# Problem Set 10

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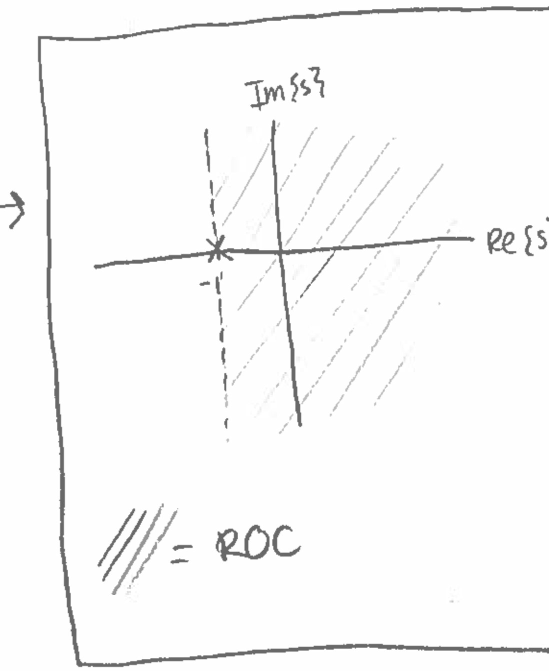
## Problem 1

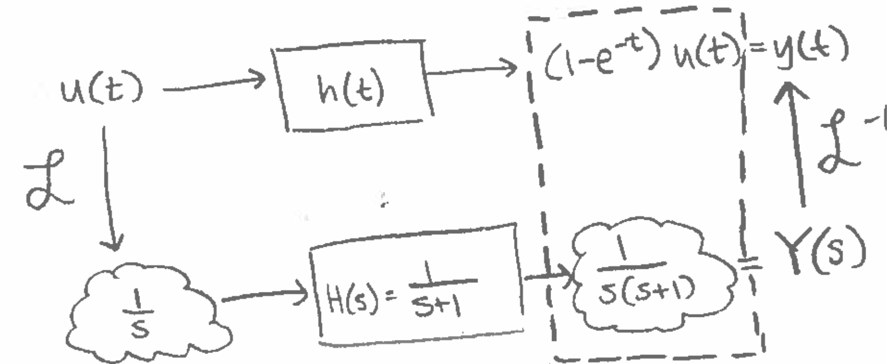
Show $\dot{y}+y=x$ has step response $y(t)=(1-e^{-t})u(t)$

$\dot{y}+y=x$ Take $\mathcal{L}$ transform

$sY+Y=X$ Get $H(s)$

$\frac{Y}{X}=\frac{1}{s+1}=H(s)$




I need to show $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}=(1-e^{-t})u(t)$

$$Y(s)=\frac{1}{s(s+1)}$$

$$=\frac{1}{s^2+s} \quad \text{add } s-s \text{ to numerator}$$

$$=\frac{1+s-s}{s^2+s}$$

$$=\frac{-s+(s+1)}{s^2+s}$$

$$=\frac{-s}{s^2+s}+\frac{s+1}{s^2+s}$$

$$=\frac{-\cancel{s}}{\cancel{s}(s+1)}+\frac{\cancel{s+1}}{s(\cancel{s+1})}$$

$$=\frac{-1}{s+1}+\frac{1}{s}$$

$$y(t)=\mathcal{L}^{-1}\left\{-\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$=-e^{-t}u(t)+u(t)$$

$$\boxed{y(t)=(1-e^{-t})u(t)}$$

Problem 2

B) $$\frac{Y}{Y_{sp}} = \frac{KH}{1+KH}$$

$$H(s) = \frac{1/\tau}{s+1/\tau} \qquad K(s) = \frac{K_I}{s}$$

$$\frac{\frac{K_I}{s}\left(\frac{1/\tau}{s+1/\tau}\right)}{1+\frac{K_I}{s}\left(\frac{1/\tau}{s+1/\tau}\right)} = \frac{K_I/\tau}{s^2+s/\tau} \div \frac{s^2+s/\tau+K_I/\tau}{s^2+s/\tau}$$

$$= \frac{K_I/\tau}{\cancel{s^2+s/\tau}} \cdot \frac{\cancel{s^2+s/\tau}}{s^2+s/\tau+K_I/\tau} = \boxed{\frac{K_I/\tau}{s^2+s/\tau+K_I/\tau}}$$

find poles/zeros:

no zeros

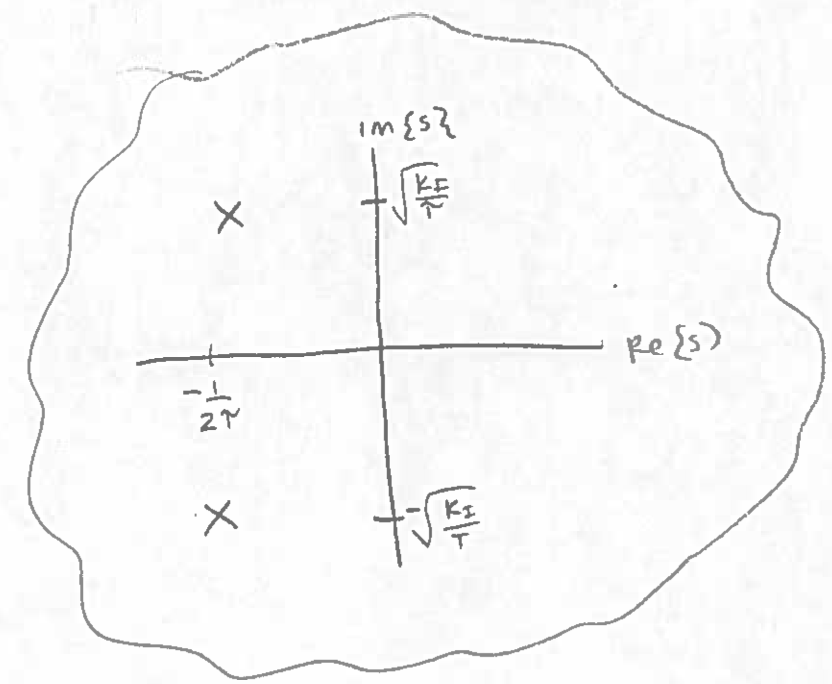
$$s^2+s/\tau+K_I/\tau=0$$

$$s = \frac{-\frac{1}{\tau} \pm \sqrt{\left(\frac{1}{\tau}\right)^2 - 4\frac{K_I}{\tau}}}{2}$$

