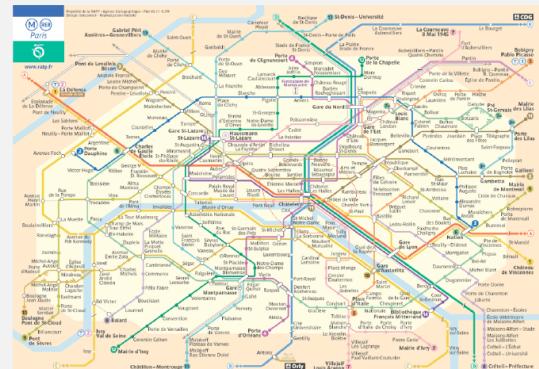
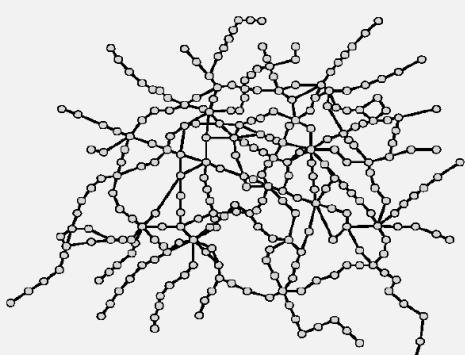


## Undirected graphs

**Graph.** Set of **vertices** connected pairwise by **edges**.

### Why study graph algorithms?

- Thousands of practical applications.
- Hundreds of graph algorithms known.
- Interesting and broadly useful abstraction.
- Challenging branch of computer science and discrete math.



## Graph applications

graph	vertex	edge
communication	telephone, computer	fiber optic cable
circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	street intersection, airport	highway, airway route
internet	class C network	connection
game	board position	legal move
social relationship	person, actor	friendship, movie cast
neural network	neuron	synapse
protein network	protein	protein-protein interaction
molecule	atom	bond

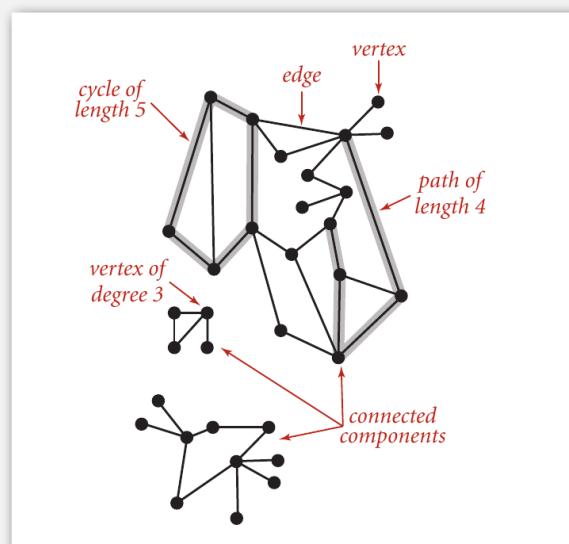
11

## Graph terminology

**Path.** Sequence of vertices connected by edges.

**Cycle.** Path whose first and last vertices are the same.

Two vertices are **connected** if there is a path between them.



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## Some graph-processing problems

**Path.** Is there a path between  $s$  and  $t$ ?

**Shortest path.** What is the shortest path between  $s$  and  $t$ ?

**Cycle.** Is there a cycle in the graph?

**Euler tour.** Is there a cycle that uses each edge exactly once?

**Hamilton tour.** Is there a cycle that uses each vertex exactly once.

**Connectivity.** Is there a way to connect all of the vertices?

**MST.** What is the best way to connect all of the vertices?

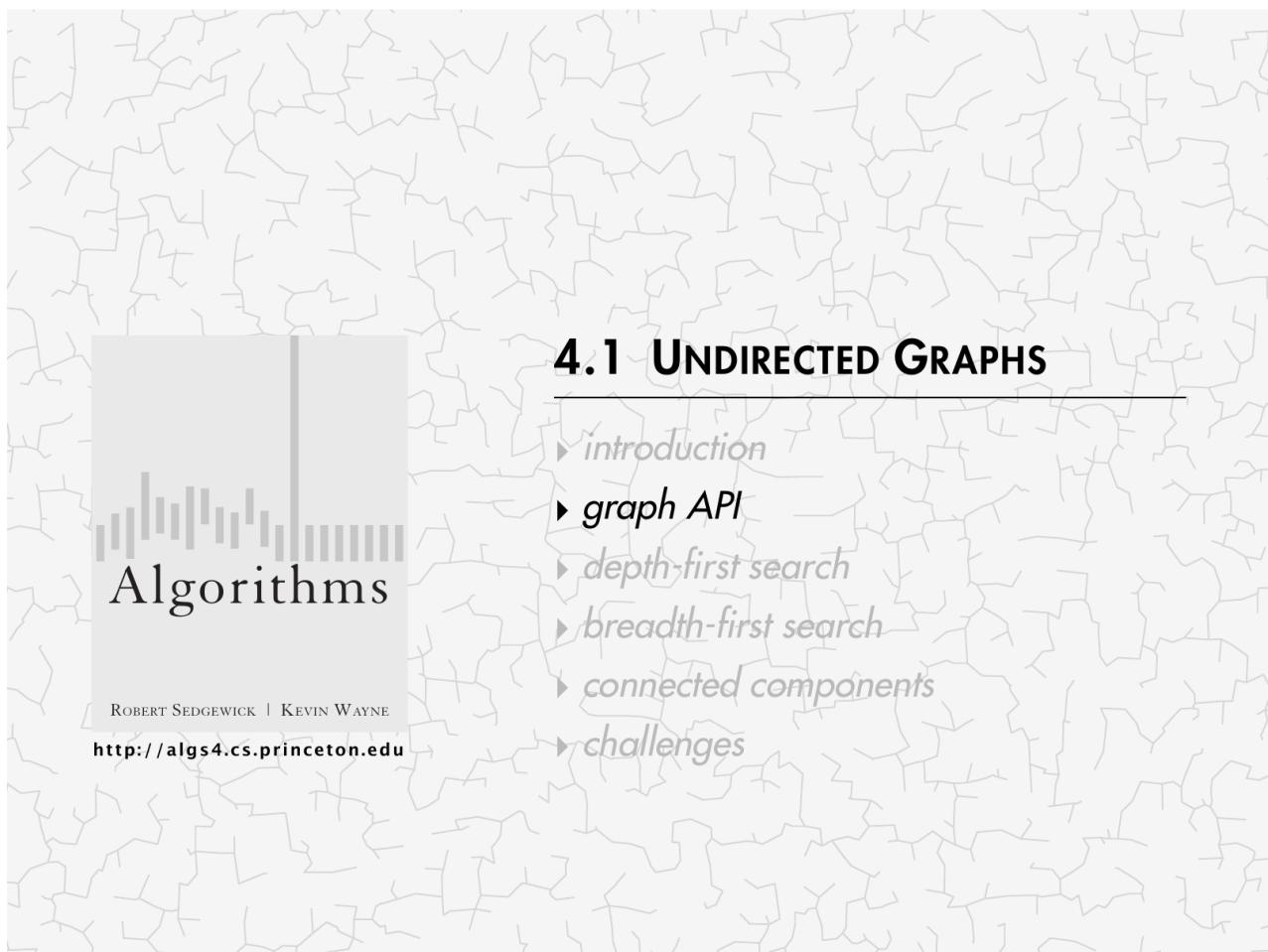
**Biconnectivity.** Is there a vertex whose removal disconnects the graph?

**Planarity.** Can you draw the graph in the plane with no crossing edges

**Graph isomorphism.** Do two adjacency lists represent the same graph?

**Challenge.** Which of these problems are easy? difficult? intractable?

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## Graph representation

Graph drawing. Provides intuition about the structure of the graph.



two drawings of the same graph

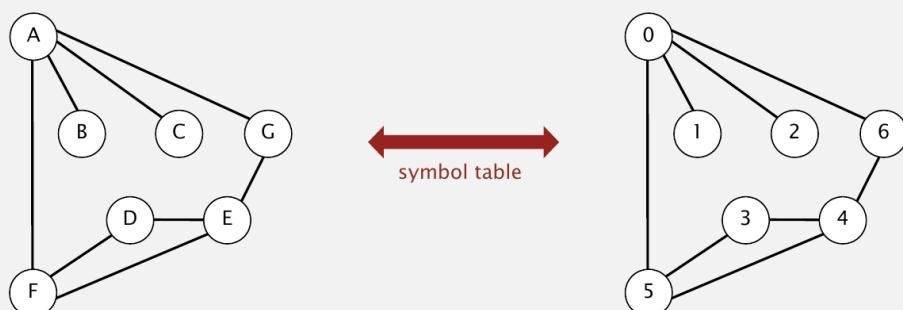
Caveat. Intuition can be misleading.

