Secants and Tangents

We defined the tangent line as a limit of secant lines. We also know that as Δx approaches 0 the secant's slope $\frac{\Delta f}{\Delta x}$ approaches the slope of the tangent line. How close to 0 does Δx have to be for $\frac{\Delta f}{\Delta x}$ to be close to the slope of the tangent line?

We'll use the Secant Approximation mathlet to look at a few examples. Use the dropdown menu in the lower left corner to select the function $f(x) = 0.5x^3 - x$. Use the red and yellow sliders to answer part (a) of each question, then use the Tangent checkbox to answer part (b). Be sure to uncheck Tangent before starting the next problem.

You may find it helps to work with a partner on this exercise.

1. Move the red slider to $x = -0.75$; we'll investigate the slopes of secant lines passing through the point $(-0.75, f(-0.75))$.

   (a) Use the yellow slider to find the value of $\frac{\Delta y}{\Delta x}$ when $x = -0.75$ and Δx has each of the following values:

   $$-0.5, -0.25, 0.25, 0.5.$$

   (b) Use the Tangent checkbox to find the (approximate) slope of the tangent line to the graph of $f(x)$ at $x = -0.75$.

   (c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

2. Now use the red slider to set $x = 0$.

   (a) Find $\frac{\Delta y}{\Delta x}$ when $x = 0$ and Δx has the values:

   $$-0.5, -0.25, 0.25, 0.5.$$

   (b) Find the slope of the tangent line to the graph of $f(x)$ at $x = 0$.

   (c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

3. Let $x = 0.75$.

   (a) Find $\frac{\Delta y}{\Delta x}$ when $x = 0.75$ and Δx has the values:

   $$-0.5, -0.25, 0.25, 0.5.$$

   (b) Find the slope of the tangent line to the graph of $f(x)$ at $x = 0.75$.

   (c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

4. Compare your answers to the previous problems.
   (a) Was your answer to part (c) the same for each problem?
   (b) For some values of x, $\frac{\Delta y}{\Delta x}$ was close to the slope of the tangent line when Δx was 0.5. For others it was not. Can you make any conjectures about when you need a very small value of Δx in order for $\frac{\Delta y}{\Delta x}$ to be close to the slope of the tangent line?

The Derivative of $|x|$

The slope of the graph of $f(x) = |x|$ changes abruptly when $x = 0$. Does this function have a derivative? If so, what is it? If not, why not?

Month	Balance ($)
January	175
February	220
March	255
$\vdots$	$\vdots$

Checking Account Balances

The derivative of a function $f(t)$ describes how the function's output changes as the value of t changes.

Suppose that a checking account has balance $f(t) = -5t^2 + 60t + 120$ dollars t months into the year, and that the derivative (rate of change) of the account balance is given by $f'(t) = -10t + 60$.

1. Compute $f'(1)$. If $f'(1)$ is approximately equal to the rate of change of the account balance during the month of January, will the account balance in February be greater or less than the balance in January?

2. Compute $f'(10)$. Does the account balance continue to increase throughout the year? How do you know?

3. During which month is the account balance greatest? What is the value of $f'(t)$ at this time?

Continuous but not Smooth

Find values of the constants a and b for which the following function is continuous but *not* differentiable.

$$f(x) = \begin{cases} ax + b, & x > 0; \\ \sin 2x, & x \leq 0. \end{cases}$$

In other words, the graph of the function should have a sharp corner at the pont $(0, f(0))$.

Limits and Discontinuity

For which of the following should one use a one-sided limit? In each case, evaluate the one- or two-sided limit.

1. $\lim_{x \to 0} \sqrt{x}$
2. $\lim_{x \to -1} \frac{1}{x+1}$
3. $\lim_{x \to 1} \frac{1}{(x-1)^4}$
4. $\lim_{x \to 0} |\sin x|$
5. $\lim_{x \to 0} \frac{|x|}{x}$

Derivatives of Sine and Cosine

Using the Creating the Derivative mathlet, select the (default) function $f(x) = \sin(x)$ from the pull-down menu in the lower left corner of the screen. Do not check any of the boxes.

Move the slider or use the >> button to display the graph of the sine function.

a) For approximately what values of x is the slope of $f(x) = \sin(x)$ equal to 0?

b) At approximately what values of x is the slope of $f(x) = \sin(x)$ largest?

c) For each of the values you listed in (b), is the slope positive or negative?

d) Use the information you have collected to sketch the graph of $f'(x)$, the derivative of the sine function.

e) Check the box next to the red $f'(x)$ to check your work.

The Function sinc(x)

The *unnormalized sinc function* is defined to be:

$$\operatorname{sinc}(x) = \frac{\sin x}{x}.$$

This function is used in signal processing, a field which includes sound recording and radio transmission.

Use your understanding of the graphs of $\sin(x)$ and $\frac{1}{x}$ together with what you learned in this lecture to sketch a graph of $\operatorname{sinc}(x) = \sin(x) \cdot \frac{1}{x}$.

