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On the Inertial-Electrostatic Confinement of a Plasma*

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A system in which electrons are projected radially inwards from a spherical surface, has been proposed for the confinement of a plasma at thermonuclear temperatures. The equilibrium, economics, and stability of such a system are discussed theoretically.

Although we conclude that it is of doubtful utility as a thermonuclear reactor, it may be possible to produce in this way small regions of thermonuclear plasma for study. The device appears to be unstable at economic densities. The stability is discussed in terms of a virial, which turns out to be mathematically tractable in this geometry.

I. INTRODUCTION

If ELECTRONS are projected with energy eV_g radially inward over the surface of a sphere, with spherical symmetry, they will be stopped by their mutual repulsion near the center and be reflected back toward the outer surface. Under the conditions described, the electrostatic potential within the sphere will be roughly as in Fig. 1. A potential well of depth V_g will exist at the center. Positive ions of energy $\epsilon_i < eV_g$ may then be confined within the well.

For thermonuclear effects, deuterium and tritium ions at a temperature of the order of 10 kev would be required, and this in turn implies a well depth $V_g \simeq 100$ kev in order that the confinement be reasonably complete.

An arrangement for realizing this geometry has been proposed (Wells).[1] The inside of an evacuated sphere is formed into a hot cathode electron emitter, and a spherical grid maintained at $+V_g$ with respect to the cathode accelerates the electrons inward (Fig. 1).

An electron can be expected to make several trips to the center before being captured by the grid, and if the electron oscillates ν times, the effective space current will be ν times the current drawn to the grid. The grid support and grid wires introduce an inevitable but perhaps small departure from spherical symmetry.

A similar proposal for realizing this geometry

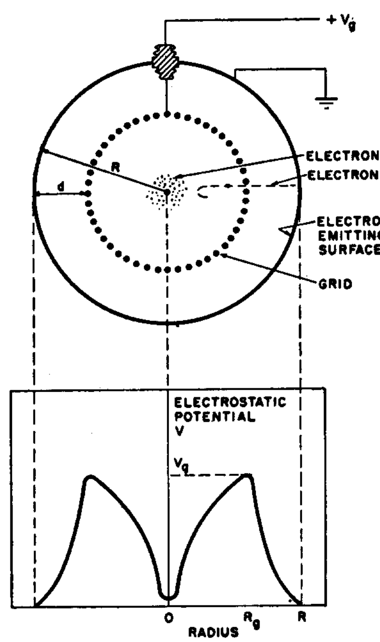

FIG. 1. The spherical device is drawn schematically.

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[1] W. H. Wells, Bendix Aviation Corporation (private communication, 1954).

has been made by Farnsworth.[2] The surface of the cathode sphere can be formed of material with a favorable secondary electron multiplication coefficient, and the positive grid, with or without a positive bias potential, driven by an oscillatory potential of frequency $f = 1/2\tau$, where τ is the round-trip transit time for an electron from the wall and back. Large secondary electron currents are known to be obtainable in dynamic multipliers using this principle.[3] And in highly evacuated spherical multipliers, light emission from the vicinity of the center has been reported.[4]

We propose first to determine the optimum conditions for plasma confinement, and then to investigate the thermonuclear economics under these optimum conditions. The basic economic requirement is that the thermonuclear power developed exceed that expended in maintaining the electron current.

Finally, we shall investigate the stability of the system. It will appear likely that it is stable for sufficiently low ion densities—but unstable for ion densities comparable to that for "optimum" operation (as mentioned in the foregoing).

II. ELECTROSTATIC EQUILIBRIUM

(A) Zero Temperature

We proceed to describe the electrostatic equilibrium of the device using Poisson's equation and assuming *zero* temperature plasmas. In Secs. II (B) and (C) we shall extend this to plasmas of finite temperature.