16

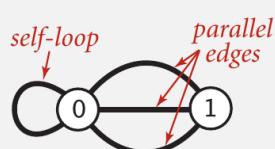
## Graph representation

### Vertex representation.

- This lecture: use integers between 0 and  $V - 1$ .
- Applications: convert between names and integers with symbol table.



Anomalies.



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## Graph API

public class Graph	
Graph(int V)	<i>create an empty graph with V vertices</i>
Graph(In in)	<i>create a graph from input stream</i>
void addEdge(int v, int w)	<i>add an edge v-w</i>
Iterable<Integer> adj(int v)	<i>vertices adjacent to v</i>
int V()	<i>number of vertices</i>
int E()	<i>number of edges</i>
String toString()	<i>string representation</i>

```
In in = new In(args[0]);
Graph G = new Graph(in);

for (int v = 0; v < G.V(); v++)
    for (int w : G.adj(v))
        StdOut.println(v + "-" + w);
```

read graph from  
input stream

print out each  
edge (twice)

18

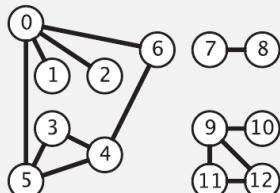
## Graph API: sample client

Graph input format.

**tinyG.txt**

V → 13    E ← 13

```
13
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```



```
% java Test tinyG.txt
0-6
0-2
0-1
0-5
1-0
2-0
3-5
3-4
...
12-11
12-9
```

```
In in = new In(args[0]);
Graph G = new Graph(in);

for (int v = 0; v < G.V(); v++)
    for (int w : G.adj(v))
        StdOut.println(v + "-" + w);
```

read graph from  
input stream

print out each  
edge (twice)

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## Typical graph-processing code

```
public static int degree(Graph G, int v)
{
    int degree = 0;
    for (int w : G.adj(v)) degree++;
    return degree;
}

public static int maxDegree(Graph G)
{
    int max = 0;
    for (int v = 0; v < G.V(); v++)
        if (degree(G, v) > max)
            max = degree(G, v);
    return max;
}

public static double averageDegree(Graph G)
{   return 2.0 * G.E() / G.V(); }

public static int numberOfSelfLoops(Graph G)
{
    int count = 0;
    for (int v = 0; v < G.V(); v++)
        for (int w : G.adj(v))
            if (v == w) count++;
    return count/2; // each edge counted twice
}
```

*compute maximum degree*

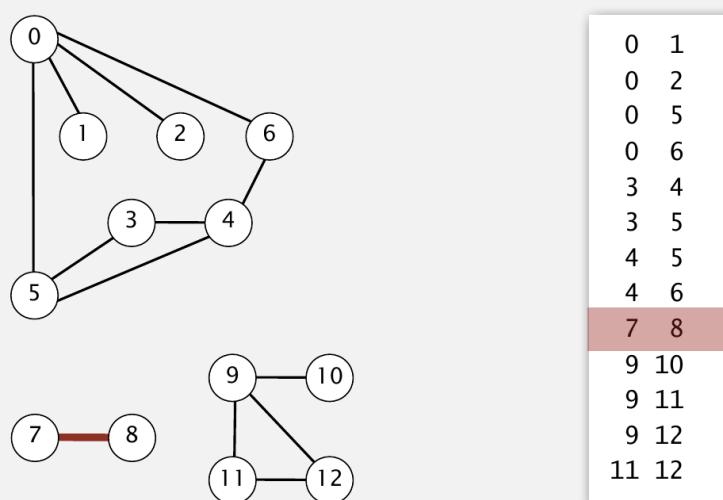
*compute average degree*

*count self-loops*

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## Set-of-edges graph representation

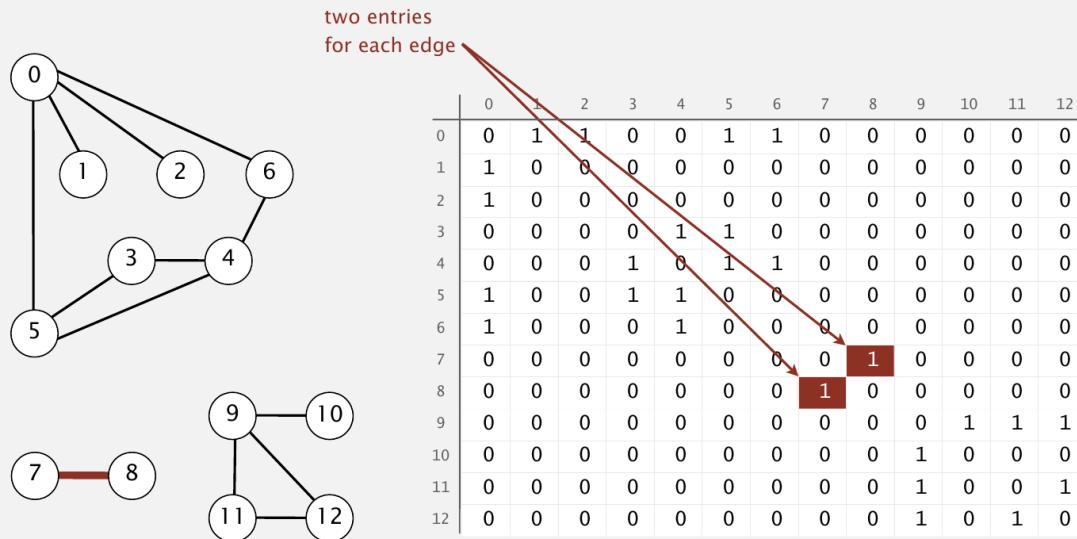
Maintain a list of the edges (linked list or array).



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## Adjacency-matrix graph representation

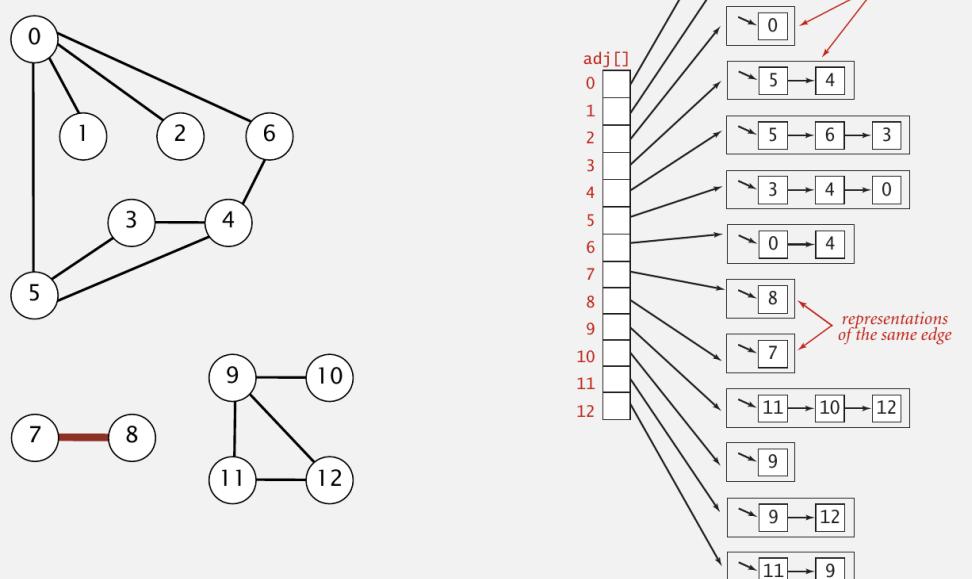
Maintain a two-dimensional  $V$ -by- $V$  boolean array;  
for each edge  $v-w$  in graph:  $\text{adj}[v][w] = \text{adj}[w][v] = \text{true}$ .



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## Adjacency-list graph representation

Maintain vertex-indexed array of lists.



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## Adjacency-list graph representation: Java implementation

