Problem Set 7

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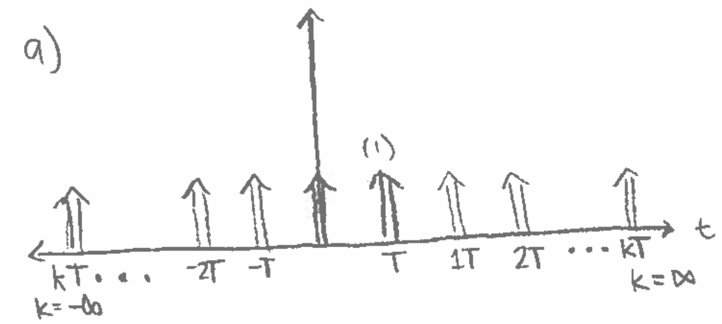
1. $p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

a)



b) Fourier series Representation of $p(t)$

Let $w_0 = \frac{2\pi}{T}$, so w_0 is the fundamental frequency of $p(t)$

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jw_0kt}$$

$$C_k = \frac{w_0}{2\pi} \int_{-T/2}^{T/2} p(t) e^{-jw_0kt} dt \qquad \left(\frac{w_0}{2\pi} \text{ because } w_0 = \frac{2\pi}{T}, \text{ so } \frac{1}{T} \text{ becomes } \frac{w_0}{2\pi}\right)$$

$$C_k = \frac{w_0}{2\pi} \int_{-T/2}^{T/2} \left(\sum_{k=-\infty}^{\infty} \delta(t-kT)\right) e^{-jw_0kt} dt$$

re arrange to use picking property $\left(\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)\right)$

$$C_k = \frac{w_0}{2\pi} \sum_{k=-\infty}^{\infty} \underbrace{\int_{-T/2}^{T/2} \delta(t-kT)\, e^{-jw_0kt} dt}_{x(t_0) = x(0)}$$

$x(t) = e^{-jw_0kt}$

$x(t_0) =$

initial impulse at $t_0 = 0$

$x(0) = e^{-jw_0k\cdot 0} = 1$

$$= \frac{w_0}{2\pi} \sum_{k=-\infty}^{\infty} x(0)$$

$$= \frac{w_0}{2\pi} \sum_{k=-\infty}^{\infty} 1$$

$$= \frac{w_0}{2\pi} = \frac{1}{T} \quad \text{because } w_0 = \frac{2\pi}{T}$$

$$\boxed{p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\frac{2\pi}{T}kt}}$$

1. cont

c) $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt}$

$$C_k = \frac{\omega_0}{2\pi}\int_{-T/2}^{T/2} x(t) e^{-j\omega_0 kt}\,dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$$

$$X(\omega) = \int_{-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt}\right) e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt - j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_k e^{jt(\frac{2\pi}{T}k - \omega)}\,dt$$

$$= \sum_{k=-\infty}^{\infty} C_k \underbrace{\int_{-\infty}^{\infty} e^{jt(\frac{2\pi}{T}k - \omega)}\,dt}_{\text{from Table 1}}$$

$$= \sum_{k=-\infty}^{\infty} C_k\, 2\pi\delta\left(\omega - \left(\frac{2\pi}{T}k - \omega\right)\right)$$

$$\boxed{X(\omega) = \sum_{k=-\infty}^{\infty} C_k\, 2\pi\delta\left(2\omega - \frac{2\pi}{T}k\right)}$$

d) $C_k = \frac{1}{T}$ → for $p(t)$

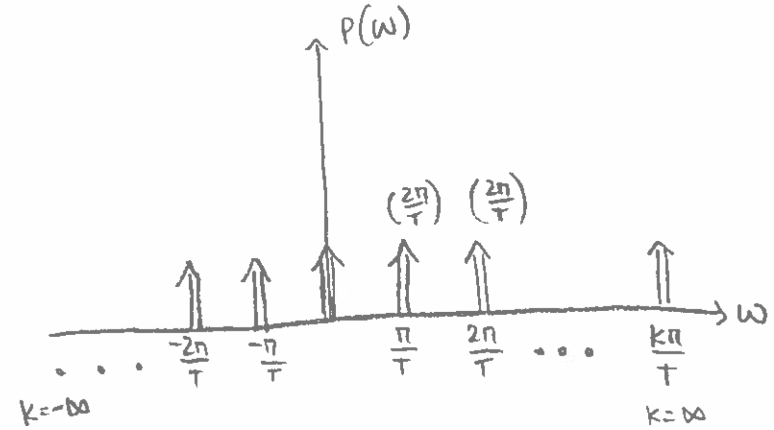
$$\boxed{P(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T}\, 2\pi\delta\left(2\omega - \frac{2\pi}{T}k\right)}$$