Smoothing a Piecewise Polynomial

For each of the following, find all values of a and b for which $f(x)$ is differentiable.

a) $f(x) = \begin{cases} ax^2 + bx + 6, & x \leq 0; \\ 2x^5 + 3x^4 + 4x^2 + 5x + 6, & x > 0. \end{cases}$

b) $f(x) = \begin{cases} ax^2 + bx + 6, & x \leq 1; \\ 2x^5 + 3x^4 + 4x^2 + 5x + 6, & x > 1. \end{cases}$

Quotient Rule Practice

Find the derivatives of the following rational functions.

a) $\dfrac{x^2}{x+1}$

b) $\dfrac{x^4+1}{x^2}$

c) $\dfrac{\sin(x)}{x}$

Do We Need the Quotient Rule?

The quotient rule can be difficult to memorize, and some students are more comfortable with negative exponents than they are with fractions. In this exercise we learn how we can use the chain and product rules together in place of the quotient rule.

a) Use the quotient rule to find the derivative of $\dfrac{x^3}{x+1}$.

b) Use the product and chain rules to find the derivative of $x^3 \cdot (x+1)^{-1}$. Note that $x^3 \cdot (x+1)^{-1} = \dfrac{x^3}{x+1}$.

c) Use the chain and product rules (and not the quotient rule) to show that the derivative of $u(x)(v(x))^{-1}$ equals $\dfrac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$.

Repeated Differentiation of Sine and Cosine

a) For which of the functions whose graphs are displayed by the Creating the Derivative mathlet is it true that $f''(x) = -f(x)$?

b) Can you think of any other functions for which $f''(x) = -f(x)$?

Implicit Differentiation and the Second Derivative

Calculate y'' using implicit differentiation; simplify as much as possible.

$$x^2 + 4y^2 = 1$$

Implicit Differentiation and the Chain Rule

The chain rule tells us that:

$$\frac{d}{dx}(f \circ g) = \frac{df}{dg}\frac{dg}{dx}.$$

While implicitly differentiating an expression like $x + y^2$ we use the chain rule as follows:

$$\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy}\frac{dy}{dx} = 2yy'.$$

Why can we treat y as a function of x in this way?

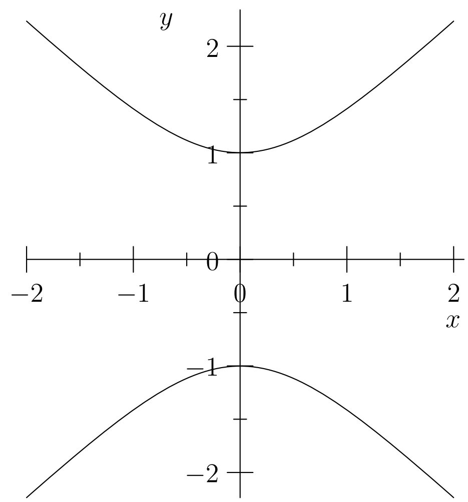


Figure 1: The hyperbola $y^2 - x^2 = 1$.

Consider the equation $y^2 - x^2 = 1$, which describes the hyperbola shown in Figure 1. We cannot write y as a function of x, but if we start with a point (x, y) on the graph and then change its x coordinate by sliding the point along the graph its y coordinate will be constrained to change as well. The change in y is *implied* by the change in x and the constraint $y^2 - x^2 = 1$. Thus, it makes sense to think about $y' = \frac{dy}{dx}$, the rate of change of y with respect to x.

Given that $y^2 - x^2 = 1$:

a) Use implicit differentiation to find y'.

b) Check your work by using Figure 1 to estimate the slope of the tangent line to the hyperbola when $y = -1$ and when $x = 1$.

c) Check your work for $y > 0$ by solving for y and using the direct method to take the derivative.

Derivative of the Square Root Function

a) Use implicit differentiation to find the derivative of the inverse of $f(x) = x^2$ for $x > 0$.

b) Check your work by finding the inverse explicitly and then taking its derivative.

Compound Interest

If you invest P dollars at the annual interest rate r, then after one year the interest is $I = rP$ dollars, and the total amount is $A = P + I = P(1 + r)$. This is *simple interest*.

For *compound interest*, the year is divided into k equal time periods and the interest is calculated and added to the account at the end of each period. So at the end of the first period, $A = P(1 + r(\frac{1}{k}))$; this is the new amount for the second period, at the end of which $A = P(1 + r(\frac{1}{k}))(1 + r(\frac{1}{k}))$, and continuing this way, at the end of the year the amount is

$$A = P\left(1 + \frac{r}{k}\right)^k .$$

The compound interest rate r thus earns the same in a year as the simple interest rate of

$$\left(1 + \frac{r}{k}\right)^k - 1;$$

this equivalent simple interest rate is in bank jargon the "annual percentage rate" or APR.[1]

1. Compute the APR of 5% compounded monthly and daily.[2]
2. As in part (a), compute the APR of 10% compounded monthly, biweekly ($k = 26$), and daily. (We have thrown in the biweekly rate because loans can be paid off biweekly.)

[1] Banks are required to reveal this so-called APR when they offer loans. The APR also takes into account certain bank fees known as points. Unfortunately, not all fees are included in it, and the true costs are higher if the loan is paid off early.

