The Theory behind PageRank

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The PageRank Algorithm: History and Facts

It was devised at Stanford by Larry Page and Sergei Brin, (the founders of Google) in 1996 as part of a research project on a new search engine.

The paper was published in 1998 [1, 2] and shortly after the authors founded Google. Named after one of the authors L. Page.

It is a *link analysis algorithm*, i.e., only the links between pages (not their content) are considered.

Other link analysis algorithms: community and spam detection, HITS, ...

Ranking Web Pages

Humans cannot make sense of billions of web pages, which have to be ranked according to their "importance".

The web pages `www.stanford.edu` and `www.johnsmith.com` have not the same importance. The former one has more than 25K Web pages linking to it, the latter one only 10.

Simple algorithm: Rank the pages according to the # of links to them.

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$\Rightarrow$ Recursive definition of importance.

Computing Importance

Web graph: a directed graph $G = (V, E)$ where nodes represent web pages, while there is a directed edge between u and v if there is a hyperlink between the corresponding web pages.

Importance of v is proportional to the importance of nodes linking to v.

It can be modeled by a system of linear equations ...

System of Linear Equations for PageRank

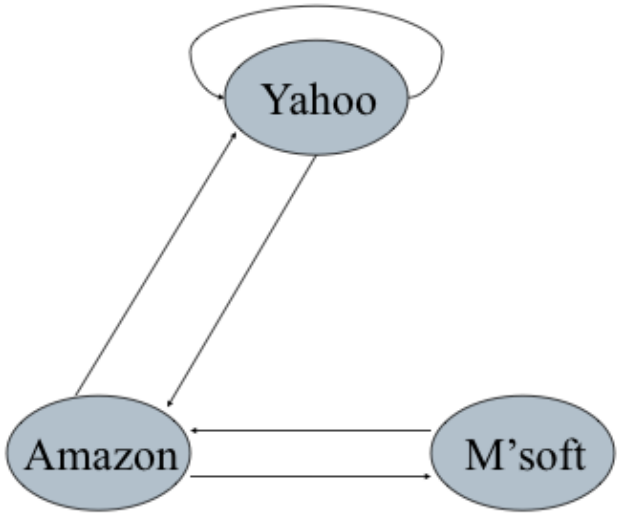
Let $G = (V, E)$ (web graph) be a directed graph, with $V = \{v_1, \ldots, v_n\}$. Let $\delta_{\text{in}}(v)$ be the in-degree of v, i.e. $\delta_{in}(v) = |\{u : (u, v) \in E\}|$, while let $\delta_{\text{out}}(v)$ be its out-degree, i.e. $\delta_{out}(v) = |\{u : (v, u) \in E\}|$.

Let M_G (M for short) be a $n \times n$ matrix with entries in $[0, 1]$ as follows:

$$M_{ij} = \begin{cases} \frac{1}{\delta_{\text{out}}(v_j)} & if\,(v_j, v_i) \in E \\ 0 & if\,(v_j, v_i) \notin E \end{cases}, \quad \forall i, j \in [1, n].$$

Any π s.t. $\pi = M\pi$ and $\sum_{i=1}^{n} \pi_i = 1$ meets our requirements.

Example



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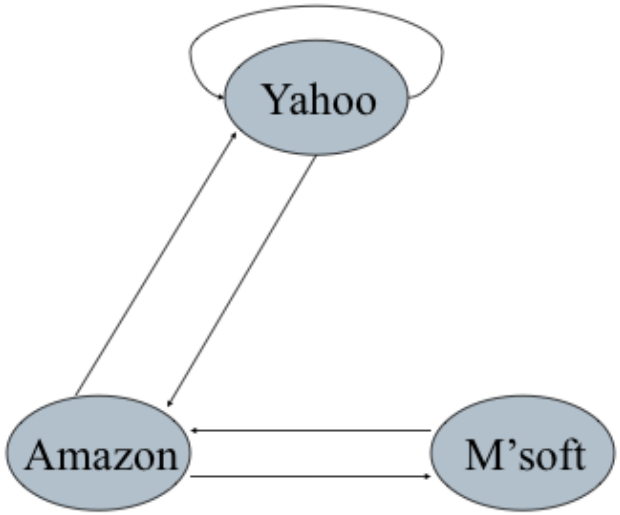