e)

$$2\omega - \frac{2\pi}{T}k = 0$$

$$2\omega = \frac{2\pi}{T}k$$

$$\omega = \frac{\pi}{T}k$$

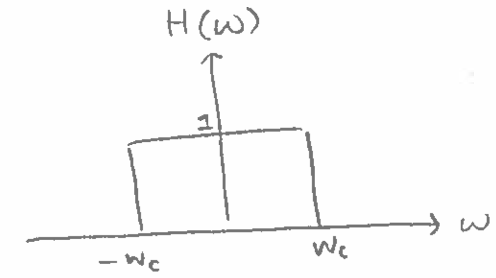


For $p(t)$, increasing T would increase space between impulses but not change the area of each impulse.

For $P(\omega)$, increasing T decreases space between impulses, and also decreases the amplitude of each impulse.

This is what I would expect

2.



a) Find $h(t)$

Inverse FT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega$$

$H(\omega) = 1$ from $-w_c$ to w_c

$$h(t) = \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{j\omega t}\,d\omega$$

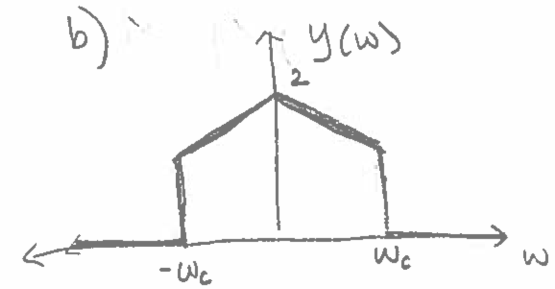
$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\left[e^{j\omega t}\right]_{-w_c}^{w_c}$$

$$= \frac{1}{2\pi j t}\left(e^{jw_c t} - e^{-jw_c t}\right)$$

$$= \frac{1}{\pi t}\left(\frac{1}{2j}e^{jw_c t} - \frac{1}{2j}e^{-jw_c t}\right)$$

$$\boxed{h(t) = \frac{\sin(w_c t)}{\pi t}}$$

b)



