Hoberman Brain Twist*

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1 Introduction

Figure 1: A Hoberman Brain Twist

A Hoberman Brain Twist is a puzzle that, like the well-known Rubik's Cube, involves mixing up colored "faces" with the intent to return them to their original position. What makes this tetrahedron shaped puzzle unique is that it can be flipped inside out to reveal another tetrahedron with different colored faces. It is possible for the outside to be solved while the inside is still mixed up and vice versa. In other words, both the outside and inside need to be take into account when trying to solve this puzzle.

2 Basic Definitions

- Group: A closed set that has an associative, binary operation, an identity element, and an inverse for each element.
- Coset: If G is a group, H is a subgroup of G, and $g \in G$, then
  $gH = \{gh : h \in H\}$ is a left coset of H in G, and
  $Hg = \{hg : h \in H\}$ is a right coset of H in G.
- Transversal: If G is a group, H is a subgroup of G, then a set T of (right) coset reprsentatives for H in G will be called a (right) transversal.
- Stabilizer: If S is a set, G is a group, $s \in S$, and G acts on S, the the stabilizer of s is

$$G_s = \{\sigma \in G : \sigma * s = s\}$$

3 Algorithm

3.1 Short Example

Let G be a group containing the sets σ and τ where $\sigma = (12345)$ and $\tau = (2354)$.

*Write-up of original poster presentation

1. Pick an element to be stabilized (in this example, 1 will be stabilized).
2. Find the transversal for G_1 by mapping 1 to each of the other elements using the generator sets (σ and τ). Note: The identity (1) is always in the transversal (Figure 2). The transversal set for the example is { 1, σ, $\sigma\tau$, $\sigma\tau\sigma$, $\sigma\tau^2$}.

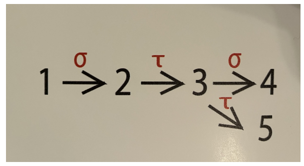

Figure 2: Mapping 1 to other elements

3. Combine one of the transversal elements (t) with one of the generators (x) and figure out what element in the transveral represents the new combination (i.e. where does the new combination send 1 and what transversal element sends 1 to the same place?)
4. Take tx and combine it with the inverse of the corresponding transversal element.
5. Repeat steps 3 and 4 until every transversal element has been combined with every generator.

t	x	tx	$\bar{tx}$	$tx\bar{tx}^{-1}$
1	σ	σ	σ	1
σ	σ	σ^2	$\sigma\tau$	(2 4 5 3)
$\sigma\tau$	σ	$\sigma\tau\sigma$	$\sigma\tau\sigma$	1
$\sigma\tau\sigma$	σ	$\sigma\tau\sigma^2$	$\sigma\tau^2$	(2 4 5 3)
$\sigma\tau^2$	σ	$\sigma\tau^2\sigma$	1	(2 5)(3 4)
1	τ	τ	1	τ
σ	τ	$\sigma\tau$	$\sigma\tau$	1
$\sigma\tau$	τ	$\sigma\tau^2$	$\sigma\tau^2$	1
$\sigma\tau\sigma$	τ	$\sigma\tau\sigma\tau$	σ	(2 5)(3 4)
$\sigma\tau^2$	τ	$\sigma\tau^3$	$\sigma\tau\sigma$	(2 5)(3 4)

Table 1: Generating the Stabilizers of 1

6. The results $tx\bar{tx}^{-1}$ are the new generators. Note: Any element in the new generators that can be written as a combination of any two or mor elements also in the new generaors can be thrown out. In Table 1, (2 5)(4 3) and (2 4 5 3) can both be written in terms of τ, so they are redundant.
7. The size of the group is the length of the transversal times the size of the stabilizer. From Table 1, $|G| = 5 \times |G_1|$ because there are five elements in the transversal and the size of the stabilizer of 1 is unknown.
8. Continue to stabilize elements in the group (using the new generators each time) until the only result is the identity, meaning the size of the stabilizer is 1 (Figure 3 and Table 2).

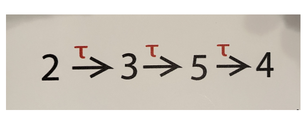

Figure 3: Mapping 2 to other unstabalized elements

t	x	tx	$\bar{tx}$	$tx\bar{tx}^{-1}$
1	τ	τ	τ	1
τ	τ	τ^2	τ^2	1
τ^2	τ	τ^3	τ^3	1
τ^3	τ	τ^4	1	1

Table 2: Generating the Stabilizers of 2

9. From Table 2, $|G| = 5 \times 4 \times |G_{1,2}| = 5 \times 4 = 20$. So, the size of the group $G = \{\sigma, \tau\}$ is 20.

3.2 The Puzzle

Applying this method to the puzzle takes a few extra steps. First, each piece was numbered 1-12. This made it so rotations could be written in cycle notation (i.e. rotating the front face clockwise was written (1 2 3) and rotating the top corner counterclockwise was written (1 7 4)). The generator sets are all of the clockwise face and corner rotations- there are eight in total, four faces and four corners.

With this knowledge and a 2-D model of the pyramid (the puzzle was not practical to work with), the same steps in the above example can be taken- just with much more work involved. Starting off with eight generators and a transversal that has twelve elements yields 96 results. Some of them were repeats or could be thrown out, leaving just sixteen for the new generator set, but the next transversal had eleven elements, which created an even longer set of results.

After going through this process a few times, the numbers finally dwindled down until the transversal only had three elements. It was at that point that the size of the group could finally be determined:

$$\begin{aligned}
G &= 12 \times |G_1| \\
&= 12 \times 11 \times |G_{1,8}| \\
&= 12 \times 11 \times 10 \times |G_{1,8,5}| \\
&= 12 \times 11 \times 10 \times 9 \times |G_{1,8,5,4}| \\
&= 12 \times 11 \times 10 \times 9 \times 8 \times |G_{1,8,5,4,2}| \\
&= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times |G_{1,8,5,4,2,10}| \\
&= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times |G_{1,8,5,4,2,10,11}| \\
&= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times |G_{1,8,5,4,2,10,11,9}| \\
&= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times |G_{1,8,5,4,2,10,11,9,12}| \\
&= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times |G_{1,8,5,4,2,10,11,9,12,3}| \\
&= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 1 \\
&= \frac{12!}{2}
\end{aligned}$$

From looking at the generators, the puzzle lives in S_{12}. Since the order is $\frac{12!}{2}$, the puzzle is in the subgroup A_{12} because it is the only subgroup with order $\frac{12!}{2}$. So, the permutation group of this puzzle is A_{12}.

4 Conclusion

This method is not the most efficient way to determine a group's size. If the group was much larger the work would be unmanageable by hand. Although with this puzzle, the order that the pieces were stabilized was random. If the order was chosen more methodically, it may have cut down on the work a bit.

5 References

Grove, Larry C. *Groups and Characters.* New York: Wiley-Interscience, 1997. Print.