[2] For daily compounding assume that the year has 365 days, not 365.25. Banks are quite careful about these subtle differences. If you look at official tables of rates from pre-calculator days you will find that they are off by small amounts because U.S. regulations permitted banks to pretend that a year has 360 days.

Solving Equations with e and $\ln x$

We know that the natural log function $\ln(x)$ is defined so that if $\ln(a) = b$ then $e^b = a$. The *common log* function $\log(x)$ has the property that if $\log(c) = d$ then $10^d = c$. It's possible to define a logarithmic function $\log_b(x)$ for any positive base b so that $\log_b(e) = f$ implies $b^f = e$. In practice, we rarely see bases other than 2, 10 and e.

Solve for y:

1. $\ln(y+1) + \ln(y-1) = 2x + \ln x$
2. $\log(y+1) = x^2 + \log(y-1)$
3. $2\ln y = \ln(y+1) + x$

Solve for x (hint: put $u = e^x$, solve first for u):

4. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$
5. $y = e^x + e^{-x}$

Evaluating an Interesting Limit

Using $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, calculate:

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$
2. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n}$
3. $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{5n}$

Evaluating an Interest Using the Limit

Recall that the formula for *compound interest* is:

$$A = P\left(1 + \frac{r}{k}\right)^k$$

and the anual percentage rate is:

$$\mathrm{APR} = \left(1 + \frac{r}{k}\right)^k - 1.$$

Here P is the principal invested, r is the annual "simple" interest rate, A is the amount in the account at a given time, and k determines the frequency with which interest is added to the account.

As k approaches infinity interest is added more and more often; in the limit we say that the interest is *compounded continuously.*

1. Use the fact that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$ to compute the APR of 5% compounded continuously.
2. Compute the APR of 10% compounded continuously.

Hyperbolic Angle Sum Formula

Find $\sinh(x+y)$ and $\cosh(x+y)$ in terms of $\sinh x$, $\cosh x$, $\sinh y$ and $\cosh y$.

Comparing Linear Approximations to Calculator Computations

In lecture, we explored linear approximations to common functions at the point $x = 0$. In this worked example, we use the approximations to calculate values of the sine function near $x = 0$ and compare the answers to those on a scientific calculator.

Find the linear approximation to $\sin(x)$ at the point $x = 0$ and use your answer to approximate the values of $\sin(.01), \sin(.1)$ and $\sin(1)$. Check your answer on a calculator.

Product of Linear Approximations

Suppose we have two complicated functions and we need an estimate of the value of their product. We could multiply the functions out and then approximate the result, or we could approximate each function separately and then find the product of the two (simple) approximations. Does it matter which we do? Not very much.

Prove that the linear approximation of $f(x) \cdot g(x)$ equals the (linear part of the) product of the linear approximations of $f(x)$ and $g(x)$.

Comparing Quadratic Approximations to Calculator Computations

In a previous worked example, we explored linear approximations to the sine function at the point $x = 0$. In this example, we use the quadratic approximation for e^x to calculate values of the exponential function near $x = 0$ and again compare the results to decimal approximations on a scientific calculator.

Find the linear approximation to e^x at the point $x = 0$ and use your answer to approximate the values of $e^{.01}, e^{.1}$ and e. Check your answer on a calculator.

Finding a Formula for the Best Degree n Approximation

Linear approximation uses the values of a function and its first derivative at x_0 — or equivalently the slope of the tangent line and the point of tangency — to find the equation of a degree 1 polynomial approximating the function. The formula for the linear approximation of a function $f(x)$ near the value $x = x_0$ is:

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0).$$

The formula for quadratic approximation:

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

describes the best degree 2 approximation of the function $f(x)$ for $x \approx x_0$. We could also describe a degree 0 approximation: $f(x) \approx f(x_0)$ when x is sufficiently close to x_0. As the degree of the approximation increases we add new terms, but the lower degree terms stay the same.

Can we define higher degree approximations, and if so, what terms should we add to do so? We'll focus on finding approximations near the value $x_0 = 0$; the calculations for the general case are very similar.

a) Find the best third degree approximation of a function $f(x)$ near $x = 0$:

$$A(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

In other words, find values for the a_i so that the first, second, and third derivatives of $f(x)$ are the same as the first, second and third derivatives of $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ when $x = 0$. (You may wish to refer to your notes on the derivation of the formula for the quadratic approximation.)

b) Use the fact that the n^{th} derivative of x^m is 0 for $m < n$ to find the n^{th} derivative of:

$$A(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

c) Use your answer to (b) to give a formula for an n^{th} degree polynomial approximation of a function $f(x)$ near $x = 0$.

Graph Features

The Graph Features mathlet allows you to choose the coefficients of a degree three polynomial and then illustrates where the graph of that polynomial is rising (increasing), falling (decreasing), concave and convex.

Find coefficient values a, b, c and d for a polynomial function:

$$f(x) = ax^3 + bx^2 + cx + d$$

whose graph is:

- convex (smile shaped) for $x < 2$
- concave (frown shaped) for $x > 2$
- falling when $x < 1$
- rising when $1 < x < 3$
- falling when $x > 3$.

Can you find two different polynomials that satisfy these requirements? Why or why not?

