle x, x \rangle}$$

If X is complete in this norm, the pair $(X, \langle , \rangle)$ is called a Hilbert space.

Examples (of inner products and Hermitian forms)

(1) $\mathbb{R}^n$, with inner product given by

$$(x,y) = y^t x = x_1 y_1 + \cdots + x_n y_n$$

(where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$)

(2) $\mathbb{C}^n$, with Hermitian product given by

$$\langle x,y \rangle = y^* x, \quad \left(\text{where } y^* = \overline{y^t}\right)$$

$$= x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}$$

Note that $\langle x,x \rangle = x_1 \overline{x_1} + \cdots + x_n \overline{x_n}$

$$= |x_1|^2 + \cdots + |x_n|^2$$

(3) Let $L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid$ Lebesgue measurable

$$\int_{\mathbb{R}} f(x)^2 dx < \infty \}$$

$$(f,g) := \int_{\mathbb{R}} f(x) g(x) dx$$

$$(f,f) = \int_{\mathbb{R}} f(x)^2 \, dx$$

This is a real Hilbert space

(4) $$L^2(\mathbb{R}, \mathbb{C}) = \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty \}$$

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} \, dx$$

(5) Consider the space of $m \times n$ real matrices, $M_{m \times n}(\mathbb{R})$ with inner product

$$(A, B) = \mathrm{Tr}(B^t A)$$

Some remarks

1. The Cauchy-Schwartz inequality:

For any X with an inner product (x, y), we have the inequality

$$(x,y) \le \sqrt{(x,x)}\sqrt{(y,y)} = \|x\|\cdot\|y\|$$

Why? Consider the polynomial

$$p(t) = (x+ty, x+ty)$$

$$\Big(\quad = (x, x+ty) + (ty, x+ty)$$

$$= (x,x) + t(x,y) + t(y,x) + t^2(y,y)$$

$$= (x,x) + 2t(x,y) + t^2(y,y) \quad \Big)$$

so since $p(t) \geq 0$, its discriminant must be ≤ 0, so

$$\big(2(x,y)\big)^2 - 4(x,x)(y,y) \leq 0$$

i.e.

$$4(x,y)^2 \leq 4(x,x)(y,y)$$

$$(x,y)^2 \leq (x,x)(y,y)$$

It also follows from the proof that

$$(x,y) = \|x\|\cdot\|y\|$$

if and only if x and y are parallel.

2 (the triangle inequality) ($\Leftarrow$ Cauchy-Schwartz ineq)

Let's show that for the norm induced by an inner product we have

$$\|x+y\| \leq \|x\| + \|y\|$$

$$\|x+y\| = \sqrt{\langle x+y, x+y\rangle}$$
$$= \sqrt{\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle}$$

By Cauchy-Schwarz

$$\leq \sqrt{\langle x,x\rangle + 2\|x\|\cdot\|y\| + \langle y,y\rangle}$$
$$= \sqrt{\|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2}$$
$$= \sqrt{(\|x\| + \|y\|)^2}$$
$$= \|x\| + \|y\|$$

So, $\|x+y\| \leq \|x\| + \|y\|$.