Let ρ_i and ρ_e be the charge densities due to ions and electrons, respectively. At a given radius r, all electrons have a radial velocity $\pm v_e$ (one-half are moving inward and one-half outward) and all ions the radial velocity $\pm v_i$. We further assume that all particles have only radial motion [this assumption will be critically studied in Sec. II(B)]. Then

$$I_e \equiv 4\pi r^2\left(\frac{\rho_e}{2}\right)v_e \tag{1}$$

= total electron current directed out (or in),

$$I_i \equiv 4\pi r^2\left(\frac{\rho_i}{2}\right)v_i$$

= total ion current directed out (or in).

[2] P. T. Farnsworth, Farnsworth Electronics Company (private communication, 1956).
[3] P. T. Farnsworth, J. Franklin Inst. **218**, 411 (1934).
[4] P. T. Farnsworth (private communication).

Poisson's equation, which describes the system in static operation, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = -4\pi[\rho_i - \rho_e], \tag{2}$$

where V is the electrostatic potential. (It should be remembered that the system has been assumed to be spherically symmetric.)

Conservation of energy requires that (m_e is the electron mass and m_i is the ion mass)

$$\tfrac{1}{2}m_e v_e^2 - eV = \epsilon_{0e},$$
$$\tfrac{1}{2}m_i v_i^2 + eV = \epsilon_{0i}, \tag{3}$$

where ϵ_{0e} and ϵ_{0i} are the total energies, respectively, of an electron or ion. A convenient variable is[3]

$$W \equiv \frac{\tfrac{1}{2}m_e v_e^2}{\Lambda}, \tag{4}$$

where Λ is a constant parameter (yet to be determined) which has the dimensions of energy.

Equations (3) and (4) lead to

$$v_e = \left(\frac{2\Lambda}{m_e}\right)^{\frac{1}{2}}W^{\frac{1}{2}},$$
$$v_i = \left(\frac{2\Lambda}{m_i}\right)^{\frac{1}{2}}[W_0 - W]^{\frac{1}{2}}, \tag{5}$$
$$W_0 \equiv (1/\Lambda)[\epsilon_{0e} + \epsilon_{0i}].$$

Equations (1) may be solved for ρ_e and ρ_i in terms of v_e and v_i, which in turn may be eliminated with Eqs. (5). Then

$$4\pi e[\rho_i - \rho_e] = -\frac{2eI_e}{\left(\frac{2\Lambda}{m_e}\right)^{\frac{1}{2}}}\frac{1}{r^2}\left[\frac{1}{W^{\frac{1}{2}}} - \frac{G_0}{[W_0 - W]^{\frac{1}{2}}}\right]$$
$$G_0 \equiv \frac{I_i}{I_e}\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}. \tag{6}$$

Returning to the Poisson equation (2), and setting $e(\partial V/\partial r) = \Lambda\ (\partial W/\partial r)$, we obtain

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial W}{\partial r}\right) = \frac{1}{W^{\frac{1}{2}}} - \frac{G_0}{[W_0 - W]^{\frac{1}{2}}}, \tag{7}$$

if we define

$$\Lambda^{\frac{3}{2}} \equiv \sqrt{2}\, eI_e(m_e)^{\frac{1}{2}}. \tag{8}$$

This equation may be re-expressed in terms of the independent variable

$$z \equiv \ln\frac{r}{r_0}. \tag{9}$$

Here r_0 is a parameter which will be considered characteristic of the "width" of the central region of the potential well. This is effected by choosing the boundary condition

$$\frac{\partial W}{\partial z} = 0 \quad \text{for} \quad z = 0. \tag{10}$$

Equation (7) now is

$$\frac{d^2W}{dz^2} + \frac{dW}{dz} = \frac{1}{W^{\frac{1}{2}}} - \frac{G_0}{[W_0 - W]^{\frac{1}{2}}}. \tag{11}$$

Solutions of this equation for the case $G_0 = 0$ indicate that a potential well of depth $V_g \simeq 100$ kev and radius $r_0 \simeq 1$ cm could be maintained in a sphere 1 m in diameter with a current $(1/\nu)\, I_e \simeq 100$ amp. This seems to be a very reasonable system from a practical point of view.