Bonus: Make up a problem similar to this one for a friend to solve.

Can Design

There are many factors to consider in food packaging, including marketing, durability, cost and materials. In this problem we minimize the cost of materials for a can.

Find the height and radius that minimizes the surface area of a can whose volume is 1 liter = 1000 cm^3.

Solving an Optimization Problem using Implicit Differentiation

Suppose you wish to build a grain silo with volume V made up of a steel cylinder and a hemispherical roof. The steel sheets covering the surface of the silo are quite expensive, so you wish to minimize the surface area of your silo. What height and radius should the silo have for a given volume V?

Although it is possible to solve this problem by the same method used in the can design question, it turns out to be much simpler to use implicit differentiation to find $\frac{d}{dr}SA$.

Answer this question for:

a) a silo with a circular floor (to keep out gophers) and

b) a silo with no built-in flooring (for use in regions with no gophers).

Cube Root of x

Show that for any non-zero starting point x_0, Newton's method will never find the exact value x for which $x^3 = 0$.

/

Generalizing the Mean Value Theorem – Taylor's theorem

We explore generalizations of the Mean Value Theorem, which lead to error estimates for Taylor polynomials. Then we test this generalization on polynomial functions.

Recall that the mean value theorem says that, given a continuous function f on a closed interval $[a, b]$, which is differentiable on (a, b), then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rearranging terms, we can make this look very much like the linear approximation for $f(b)$ using the tangent line at $x = a$:

$$f(b) = f(a) + f'(c)(b - a)$$

except that the term $f'(a)$ has been *replaced* by $f'(c)$ for some point c in order to achieve an *exact* equality. Remember that the Mean Value Theorem only gives the existence of such a point c, and not a method for how to find c.

We understand this equation as saying that the difference between $f(b)$ and $f(a)$ is given by an expression resembling the next term in the Taylor polynomial. Here $f(a)$ is a "0-th degree" Taylor polynomial.

Repeating this for the first degree approximation, we might expect:

$$f(b) = \left[f(a) + f'(a)(b - a)\right] + \frac{f''(c)}{2}(b - a)^2$$

for some c in (a, b). The term in square brackets is precisely the linear approximation.

Question: Guess the formula for the difference between $f(b)$ and its n-th order Taylor polynomial at $x = a$. Test your answer using the cubic polynomial $f(x) = x^3 + 2x + 1$ using a quadratic approximation for $f(3)$ at $x = 1$.

$|\sin(b) - \sin(a)|$ vs. $|b - a|$

Use what we've learned about the mean value theorem to compare the values of $|\sin(b) - \sin(a)|$ and $|b - a|$.

Using Differentials to Study Population Dynamics

We have seen that differentials give a convenient way for expressing linear approximations. In this example, we explore population dynamics in the language of differentials.

A simple generational model of population dynamics says that an initial population x will yield a next generation with population given by a function $P(x)$. The next generation after that is given by "iterating" the function P, that is, $P(P(x))$. We can keep applying P to the result to find the population of successive generations. Note in particular that population will be stable over generations at any x such that $P(x) = x$. Such an x is known as a "fixed point."

We say that a fixed point x_0 is "attracting" if, given an initial population value $x_0 + \Delta x$ with Δx sufficiently small, the successive generations have size closer and closer to x_0. More formally, the sequence of values

$$x_0 + \Delta x, P(x_0 + \Delta x), P(P(x_0 + \Delta x)), P(P(P(x_0 + \Delta x))), \ldots$$

gets closer and closer to x_0.

Question:

- Show that if x_0 is a fixed point of $P(x)$ and $|P'(x_0)| < 1$, then x_0 is attracting.
- Given fixed positive constants a, b with $ab > 1$, find the fixed points of $P(x) = ax(b - x)$ and determine if they are attracting.

Exploiting Derivative Rules

Every differentiation rule $F'(x) = f(x)$ corresponds to a rule for finding the anti-derivative $F(x)$ of some function f.

a) Find an anti-derivative rule that is the inverse of the sum rule $(f + g)'(x) = f'(x) + g'(x)$.

b) Find an anti-derivative rule that is the inverse of the product rule $(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x)$.

Antiderivative of $\tan x \sec^2 x$

Compute $\displaystyle\int \tan x \sec^2 x \, dx$ in two different ways:

a) By substituting $u = \tan x$.

b) By substituting $v = \sec x$.

c) Compare the two results.

How to Check Your Answer

While it may be difficult to solve a differential equation, it is fairly easy to see if a proposed solution is correct. Check the following results by plugging the proposed answer into the original equation.

a) $y = \frac{1}{3}e^x$ is a solution to $4y'' - y = e^x$.

b) $y = \frac{1}{x}$ is a solution to $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$.

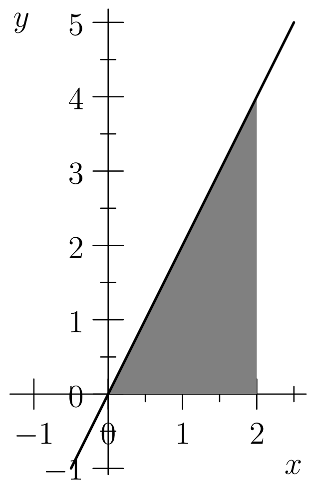
Exponential Growth and Inhibited Growth

a) The differential equation $\dfrac{dy}{dx} = ry$ describes a situation in which a population size y increases at a rate proportional to its size. Use separation of variables to find a solution to this equation.