```
public class Graph
{
    private final int V;
    private Bag<Integer>[] adj;

    public Graph(int V)
    {
        this.V = V;
        adj = (Bag<Integer>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Integer>();
    }

    public void addEdge(int v, int w)
    {
        adj[v].add(w);
        adj[w].add(v);
    }

    public Iterable<Integer> adj(int v)
    { return adj[v]; }
}
```

adjacency lists  
( using Bag data type )

create empty graph  
with  $V$  vertices

add edge  $v-w$   
(parallel edges and  
self-loops allowed)

iterator for vertices adjacent to  $v$

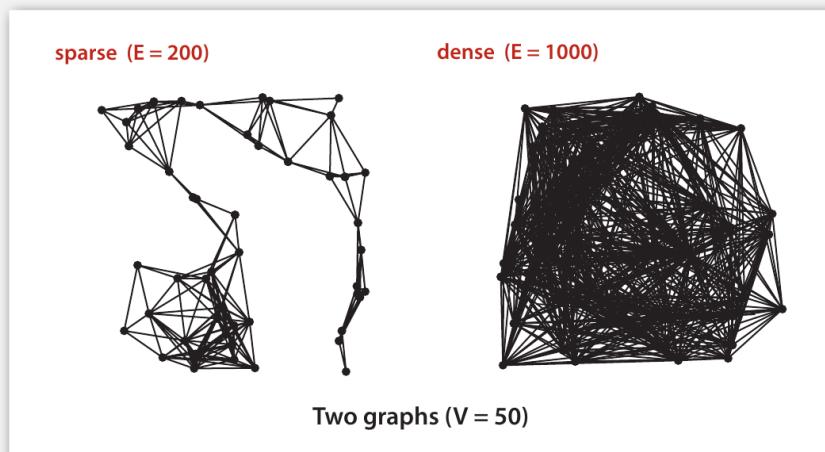
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## Graph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent to  $v$ .
- Real-world graphs tend to be **sparse**.

huge number of vertices,  
small average vertex degree



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## Graph representations

In practice. Use adjacency-lists representation.

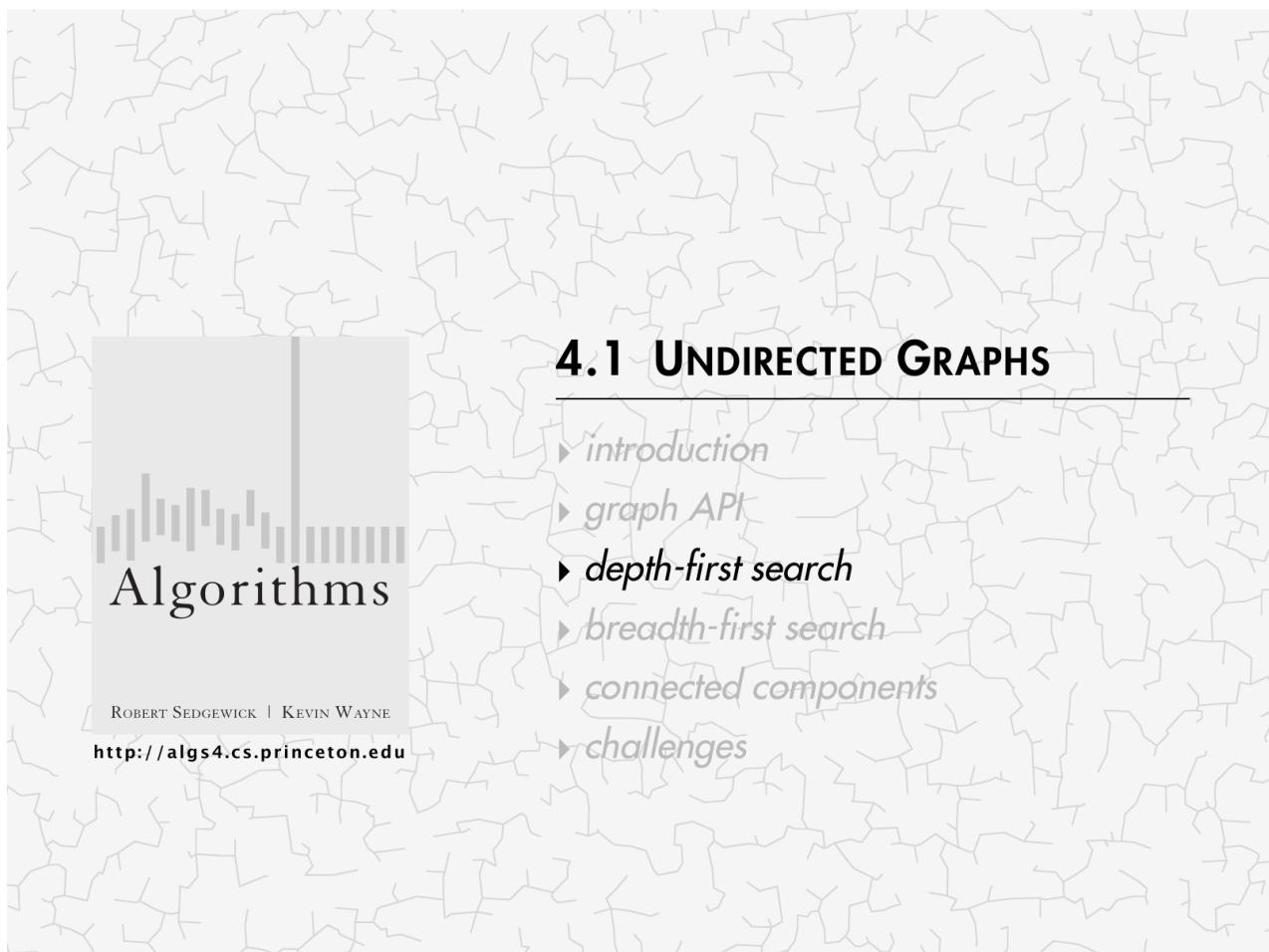
- Algorithms based on iterating over vertices adjacent to  $v$ .
- Real-world graphs tend to be **sparse**.

huge number of vertices,  
small average vertex degree

representation	space	add edge	edge between $v$ and $w$ ?	iterate over vertices adjacent to $v$ ?
list of edges	$E$	1	$E$	$E$
adjacency matrix	$V^2$	1 *	1	$V$
adjacency lists	$E + V$	1	$\text{degree}(v)$	$\text{degree}(v)$

\* disallows parallel edges

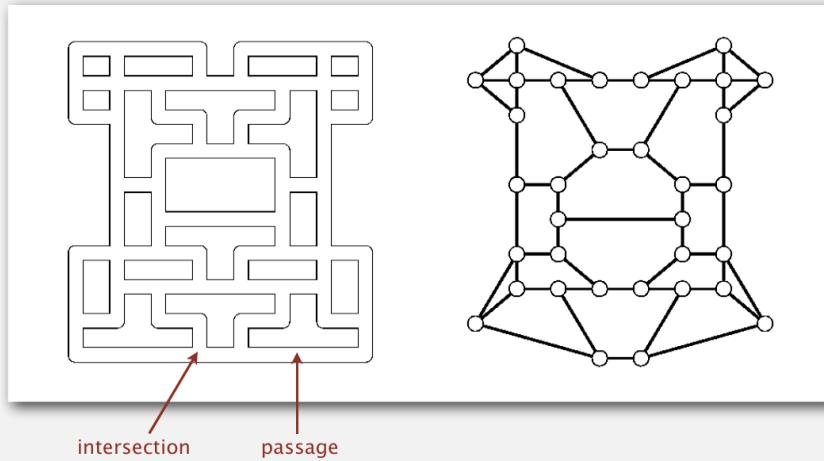
26



## Maze exploration

### Maze graph.

- Vertex = intersection.
- Edge = passage.



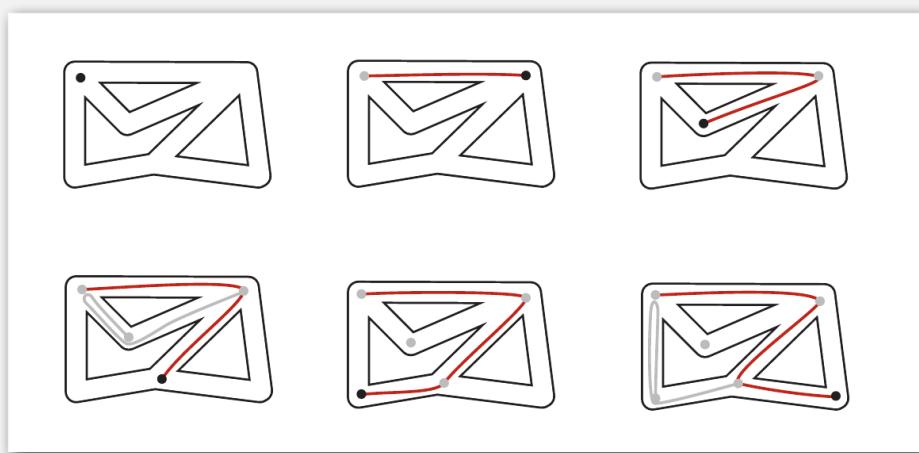
**Goal.** Explore every intersection in the maze.

29

## Trémaux maze exploration

### Algorithm.

- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.



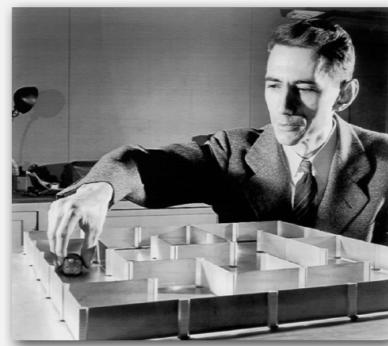
30

## Trémaux maze exploration

### Algorithm.

- Unroll a ball of string behind you.
- Mark each visited intersection and each visited passage.
- Retrace steps when no unvisited options.

**First use?** Theseus entered Labyrinth to kill the monstrous Minotaur; Ariadne instructed Theseus to use a ball of string to find his way back out.



Claude Shannon (with Theseus mouse)

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## Depth-first search

**Goal.** Systematically search through a graph.

**Idea.** Mimic maze exploration.

### DFS (to visit a vertex v)

---

Mark v as visited.  
Recursively visit all unmarked  
vertices w adjacent to v.

---

**Typical applications.**

- Find all vertices connected to a given source vertex.
- Find a path between two vertices.

**Design challenge.** How to implement?

## Design pattern for graph processing

**Design pattern.** Decouple graph data type from graph processing.

- Create a Graph object.
- Pass the Graph to a graph-processing routine.
- Query the graph-processing routine for information.