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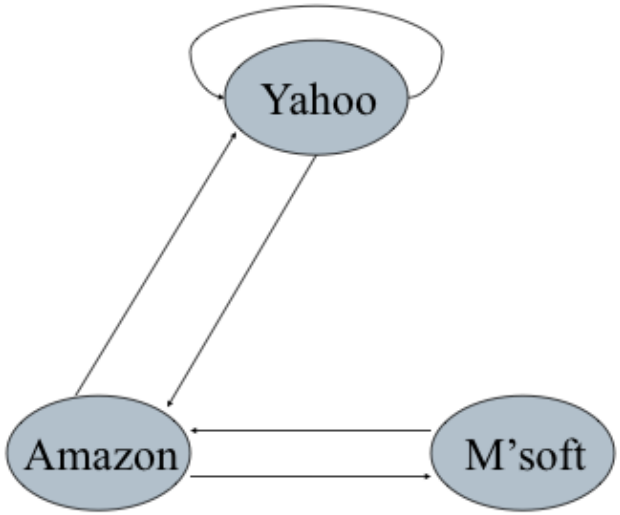

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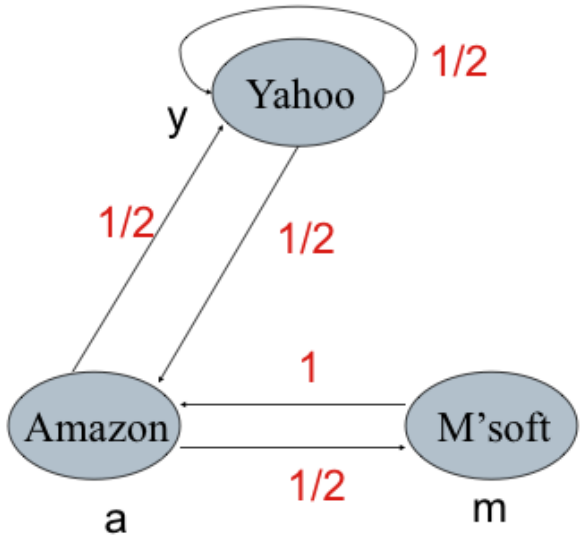

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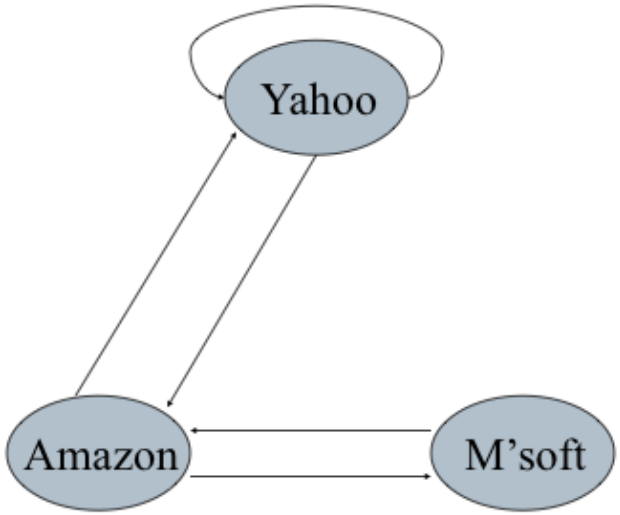

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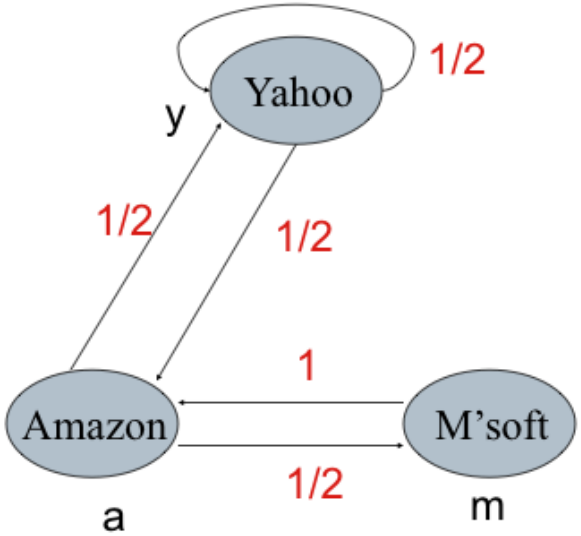


$$\begin{aligned} y &= \frac{y}{2} + \frac{a}{2} \\ a &= \frac{y}{2} + m \\ m &= \frac{a}{2} \\ y+a+m &= 1 \end{aligned}$$

PageRank

The importance of a web page can be computed by solving the corresponding system of linear equations.