$$\approx \frac{-\frac{1}{\tau} \pm \sqrt{-4\frac{K_I}{\tau}}}{2}$$

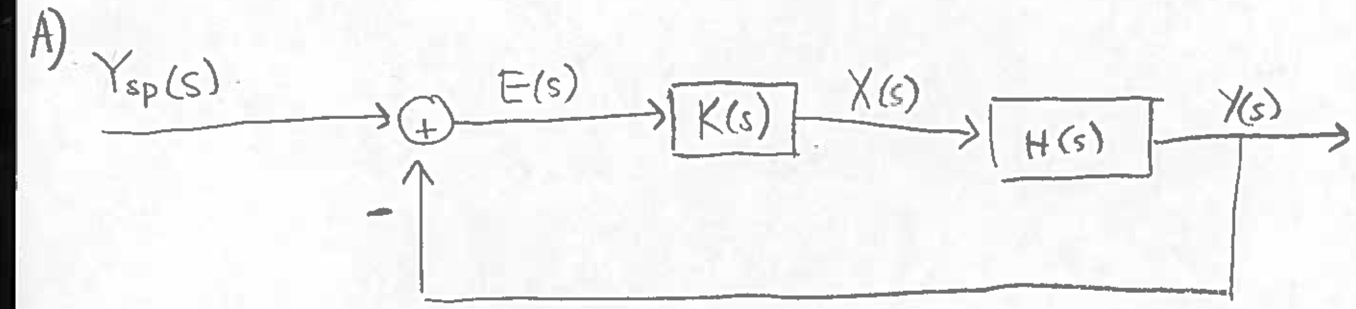
$$= \frac{-\frac{1}{\tau} \pm 2j\sqrt{\frac{K_I}{\tau}}}{2}$$

$$= \boxed{-\frac{1}{2\tau} \pm j\sqrt{\frac{K_I}{\tau}}}$$



Problem 2

A)




DC gain of the system

First, figure out the system:

$$K(Y_{sp}-Y)=X$$

$$K(Y_{sp}-HX)=X$$

$$\frac{X}{Y_{sp}}=\frac{K}{1+KH}$$

$$\left.\frac{Y}{Y_{sp}}=\frac{KH}{1+KH}\right\} \longrightarrow \text{Black's formula}$$

$$\frac{Y}{Y_{sp}}=\frac{\frac{k_I}{s}H}{1+\frac{k_I}{s}H}$$

$\downarrow$

DC gain is $\lim_{s\to 0}$ $\boxed{\text{Transfer func} = \frac{Y}{Y_{sp}}}$

$$\lim_{s\to 0}\frac{\frac{k_I}{s}H}{1+\frac{k_I}{s}H}=\frac{\infty}{1+\infty}\approx\boxed{1}\longrightarrow \text{DC gain}$$

# Problem 3

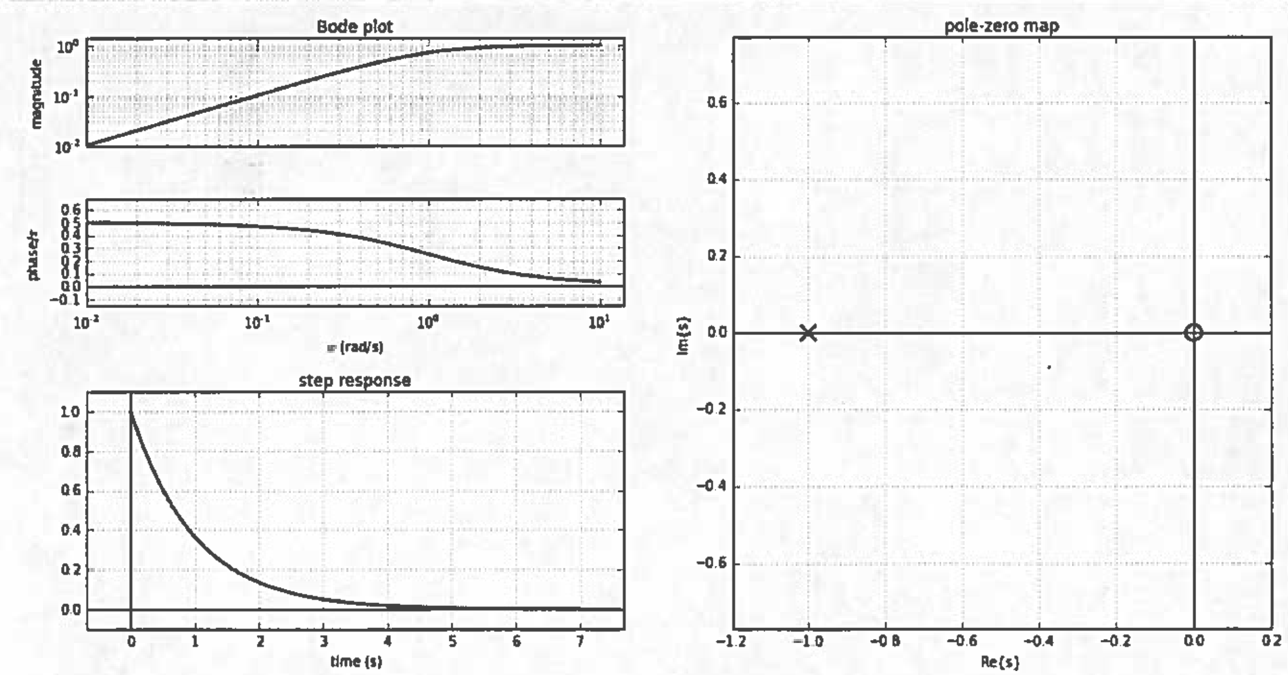
In [14]:
```
%matplotlib inline
%run convenience.ipynb
from __future__ import print_function
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal

np.set_printoptions(precision=2,suppress=True) # numpy output options

pi=np.pi
j=1j
```

## Part A

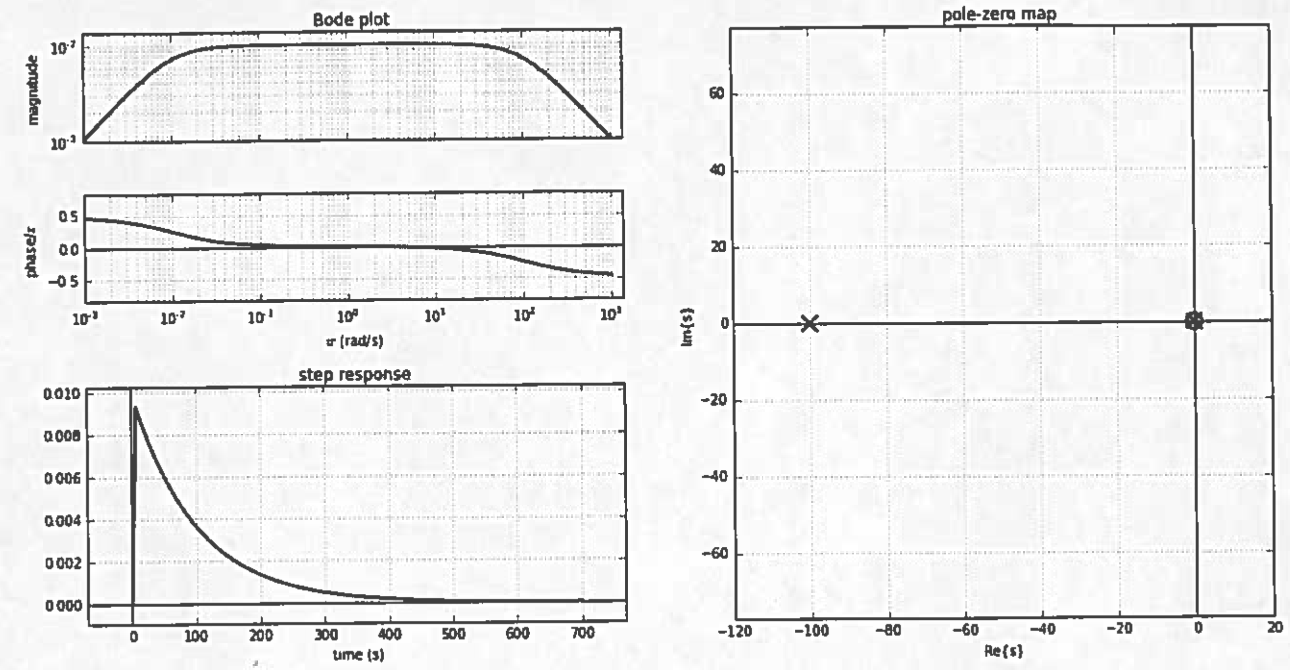
In [15]:
```
sys = signal.lti([1,0], [1,1])
combinedplot(sys)
```




