Whispering Gallery Modes

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## Contents



# 1 Abstract

In this paper Whispering gallery modes and their causes are studied. The basic equations required to solve for such waves are presented. Dielectric fields and whispering gallery modes are discussed at length. The numerous fields of applications of whispering-gallery modes are also discussed.

# 2 Introduction

The second largest dome ever built, the Gol Gumbaz, is a $17^{th}$ century mausoleum located in Bijapur, Karnataka, India. It has a very unique feature. When a sound it whispered at one end of the dome, it is clearly heard in other parts of the dome. Whether or not the architects of that time knew the science behind such a phenomenon, The modern physical explanation of this effect was proposed by Rayleigh as early as over a century ago [6] .
Before Rayleigh, the effect was thought to have happened due to the reflection of acoustic rays from the surface near the dome apex. It was anticipated that only

the region directly opposite a sound source would experience a concentration of the rays travelling over the several big arcs of the dome's hemisphere-shaped surface.
However, Rayleigh found that, along with this effect, another effect exists: sound 'clings' to the wall surface and 'creeps' along it.
The concave surface of the dome prevents the beam cross section from expanding as quickly as it would during free-space propagation. The radiation in the whispering gallery propagates inside a narrow layer close to the wall surface, unlike the former scenario when the beam cross section grows and the radiation intensity drops proportionately to the square of distance from a source. Because of this, the sound intensity within of this layer diminishes directly in proportion to the distance, which is much slower than in free space[5].
In the $20^{th}$ century, it was discovered that that even electromagnetic waves that had the same spatial structure as these sound waves, can exist[1][4].

## 3 Whispering Gallery modes

Electromagnetic surface modes are present at all surfaces and interfaces between material of different dielectric properties. They are a special type of wave localized at the interface separating two different media[7].
Whispering gallery modes are surface modes that propagate "azimuthally" around resonators with rotational symmetry, generally a dielectric[2].
Optical Whispering gallery modes occur when light waves confined by total internal reflection in a dielectric structure are reflected back on the same optical path where they interfere constructively. Interference generates the optical resonance signal, which occurs whenever exactly an integer number of wavelengths fit on the confined (circular) light path[8].

## 4 Basic Equations

We start with the basic Maxwell's equations and get the formulas:

$$(\nabla \times \mathbf{E})_l = \frac{1}{L_m L_n}\left(\frac{\partial}{\partial \xi_n}(L_m E_m) - \frac{\partial}{\partial \xi_m}(L_n E_n)\right) = ikH_l \tag{1}$$

$$(\nabla \times \mathbf{H})_l = \frac{1}{L_m L_n}\left(\frac{\partial}{\partial \xi_n}(L_m H_m) - \frac{\partial}{\partial \xi_m}(L_n H_n)\right) = -ikE_l \tag{2}$$

$$(m, n, l) = \{(3,2,1), (1,3,2), (2,1,3)\}$$

We consider an E type wave which means $H_3 = 0$. This is called as a transverse magnetic mode. Conversely we also have a **H** type wave such that $E_3 = 0$ and such wave is called transverse electric mode.

Introducing a function W such that

$$\nabla(W) = \mathbf{E} \tag{3}$$

If we look at above function $W$ it takes the form of a very familiar quantity (negative of potential)! This gives us

$$L_1E_1 = \frac{\partial W}{\partial \xi_1} \; , \; L_2E_2 = \frac{\partial W}{\partial \xi_2} \tag{4}$$

Upon substituting equation (4) in equation (2) to indices $l = 1, 2$ we get:

$$\frac{\partial}{\partial \xi_3}(L_2H_2) = ik\frac{L_2L_3}{L_1}\frac{\partial W}{\partial \xi_1} \; , \; \frac{\partial}{\partial \xi_3}(L_1H_1) = ik\frac{L_1L_3}{L_2}\frac{\partial W}{\partial \xi_2} \tag{5}$$