However there are two main issues:

- The solution might not be unique!
- It is expensive to solve large system of linear equations. *Gaussian elimination* requires $\Omega(n^3)$ operations.

PageRank: Eigenvector Computation

Definition 1

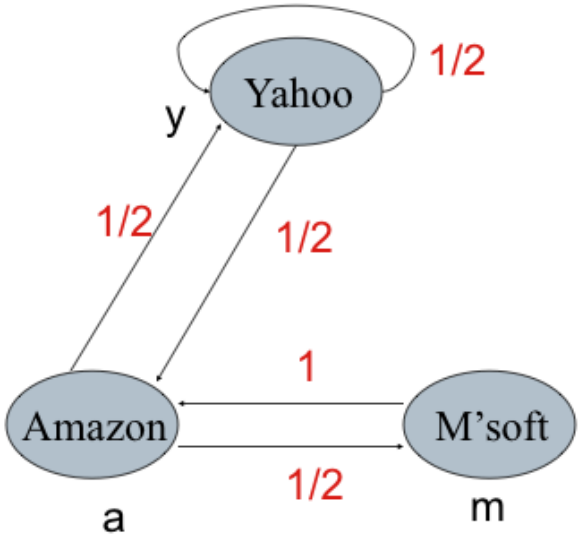
The vector x is an eigenvector of the matrix A with eigenvalue λ if

$$Ax = \lambda x.$$

Therefore, one could compute the importance of web pages by computing an eigenvector with eigenvalue 1 of M_G, which is also very expensive!

PageRank: Random Surfer

A *random surfer* starts surfing the web from a random page (step 0). At step t, let u be the web page currently visited by the random surfer. At step $t+1$, the random surfer visits a web page v being picked uniformly at random from the $\delta_{\text{out}}(u)$ neighbors.



$$M_G = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

M_{ij} is the probability that the random surfer moves from page u_j to u_i.

PageRank = probability of visiting web pages

We would like to compute the importance of a web page u as the probability that the random surfer visits u at step $t > 0$ for some large t.

Issues:

- Which value of t?
- The probability of visiting u might depend on the starting page.
- Can such a probability be computed efficiently?
- So called *spider traps* and dead-ends might give non-interesting results. A spider trap is a set of web pages S not containing any link to $V \setminus S$. *Dead-ends* are web pages not having any link to any other web page. We assume for simplicity that there are no dead-ends.

Random Surfer 2.0

The random surfer starts from one page chosen uniformly at random. Then, at step t it either 1) follows one random link from the current page with prob. β or 2) "jumps" to any of the n pages uniformly at random.

Common values for β are in $[0.8, 0.9]$.

Let A be the $n \times n$ matrix where $A_{ij} = \beta M_{ij} + \frac{1-\beta}{n}$, $i, j \in [1, n]$. A_{ij} denotes the probability that the random surfer 2.0 moves from page u_j to page u_i.

PageRank: Interpretation

Given a web graph $G = (V, E)$ the following facts hold:

- Let $P(X_t = u)$ be the probability that the random surfer visits page u at step t. The PR of page u is equal to $\lim_{t\to\infty} P(X_t = u)$.
- The PR vector π is the (unique) eigenvector of A with eigenvalue 1.
- The PageRank vector π satisfies $\pi = A\pi$ and $\sum_{i=1}^{n} \pi_i = 1$.

To be proved later...

The PageRank algorithm

Input: A directed graph G with n nodes (Web pages), $0 < \beta < 1, \epsilon > 0$.
Output: The PageRank vector r of the web pages in G.

1: Remove *dead ends* iteratively from G;
2: Build the stochastic matrix M_G (M for short);
3: Let $\pi^{(0)} = [\frac{1}{n}, \ldots \frac{1}{n}]^T$
4: **while** (true) **do**
5: $\quad t = t + 1$;
6: $\quad \pi^{(t)} = A\pi^{(t-1)}$;
7: $\quad$ If $||\pi^{(t)} - \pi^{(t-1)}||_1 < \epsilon$ **break**;
8: **end while**
9: **return** $\pi^{(t)}$.

Efficiency issues

A is a dense $n \times n$ matrix. $n \gg 10^9$ which implies A contains $\gg 10^{18}$ non-zero entries. Lemma 2 allows us to deal with M_G which is sparse.

Lemma 2

Let $x \in \mathbb{R}$ and let $[x]_n$ be the vector with n entries equal to x. Let $G = (V, E)$ a directed graph, let $\beta > 0$, let M_G and A be the $n \times n$ matrices as defined above. For any π with $||\pi||_1 = 1$, we have:

$$A\pi = \beta M_G \pi + \left[\frac{1-\beta}{n}\right]_n$$

Proof of Lemma 2

Proof.