**This system acts as a high pass filter. You can see in the Bode plot that the system cuts off lower frequencies and keeps higher frequencies. Also, this system is stable. The step response shows decaying. Also, the pole in the pole zero map is to the left of the jw axis, so we know that this system is stable. The pole of this system is real, and the system decays in a 1st order system fashion.**

## Part B

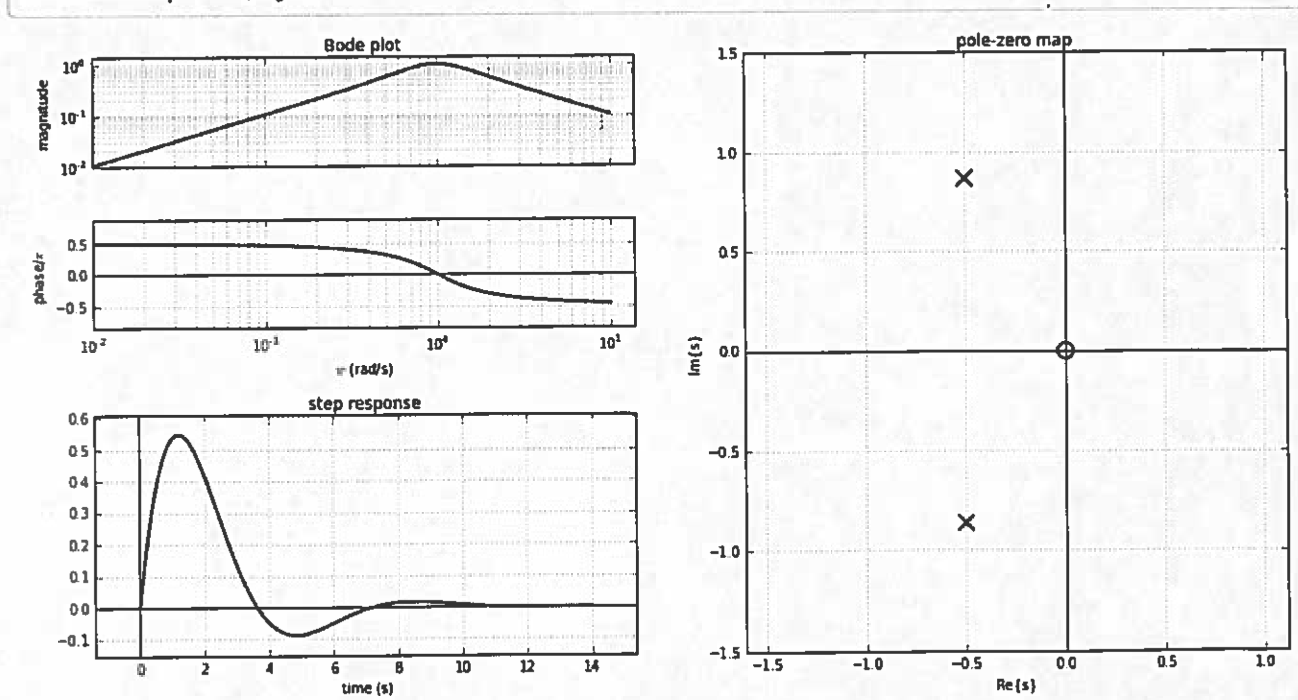
```
In [16]: sys = signal.lti([1,0], [1,100,1])
         combinedplot(sys)
```




**This system acts as a bandpass filter with cutoff frequencies at 1/100 Hz and 100 Hz. The system is stable becasue the step response decays. Also, both poles are to the left of the jw axis, although the rightmost one is just barely so. The poles here are also real, so the system decays without oscillations.**

## Part C

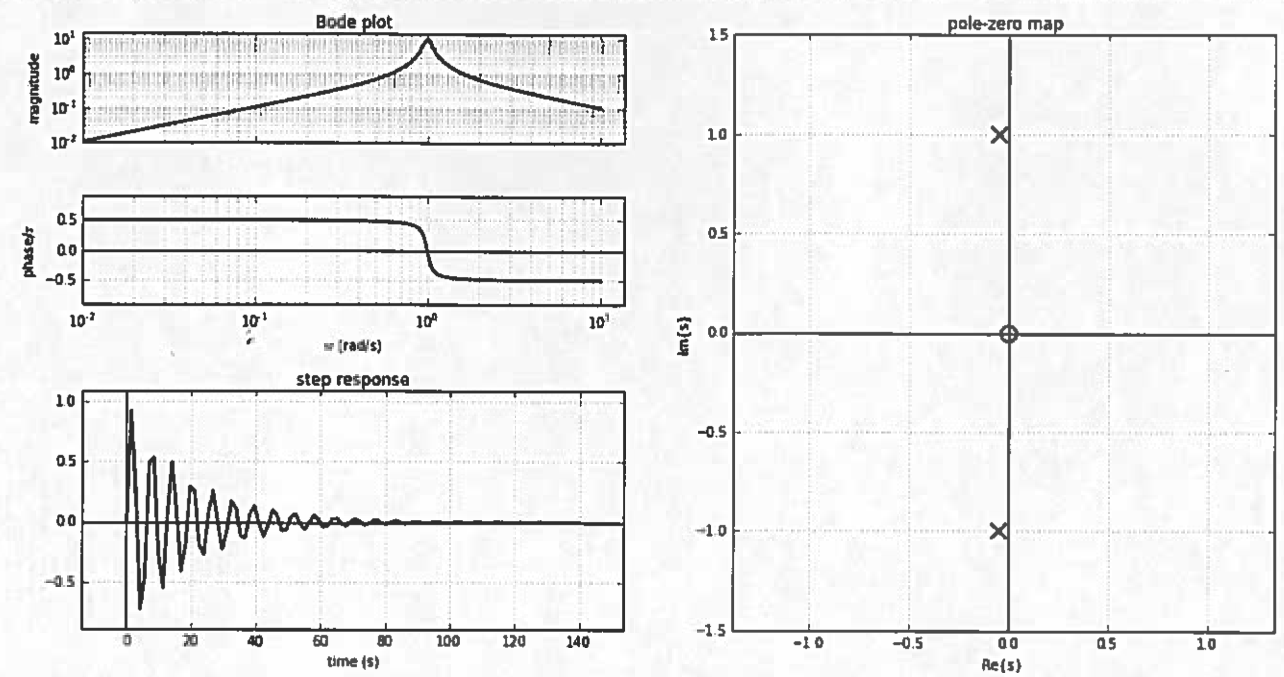
```
In [17]: sys = signal.lti([1,0], [1,1,1])
         combinedplot(sys)
```