Two questions immediately arise concerning this conclusion. First, the center of the potential well is restricted to a very small volume compared with the total available within the sphere. This is achieved, however, by assuming that the electrons have *no* angular momentum about the center of the sphere. The electrons will develop transverse motion from scattering by the accelerating grid and from scattering against the ions. In part (B) we shall investigate these limitations.

A more serious difficulty occurs when $G_0 \neq 0$ $[G_0 \geq 0]$. If we integrate Eq. (11) out from $z = 0$, W will increase from its value W_L at $z = 0$. As $W \rightarrow W_0$, the second term will dominate on the right-hand side of (11), tending to cause W to start *decreasing* again. (Clearly W never reaches the value W_0.) This means that the potential *never becomes large enough* to confine the ions [see Eq. (5)]. Physically, this means that the ion charge density always dominates the electron charge density when the ions have "slowed down" sufficiently.

This difficulty is not inherent in the device described, however. It arises because the ions have all been given a single energy ϵ_{0i}, as specified by Eq. (3). On the other hand, if the ions are injected with a distribution of values ϵ_{0i}, or if they are injected sufficiently slowly for a distribution to develop due to gas scattering, then it may be possible to have a system for which W is not bounded from above by the ion charge density. In part (C) this situation will be investigated quantitatively.

(B) Electron Angular Momentum

An approximate calculation of the average deflection of an electron in passing through a grid, whose wires are spaced a cm apart, is made easily. This angular deflection is

$$\bar{\phi}_e \simeq \frac{a}{16\pi\, d}, \tag{12}$$

where d is the distance of the grid from the emitting surface (see Fig. 1). If an electron moved inward without further deflection, it would miss the center by an amount

$$X = R\bar{\phi}_e = a\left(\frac{R}{16\pi\, d}\right). \tag{13}$$

Since the average number of transversals of its orbit is ν for the electron, we find a net deflection from the center of

$$X_{\text{Av}} \simeq a\left(\frac{R}{16\pi\, d}\right)\nu^{\frac{1}{2}}. \tag{14}$$

For $\nu \simeq 10$ and a fine grid this seems to be a negligible effect.

The cumulative effect of small-angle electron-ion collisions will also give the electrons angular momentum about the center of the sphere. The rms angular deflection of an electron after time t is [n_i = ion density, ϵ_e = electron kinetic energy, b = impact parameter]

$$\phi_e \simeq \frac{e^2}{\epsilon_e}\left[\frac{2\pi}{3}\, n_i v_e \ln\left(\frac{b_{\max}}{b_{\min}}\right)\right]^{\frac{1}{2}} t^{\frac{1}{2}}. \tag{15}$$

The "displacement"

$$X = r\phi_e, \tag{16}$$

for $r = 1$ cm, $t = \nu r/v_e$, $n_i = 10^{17}/\text{cm}^3$, $\epsilon_e = 10$ kev, $\ln(b_{\max}/b_{\min}) \simeq 10$, is $X \simeq 0.1$ cm. This is probably not large enough to be serious, but the preceding analysis should be made in more quantitative detail, however, for a specific device.

The discussion at the end of part (A) indicated the importance of collisions in giving the ions a velocity distribution. The angular deflection ϕ_i of an ion from its original orbit is given approximately by

$$\phi_i \simeq \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \phi_e, \tag{17}$$

where ϕ_e is the electron deflection as given by Eq. (15). The "equilibration time" for an ion may be taken as that time for which $\phi_i \simeq 1$. Upon using Eqs. (15) and (17), we find that

$$t \simeq 10^{-3}\ \text{sec} \tag{18}$$

for an equilibrium distribution to develop for the parameters used in the foregoing. This is somewhat less than thermonuclear "burning times" so we may expect the velocity distribution of ions to develop approximately a Maxwellian form.