```
public class Paths  
  
    Paths(Graph G, int s)           find paths in G from source s  
    boolean hasPathTo(int v)         is there a path from s to v?  
    Iterable<Integer> pathTo(int v)  path from s to v; null if no such path
```

```
Paths paths = new Paths(G, s);  
for (int v = 0; v < G.V(); v++)  
    if (paths.hasPathTo(v))  
        StdOut.println(v);
```

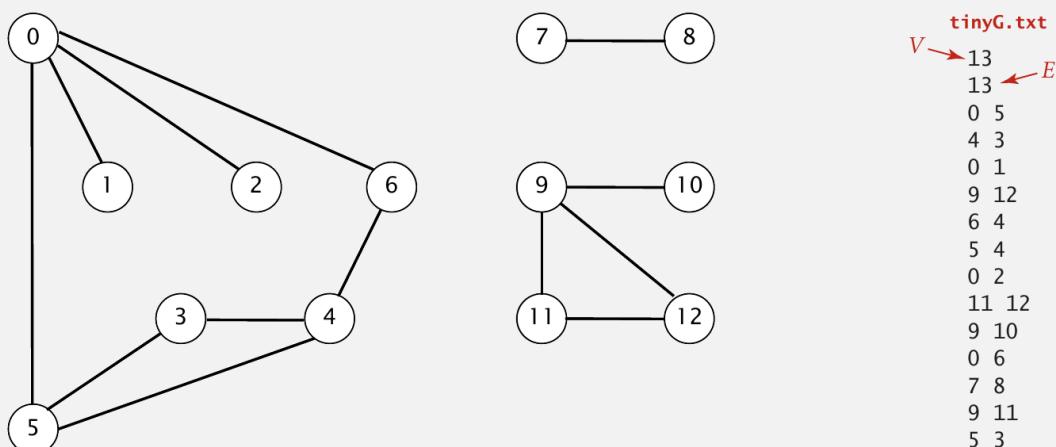
print all vertices  
connected to s

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## Depth-first search demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .

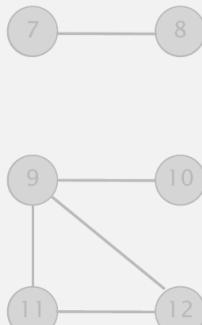
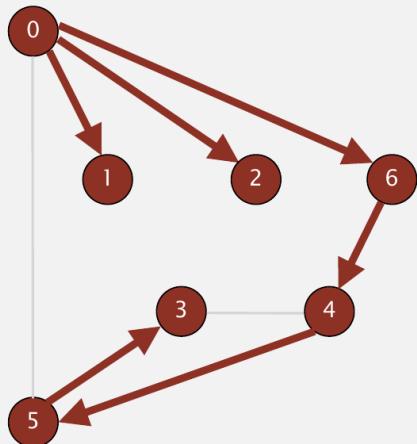


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## Depth-first search demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



$v$	marked[]	edgeTo[ $v$ ]
0	T	-
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	-
8	F	-
9	F	-
10	F	-
11	F	-
12	F	-

vertices reachable from 0

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## Depth-first search

**Goal.** Find all vertices connected to  $s$  (and a corresponding path).

**Idea.** Mimic maze exploration.

### Algorithm.

- Use recursion (ball of string).
- Mark each visited vertex (and keep track of edge taken to visit it).
- Return (retrace steps) when no unvisited options.

### Data structures.

- `boolean[] marked` to mark visited vertices.
- `int[] edgeTo` to keep tree of paths.  
( $\text{edgeTo}[w] == v$ ) means that edge  $v-w$  taken to visit  $w$  for first time

## Depth-first search

```
public class DepthFirstPaths
{
    private boolean[] marked;
    private int[] edgeTo;
    private int s;

    public DepthFirstPaths(Graph G, int s)
    {
        ...
        dfs(G, s);
    }

    private void dfs(Graph G, int v)
    {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w])
            {
                dfs(G, w);
                edgeTo[w] = v;
            }
    }
}
```

marked[v] = true  
if v connected to s  
edgeTo[v] = previous vertex on path from s to v

initialize data structures  
find vertices connected to s

recursive DFS does the work

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## Depth-first search properties

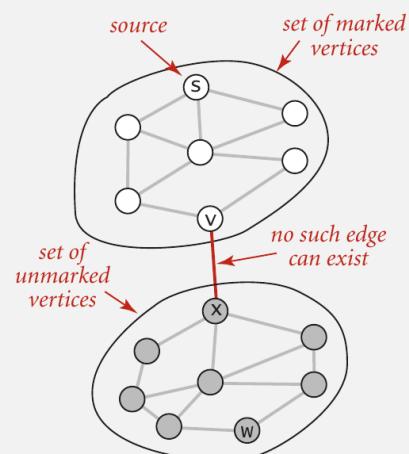
**Proposition.** DFS marks all vertices connected to  $s$  in time proportional to the sum of their degrees.

**Pf.** [correctness]

- If  $w$  marked, then  $w$  connected to  $s$  (why?)
- If  $w$  connected to  $s$ , then  $w$  marked.  
(if  $w$  unmarked, then consider last edge on a path from  $s$  to  $w$  that goes from a marked vertex to an unmarked one).

**Pf.** [running time]

Each vertex connected to  $s$  is visited once.



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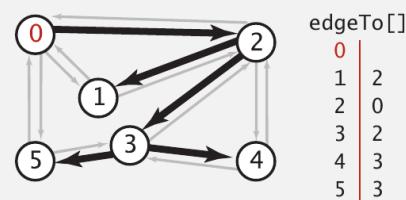
## Depth-first search properties

**Proposition.** After DFS, can find vertices connected to  $s$  in constant time and can find a path to  $s$  (if one exists) in time proportional to its length.

**Pf.** `edgeTo[]` is parent-link representation of a tree rooted at  $s$ .