This system acts as a bandpass filter, with not very steep cutoffs. It begins to cutoff at 1Hz on both sides, but not very steeply. The system is stable because the step response oscillates just a little and decays. The 2 poles are on the left of the jw axis, so a stable system is what I expect. The poles are also complex, so there's oscillations that decay slowly.

## Part D

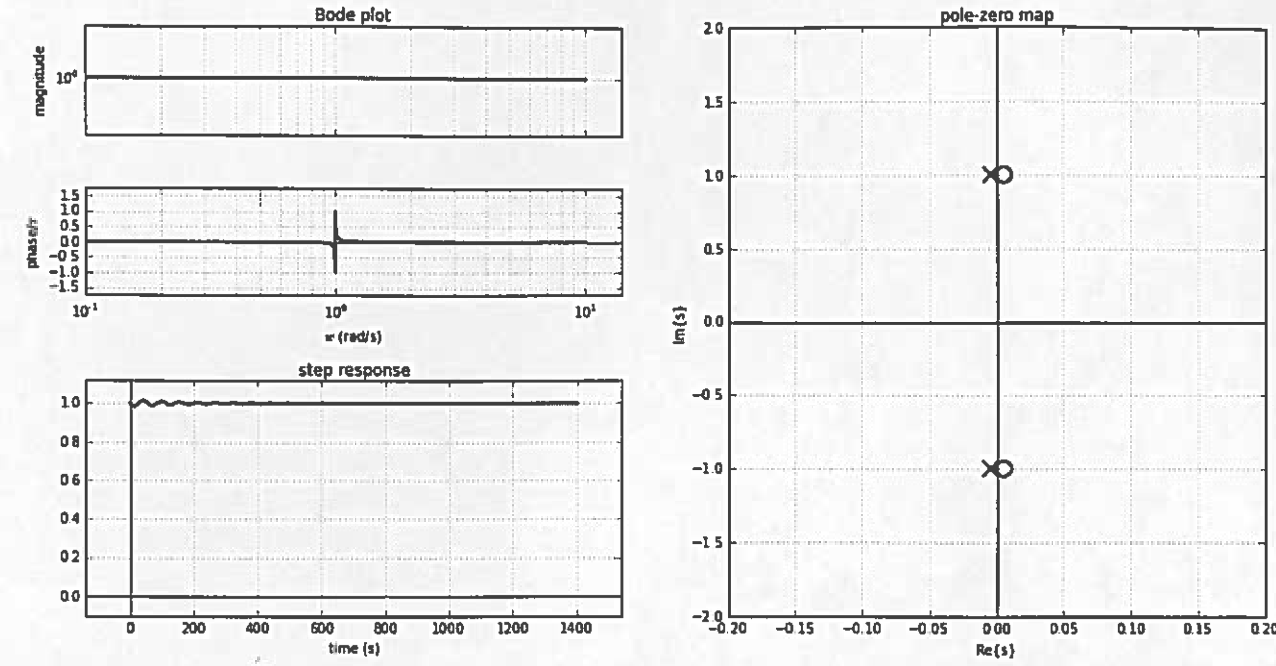
In [18]:
```
sys = signal.lti([1,0], [1,0.1,1])
combinedplot(sys)
```




This system amplifies 1Hz, and sharply cuts off on either side of 1Hz, so it's like a bandpass filter. The system is stable, but oscillates a lot while decaying slowly to become stable. The poles are slightly to the left of the jw axis. I imagine that it takes so long for the step response to stabilize because the poles are so close to the jw axis. There are 2 complex poles, that are closer to the jw axis so there is more of an imaginary component than a real component so the step response oscillates more than in part c.

## Part E

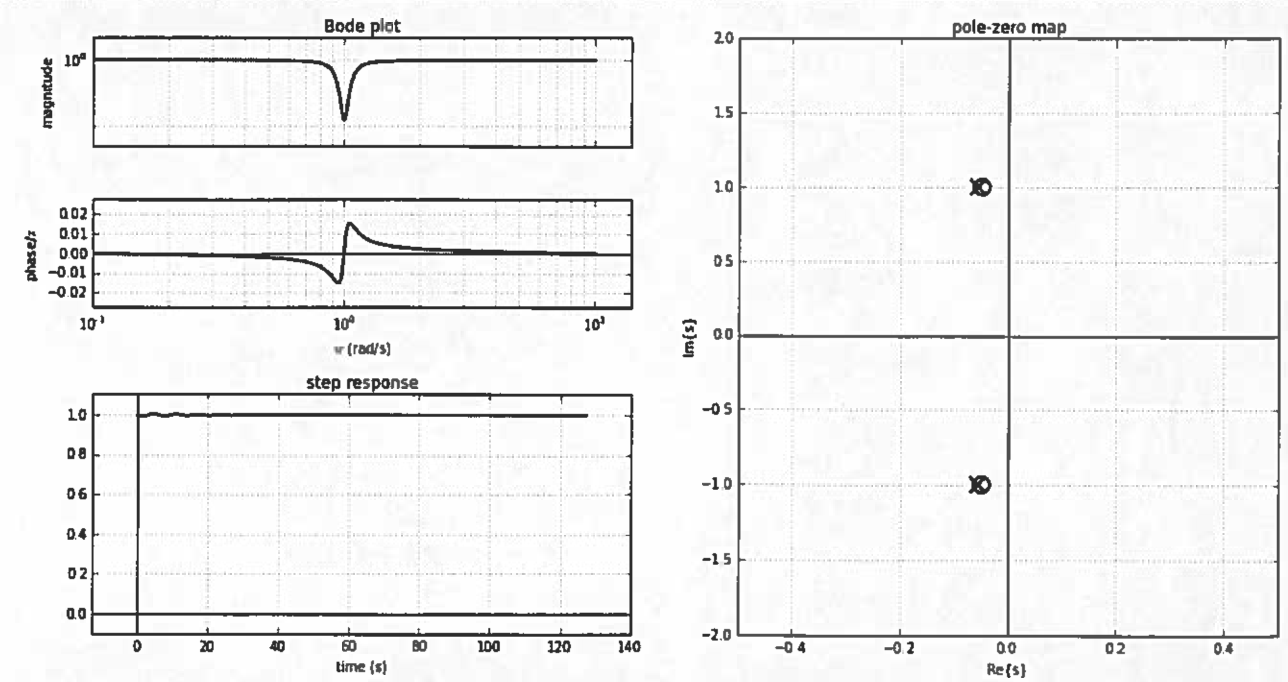
In [19]:
```
sys = signal.lti([1,-.01,1], [1,.01,1])
combinedplot(sys, axis=[-0.2,0.2,-2,2])
```