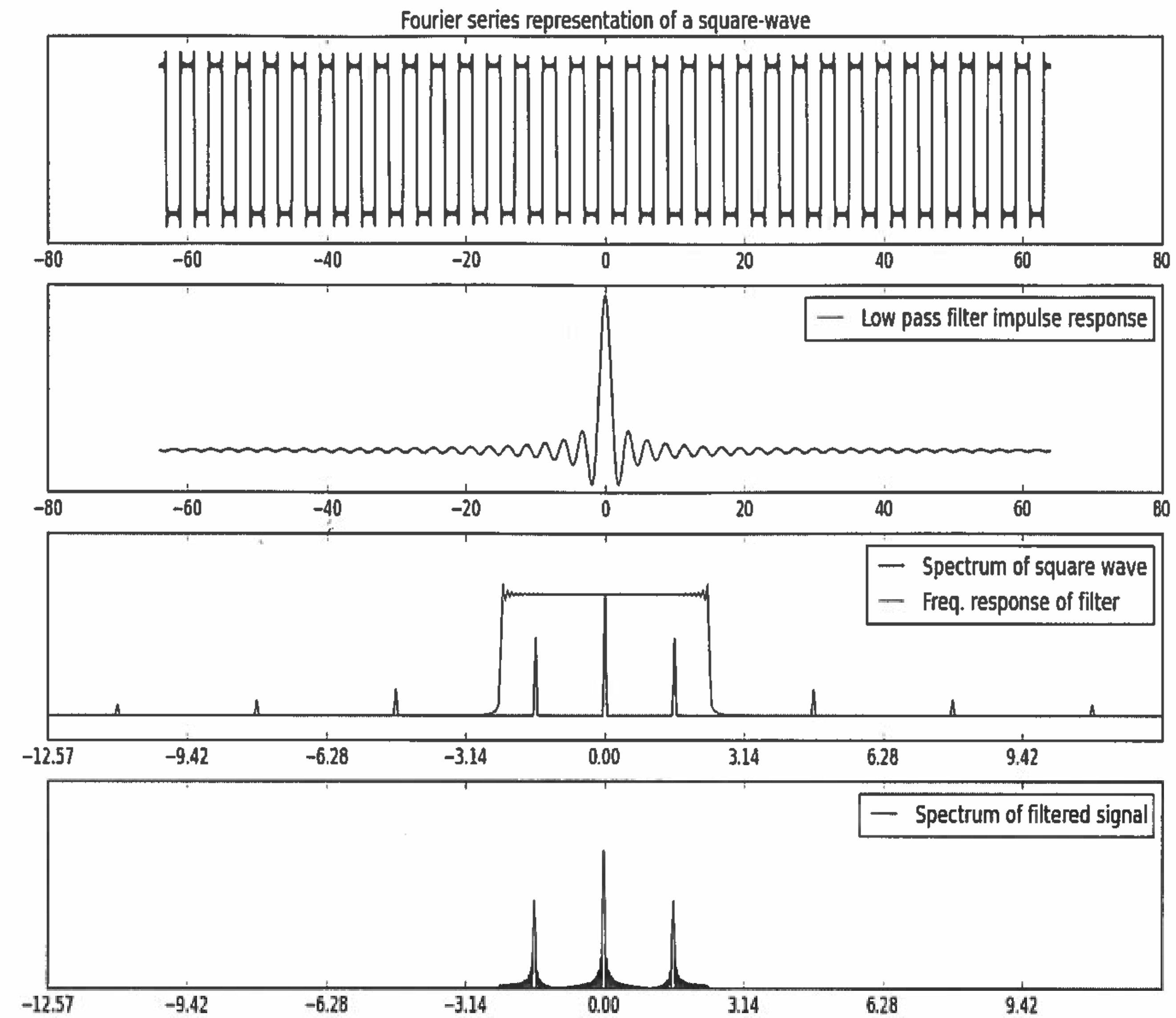
$Y(\omega) = X(\omega)\cdot H(\omega)$

because convolution in time domain is multiplying in frequency domain.

c) $H(\omega)$ acts as a low pass filter because it multiplies any frequencies below w_c by 1, which means keeping those frequencies perfectly, and any frequency above w_c by 0, which completely removes them. The low frequencies pass and the higher ones are cut off. This is what we saw in $Y(\omega)$, where any frequency above w_c (or below $-w_c$) was completely cut off.

