Evaluating an Interesting Limit

Using $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, calculate:

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$
2. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n}$
3. $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{5n}$

Solution

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$

   The key to all of these problems is forcing them into the form $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. In this problem we do this by using rules of exponents to remove the 3 from the exponent.

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\cdot 3}\\ &= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^3\\ &= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^3\\ &= e^3\end{aligned}$$

   How do we know that $\lim_{n\to\infty}(f(n)^3) = (\lim_{n\to\infty} f(n))^3$? This works because the function $g(x) = x^3$ is continuous; we could also justify it using what we know about limits of products.

2. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n}$

   In this problem we could easily remove the 5 from the exponent but there's no easy way to remove the numerator of 2. We must apply a change of variables to rewrite $\frac{2}{n}$ in the form $\frac{1}{m}$.

$$\begin{aligned}\frac{2}{n} &= \frac{1}{m}\\ 2 &= \frac{n}{m}\end{aligned}$$

$$n = 2m$$

Note that $\lim_{n\to\infty} n = \lim_{m\to\infty} 2m = \infty$. (Does it matter that m goes to infinity half as fast as n does? Why or why not?)

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n} &= \lim_{m\to\infty}\left(1+\frac{2}{2m}\right)^{5\cdot 2m} \\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{10m} \\
&= \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^{m}\right]^{10} \\
&= e^{10}
\end{aligned}$$

3. $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{5n}$

This problem is very similar to the previous one.

$$\begin{aligned}
\frac{1}{2n} &= \frac{1}{m} \\
m &= 2n \\
n &= \frac{m}{2}
\end{aligned}$$

Again, $\lim_{n\to\infty} n = \lim_{m\to\infty} \frac{m}{2} = \infty$.

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{5n} &= \lim_{n\to\infty}\left(1+\frac{1}{2(\frac{m}{2})}\right)^{5\frac{m}{2}} \\
&= \lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{\frac{5}{2}m} \\
&= e^{5/2}
\end{aligned}$$

Evaluating an Interest Using the Limit

Recall that the formula for *compound interest* is:

$$A = P\left(1 + \frac{r}{k}\right)^k$$

and the anual percentage rate is:

$$\text{APR} = \left(1 + \frac{r}{k}\right)^k - 1.$$

Here P is the principal invested, r is the annual "simple" interest rate, A is the amount in the account at a given time, and k determines the frequency with which interest is added to the account.

As k approaches infinity interest is added more and more often; in the limit we say that the interest is *compounded continuously*.

1. Use the fact that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$ to compute the APR of 5% compounded continuously.
2. Compute the APR of 10% compounded continuously.

Solution

1. Use the fact that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$ to compute the APR of 5% compounded continuously.

   We take the limit of the expression for APR as k goes to infinity, then manipulate the expression until it matches one we know.

$$\begin{aligned}\text{APR of 5\% compounded continuously} &= \lim_{k\to\infty}\left[\left(1 + \frac{r}{k}\right)^k - 1\right] \\ &= \lim_{k\to\infty}\left(1 + \frac{0.05}{k}\right)^k - 1\end{aligned}$$

   We solve $\frac{0.05}{k} = \frac{1}{n}$ to find $k = 0.05n$. Our equation then becomes:

$$\begin{aligned}\text{APR of 5\% compounded continuously} &= \lim_{k\to\infty}\left(1 + \frac{0.05}{k}\right)^k - 1 \\ &= \lim_{n\to\infty}\left(1 + \frac{0.05}{0.05n}\right)^{0.05n} - 1 \\ &= \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{0.05n} - 1\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{0.05} - 1 \\
&= e^{0.05} - 1 \\
&\approx 0.0513 \\
&\approx 5.13\%
\end{aligned}
$$

2. Compute the APR of 10% compounded continuously.

   This problem is nearly identical to the previous one; here we solve $\frac{0.10}{k} = \frac{1}{n}$ to find $k = 0.10n$.

$$
\begin{aligned}
\text{APR of 10\% compounded continuously} &= \lim_{k\to\infty}\left(\left(1+\frac{0.10}{k}\right)^{k} - 1\right) \\
&= \lim_{n\to\infty}\left(1+\frac{0.10}{0.10n}\right)^{0.10n} - 1 \\
&= e^{0.10} - 1 \\
&\approx 0.10517 \\
&\approx 10.52\%
\end{aligned}
$$

Comparing these answers to our previous calculations we see that the equation for continuously compounded interest is simpler than that for interest compounded monthly, and that the two formulas yield comparable results.