For any $i \in [1, n]$:

$$\begin{aligned}\sum_{j=1}^{n} A_{ij}\pi_j &= \sum_{j=1}^{n} \left(\beta M_{ij} + \frac{1-\beta}{n}\right)\pi_j \\ &= \beta \sum_{j=1}^{n} M_{ij}\pi_j + \frac{1-\beta}{n}\sum_{j=1}^{n}\pi_j \\ &= \beta \sum_{j=1}^{n} M_{ij}\pi_j + \frac{1-\beta}{n}.\end{aligned}$$

The PageRank algorithm (improved)

Input: A directed graph G with n nodes (Web pages), $0 < \beta < 1, \epsilon > 0$.
Output: The PageRank vector π of the web pages in G.

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6: $\quad \pi^{(t)} = \beta M \pi^{(t-1)} + [\frac{1-\beta}{n}]_n$;
7: $\quad$ If $||\pi^{(t)} - \pi^{(t-1)}||_1 < \epsilon$ **break**;
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Events and Probability

Consider a stochastic process (e.g. throw a dice, pick a card from a deck)

- Each possible outcome is a *simple event*.
- The sample space Ω is the set of all possible simple events.
- An event is a set of simple events (a subset of the sample space).
- With each simple event E we associate a real number $0 \leq P(E) \leq 1$, which is the probability that event E happens.

Probability Space

Definition 3

A *probability space* has three components:

- A *sample space* Ω, which is the set of all possible outcomes of the random process modeled by the probability space;
- A family of sets $\mathcal{F}$ representing the allowable events, where each set in $\mathcal{F}$ is a subset of the sample space in Ω;
- a *probability function* $P : \mathcal{F} \to \mathbb{R}$, satisfying the definition below (next slide).

Probability Function

Definition 4

A *probability function* is any function $P : \mathcal{F} \to \mathbb{R}$ that satisfies the following conditions:

- for any event E, $0 \leq P(E) \leq 1$;
- $P(\Omega) = 1$;
- for any finite or countably infinite sequence of pairwise mutually disjoint events $E_1, E_2, E_3, \ldots$

$$P\left(\cup_{i \geq 1} E_i\right) = \sum_{i \geq 1} P(E_i). \tag{1}$$

The Union Bound

Theorem 5

$$P(\cup_{i=1}^{n} E_i \leq \sum_{i=1}^{n} P(E_i)). \tag{2}$$

Example: roll a dice:

- let $E_1 =$ "result is odd"
- let $E_2 =$ "result is ≤ 2"

Independent Events

Definition 6

Two events E_1 and E_2 are *independent* if and only if

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) \tag{3}$$

Conditional Probability: Example

What is the probability that a random student at Telecom ParisTech was born in Paris?
E_1 = the event "born in Paris".
E_2 = the event "student at Telecom ParisTech".
The conditional probability that a a student at Telecom ParisTech was born in Paris is written:

$$P(E_1|E_2).$$

Conditional Probability: Definition

Definition 7

The *conditional probability* that event E_1 occurs given that event E_2 occurs is

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \tag{4}$$

The conditional probability is only well-defined if $P(E_2) > 0$.

By conditioning on E_2 we restrict the sample space to the set E_2.

Law of Total Probability

Theorem 8

Let $B_1, \ldots, B_k$ be a partition of the sample space Ω, with $P(B_i) > 0$, $i = 1, \ldots, k$. Then, for any event $A \subseteq \Omega$:

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(A|B_i)P(B_i). \tag{5}$$

Random Variable

Definition 9

A *random variable* X on a sample space Ω is a function on Ω; that is, $X : \Omega \to \mathbb{R}$.

A *discrete random variable* is a random variable that takes on only a finite number of values.

Examples

In practice, a random variable is some random quantity that we are interested in:

- I roll a die, $X =$ result. E.g. $X = 6$.
- I pick a card, $X = 1$ if card is an Ace, 0 otherwise.
- I roll a dice two times. $X_1 =$ result of the first experiment, $X_2 =$ result of the second experiment. What is $P(X_1 + X_2 = 2)$?

Stochastic Processes

Definition 10

A stochastic process in discrete time $n \in \mathbb{N}$ is a sequence of random variables $X_0, X_1, X_2 \ldots$ denoted by $\mathbf{X} = \{X_n\}$.