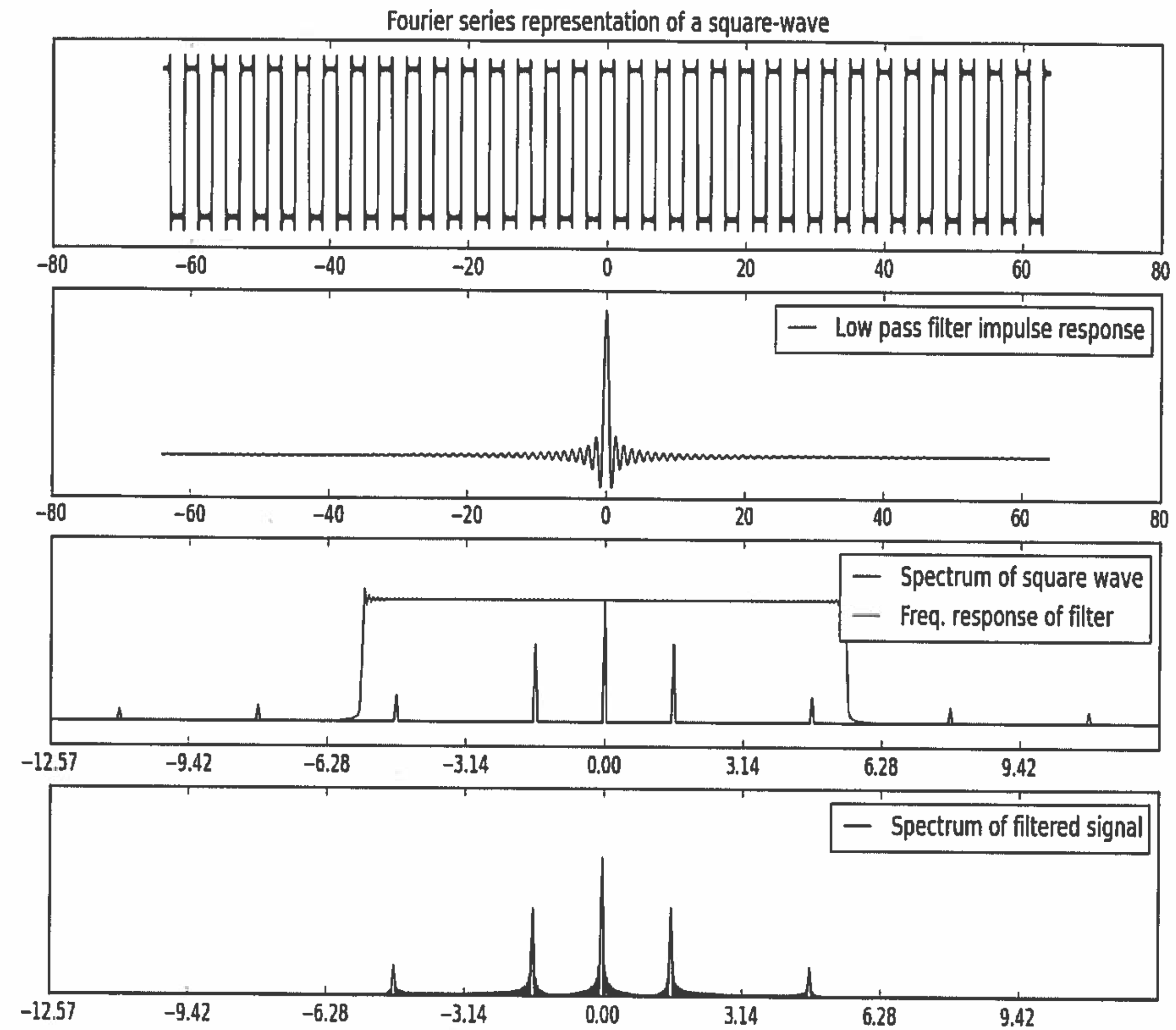
2d.

$W_c = 0.75\pi$



Fourier series representation of a square-wave
−80
−60
−40
−20
0
20
40
60
80
Low pass filter impulse response
Spectrum of square wave
Freq. response of filter
−12.57
−9.42
−6.28
−3.14
0.00
3.14
6.28
9.42
Spectrum of filtered signal

$W_c = 1.75\pi$



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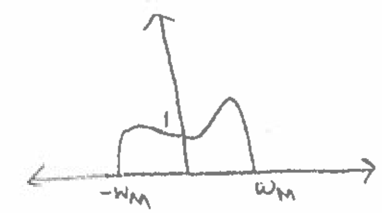
Spectrum of filtered signal

3.



$$y(t) = x(t)\overbrace{\cos(\omega_c t)}^{h(t)}$$

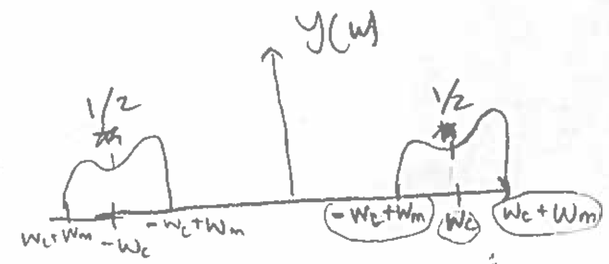
$$Y(\omega) = \frac{1}{2\pi} X * H(\omega)$$

$$= \frac{1}{2\pi}\left(X * H(\omega)\right)$$

$$= \frac{1}{2\pi}\left(X * \underbrace{\left(\pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)\right)}\right)$$

$h(t) = \cos(\omega_c t)$

$H(\omega) = \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$



I think convolving $X(\omega)$ with $H(\omega)$ shifts $Y(\omega)$ by ω_c and scales it by 1/2