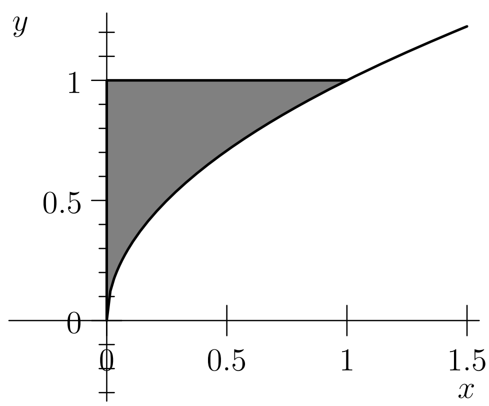
b) The differential equation $\dfrac{dy}{dx} = ry(s-y)$ $(s > 0)$ describes change in a population which tends toward a fixed size s. For example, this might describe a population in which food or space is limited. Use separation of variables and the fact that $\displaystyle\int \frac{dy}{y(s-y)} = \frac{1}{s}\ln\left|\frac{y}{s-y}\right| + c$ to find a solution to this equation.

Integration Intuition

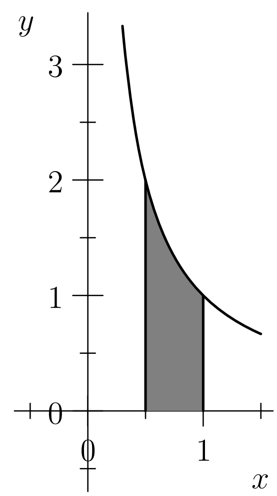
When calculating areas, it's a good idea to check your answer against a rough visual estimate of the region's area. For each graph shown below, select the value that's closest to the shaded area.



Approximate area: a) 2 b) 4 c) 8 d) 16



Approximate area: a) $\frac{1}{4}$ b) $\sqrt{2}$ c) $\frac{1}{2}$ d) $\frac{2}{3}$



Approximate area: a) $\frac{1}{2}$ b) $\frac{3}{4}$ c) $\frac{3}{2}$ d) 2

Summation

Compute the following sums:

a) $\displaystyle\sum_{k=1}^{5} k^2$

b) $\displaystyle\sum_{k=1}^{3} (2k)^2$

c) $\displaystyle\sum_{n=1}^{4} (-1)^n n$

d) $\displaystyle\sum_{k=0}^{5} 2^k$

Integral of $|x|$

Use the geometric definition of the definite integral to compute:

$$\int_{-1}^{2} |x| \, dx.$$

Riemann Sum Practice

Use a Riemann sum with $n = 6$ subdivisions to estimate the value of $\int_0^2 (3x + 2)\, dx$.

Practice with Definite Integrals

Use antidifferentiation to compute the following definite integrals. Check your work using the geometric definition of the definite integral, graphing and estimation.

a) $\displaystyle\int_0^2 x^2\,dx$

b) $\displaystyle\int_1^e \frac{1}{x}\,dx$

c) $\displaystyle\int_{-\pi/4}^0 \sin x\,dx$

Integral of $\sin(x) + \cos(x)$

Consider the following integral:

$$\int_0^\pi \sin(x) + \cos(x)\, dx.$$

a) Use what you have learned about definite integrals to guess the value of this integral.

b) Find antiderivatives of $\cos(x)$ and $\sin(x)$. Check your work.

c) Use the addition property of integrals to compute the value of:

$$\int_0^\pi \sin(x) + \cos(x)\, dx.$$

Check your work by comparing to your answer from part a.

Integration by Change of Variables

Use a change of variables to compute the following integrals. Change both the variable and the limits of substitution.

a) $\displaystyle\int_0^4 \sqrt{3x+4}\,dx$

b) $\displaystyle\int_1^3 \frac{x}{x^2+1}\,dx$

c) $\displaystyle\int_0^{\pi/2} \sin^5 x \cos x\,dx$

Estimating $\ln(5)$

a) Use the mean value theorem and the fundamental theorem of calculus to find upper and lower bounds on $\displaystyle\int_1^5 \frac{1}{x}\,dx$.

b) Compute $\displaystyle\int_1^5 \frac{1}{x}\,dx$.

c) Does your answer to (a) provide a good estimate of the value of $\ln(5)$?

Logs and Exponents

a) Prove that for $x > 1$:

$$a \int_{1/x}^{1} \frac{1}{t}\, dt = \int_{(1/x)^a}^{1} \frac{1}{t}\, dt.$$

b) Assume $x > 1$. What is the geometric interpretation of the result of part a?

c) What does this tell you about the area between the x-axis and the graph of $\frac{1}{x}$ over the interval from 0 to 1?