We define a a new scalar function U where U is an unknown function yet, such that

$$\frac{\partial U}{\partial \xi_3} = W \tag{6}$$

Now from equation (5) and equation (6) we will get,

$$H_1 = -ik\frac{1}{L_2}\frac{\partial U}{\partial \xi_2} \; , \; H_2 = -ik\frac{1}{L_1}\frac{\partial U}{\partial \xi_1} \tag{7}$$

Upon using equation (7) in equation (2) we can determine $E_3$ to be

$$E_3 = -\frac{1}{L_1L_2}\left[\frac{\partial}{\partial \xi_1}\left(\frac{L_2}{L_1}\frac{\partial U}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{L_1}{L_2}\frac{\partial U}{\partial \xi_2}\right)\right] \tag{8}$$

We impose conditions such as:

- $L_3 = 1$
- $\frac{L_1}{L_2}$ is independent of $\xi_3$

We algebraically manipulate equations (2), (4), (7) and (8) and we get a master equation in U :

$$\Rightarrow \frac{\partial^2 U}{\partial \xi_3^2} + \frac{1}{L_1L_2}\left[\frac{\partial}{\partial \xi_1}\left(\frac{L_2}{L_1}\frac{\partial U}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{L_1}{L_2}\frac{\partial U}{\partial \xi_2}\right)\right] + k^2U = 0 \tag{9}$$

Fields of dielectric sphere implies spherical co-ordinate system. Hence converting our problem into spherical coordinates in the following manner:

- $\xi_3 = r, \xi_2 = \theta, \; \xi_1 = \phi$
- $L_3 = 1, L_2 = r, L_1 = r sin\theta$

This in-turn changes the equations to:

$$\Rightarrow \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2 sin\theta}\left[\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{sin\theta}\frac{\partial U}{\partial \phi}\right)\right] + k^2U = 0$$

$$\Rightarrow \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} + k^2U = 0 \tag{10}$$

Writing U as product of independent variable functions,

$$U = R(r)\Theta(\theta)\Phi(\phi)$$

where $(r, \phi, \theta)$ are three independent variables.

This gives us three independent differential equations in $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$ respectively

$$\frac{d^2R}{dr^2} + \left(k^2 - \frac{c_1}{r^2}\right) R = 0 \tag{11}$$

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Phi}{d\theta}\right) + \left(c_2 - \frac{c_3}{sin^2\theta}\right)\Theta = 0 \tag{12}$$

$$\frac{d^2\Phi}{d\phi^2} + c_3\Phi = 0 \tag{13}$$

Physical requirement of uniqueness enforces constant values to be $c_2 = n(n+1)$ and $c_3 = m^2$ So the equations above become:

$$\frac{d^2R}{dr^2} + \left(k^2 - \frac{c_1}{r^2}\right) R = 0 \tag{14}$$

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Phi}{d\theta}\right) + \left(n(n+1) - \frac{m^2}{sin^2\theta}\right)\Theta = 0 \tag{15}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{16}$$

The Helmholtz wave equation in Cartesian co-ordinates describes a plane wave moving on a unilateral direction (say x). So resonance condition (constructive interference) for these waves look like

$$\Delta x = n\lambda \ n \in Z$$

Since the wave is moving along the surface of sphere dome clinging to it and Rayleigh notes only the radial part needs to accounted to understand WGM, we can try to comprehend resonance mode of WGMs as

$$m\lambda_m = 2\pi rn \tag{17}$$

where m is azhimuthal mode number and n is radial mode number.
Now back to trying to solve equations (14), (15) and (16).

The equation

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Phi}{d\theta}\right) + \left(n(n+1) - \frac{m^2}{sin^2\theta}\right)\Theta = 0$$

has general solution in the form of adjoint Legendre polynomial $P_n^m(cos\theta)$ whereas

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{18}$$

has a sinusoidal solution $e^{\pm im\phi}$

But the equation

$$\frac{d^2R}{dr^2} + \left(k^2 - \frac{c_1}{r^2}\right) R = 0$$

is not in an easily recognisable form. In order to make it recognisable let us define a new variable $Z(kr)$ as:

$$R(r) = \sqrt{kr}Z(kr)$$

This transforms equation (14) into

$$\implies \frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z = 0 \tag{19}$$

where $z = kr$ and $\nu = n + \frac{1}{2}$.