```
public boolean hasPathTo(int v)
{   return marked[v]; }

public Iterable<Integer> pathTo(int v)
{
    if (!hasPathTo(v)) return null;
    Stack<Integer> path = new Stack<Integer>();
    for (int x = v; x != s; x = edgeTo[x])
        path.push(x);
    path.push(s);
    return path;
}
```



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## Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

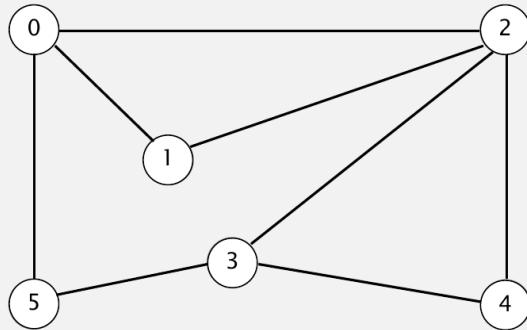
## 4.1 UNDIRECTED GRAPHS

- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *connected components*
- ▶ *challenges*

## Breadth-first search demo

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices adjacent to  $v$  and mark them.



**tinyCG.txt**

$V \rightarrow$  6  
8  $\leftarrow E$   
0 5  
2 4  
2 3  
1 2  
0 1  
3 4  
3 5  
0 2

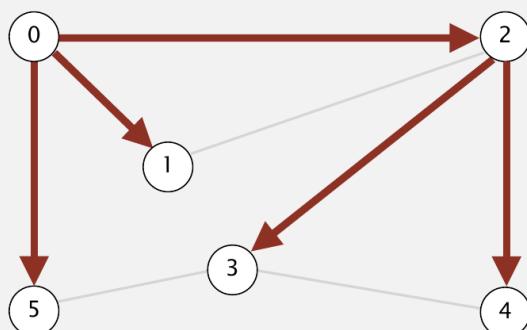
**graph G**

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## Breadth-first search demo

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices adjacent to  $v$  and mark them.



$v$	edgeTo[]	distTo[]
0	-	0
1	0	1
2	0	1
3	2	2
4	2	2
5	0	1

**done**

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## Breadth-first search

Depth-first search. Put unvisited vertices on a **stack**.

Breadth-first search. Put unvisited vertices on a **queue**.

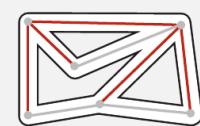
Shortest path. Find path from  $s$  to  $t$  that uses **fewest number of edges**.

### BFS (from source vertex $s$ )

Put  $s$  onto a FIFO queue, and mark  $s$  as visited.

Repeat until the queue is empty:

- remove the least recently added vertex  $v$
- add each of  $v$ 's unvisited neighbors to the queue,  
and mark them as visited.



Intuition. BFS examines vertices in increasing distance from  $s$ .

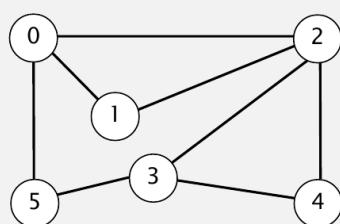
48

## Breadth-first search properties

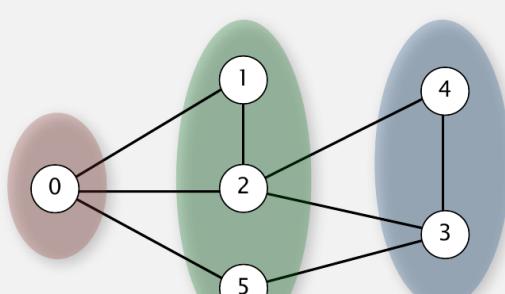
**Proposition.** BFS computes shortest paths (fewest number of edges) from  $s$  to all other vertices in a graph in time proportional to  $E + V$ .

**Pf.** [correctness] Queue always consists of zero or more vertices of distance  $k$  from  $s$ , followed by zero or more vertices of distance  $k + 1$ .

**Pf.** [running time] Each vertex connected to  $s$  is visited once.



graph



dist = 0

dist = 1

dist = 2

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## Breadth-first search

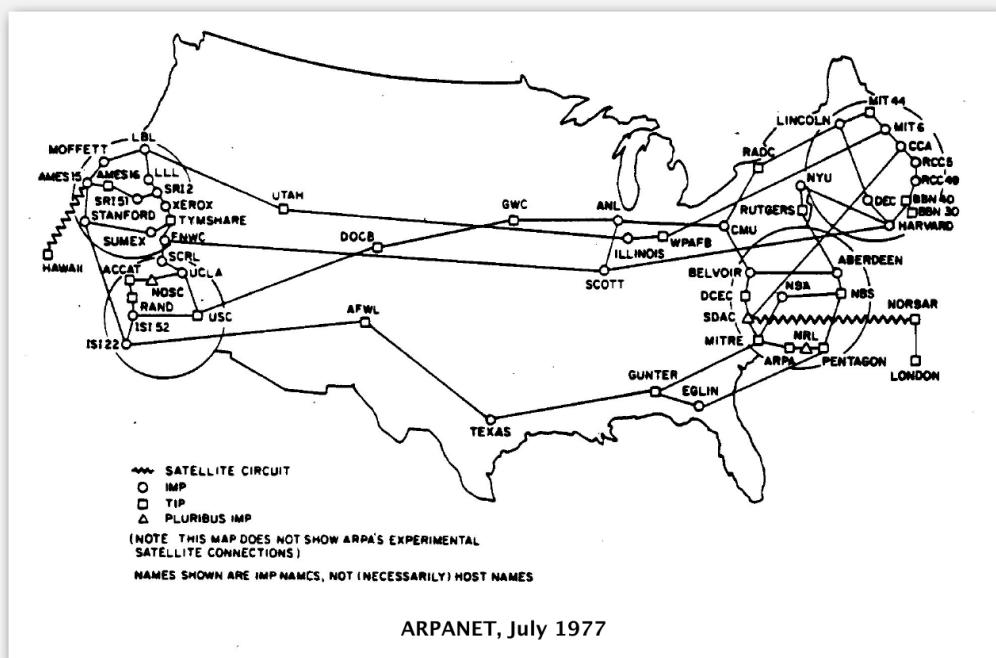
```
public class BreadthFirstPaths
{
    private boolean[] marked;
    private int[] edgeTo;
    ...