**There's such a small difference between the top and the bottom of the fraction in this system, so it makes sense that the Bode plot shows the system just keeping all the frequencies. Since the top subtracts 0.01 s but the bottom adds it, the system decays just so slightly, enough to stabilize the system. The poles are just barely to the left of the jw axis. The poles are complex with a higher imaginary part so there are oscilations before it becomes stable, but this system barely does much anyways.**

## Part F

In [20]:
```
sys = signal.lti([1,0.1,1], [1,.11,1])
combinedplot(sys, axis=[-0.5,0.5,-2,2])
```



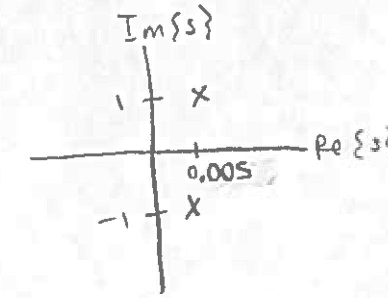
This system is a band stop filter. It cuts off frequencies right next to 1Hz and keeps everything else. The poles of this system are to the left of the jw axis, so you can see that in the step response, the signal decays and stabilizes after oscillating a bit. It stabilizes faster than in part e because there is a higher real part.

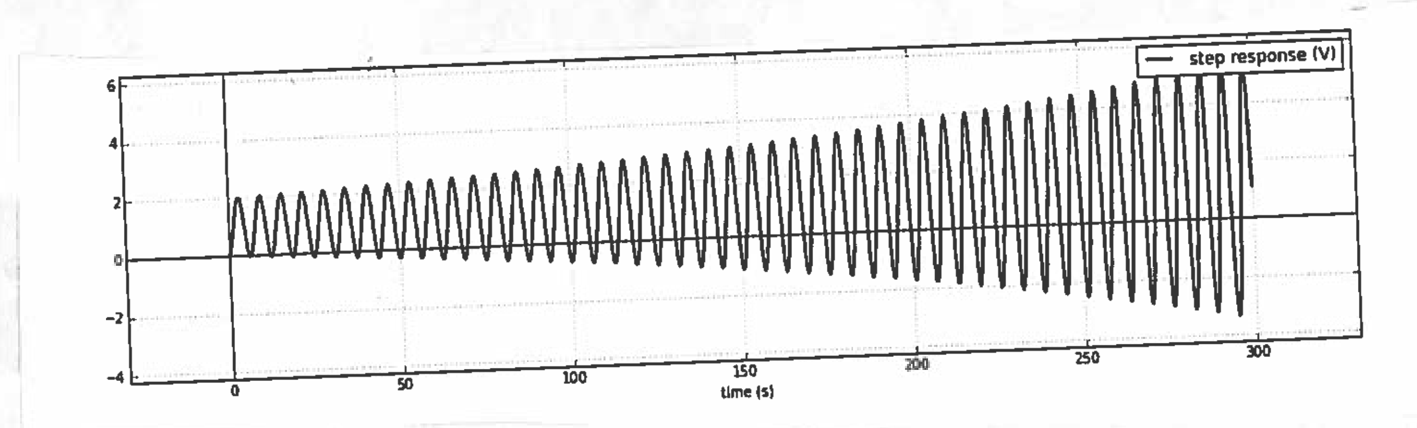
Problem 4

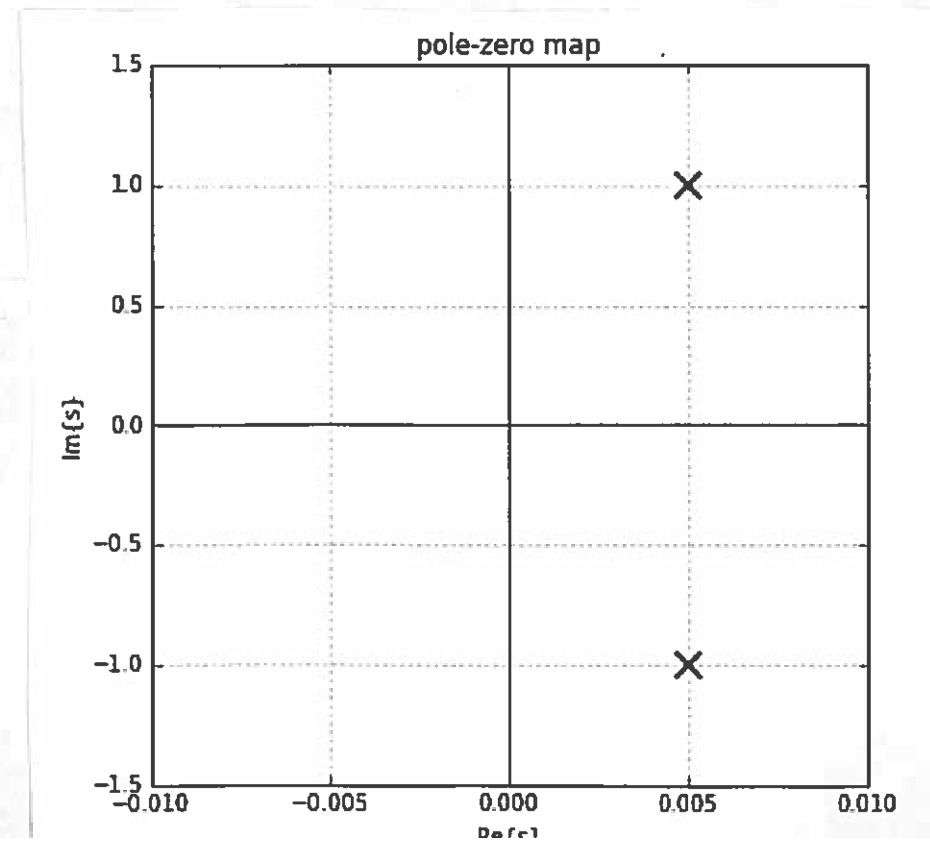
A) $H(s) = \frac{1}{s^2 - 0.01s + 1}$

poles

$$s = \frac{0.01 \pm \sqrt{0.0001 - 4}}{2} = \frac{0.01 \pm \sqrt{-4}}{2} = \frac{0.01 \pm 2j}{2} = \boxed{0.005 \pm j}$$



Proportional Control

$$\frac{Y}{Y_{sp}} = \frac{KH}{1+KH} \qquad K = k \qquad H = \frac{1}{s^2 - 0.01s + 1}$$

$$\frac{\frac{k}{s^2-0.01s+1}}{1+\frac{k}{s^2-0.01s+1}} = \frac{k}{s^2-0.01s+1} \div \underbrace{1+\frac{k}{s^2-0.01s+1}}_{\frac{s^2-0.01s+1+k}{s^2-0.01s+1}}$$

$$= \frac{k}{\cancel{(s^2-0.01s+1)}} \times \frac{\cancel{(s^2-0.01s+1)}}{s^2-0.01s+1+k} = \boxed{\frac{k}{s^2-0.01+1+k}}$$