Probability Function

A Poisson process is a situation in which a phenomenon occurs at a constant average rate. Each occurrence is independent of all other occurrences; in a Poisson process, an event does not become more likely to occur just because it's been a long time since its last occurrence. The location of potholes on a highway or the emission over time of particles from a radioactive substance may be Poisson processes.

The probability density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

describes the relative likelihood of an occurrence at time or position x, where λ describes the average rate of occurrence.

The probability $P(a < x < b)$ of an event occurring in the interval between a and b is given by:

$$\int_a^b f(x)\,dx.$$

Compute this integral:

a) for the case in which a and b are both positive (assume $a < b$),

b) for the case in which $a \leq 0$ and $b > 0$,

c) for the case in which $a \leq 0$ and $b \leq 0$.

Area of a Smile

The region between the curves $y = \frac{1}{2}x^2 - \frac{1}{2}$ and $y = x^4 - 1$ is smile shaped. Find the area of that region.

Revolution About the x-axis

Find the volume of the solid of revolution generated by rotating the regions bounded by the curves given around the x-axis.

a) $y = 3x - x^2, y = 0$

b) $y = \sqrt{ax}, y = 0, x = a$

Volume of a Spheroid

The solid of revolution generated by rotating (either half of) the region bounded by the curves $x^2+4y^2 = 4$ and $x = 0$ about the y-axis is an example of an oblate spheroid. Compute its volume.

Volume of Revolution Via Washers

Problem: By integrating with respect to the variable y, find the volume of the solid of revolution formed by rotating the region bounded by $y = 0$, $x = 4$ and $y = \sqrt{x}$ about the line $x = 6$.

Average Bank Balance

An amount of money A_0 compounded continuously at interest rate r increases according to the law:

$$A(t) = A_0 e^{rt} \qquad (t\text{=time in years.})$$

a) What is the average amount of money in the bank over the course of T years?

b) Check your work by plugging in $A_0 = \$100$, $r = .05$ and $T = 1$; does the result seem plausible?

Weighted Average

The centroid or center of mass of a planar region is the point at which that region balances perfectly, like a plate on the end of a stick. The coordinates of the centroid are given by weighted averages.

The x coordinate of the centroid is $\bar{x} = \dfrac{\int x\,dA}{\int dA}$, where dA is an infintessimal portion of area; the weighting function in this average is just x.

Similarly, the y coordinate of the centroid is $\bar{y} = \dfrac{\int y\,dA}{\int dA}$.

Find the centroid $(\bar{x}, \bar{y})$ of the parabolic region bounded by $x = -1$, $x = 3$, $y = (x-1)^2$ and $y = 4$.

Numerical Integration

Compare the trapezoidal rule to the left Riemann sum. The area of each trapezoid is calculated using twice as much information (left *and* right endpoints) as the area of each rectangle. This leads one to expect that applying the trapezoidal rule with $n = 6$ should produce a result comparable to the one obtained from a Riemann sum with $n = 12$.

a) Open the Riemann Sums mathlet. Set the function to $x^3 - 2x$ and select the trapezoidal rule. Make sure $n = 6$. Record the mathlet's estimate of the integral.

b) Select "Evaluation point" to change to the Riemann sum approximation. Move the slider to 0.5, setting the evaluation point to be the midpoint of the interval. Set n equal to 12 and record the mathlet's estimate of the integral.

c) Calculate $\displaystyle\int_{-1}^{2} x^3 - 2x\,dx$ by hand. Was the accuracy of the Riemann sum with $n = 12$ comparable to that of the trapezoidal rule with $n = 6$? Why or why not?

Using Simpson's Rule for the normal distribution

This problem uses Simpson's rule to approximate a definite integral important in probability.

In our probability unit, we learned that when given a probability density function $f(x)$, we may compute the probability P that an event x is between a and b by calculating the definite integral:

$$P(a \leq x \leq b) = \int_a^b f(x)\, dx.$$

Here we're assuming that a probability density function $f(x)$ has the property that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

In the next session, we will show that $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ is a probability density function with this property. For now, we assume this property.

Question: Suppose the probability density function for American male height is roughly (in inches x)

$$h(x) = \frac{1}{2.8\sqrt{2\pi}} e^{-(x-69)^2/5.6}.$$

- Use Simpson's rule to estimate the probability that an American male is between 5 and 6 feet tall.
- Use Simpson's rule to estimate the probability that an American male is over 8 feet tall.

$$\int \cos^3(2x)\,dx$$

Compute $\int \cos^3(2x)\,dx$.

$$\int \sin^4(x)\cos^2(x)\,dx$$

Compute $\int \sin^4(x)\cos^2(x)\,dx$.

An Alternate Solution

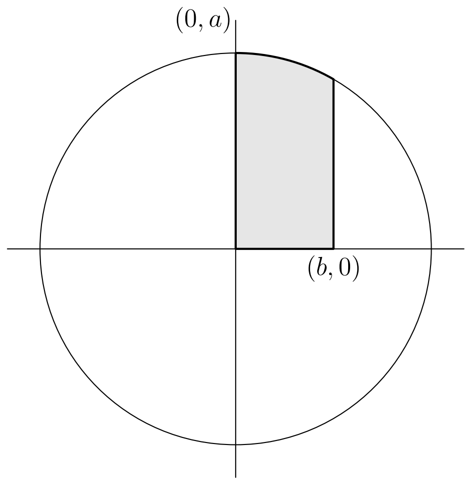


Figure 1: The area of the shaded region is $\int_0^b \sqrt{a^2 - x^2}\,dx$.