    private void bfs(Graph G, int s)
    {
        Queue<Integer> q = new Queue<Integer>();
        q.enqueue(s);
        marked[s] = true;
        while (!q.isEmpty())
        {
            int v = q.dequeue();
            for (int w : G.adj(v))
            {
                if (!marked[w])
                {
                    q.enqueue(w);
                    marked[w] = true;
                    edgeTo[w] = v;
                }
            }
        }
    }
}
```

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## Breadth-first search application: routing

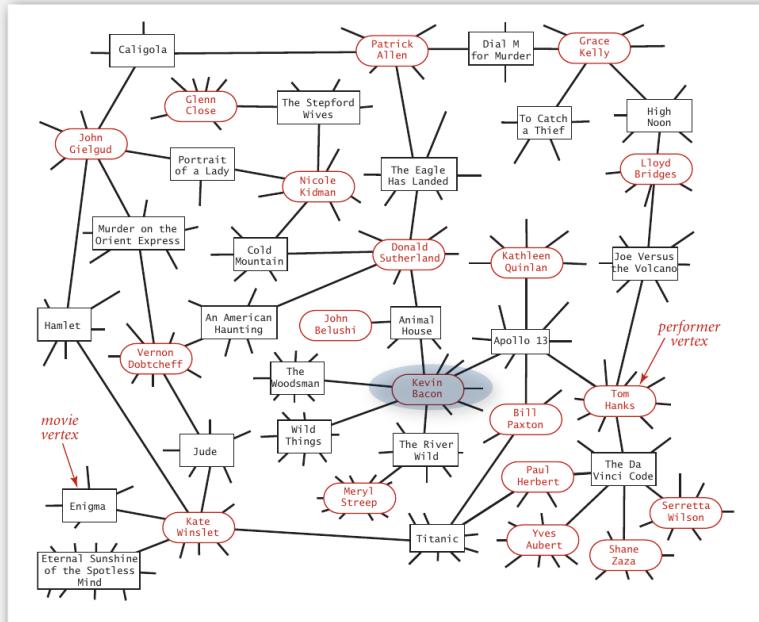
Fewest number of hops in a communication network.



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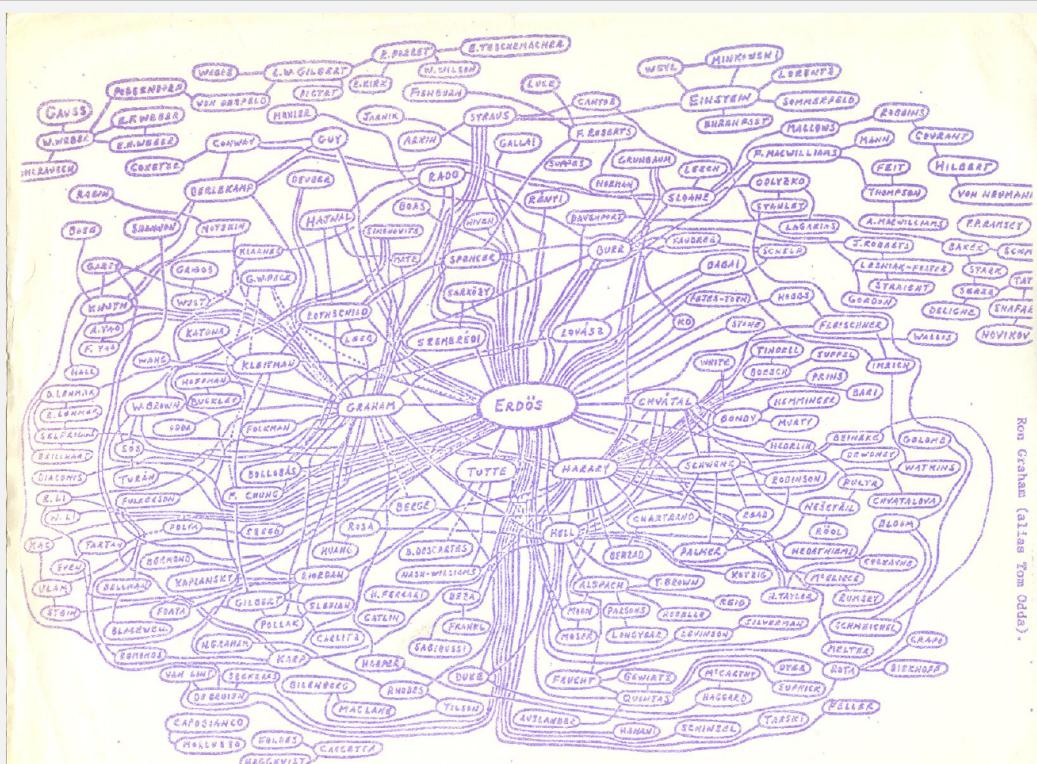
## Kevin Bacon graph

- Include one vertex for each performer and one for each movie.
- Connect a movie to all performers that appear in that movie.
- Compute shortest path from  $s = \text{Kevin Bacon}$ .



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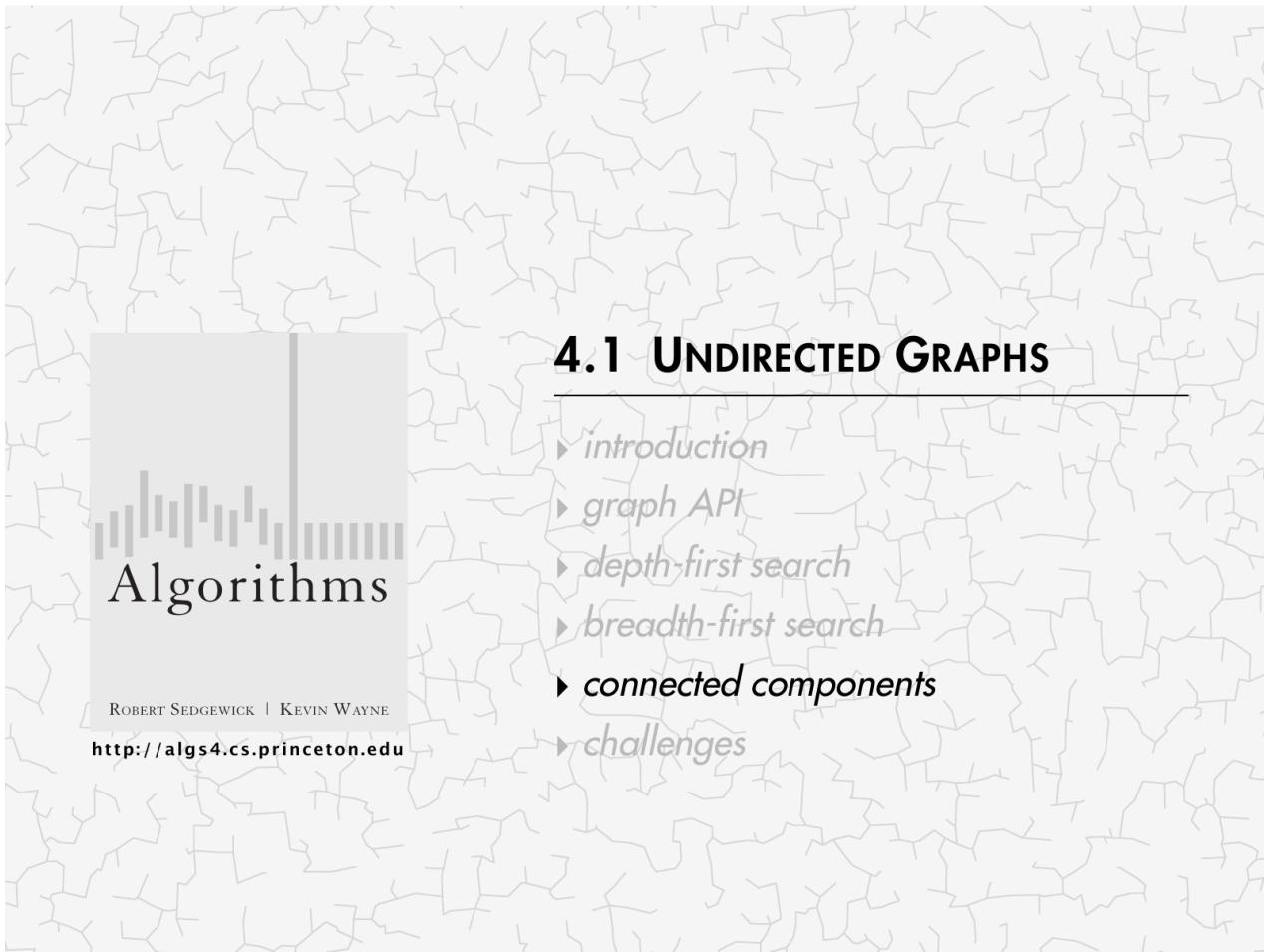
## Breadth-first search application: Erdős numbers



Ron Graham (alias Tom Odd).

hand-drawing of part of the Erdős graph by Ron Graham

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## Connectivity queries

**Def.** Vertices  $v$  and  $w$  are **connected** if there is a path between them.

**Goal.** Preprocess graph to answer queries of the form *is  $v$  connected to  $w$ ?* in **constant** time.

```
public class CC
    CC(Graph G)                      find connected components in G
    boolean connected(int v, int w)    are v and w connected?
    int count()                       number of connected components
    int id(int v)                     component identifier for v
```