(C) Ion Velocity Distribution

If the ions are distributed according to a Maxwell distribution with temperature θ (as has been indicated), then the ion density as a function of r will be

$$n_i = n^0 \exp\left(-\frac{eV}{\theta}\right) \equiv \left[n_0' \exp\left(\frac{W_L}{\rho}\right)\right] \exp\left(-\frac{W}{\rho}\right), \tag{19}$$

with

$$\rho \equiv \theta/\Lambda \tag{20}$$

and W_L the value of W at $r = r_0$. Here n^0 and n_0' are constants, n_0' being the density at $r = r_0$.

On the other hand, for only radial velocities and for the distribution of ϵ_{0i} values $P(\epsilon_{0i})\, d\epsilon_{0i}$, the distribution function in r and v_i space is

$$f(r, v_i) = \frac{C}{r^2} P(\tfrac{1}{2}m_i v_i^2 + eV), \tag{21}$$

with

$$n_i(r) = \int f(r, v_i)\, dv_i. \tag{22}$$

We shall actually take

$$n(r) = n_0 r_0^2 \frac{\exp\left(-\frac{W}{\rho}\right)}{r^2} \tag{23}$$

for our subsequent discussion. This corresponds roughly to a "radial Maxwell distribution."

The distribution (23) will be used rather than (19) for our calculations. The reason for this is mathematical, since with (23) we can obtain in closed form the parameters of an "optimum system." Since the ratio $(r_0/r)^2$ is limited in its variation in a practical device, we do not expect any appreciable difference in the operation of the two systems (19) and (23). Finally, Eq. (23) is *in any case optimistic* for the behavior of the device. That is, the intrinsic difficulty is to obtain a sufficient ion density for thermonuclear reactions at the center and, at the same time, a low enough ion density elsewhere, that the charge density remains *negative*.

In discussing the optimum possible performance of the electrostatic system, we are evidently being conservative in using Eq. (23).

With the new distribution function (23) we may redevise the appropriate Poisson equation (2). In place of Eq. (11), we now obtain

$$\frac{d^2W}{dz^2} + \frac{dW}{dz} = \frac{1}{W^{\frac{1}{2}}} - G \exp\left(-\frac{W}{\rho}\right), \tag{24}$$

$$G \equiv \frac{2\pi n_0 e r_0^2}{I_e}\left(\frac{2\Lambda}{m_e}\right)^{\frac{1}{2}} \tag{25}$$

[Λ is defined by Eq. (8)].

III. CONDITION FOR OPTIMUM THERMONUCLEAR PERFORMANCE

(A) General Statement of Point Balance

The power input to the device is the grid current times the grid voltage, divided by the number of times an average electron goes into the control region of the device, or

$$P_i = \frac{1}{\nu} I_e V_g, \tag{26}$$

where I_e and V_g are in esu.

The thermonuclear power output is

$$P_0 = \left[\frac{4\pi}{3} r_0^3\right]\left[\frac{n_0^2}{4} \langle\sigma_{\mathrm{DT}} v\rangle_{\mathrm{Av}}\, \Delta\epsilon\right]. \tag{27}$$

Here $[(4\pi)/3]\, r_0^3$ is considered as the active volume in which reactions occur. $\langle\sigma_{\mathrm{DT}}\; v\rangle_{\mathrm{Av}}$ is the D-T reaction cross section times relative velocity averaged over a Maxwellian velocity distribution for the ions. [The D-T is more favorable than the D-D reaction.] $\Delta\epsilon$ is the energy liberated in a D-T reaction and is taken as

$$\Delta\epsilon = 15 \text{ Mev}. \tag{28}$$

Finally, we take

$$\frac{n_0^2}{4} = n_{\mathrm{D}} n_{\mathrm{T}}, \tag{29}$$

where n_{D} and n_{T} are the respective densities or deuterons and tritons.

Equation (27) is subject to a geometrical correction factor of order unity, since $[(4\pi)/3]\, r_0^3$ gives only an approximate value for the reaction volume.