As Professor Miller explained in lecture, the area of the region shown in Figure 1 is $\int_0^b \sqrt{a^2 - x^2}\,dx$. Use the substitution $x = a\cos\theta$ to solve this integral. Hint: pay particular attention to your limits of integration.

Integral of $\tan^3(x)$

Use what you have learned to integrate the function $\tan^3(x)$.

Substitution Practice

Use a trigonometric substitution to integrate the function $f(x) = x\sqrt{x^2 - 9}$. Check your work by integration using the substitution $u = x^2$.

Complete the Square

Compute the integral $\int \frac{dx}{\sqrt{2x - x^2}}$ by completing the square.

Integrate by Partial Fractions

Use the method of partial fractions to compute the integral:

$$\frac{x^2 + 2x + 3}{(x+1)(x+2)(x+3)}\, dx.$$

$$\int \frac{x-11}{(x^2+9)(x+2)}\,dx$$

Compute the integral:

$$\int \frac{x-11}{(x^2+9)(x+2)}\,dx.$$

Integral of $\dfrac{x^3}{x^2-1}$

Express the integrand as a sum of a polynomial and a proper rational function, then integrate:

$$\int \frac{x^3}{x^2-1}\,dx.$$

Integral of $x^4 \cos x$

This problem provides a lot of practice with integration by parts.

Compute the integral of $x^4 \cos x$.

Arc Length of $y = \ln(x)$

Express the arc length of the graph of $y = \ln x$ between $x = 1/10$ and $x = 1$ as an integral. (Do not evaluate.)

Surface Area of a Wine Glass

Professor Jerison found the volume of a "wine glass" shape formed by revolving the graph of $y = e^x$ $(0 \leq x \leq 1)$ about the y-axis. Set up but do not evaluate an integral to compute the surface area of that shape.

Exploring a Parametric Curve

a) Describe the curve traced out by the parametrization:

$$\begin{aligned} x &= t\cos t \\ y &= t\sin t, \end{aligned}$$

where $0 \le t \le 4\pi$.

b) Set up and simplify, but do not integrate, an expression for the arc length $\displaystyle\int_0^{4\pi} \frac{ds}{dt}\,dt$ of this curve.

Path of a Falling Object

A teenager throws a ball off a rooftop. Assume that the x coordinate of the ball is given by $x(t) = t$ meters and its y coordinate satisfies the following properties:

$$\begin{aligned} y''(t) &= -9.8 \text{ meters/second} \\ y'(0) &= 0 \\ y(0) &= 5 \text{ meters.} \end{aligned}$$

a) Find an equation directly describing y in terms of t.

b) Find a parametrization $(x(t), y(t))$ which describes the path of the ball.

c) Find the speed $\frac{ds}{dt}$ of the ball (this answer will only be valid for times before the ball hits the ground.)

Lemniscate

The curve described in polar coordinates by $r^2 = \cos(2\theta)$ is called a *lemniscate*.

a) For what values of θ does there exist such a point (r, θ)?

b) For what values of θ is r at its minimum length?

c) For what values of θ is r at its maximum length?

d) Use the information you have gathered to sketch a rough graph of this curve.

$$\lim_{x \to 0} \frac{\sin x}{1 - \cos x}$$

In this problem attempt to evaluate:

$$\lim_{x \to 0} \frac{\sin x}{1 - \cos x}$$

using approximation.

a) Substitute linear approximations for $\sin x$ and $\cos x$ into this expression. Can you tell what happens in the limit?

b) Substitute quadratic approximations for $\sin x$ and $\cos x$ into this expression. Can you tell what happens in the limit?

Limits at Infinity of $\dfrac{e^x}{x}$ and $\dfrac{x}{e^x}$

As x approaches infinity, the rational expressions $\dfrac{e^x}{x}$ and $\dfrac{x}{e^x}$ take on the form $\frac{\infty}{\infty}$. Use the extended version of l'Hopital's rule to evaluate the following limits, if they exist.

a) $\lim\limits_{x \to +\infty} \dfrac{e^x}{x}$

b) $\lim\limits_{x \to +\infty} \dfrac{x}{e^x}$

$$\lim_{x\to\infty}(x^{1/x})$$

Use an extension of l'Hôpital's rule to compute $\lim_{x\to\infty}(x^{1/x})$.

Integrating $\dfrac{1}{(5x+2)^2}$ from 1 to infinity

Compute: $\displaystyle\int_1^\infty \frac{dx}{(5x+2)^2}$

Confirming an Integral Converges

Use limit comparison to show that $\displaystyle\int_1^\infty \frac{dx}{(5x+2)^2}$ is finite.

Summing the Geometric Series

In lecture we saw a geometric argument that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. By answering the questions below, we complete an algebraic proof that this is true.