**Union-Find?** Not quite.

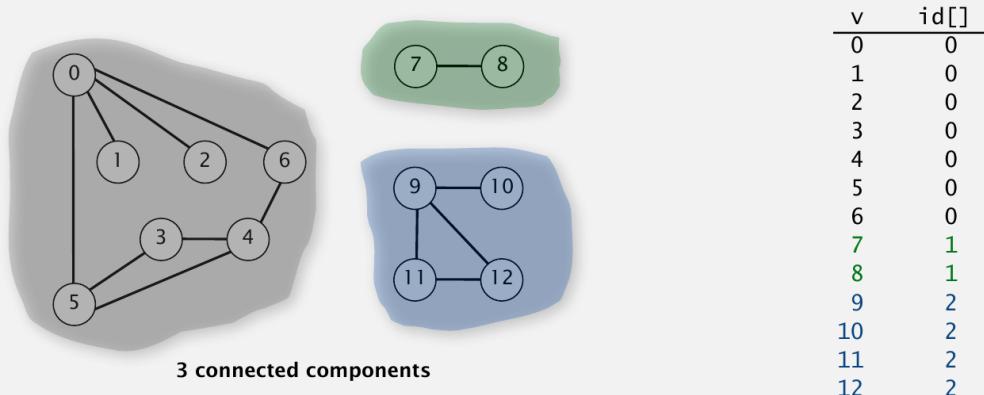
**Depth-first search.** Yes. [next few slides]

## Connected components

The relation "is connected to" is an **equivalence relation**:

- Reflexive:  $v$  is connected to  $v$ .
- Symmetric: if  $v$  is connected to  $w$ , then  $w$  is connected to  $v$ .
- Transitive: if  $v$  connected to  $w$  and  $w$  connected to  $x$ , then  $v$  connected to  $x$ .

**Def.** A **connected component** is a maximal set of connected vertices.

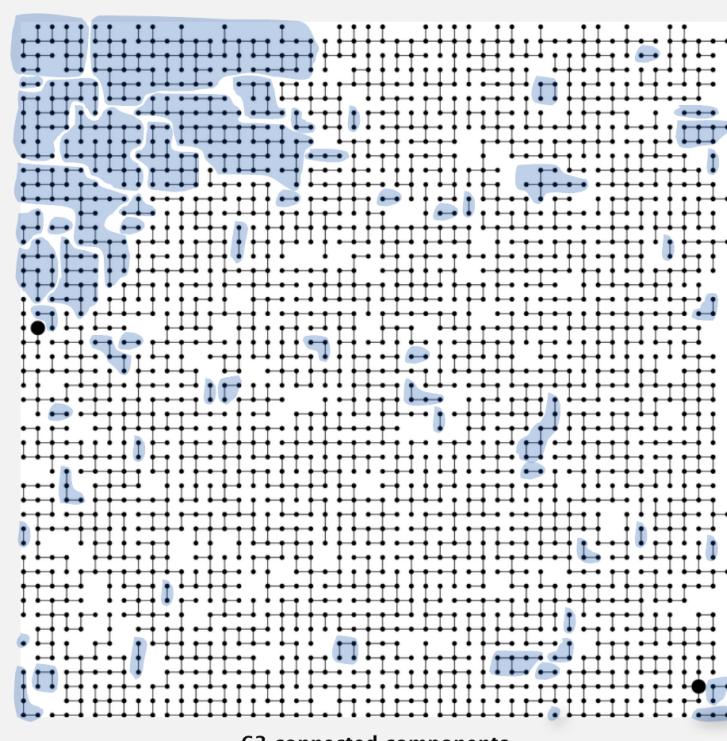


**Remark.** Given connected components, can answer queries in constant time.

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## Connected components

**Def.** A **connected component** is a maximal set of connected vertices.



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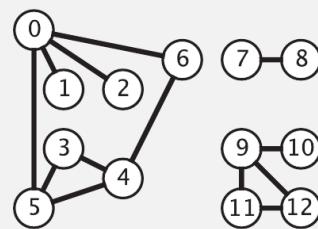
## Connected components

Goal. Partition vertices into connected components.

### Connected components

Initialize all vertices  $v$  as unmarked.

For each unmarked vertex  $v$ , run DFS to identify all vertices discovered as part of the same component.



**tinyG.txt**

$V \rightarrow 13$   
 $E \rightarrow 13$

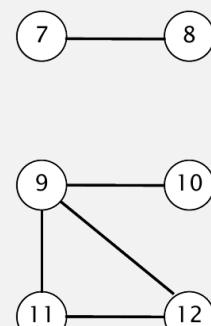
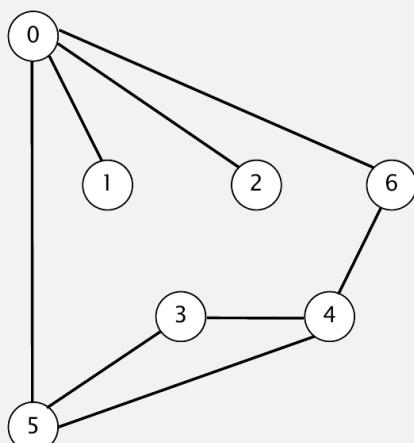
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3

60

## Connected components demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



$v$	marked[]	id[]
0	F	-
1	F	-
2	F	-
3	F	-
4	F	-
5	F	-
6	F	-
7	F	-
8	F	-
9	F	-
10	F	-
11	F	-
12	F	-

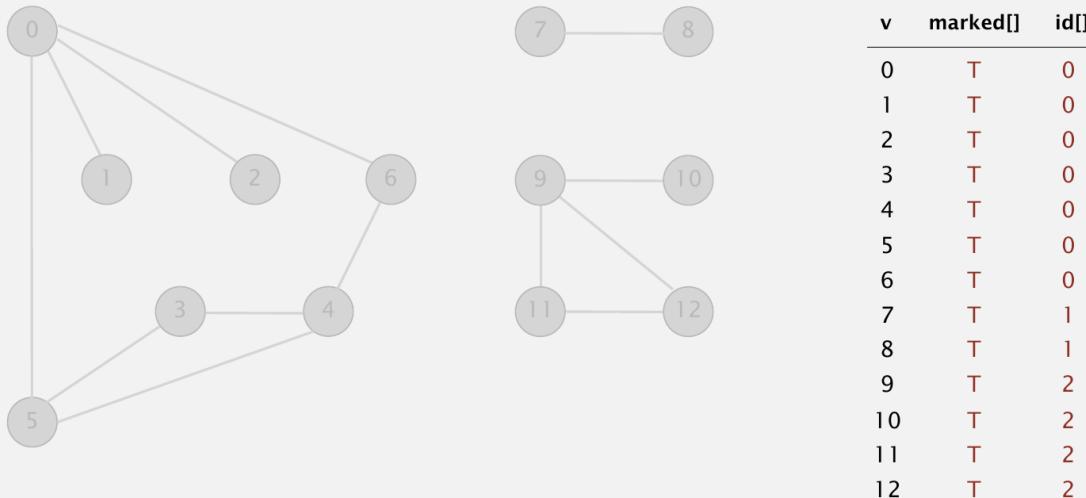
graph G

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## Connected components demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices adjacent to  $v$ .



done

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## Finding connected components with DFS