The fundamental condition for a successful device is

$$\frac{P_0}{P_i} = \frac{n_0^2 r_0^3\left[\frac{\pi}{3}\,\Delta\epsilon\langle\sigma v\rangle_{\mathrm{Av}}\right]}{I_e V_g}\,\nu > 1. \tag{30}$$

(B) Optimum Electrostatic System

We now seek the largest ion density for which ion containment is possible. In mathematical terms, we seek the largest value of G in Eq. (24) for which W is not bounded from above. This is illustrated in Fig. 2 where the ions are shown as confined to the bottom of a potential well.

Equation (24) is multiplied by $p \equiv (dW/dz)$, so it reads

$$\frac{d}{dz}\left(\frac{p^2}{2}\right) + p^2 = \frac{dW}{dz}\left[\frac{1}{W^{\frac{1}{2}}} - G \exp\left(-\frac{W}{\rho}\right)\right]. \quad (31)$$

With $W = W_L$ and $p = 0$ at $z = 0$, we integrate out from $z = 0$ to obtain

$$\begin{aligned} \frac{p^2}{2} + \int_0^z p^2\,dz &= \int_{W_L}^{W}\left[\frac{1}{W^{\frac{1}{2}}} - G\exp\left(-\frac{W}{\rho}\right)\right]dW \\ &= \Big\{2[W^{\frac{1}{2}} - (W_L)^{\frac{1}{2}}] \\ &\quad - \rho G\Big[\exp\left(-\frac{W_L}{\rho}\right) - \exp\left(-\frac{W}{\rho}\right)\Big]\Big\}. \end{aligned} \quad (32)$$

Since $\int p^2 dz > 0$,

$$\begin{aligned} p^2 \le 2\Big\{2[W^{\frac{1}{2}} - (W_L)^{\frac{1}{2}}] - \rho G\Big[\exp\left(-\frac{W_L}{\rho}\right) \\ - \exp\left(-\frac{W}{\rho}\right)\Big]\Big\} \equiv F(W). \end{aligned} \quad (33)$$

Clearly, $F(W)$ must be positive for all $W_L < W < W_{\max}$. This implies an *upper limit* on G. Because of the range of parameters involved (for confinement $W_{\max} \gg \rho$) it will suffice to take

$$\begin{aligned} W_L &= 0, \\ W_{\max} &= \infty. \end{aligned} \quad (34)$$

$$F(W) = 2\Big\{2W^{\frac{1}{2}} - \rho G\Big[1 - \exp\left(-\frac{W}{\rho}\right)\Big]\Big\}. \quad (35)$$

Now the least value of F is given by

$$\frac{dF}{dW} = 0, \quad (36)$$

defining a root $W = W_R$. Condition (33) implies that

$$F(W_R) \ge 0. \quad (37)$$

If we write

$$Q \equiv \rho^{\frac{1}{2}} G, \quad (38)$$

Eq. (36) becomes

$$1 = Qx \exp(-x^2), \quad (39)$$

where $x^2 \equiv (1/\rho)\, W_R$. The equation $F(W_R) = 0$ is

$$Q = \frac{2x}{1 - \exp(-x^2)}. \quad (40)$$

The solution of Eqs. (39) and (40) is

$$Q = \rho^{\frac{1}{2}} G_{\max} \simeq \pi,$$

or

$$G \le \left(\frac{\Lambda}{\theta}\right)^{\frac{1}{2}}\pi. \quad (41)$$

If Eq. (25) for G is substituted into Eq. (41), there results

$$n_0 \le \left(\frac{m_e}{\theta}\right)^{\frac{1}{2}} \frac{I_e}{2^{\frac{1}{2}} e r_0^{\,2}}. \quad (42)$$

This provides an *upper limit* on the ion density n_0 in the central region.

The power ratio (30) will be *greatest* if we use the *largest possible* ion density, as given by making (42) an equality. Substituting, then, into Eq. (30) for n_0, one obtains

$$\frac{m_e}{\theta}\frac{I_e^{\,2}}{e^2 r_0}\frac{\left[\dfrac{\pi}{24}\,\Delta\epsilon\langle\sigma v\rangle_{\mathrm{Av}}\right]}{I_e V_g} > \frac{1}{\nu}. \quad (43)$$

Next, we set

$$eV_g \simeq 5\theta. \quad (44)$$

The value of eV_g cannot be much less than this, or the ions would not be confined. Again one does not wish to use a larger value because of the inequality (43).