We start by proving by induction that:

$$S_N = \sum_{n=0}^{N} \frac{1}{2^n} = \frac{2^{N+1} - 1}{2^N}.$$

Finally we show that $\lim_{N\to\infty} S_N = 2$.

a) (Base case) Prove that $S_0 = \frac{2^1-1}{2^0} = 1$.

b) (Inductive hypothesis and inductive step) Assume that:

$$S_{N-1} = \frac{2^{(N-1)+1} - 1}{2^{N-1}} = \frac{2^N - 1}{2^{N-1}}.$$

Add $\frac{1}{2^N}$ to both sides to prove that:

$$S_N = \frac{2^{N+1} - 1}{2^N}.$$

This completes the inductive proof.

c) Show that if $S_N = \dfrac{2^{N+1} - 1}{2^N}$, then $\lim_{N\to\infty} S_N = 2$.

Using the Ratio Test

The ratio test for convergence is another way to tell whether a sum of the form $\sum_{n=n_0}^{\infty} a_n$, with $a_n > 0$ for all n, converges or diverges. To perform the ratio test we find the ratio $\frac{a_{n+1}}{a_n}$ and let:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

The test has three possible outcomes:

$L < 1$	$\Rightarrow$	The series converges.
$L > 1$	$\Rightarrow$	The series diverges.
$L = 1$		No conclusion; the series may converge or diverge.

Apply the ratio test to each of the following series. Note that not all series satisfy the conditions needed to apply this form of the ratio test.

a) $\sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n}}$

b) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

c) $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

Finding the Radius of Convergence

Use the ratio test to find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

When does a function equal its Taylor series?

We have computed the Taylor series for a differentiable function, and earlier in the course, we explored how to use their partial sums, i.e. Taylor polynomials, to approximate the function. So we know that the Taylor series can be quite useful. But we haven't addressed the question of when a function is equal to its Taylor series. We do this now.

Recall that when we write down an infinite series with upper bound "∞," we mean the following:

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$$

In the context of Taylor series for a function f, this means that the Taylor series $T_f(x)$ is expressible as a limit of Taylor polynomials $P_N(x)$ as $N \to \infty$. That is, the notation

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \text{means} \quad T_f(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n = \lim_{N\to\infty} P_N(x),$$

where $P_N(x)$ is the N-th degree Taylor polynomial to $f(x)$ at $x = c$.

Hence $f(x)$ will equal $T(x)$ provided

$$f(x) - \lim_{N\to\infty} P_N(x) = \lim_{N\to\infty} (f(x) - P_N(x)) = 0.$$

For each N, the function $(f(x) - P_N(x))$ is often called the "remainder" (i.e., what remains after taking the N-th Taylor polynomial from f) and denoted $R_N(x)$. Thus we must find all x for which

$$\lim_{N\to\infty} R_N(x) = 0.$$

A very useful bound on $R_N(x)$ is given by Taylor's inequality:

Taylor's Inequality. *If $|f^{(N+1)}(x)| \leq B$ for all x in the interval $[c-d, c+d]$, then the remainder $R_N(x)$ (for the Taylor polynomial to $f(x)$ at $x = c$) satisfies the inequality*

$$|R_N(x)| \leq \frac{B}{(N+1)!}|x-c|^{N+1} \quad \textit{for all } x \textit{ in } [c-d, c+d].$$

If the right-hand side of Taylor's inequality goes to 0 as $N \to \infty$, then the remainder must go to 0 as well, and hence for those x values, the function matches its Taylor series.

Question: Use Taylor's inequality to show that e^x converges to its Taylor series at 0 for all real x.

Operations on Power Series Related to Taylor Series

In this problem, we perform elementary operations on Taylor series – term by term differentiation and integration – to obtain new examples of power series for which we know their sum. Suppose that a function f has a power series representation of the form:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x-c)^n$$

convergent on the interval $(c-R, c+R)$ for some R. The results we use in this example are:

- (Differentiation) Given f as above, $f'(x)$ has a power series expansion obtained by by differentiating each term in the expansion of $f(x)$:

$$f'(x) = a_1 + a_2(x-c) + 2a_3(x-c) + \cdots = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

- (Integration) Given f as above, $\int f(x)\,dx$ has a power series expansion obtained by by integrating each term in the expansion of $f(x)$:

$$\int f(x)\,dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

  for some constant C depending on the choice of antiderivative of f.

Questions:

1. Find a power series representation for the function $f(x) = \arctan(5x)$. (Note: $\arctan x$ is the inverse function to $\tan x$.)

2. Use power series to approximate

$$\int_0^1 \sin(x^2)\,dx$$

   (Note: $\sin(x^2)$ is a function whose antiderivative is not an elementary function.)

Hyperbolic Sine

In this problem we study the hyperbolic sine function:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

reviewing techniques from several parts of the course.

a) Sketch the graph of $y = \sinh x$ by finding its critical points, points of inflection, symmetries, and limits as $x \to \infty$ and $-\infty$.

b) Give a suitable definition for $\sinh^{-1} x$ (the inverse hyperbolic sine) and sketch its graph, indicating the domain of definition.

c) Find $\dfrac{d}{dx} \sinh^{-1} x$.

d) Use your work to evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}}$.