```
public class CC
{
    private boolean[] marked;
    private int[] id;
    private int count;

    public CC(Graph G)
    {
        marked = new boolean[G.V()];
        id = new int[G.V()];
        for (int v = 0; v < G.V(); v++)
        {
            if (!marked[v])
            {
                dfs(G, v);
                count++;
            }
        }
    }

    public int count()
    public int id(int v)
    private void dfs(Graph G, int v)
}
```

id[v] = id of component containing v  
number of components

run DFS from one vertex in each component

see next slide

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## Finding connected components with DFS (continued)

```
public int count()
{ return count; }

public int id(int v)
{ return id[v]; }

private void dfs(Graph G, int v)
{
    marked[v] = true;
    id[v] = count;
    for (int w : G.adj(v))
        if (!marked[w])
            dfs(G, w);
}
```

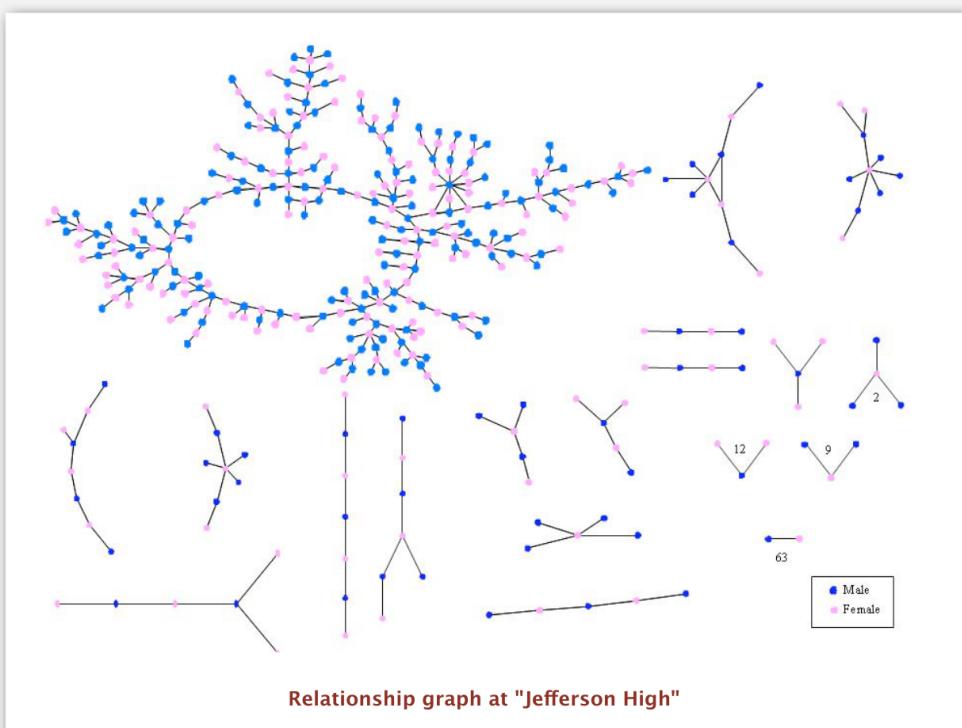
number of components

id of component containing v

all vertices discovered in same call of dfs have same id

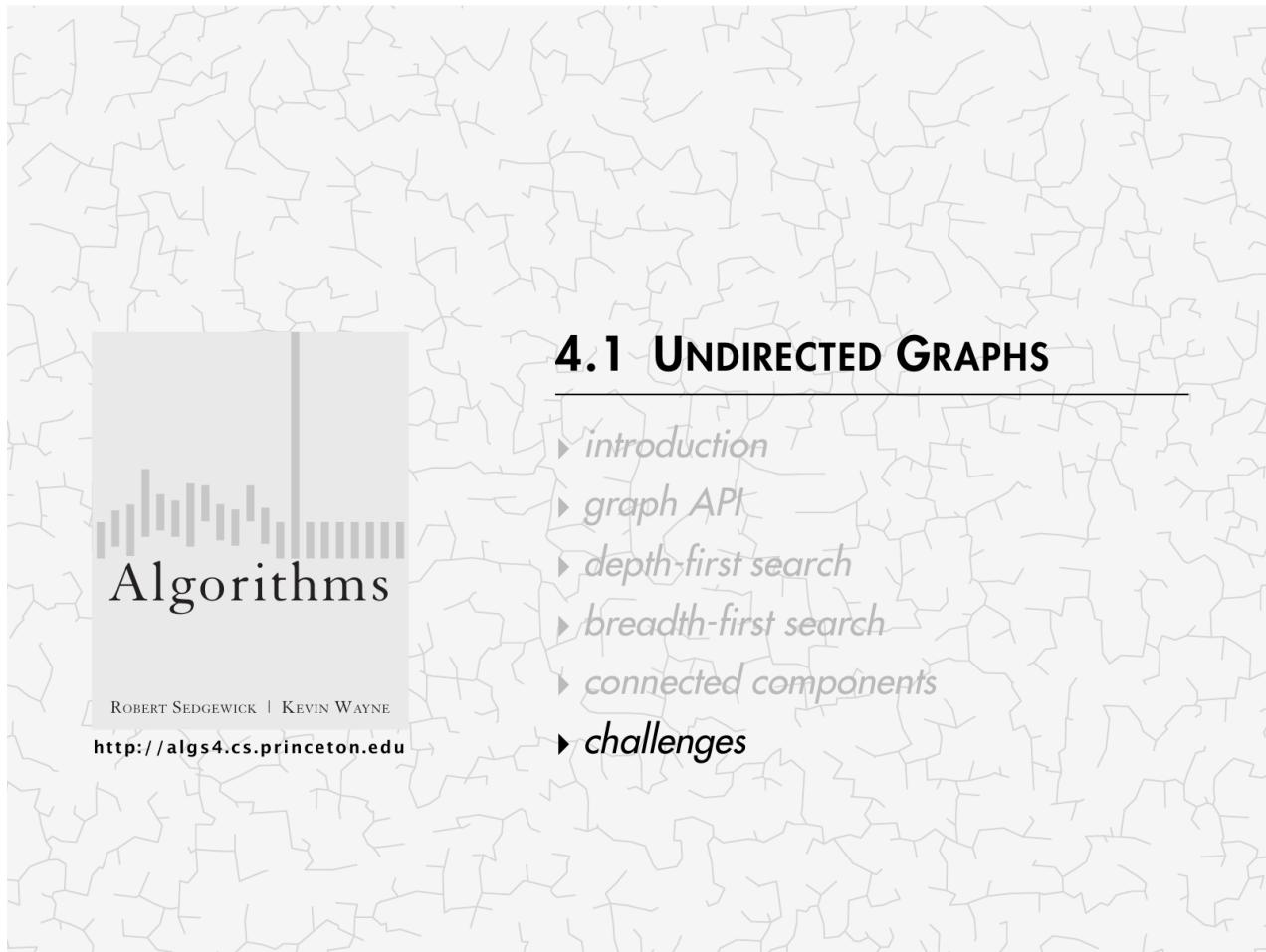
64

## Connected components application: study spread of STDs



Peter Bearman, James Moody, and Katherine Stovel. **Chains of affection: The structure of adolescent romantic and sexual networks.** American Journal of Sociology, 110(1): 44–99, 2004.

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## 4.1 UNDIRECTED GRAPHS

- ▶ *introduction*
- ▶ *graph API*
- ▶ *depth-first search*
- ▶ *breadth-first search*
- ▶ *connected components*
- ▶ **challenges**

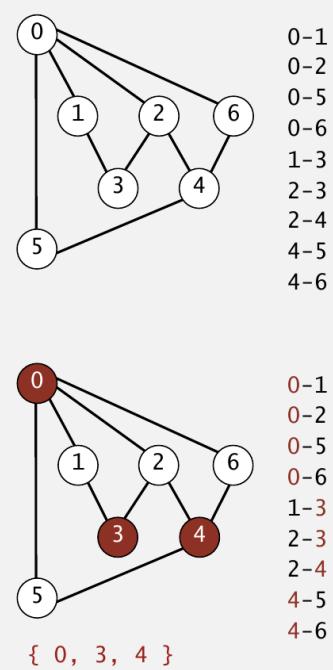
### Graph-processing challenge 1

**Problem.** Is a graph bipartite?

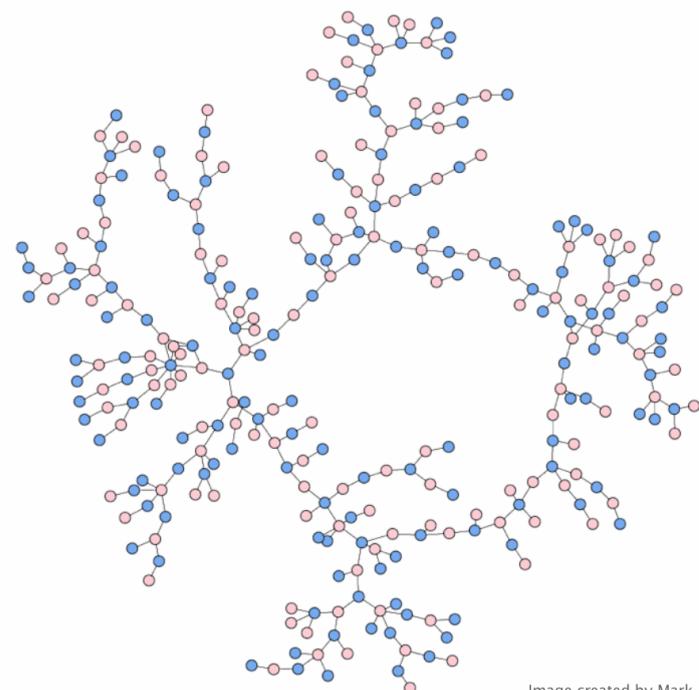
**How difficult?**

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

simple DFS-based solution  
(see textbook)



## Bipartiteness application: is dating graph bipartite?



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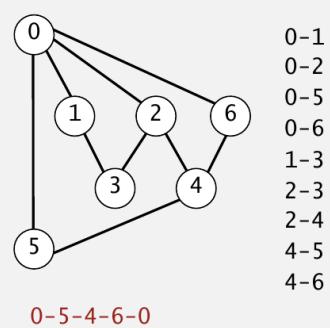
## Graph-processing challenge 2

**Problem.** Find a cycle.

**How difficult?**

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
  - Hire an expert.
  - Intractable.
  - No one knows.
  - Impossible.

simple DFS-based solution  
(see textbook)

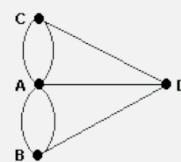
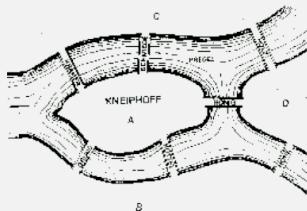


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## Bridges of Königsberg

The Seven Bridges of Königsberg. [Leonhard Euler 1736]

“... in Königsberg in Prussia, there is an island A, called the Kneiphof; the river which surrounds it is divided into two branches ... and these branches are crossed by seven bridges. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he could cross each bridge once and only once.”



**Euler tour.** Is there a (general) cycle that uses each edge exactly once?

**Answer.** A connected graph is Eulerian iff all vertices have **even** degree.

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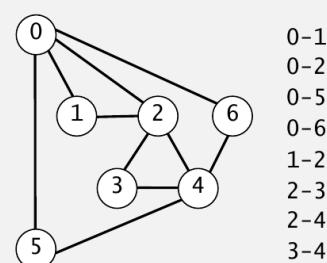
## Graph-processing challenge 3

**Problem.** Find a (general) cycle that uses every edge exactly once.

**How difficult?**

- Any programmer could do it.
- ✓ • Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- No one knows.
- Impossible.

Eulerian tour  
(classic graph-processing problem)



0-1-2-3-4-2-0-6-4-5-0

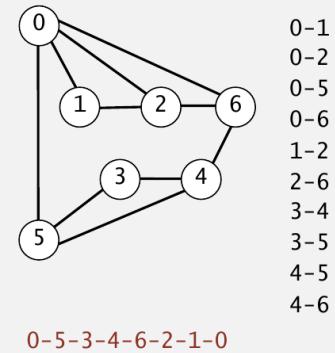
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## Graph-processing challenge 4

**Problem.** Find a cycle that visits every vertex exactly once.

**How difficult?**

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- Hire an expert.
- ✓ • Intractable. ←
- No one knows.      Hamiltonian cycle  
(classical NP-complete problem)
- Impossible.



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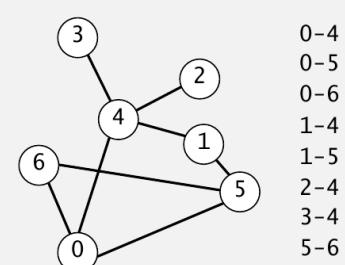
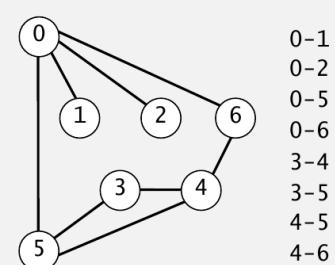
## Graph-processing challenge 5

**Problem.** Are two graphs identical except for vertex names?

**How difficult?**

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- Hire an expert.
- Intractable.
- ✓ • No one knows. ←
- Impossible.

graph isomorphism is  
longstanding open problem



0↔4, 1↔3, 2↔2, 3↔6, 4↔5, 5↔0, 6↔1

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## Graph-processing challenge 6

**Problem.** Lay out a graph in the plane without crossing edges?

### How difficult?

- Any programmer could do it.
- Typical diligent algorithms student could do it.
- ✓ • Hire an expert.
- Intractable.
- No one knows.
- Impossible.

linear-time DFS-based planarity algorithm  
discovered by Tarjan in 1970s  
(too complicated for most practitioners)