On substituting Eq. (44) into (43) we find

$$\frac{\pi}{120}\,\frac{m_e I_e(\Delta\epsilon)}{e r_0}\,\frac{\langle\sigma v\rangle_{\mathrm{Av}}}{\theta^2} > \frac{1}{\nu}. \quad (45)$$

The next step is evidently to find that temperature, $\theta = \theta_0$, which maximizes the ratio

$$\frac{\langle\sigma v\rangle_{\mathrm{Av}}}{\theta^2}.$$

This occurs for[5]

$$\begin{aligned} \theta_0 &\simeq 40\ \mathrm{kev} \\ \langle\sigma v\rangle_{\mathrm{Av}} &\simeq 7.2(10)^{-16}\,\frac{\mathrm{cm}^3}{\mathrm{sec}}. \end{aligned} \quad (46)$$

Putting numerical values into the inequality (45) now gives

$$I_e > 10^{25}\,\frac{r_0}{\nu}\ \mathrm{esu}, \quad (47)$$

or

$$I_{\mathrm{amp}} > 10^{16}\,\frac{r_0}{\nu}\ \mathrm{amp}. \quad (48)$$

From this it appears that currents of the order of

[5] J. L. Tuck, Los Alamos Rept. No. 1190, Los Alamos Manuscript No. 1640.

10^{14} amp would be required for successful operation of the device described.

IV. STABILITY

We have seen that static solutions for which ions are contained do not exist for sufficiently high ion densities. Even when static solutions do exist, however, the question arises of stability of the system against small perturbations from the static condition. Although it is not *directly* applicable to the present case, Earnshaw's theorem strongly suggests that a plasma confined by electrostatic means may be unstable.

In the present section we shall investigate the stability of the system under consideration. It will appear likely that it is quite unstable for ion densities high enough to give appreciable thermonuclear yield.

To describe the behavior in time, we shall assume that the *ion* component of the plasma satisfies the usual hydrodynamic equations:

$$\begin{aligned} n_i m_i \frac{d\mathbf{v}_i}{dt} &= -\nabla p_i + e n_i \mathbf{E}, \\ \frac{\partial n_i}{\partial t} + \nabla\cdot(n_i \mathbf{v}_i) &= 0, \\ \frac{d}{dt}\,[p_i n_i^{-\gamma}] &= 0. \end{aligned} \tag{49}$$

Here p_i is the ion pressure, $\mathbf{E}$ is the electric field, and $\gamma = \frac{5}{3}$ is the ratio of specific heats for a monoatomic gas. Equations (49) are consistent with our conclusion in Sec. II that the ions will achieve, at least approximately, a Maxwellian velocity distribution.

We now linearize the quantities in Eqs. (49) about a static state:

$$\begin{aligned} n_i &= n_0 + n_i', \\ p_i &= p_0 + p_i', \\ \mathbf{v}_i &= \mathbf{v}_i', \\ \mathbf{E} &= \mathbf{E}_0 + \mathbf{E}'. \end{aligned} \tag{50}$$

Here the subscript zero represents the static value of the quantity in question and a superscript prime an infinitesimal perturbation from the static value. We have

$$\begin{aligned} p_0 &= P_0 \exp\left(-\frac{eV_0}{\theta}\right), \\ n_0 &= N_0 \exp\left(-\frac{eV_0}{\theta}\right), \\ \mathbf{E}_0 &= -\nabla V_0, \end{aligned} \tag{51}$$

where $V_0(r)$ is the static potential.

