5374 Fall '22

Numerical Linear Algebra

Orthogonal Polynomials

Consider $\mathbb{R}[x]$ or $\mathbb{C}[x]$, the vector spaces of polynomials with coefficients in $\mathbb{R}$ or $\mathbb{C}$.

Given an interval (a,b) and a weight function $w(x)$ (= non-negative function in $L^1(a,b)$) we can define an inner (resp. Hermitian) product in $\mathbb{R}[x]$ (resp $\mathbb{C}[x]$) to:

$$(p,q) := \int_a^b p(x)\, q(x)\, w(x)\, dx$$

$$\left(\text{resp.} \quad \langle p,q \rangle := \int_a^b p(x)\, \overline{q(x)}\, w(x)\, dx \right)$$

Using the Gram-Schmidt with such an inner product one can take the sequence of polynomials $\{1, x, x^2, \dots\}$ and generate a sequence of orthogonal polynomials $\{P_0, P_1, P_2 \dots\}$, where $P_n(x)$ is a polynomial of degree n.

Different choices of the interval (a,b) and weight function $w(x)$ lead to many important classes of polynomials $\{P_n\}_{n=1}^{\infty}$.

1. Legendre Polynomials

$$(a,b) = (0,1), \qquad w(x) = 1$$

Let's see what the first few terms of the sequence would look like.

$$\tilde{P}_0(x) = 1, \qquad P_0(x) = \frac{\tilde{P}_0(x)}{(\tilde{P}_0, \tilde{P}_0)^{1/2}}$$

$$\Rightarrow \quad P_0(x) = \frac{1}{\left(\int_{-1}^{1} 1\, dx\right)^{1/2}} = \frac{1}{\sqrt{2}}.$$

$$\begin{aligned}
\tilde{P}_1(x) &= x - (x, P_0(x))\, P_0(x) \\
&= x - \left(\int_{-1}^{1} y \tfrac{1}{\sqrt{2}}\, dy\right) \tfrac{1}{\sqrt{2}} \\
&= x \quad \left(\text{since } \int_{-1}^{1} y\, dy = 0\right).
\end{aligned}$$

$$\Rightarrow \quad P_1(x) = \frac{\hat{P}_1(x)}{(\hat{P}_1, \hat{P}_1)^{1/2}} = \frac{x}{\left(\int_{-1}^{1} x^2 dx\right)^{1/2}} = \frac{x}{\left(\frac{2}{3}\right)^{1/2}} = \sqrt{\frac{3}{2}}\, x$$

$$\begin{aligned}
\tilde{P}_2(x) &= x^2 - (x^2, P_1)\, P_1 - (x^2, P_0)\, P_0 \\
&= x^2 - \int_{-1}^{1} y^3 \sqrt{\tfrac{3}{2}}\, dy \sqrt{\tfrac{3}{2}}\, x - \int_{-1}^{1} y^2 \tfrac{1}{\sqrt{2}}\, dy\, \tfrac{1}{\sqrt{2}} \\
&= x^2 - 0 - \left(\tfrac{1}{3} y^3 \Big|_{-1}^{1}\right) \tfrac{1}{2} \\
&= x^2 - \tfrac{1}{3}
\end{aligned}$$

$$P_2(x) = \frac{x^2 - \frac{1}{3}}{\left(\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx\right)^{1/2}}$$

Note: $$\int_{-1}^{1} \left(x^2 - \tfrac{1}{3}\right)^2 dx = \int_{-1}^{1} x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9}\, dx$$
$$= 2\int_{0}^{1} x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9}\, dx$$
$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9}$$
$$= \frac{2}{5} + \frac{4}{9} = \frac{38}{45}$$

$$P_2(x) = \sqrt{\frac{45}{38}}\left(x^2 - \frac{1}{3}\right)$$

...

A more common normalization is to ask that

$$P_n(1) = 1 \quad \text{for each } n$$

With this normalization, the Legendre polynomials (at least the first few) take the form

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$
$$P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

Hermite Polynomials

$$(a,b) = (-\infty, +\infty), \quad w(x) = e^{-x^2}$$

$$\left(\begin{array}{ll} e^{-x^2/2} & \text{probabilist convention} \\ e^{-x^2} & \text{physicist convention} \end{array} \right)$$

There are several ways of describing Hermite polynomials, often they are described in terms of the following differential operator

$$L\phi(x) := -\frac{1}{2}\frac{d^2}{dx^2}\phi(x) + x^2\phi(x)$$

This operator admits a family of eigenvectors $\psi_n(x)$ such that

$$L\psi_n(x) = \left(n + \frac{1}{2}\right)\psi_n(x) \quad \text{for } n=0,1,2,\ldots$$

The eigenfunctions $\psi_n(x)$ have a special form, for each n, we have

$$\psi_n(x) = e^{-x^2/2} H_n(x)$$

where $H_n(x)$ is a polynomial of degree n, and this we call the n-th Hermite polynomial.