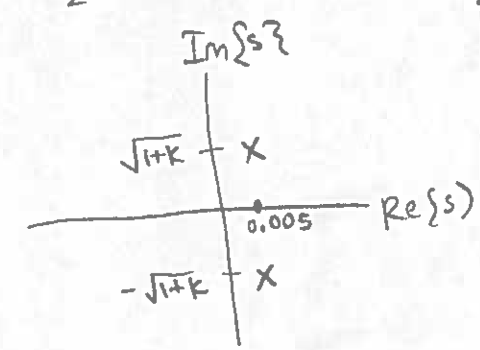
This describes the system (transfer function)

Poles and zeroes

No zeros

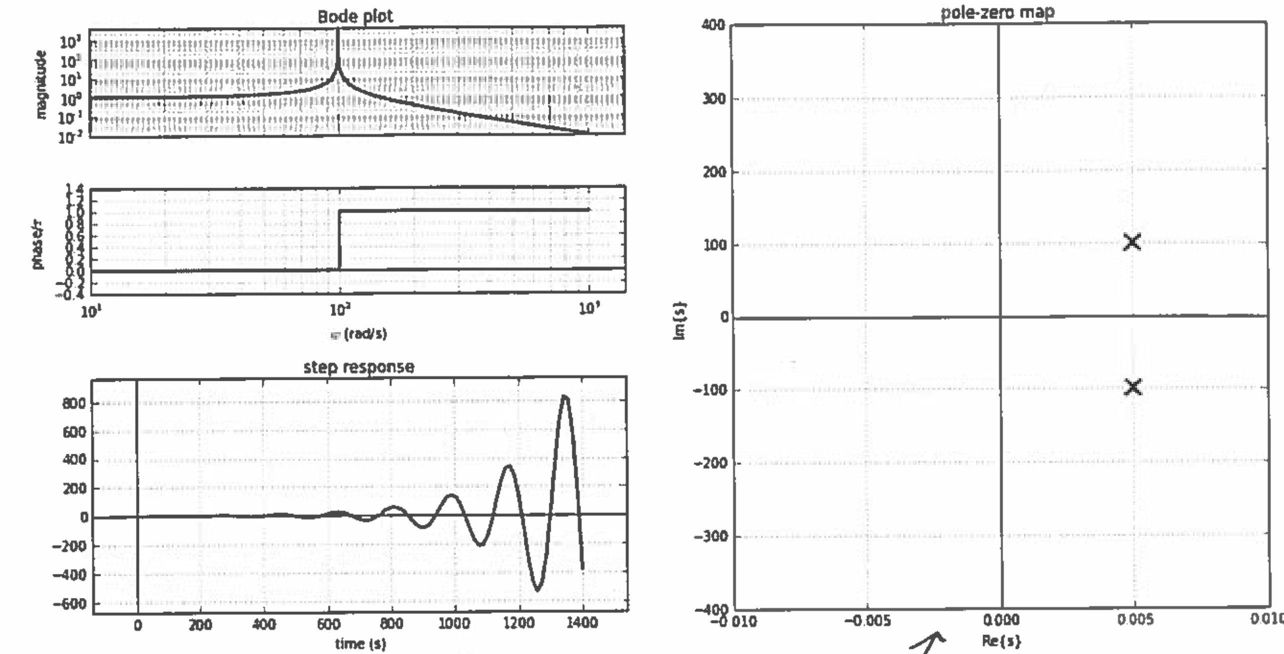
poles $\longrightarrow$ $s^2 - 0.01 + 1 + k = 0$

$$s = \frac{0.01 \pm \sqrt{.0001-(4+4k)}}{2} = \frac{0.01 \pm 2j\sqrt{1+k}}{2} = \boxed{0.005 \pm j\sqrt{1+k}}$$



```
In [31]: k=10000
sys = signal.lti([k], [1,-.01,1+k])
combinedplot(sys, axis=[-.01,.01,-400,400])
```

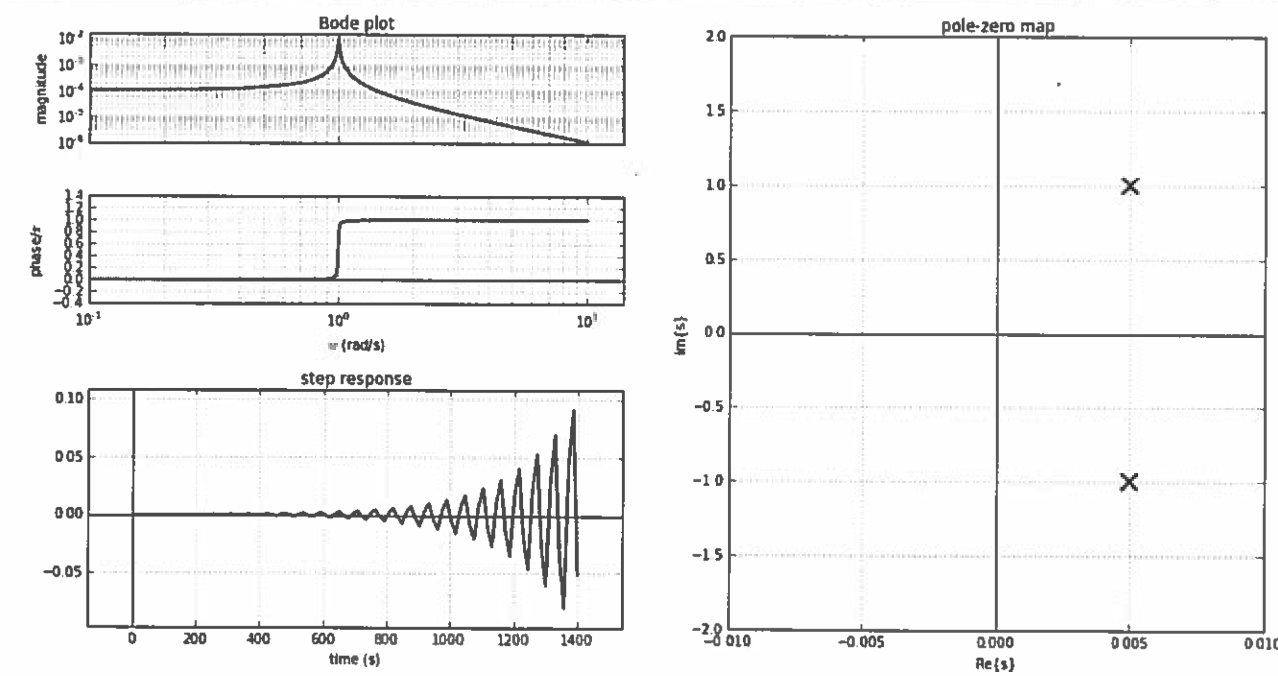
Proportional Control with large K




The system is unstable w/ high or low values of k. The poles are on the right of jw axis

```
In [34]: k=.0001
sys = signal.lti([k], [1,-.01,1+k])
combinedplot(sys, axis=[-.01,.01,-2,2])
```