The linearized Eqs. (49) are

$$\begin{aligned} n_0 m_i \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= -\nabla p_i' + e n_i' \mathbf{E}_0 + e n_0 \mathbf{E}', \\ \frac{\partial n_i'}{dt} + \nabla\cdot\left(n_0 \frac{\partial \boldsymbol{\xi}}{\partial t}\right) &= 0, \\ \frac{\partial}{\partial t}\,[p_i' n_0^{-\gamma} - \gamma p_0 n_0^{-(\gamma+1)} n_i'] &+ \frac{\partial \boldsymbol{\xi}}{\partial t}\cdot\nabla[p_0 n_0^{-\gamma}] = 0. \end{aligned} \tag{52}$$

Here we have introduced a new *position* variable $\boldsymbol{\xi}$, defined by

$$\mathbf{v}_i' \equiv \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{53}$$

The last two equations of (52) may be integrated immediately to give

$$n_i' = -\nabla\cdot(n_0 \boldsymbol{\xi}) \tag{54}$$

and

$$p_i' = -\gamma p_0 \nabla\cdot\boldsymbol{\xi} - p_0 \frac{e}{\theta}\, \boldsymbol{\xi}\cdot\mathbf{E}_0. \tag{55}$$

To describe the perturbed electron motion, we linearize the equations [see Eqs. (1) and (3)]

$$\begin{aligned} \frac{m_e}{2} v_e^2 - eV &= \epsilon_{0e}, \\ I_e &= 2\pi r^2 \rho_e v_e, \end{aligned} \tag{56}$$

to obtain

$$\rho_e' = -\frac{eI_e}{2\pi r^2 v_{0e}^3 m_e}\, V', \tag{57}$$

where we use the notation of Eq. (50) to write

$$\begin{aligned} \rho_e &= \rho_{0e} + \rho_e', \\ v_e &= v_{0e} + v_e', \\ V &= V_0 + V'. \end{aligned} \tag{58}$$

It should be noted that in using Eqs. (56) we are assuming that the electrons can adjust their motion adiabatically to the perturbation and also that the electrons continue to move radially even when perturbed. The first assumption is evidently valid for the type of perturbation considered, because $m_e/m_i \ll 1$. The second assumption amounts to a constraint on the electron motion. Such a constraint is expected to make the system *appear* more stable than it actually is. On the other hand, the constraint is reasonable because of the high radial velocities which the electrons have.

Using Eqs. (54) and (57), Poisson's equation,

$$\nabla^2 V' = -4\pi[en_i' - \rho_e'],$$

becomes

$$\nabla^2 V' + k^2 V' = 4\pi_e \nabla\cdot(n_0 \xi), \tag{59}$$

where

$$k^2 = \frac{2I_e e}{r^2 v_{0e}{}^3 m_e}. \tag{60}$$

For the system satisfying Eq. (47), $k^2 \gg 10^3$ cm^{-2}. For perturbations of wavelength greater than 0.1 cm, then Eq. (59) becomes just

$$V' = \frac{4\pi e}{k^2}\nabla\cdot(n_0\xi). \tag{61}$$

For the other extreme of perturbations a very short radial wavelength $k_0{}^{-1} \ll k^{-1}$,

$$V' \simeq \frac{4\pi e}{k_0{}^2}\nabla\cdot(n_0\xi). \tag{61a}$$

We shall actually develop the case (61), the other case (61a) being obtainable from the first by a trivial modification of the final equations.

The complete description of the motion of the system for small perturbations is given now by the first of Eqs. (52), substituting Eq. (55) for p_i', Eq. (54) for n_i', and $\mathbf{E}' = -\nabla V'$ from Eq. (61). Rather than solve the resulting equation directly, we prefer to give a qualitative discussion based on the virial. The virial is defined as

$$\mathcal{V} \equiv -m_i \int n_{0i}\xi\cdot\frac{\partial^2\xi}{\partial t^2}\,d\tau, \tag{62}$$

where the volume integral is extended over the interior of the region containing plasma. We may interpret $\mathcal{V}$ as follows.

If the integrand in Eq. (62) is positive for a certain choice of ξ, this means that the acceleration $\partial^2\xi/\partial t^2$ is in the *same direction* as the *displacement* ξ. Thus the displacement ξ will grow with increasing time and the system is unstable with respect to that perturbation. We note that if there exists at least one choice of ξ which makes the system unstable, then the system is generally an unstable one.