The functions $\{\psi_n\}$ are orthogonal, i.e.

$$\int_{-\infty}^{+\infty} \psi_n(x)\,\psi_m(x)\,dx = 0 \qquad \text{if } n \neq m$$

Indeed, the operator L is such that

$$\int_{-\infty}^{+\infty} (L\psi)\,\phi\,dx = \int_{-\infty}^{+\infty} \psi\, L\phi\,dx$$

(this follows easily from integration by parts)
This mean L is a "symmetric operator" with respect to the inner product

$$\langle \phi, \psi \rangle = \int_{-\infty}^{+\infty} \phi(x)\,\psi(x)\,dx$$

and therefore the eigenfunct $\{\psi_n\}$, having all different eigenvalues of a symmetric operator, must be orthogonal.

Since $\psi_n(x) = e^{-x^2/2} H_n(x)$,

$$\int_{-\infty}^{+\infty} \psi_n(x)\,\psi_m(x)\,dx = \int_{-\infty}^{+\infty} H_n(x)\,H_m(x)\,e^{-x^2}\,dx$$

Therefore

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x)\, e^{-x^2}\, dx = 0 \qquad \text{if } n \neq m.$$

In this way we see the polynomials $\{H_n\}$ are orthogonal with respect to the inner product with weight $w = e^{-x^2}$:

$$\langle \phi, \psi \rangle_w = \int_{-\infty}^{+\infty} \phi(x) \psi(x)\, e^{-x^2}\, dx$$

First few Hermite polynomials:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x$$

Chebyshef Polynomials

$$(a,b) = (-1,1), \quad w(x) = \frac{1}{\sqrt{1-x^2}}$$

Let $n = 0, 1, 2, \dots$. $T_n(x)$ is the polynomial defined by the relation:

$$(*) \qquad \cos(n\theta) = T_n(\cos(\theta)), \quad 0 \le \theta \le 2\pi$$

As we know, if $n, m \in \mathbb{Z}$ and $n \neq m$ then

$$\int_0^{2\pi} \cos(n\theta)\cos(m\theta)\, d\theta = 0$$

Then, from the definition of T_n, this is the same as

$$\int_0^{2\pi} T_n(\cos(\theta))\, T_m(\cos(\theta))\, d\theta = 0$$

Changing variables, $\theta = \arccos(x)$, we have

$$\left|\frac{d\theta}{dx}\right| = \frac{1}{\sqrt{1-x^2}}$$

and thus,

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0 \quad \text{if } n \neq m$$

Remark: To convince ourselves there really exists a unique polynomial T_n such that (*) holds, we can use complex multiplication to note that for any $n \geq 1$

$$\cos(n\theta) + i\sin(n\theta) = \left(\cos(\theta) + i\sin(\theta)\right)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\cos\theta)^{n-k} (\sin(\theta))^{k} \, i^{k}$$

$$= \sum_{\substack{k=0 \\ (k \text{ even})}}^{n} \binom{n}{k} \cos(\theta)^{n-k} (\sin(\theta))^{k} (-1)^{k/2}$$

$$+ \, i \left(\sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} \cos(\theta)^{n-k} (\sin\theta)^{k} (-1)^{\frac{k-1}{2}} \right)$$

So,

$$\cos(n\theta) = \sum_{\substack{k=0 \\ (k \text{ even})}}^{n} \binom{n}{k} (\cos\theta)^{n-k} (\sin\theta)^{k} (-1)^{k/2}$$

For k even,

$$(\sin\theta)^{k} = \left(1 - (\cos\theta)^{2}\right)^{k/2}$$

so $(\sin\theta)^{k}$ is a polynomial in $(\cos\theta)$,

$$\cos(n\theta) = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} (\cos(\theta))^{n-k} \left(1 - (\cos\theta)^{2}\right)^{k/2} (-1)^{k/2}$$

In conclusion ($x = \cos\theta$)

$$T_n(x) = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} x^{n-k} (x^2 - 1)^{k/2}$$

which is a polynomial since each $x^{n-k}(x^2-1)^{k/2}$ is a polynomial when $0 \le k \le n$ and k is even.