Boundary condition for this problem will be:

- $k = k_0\sqrt{\mu_r \epsilon_r}$ for $r \leq a$ for case of inside the sphere $\implies$ solution is a Bessel function
- $k = k_0 = \frac{\omega}{c}$ for $r > a$ for case of outside the sphere $\implies$ solution is Hankel function.

Hence the final solutions for the dielectric sphere will look like:

**For inside the sphere:**

$$U^i_{mn}(r,\theta,\phi) = C_i P^m_n(cos\theta)(kr)^{\frac{1}{2}} J_\nu(kr)e^{\pm im\phi} \tag{20}$$

**For outside the sphere:**

$$U^e_{mn}(r,\theta,\phi) = C_e P^m_n(cos\theta)(kr)^{\frac{1}{2}} H_\nu(k_0r)e^{\pm im\phi} \tag{21}$$

# 5 Graphs and Interpretation

The Hankel functions are used to express outward- and inward-propagating cylindrical-wave solutions of the cylindrical wave equation, respectively (or vice versa, depending on the sign convention for the frequency).
For WGM to occur we should have very high value of n (number of modes). This is because only at very high values of n the movement of standing waves in a closed n sided polygon loop can be approximated to the waves moving along the curved surface of the sphere. This can be clearly observed in the Figures 5 and 6 in section 6. At lower n values the wave oscillating field almost fills the entire volume of the sphere and hence cannot be whispering gallery modes. Figure 1 shows the simulation of bessel function for different values of n and Figure 2 for $n = 1000$
Legendre polynomial solution $[P^n_m(cos\theta)]^2 \propto sin^{2n}\theta$ for n=m has a maxima