On the other hand, if the integrand (62) is negative for a given displacement, the acceleration is in the opposite direction to the displacement. This means that the displacement tends to be *reduced* by the forces acting on the system and thus the system is stable, except in certain cases.[6]

[6] If it is "overstable," it is unstable because of growing oscillations even though the integrand in Eq. (62) is negative.

In view of the discussion just given, we shall call the system *unstable* if the virial

$$\mathcal{V} < 0 \tag{63}$$

for some ξ. A more complete discussion of stability might be carried out with the first of Eqs. (52) but is beyond our present scope. If we substitute the first of Eqs. (52) for $m_i n_{0i}\,(\partial^2\xi/\partial t^2)$ into the expression for $\mathcal{V}$, there results

$$\mathcal{V} = -\int \{p_i'(\nabla\cdot\xi) + eV'\nabla\cdot(n_{0i}\xi) - e\xi\cdot\mathbf{E}_0\nabla\cdot(n_{0i}\xi)\}\,d\tau. \tag{64}$$

Here we have performed a partial integration and discarded surface integrals on the assumption that the normal component of ξ vanishes on the surface of the system. Substituting from Eqs. (54), (55), and (51), we find

$$\mathcal{V} = +\int p_0\{\gamma z^2 + 2zy + y^2 - P(z+y)^2\}\,d\tau, \tag{65}$$

$$P \equiv \frac{4\pi e^2}{k^2}\frac{n_0}{\theta}. \tag{66}$$

$$z \equiv \nabla\cdot\xi, \qquad y = \frac{e}{\theta}\,\xi\cdot\mathbf{E}_0. \tag{67}$$

We may interpret z as $\simeq \xi/\lambda$, where λ is the wavelength of the disturbance and $y \simeq (e\lambda\mathbf{E}_0/\theta)z$, which is the change in the potential energy over the distance λ, divided by θ and multiplied by z.

Now,

$$\gamma z^2 + 2zy + y^2 \geq 0 \tag{68}$$

for all (z, y). Thus this term is *stabilizing*. It represents the fact that a Maxwellian distribution is intrinsically stable (in the absence of destabilizing effects). Since

$$P(z+y)^2 \geq 0, \tag{69}$$

the second term in the integrand is always destabilizing. It represents the result of electrostatic forces for a self-confined system, which by Earnshaw's theorem are expected to constribute to *instability*. Whether the net effect leads to stability or instability depends upon the size of P, or upon the ratio of ion density to electron density, etc.

For the "optimum" system described in Sec. III, we have from Eq. (42)

$$I_e = 2^{\frac{1}{2}} e n_0 r_0{}^2\left(\frac{\theta}{m_e}\right)^{\frac{1}{2}}. \tag{70}$$

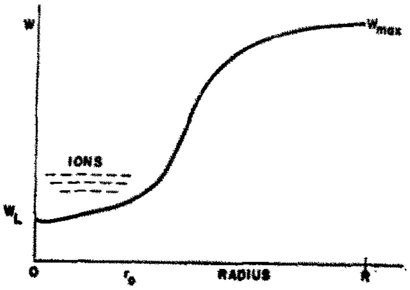

FIG. 2. The confinement of ions near the center is shown.

Then we easily find that [$\epsilon_e \equiv (m_e/2)\, v_{0e}{}^2$]

$$P = 2\pi\left(\frac{\epsilon_e}{\theta}\right)^{\frac{3}{2}}\left(\frac{r}{r_0}\right)^2. \tag{71}$$

Since $r/r_0 \gg 1$ and $\epsilon_e/\theta \gg 1$ over most of the system, we can easily find displacements ξ which made the virial $\mathcal{V}$ strongly negative. We conclude then that the system is unstable for ion densities sufficiently high that appreciable thermonuclear yield is expected.

Admittedly, a more thorough investigation is required to obtain a complete understanding of the stability of this electrostatic device.