Proportional Control with Small k



Integral Control

$$\frac{Y}{Y_{sp}} = \frac{KH}{1+KH} \qquad K = \frac{k}{s} \qquad H = \frac{1}{s^2 - 0.01s + 1}$$

$$\frac{\frac{k/s}{s^2-0.01s+1}}{1+\frac{k/s}{s^2-0.01s+1}} = \frac{k/s}{s^2-0.01s+1} \div \frac{s^2-0.01s+1+k/s}{s^2-0.01s+1}$$

$$= \frac{k/s}{s^2-0.01s+1+k/s} = \boxed{\frac{k}{s^3-0.01s^2+s+k}}$$

transfer function

Poles and zeros

No zeros

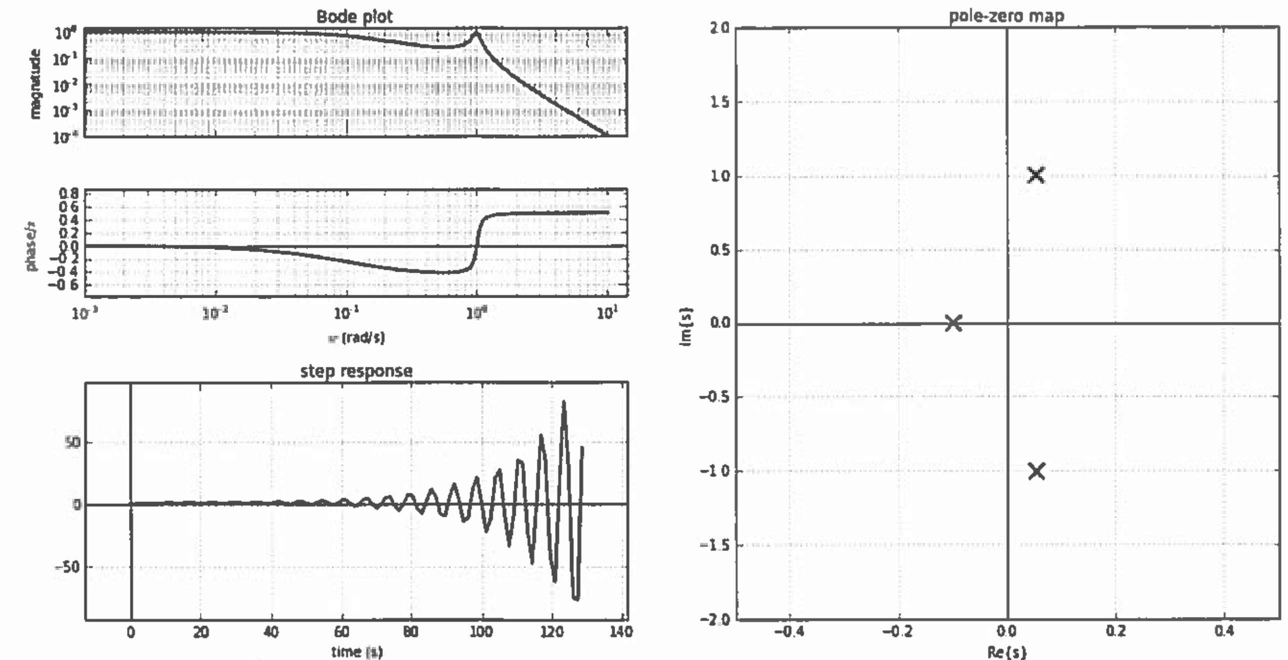
$$s^3 - 0.01s^2 + s + k = 0$$

s = a really long thing

pole zero map plotted computationally →

In [39]:
```
k=.1
sys = signal.lti([k], [1,-.01,1, k])
combinedplot(sys,axis=[-.5,.5,-2,2])
```

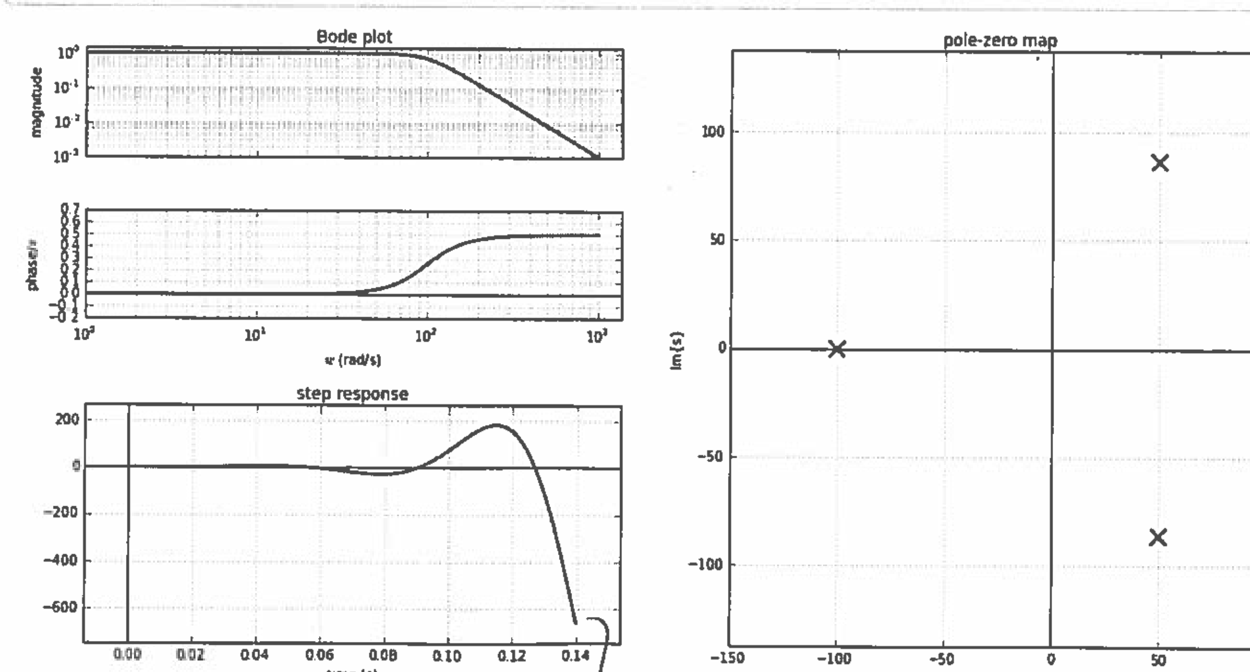
Integral Control with small k




unstable because some poles are on the right.