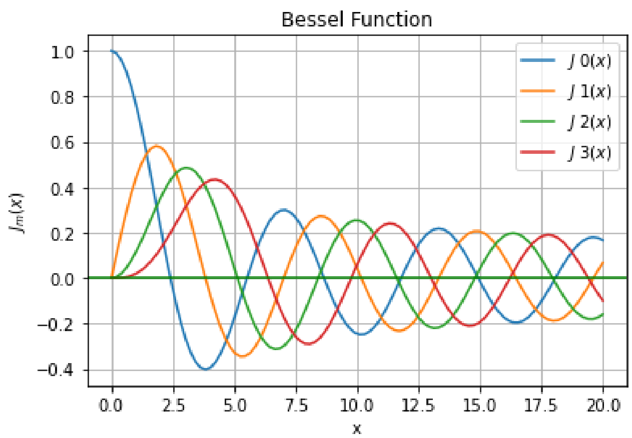


Figure 1: Bessel function with smaller indices

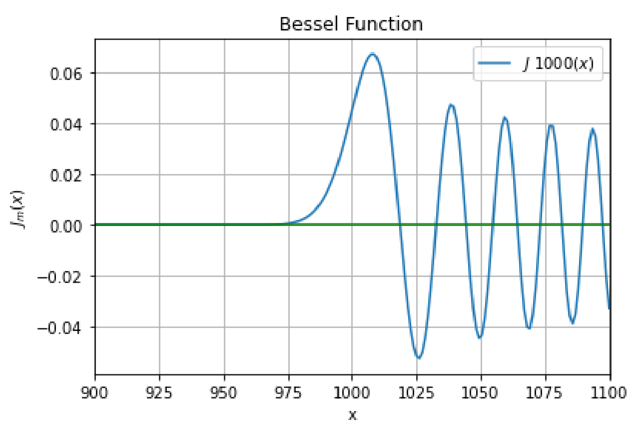


Figure 2: Bessel function with large index = 1000

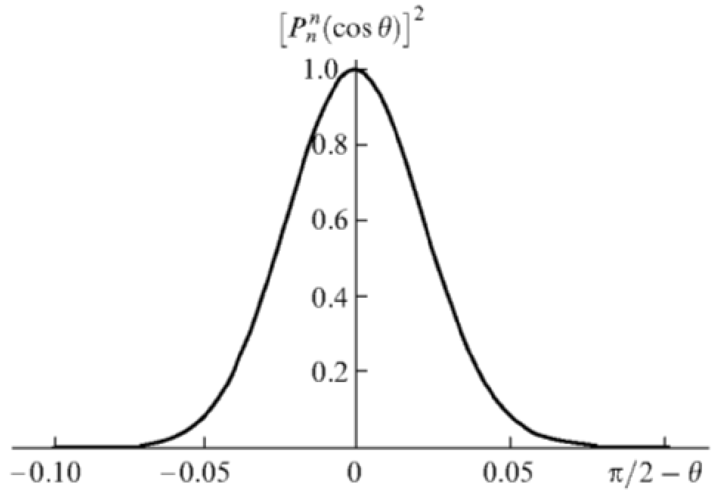


Figure 3: Function $[P_n^n(\cos\theta)]^2 \propto \sin^{2n}\theta for n = 1000$

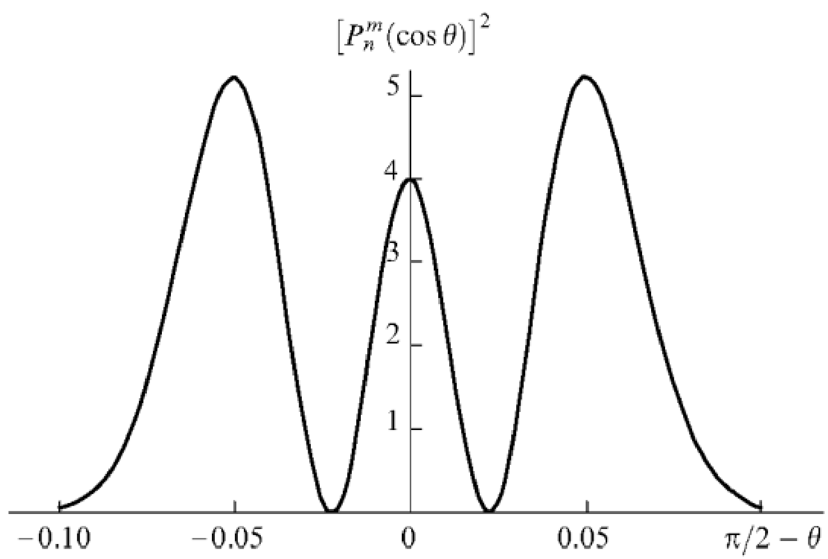


Figure 4: Function $[P_n^m(\cos\theta)]^2$for $m = n - 2$ and $n = 1000$

exactly at the resonant frequency of the system (observed in Figure 3).
For cases where $m \neq n$ we do not observe such a clear maxima at resonant frequency, in fact we observe multiple peaks at multiple frequencies (seen in Figure 4). While for an n value close to m we can approximate the system to be a WGM it is not exactly WGM.

# 6 Understanding WGMs physically

Optical Whispering gallery modes occur when waves confined by total internal reflection in a dielectric structure are reflected back on the same optical path where they constructively interfere. This interference generates an optical resonance signal which occur when there are integral number of wavelengths confined in the dielectric sphere. The more the number of modes, more it will cling to the wall. This can be understood using ray optics as well. Once the

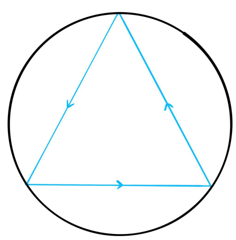

Figure 5: 3 modes

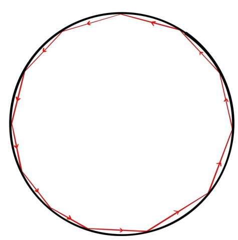

Figure 6: n modes

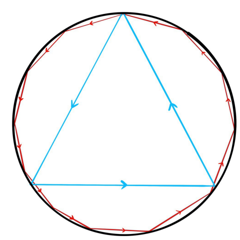

Figure 7: both modes

waves constructively interfere, they will stay in constructive interference as the frequency will remain the same.
We can find the critical angle for total internal reflection using Snell's law.