We refer to the value X_n as the *state* of the process at time n, with X_0 denoting the initial state.

The set of possible values that each random variable can take is denoted by S. Here, we shall assume that S is finite and $S \subseteq \mathbb{N}$.

Markov Chains

Definition 11

A *stochastic process* $\{X_n\}$ is called a *Markov chain* if for any $n \geq 0$ and any value $j_0, j_1, \ldots, i, j \in S$,

$$P(X_{n+1} = i | X_n = j, X_{n-1} = j_{n-1}, \ldots, X_0 = j_0) = P(X_{n+1} = i | X_n = j),$$

which we denote by P_{ij}.

This can be stated as *the future is independent of the past given the present state*. In other words, the probability of moving to the next state does not depend on what happened in the past. Note that $P_{ij} \neq P_{ji}$.

One-step Transition Matrix

P_{ij} denotes the probability that the chain, whenever in state j, moves next into state i.

The square matrix $\mathbf{P} = (P_{ij})$, $i, j \in S$, is called the *one-step transition matrix*. Note that for each $j \in S$ we have:

$$\sum_{i \in S} P_{ij} = 1. \tag{6}$$

n-step Transition Matrix

The *n-step transition matrix* $\mathbf{P}^{(n)}$, $n \geq 1$, where

$$P_{ij}^{n} = P(X_n = i | X_0 = j) = P(X_{m+n} = i | X_m = j), \quad \forall m \tag{7}$$

denotes the probability that n steps later the Markov chain will be in state i given that at step m is in state j.

Theorem 12

$$\mathbf{P}^{(n)} = \mathbf{P}^n = \mathbf{P} \times \mathbf{P} \times \cdots \times \mathbf{P}, n \geq 1.$$

Stationary Distribution

Definition 13

A probability distribution π over the states of the Markov chain ($\sum_{j \in S} \pi_j = 1$) is called a *stationary distribution*[a] if

$$\pi = P\pi. \tag{8}$$

[a] in literature the transpose of P is often used, in that case $\pi = \pi P$.

Irreducible Markov Chains

Definition 14

A Markov chain is called *irreducible*[a] iff for any $i, j \in S$, there is $n \geq 1$ s.t.

$$P_{ij}^n > 0. \tag{9}$$

[a] definition slightly different when S is not finite.

That is, we can move from any state i to any state j, in one or more steps. If a Markov chain is irreducible then there must be n such that $P_{ii}^n > 0$.

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Theorem 15

If a Markov chain is irreducible, there is a unique stationary distribution.

Aperiodic Markov Chains

A state i has period k if any return to i occurs at step $k \cdot l$, for some $l > 0$. Formally,

$$k = gcd\{n : P(X_n = i | x_0 = i) > 0\} \tag{10}$$

where gcd denotes the *greatest common divisor*. If $k = 1$ then state i is said to be *aperiodic*.

Definition 16

A Markov chain is called *aperiodic* if every state is aperiodic.

Main Theorem

Theorem 17

If a Markov chain is irreducible and aperiodic[a]*, then the Markov chain converges to its (unique) stationary distribution, that is,*

$$\pi_j = \lim_{n\to\infty} P(X_n = j) = \lim_{n\to\infty} P(X_n = j | X_0 = i), \quad \forall i, j \in S. \tag{11}$$

[a] in this case the Markov chain is called *ergodic*

Note: Equation (11) holds for any initial state i of the Markov chain.

Markov chains and the Random Surfer

Consider the Markov chain (MC) of the random surfer.

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- If there are no random jumps, is the MC irreducible/aperiodic ? Not necessarily (because of spider traps/dead ends).
- What if we add random jumps? It is both irreducible and aperiodic, which implies that the PageRank vector converges to the unique stationary distribution π of MC (see next slide).

The Random Surfer and its Stationary Distribution

Fact: The stationary distribution π of the MC is the PageRank vector!

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Sketch: At each step n of the PageRank algorithm we compute $\pi^{(n)} = P\pi^{(n-1)} = P^n\pi^{(0)} = P^{(n)}\pi^{(0)}$. From the law of total probability (Theorem 8) and from the fact that $\pi^{(n)} = P^{(n)}\pi^{(0)}$ it follows that $\pi_j^{(n)} = P(X_n = j)$, $\forall j$, which converges to π (Theorem 17).

Therefore, the PageRank algorithm computes for every page j the probability that the random surfer visits page j, i.e. $\lim_{n\to\infty} P(X_n = j)$.

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