In [48]:
```
k=1000000
sys = signal.lti([k], [1,-.01,1, k])
combinedplot(sys)
```

Integral control with large k




still unstable, goes to $-\infty$

Derivative Control

$$\frac{ks}{s^2 - 0.01s + 1 + ks} = \boxed{\frac{ks + 0}{s^2 + (-0.01 + k)s + 1}}$$

transfer function

Poles and zeros

zero at $\boxed{s=0}$

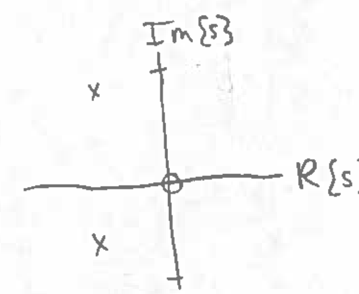
Poles

$$(-0.01 + k)(-0.01 + k) = .0001 - .02k + k^2 \approx k^2$$

$$s^2 + (-0.01 + k)s + 1 = 0$$

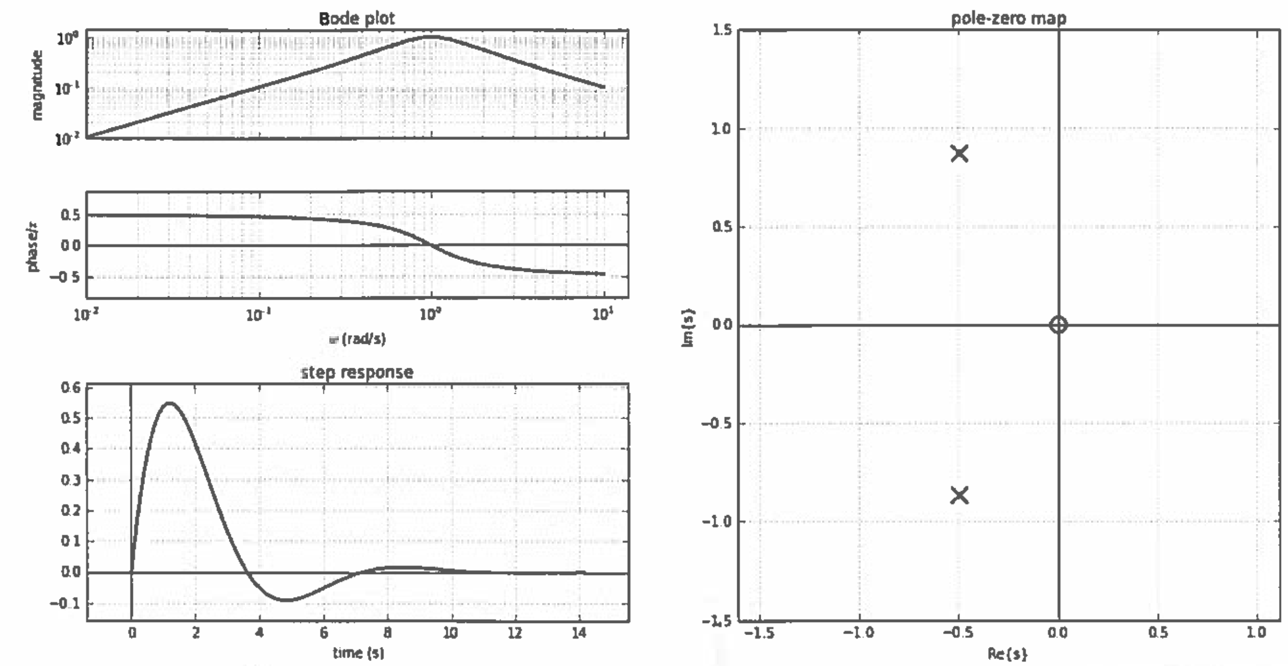
$$s = \frac{0.01 - k \pm \sqrt{k^2 - 4}}{2}$$

$$= \frac{0.01 - k}{2} \pm \frac{\sqrt{k^2 - 4}}{2}$$



In [56]:
```
k=1
sys = signal.lti([k,0], [1,-.01+k,1])
combinedplot(sys)
```

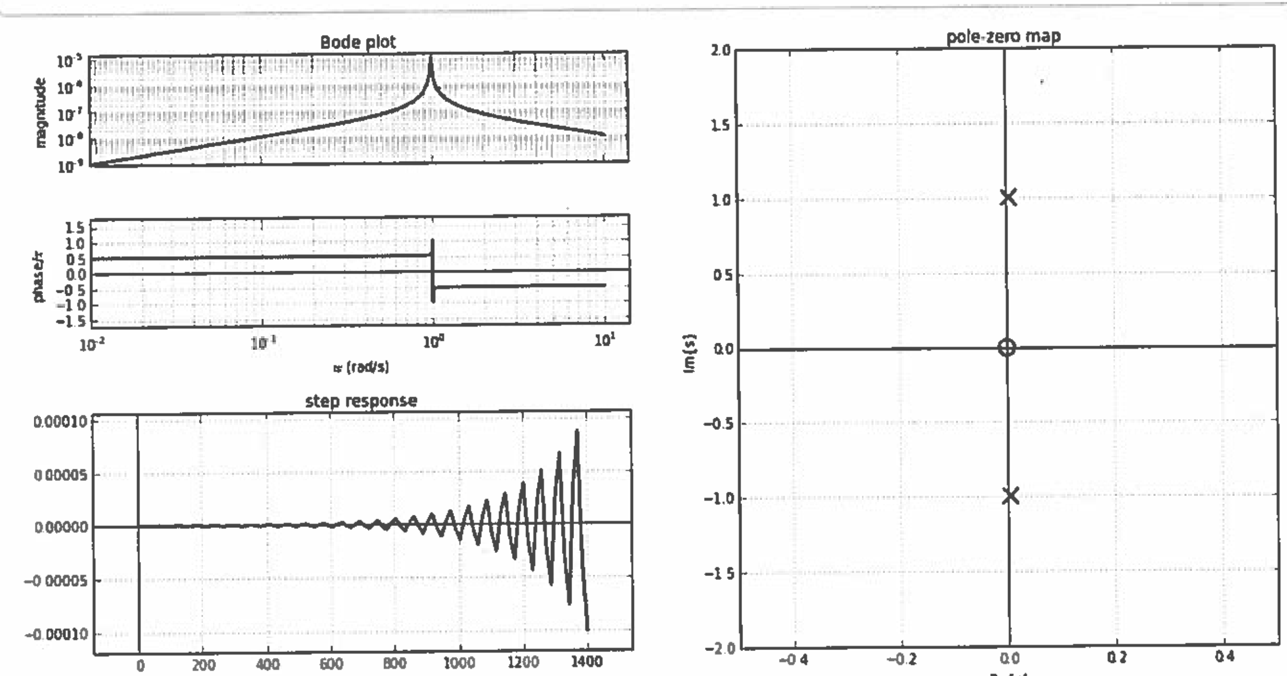
Derivative control with large k




With a large enough k, the derivative control is able to stabilize the system. The poles are to the left.

In [58]:
```
k=.0000001
sys = signal.lti([k,0], [1,-.01+k,1])
combinedplot(sys,axis=[-.5,.5,-2,2])
```

Derivative Control with small k




Without enough derivative control, the system is still unstable.