# 7 Application of WGMs

The most important aspect of studying optics is to study light matter interactions since this is what is of great practical use to us. Confining light in a small space along with matter significantly increases the light- matter interaction that we sought to explore which leads us to observe new phenomena and functionalities. The simplest structure to confine light is a resonator built by two reflecting surfaces.

However there are limitations to a normal resonator. It is not possible to make extremely reflecting small surface. Hence, making a resonator for nanoparticles becomes extremely expensive. In such cases, WGMs can be used as optical resonators to view bacteria, viruses and other micro and nano- particles. They are less expensive and robust compared to normal resonators as materials

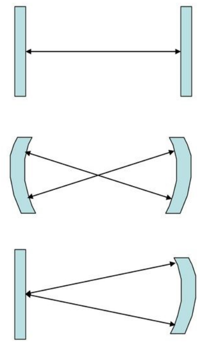

Figure 8: Resonator

such as castor oil, silicon etc are used.
Another important aspect of using WGM resonators is their high Q-factor. We define Q-factor as :

$$\text{Q-factor} = 2\pi \times \frac{\text{Stored Energy}}{\text{Energy losy per cycle}}$$

In order to visualise the sheer magnitude of light matter interactions let us define another quantity called Cavity photon lifetime $\tau$

$$\tau = \frac{Q}{\omega} \tag{22}$$

Q-factor value obtained experimentally is around the order of $10^8$ while frequency of light is around $10^15$ Hz. Upon substituting these values in the above equation we get $\tau$ = 100ns which might seem very low but in this time light travels a distance of 30m! This distance inside a small dielectric sphere is very significant and we can clearly observe higher number of light matter interactions. However, it is important to note that light is not shone directly on the WGM optical resonator. "Evanescent Waves" are used to view the WGMs in these dielectric spheres. Evanescent waves are formed when sinusoidal waves are (internally) reflected off an interface at an angle greater than the critical angle so that total internal reflection occurs.

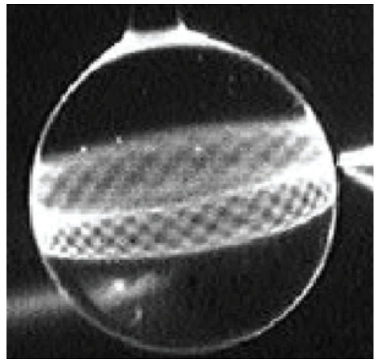

Figure 9: WGM in sphere

We can interpret the light in the dielectric sphere and the size of the particle we are trying to view through a transmission spectrum. The wavelength at which we see a dip is the wavelength of light which underwent total internal reflection and is trapped inside the dielectric sphere. This light forms the optical WGMs as shown in the figure above. Hence that light will be "missing" from the transmission spectrum. These properties can be used to light up and understand the particle we wish to observe.

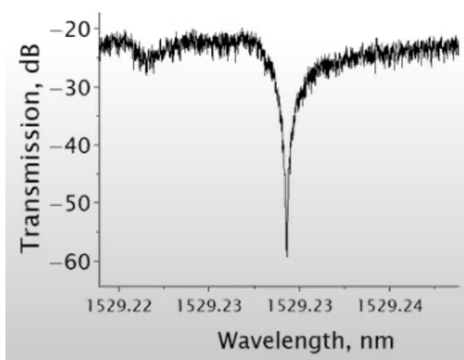


Figure 10: Transmission spectrum of WGM in dielectric sphere

WGMs can also used in optical fiber resonators for efficient energy transfer. The WGMs are propagated along the cylindrical cross section of the optical fibre[3].

Optical whispering gallery modes is a very new field of research. Novel methods to observe and learn about it are being discovered and researched at a rapid rate. In the future we will see many interesting problems related to spherical modes and optical whispering gallery modes. We can expect new and ingenious proposals for applications of WGMs.

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