cBART - Game theoretic analysis

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Task

- Two players are playing a competitive version of the BART (cBART), where each balloon is equally likely to burst anywhere from 1 to 10 pumps.
- Players play N balloons. Whichever player has the most points after N balloons wins an all-or-nothing bonus. If players tie, then one is randomly determined to win.
- For example, consider two players x and y playing a game with $N = 5$ balloons. Here are their point earnings across all 5 balloons:

Balloon	Player x: $x_p = 2$	Player y: $y_p = 5$
1	0	5
2	2	0
3	2	0
4	2	5
5	0	5
Total	**6**	**15**

- Here, player x always pumps 2 times, while player y always pumps 5 times.
- Because player y earned more points across all $N = 5$ balloons than player x, player y wins a (large) bonus while player x earns nothing.

Questions

- How should an optimal player adjust their pumping strategy (aka, risk level) when playing against a competitor with either a known, or unknown strategy?
- How does the optimal risk strategy change as a function of the number of games that are played?

Conclusions

- When the number of balloons is low (i.e.; $N = 1$), then the best response depends on one's expectations of their opponent.
  - If you believe that your opponent will pump 1 time, you should pump 2 times – if you expect he will pump 2 times you should pump 3 times. This continues until your expect your opponent to pump 5 times or more. In these cases, you should only pump once. This is because when an opponent pumps say 7 times, then it's better to pump once and hope that his pops.
- As the number of balloons increases, then the best response becomes closer to 5, *regardless* of how much you expect your opponent to pump.
  - For example, if you play $N = 100$ balloons, then you should always pump 5 times. The reason for this is that, if one plays infinitely many balloons, then one should simply try to maximize their expected rewards, regardless of the other player's strategy.
- It is (almost) never optimal to pump more than 50% of the time (i.e.; 5 out of 10 is the maximum).
- These results suggest that the *time horizon* (i.e. number of game opportunities) matters when deciding how to adjust your risk in a competitive task. In a game with a very short time horizon (i.e. $N = 1$), then only pumping a few times can be optimal given certain expectations of one's opponent. However, in games with very long time horizons (i.e., $N = 100$, then it is almost always optimal to use an individual payoff maximization strategy.)

Definitions

Parameter	Definition
N	Total number of ballooons
max	Maximum number of pumps per balloon
x_p	Pumps by player x
y_p	Pumps by player y

- Each balloon is equally likely to pop anywhere from 1 to max.
- If a balloon pops, the player earns no points. If the balloon does not pop, the player earns p points, where p is the number of pumps he made.
- Whichever player has the most points after N balloons wins an all-or-nothing bonus. If players tie, one is randomly determined to be the winner.

Formulas

- A player's earnings on each balloon is x_p with probability $1 - \frac{x_p}{max}$, or 0 with probability $\frac{x_p}{max}$
  - For example, given a balloon with a maximum of 10 pumps, a player who pumps 5 times will earn 5 with probability $1 - 5/10 = .5$, and 0 with probability $5/10$
- A player's distribution of earnings across N balloons is given by a binomial distribution with $p = 1 - \frac{x_p}{max}$, and $N = N$, where each success has a value of $x_p\}$

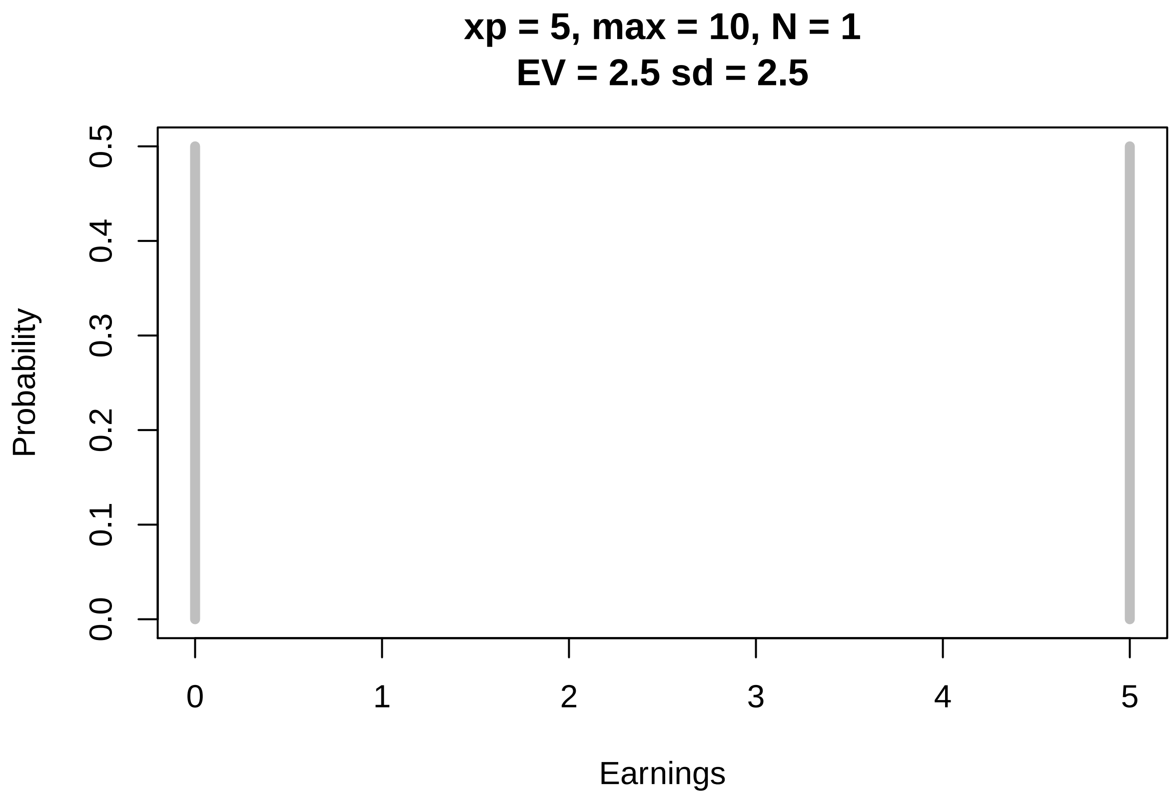


Figure 1: The distribution of reward outcomes when N = 1, max = 10, x.p = 50

Ex: N = 1, max = 10, x.p = 5

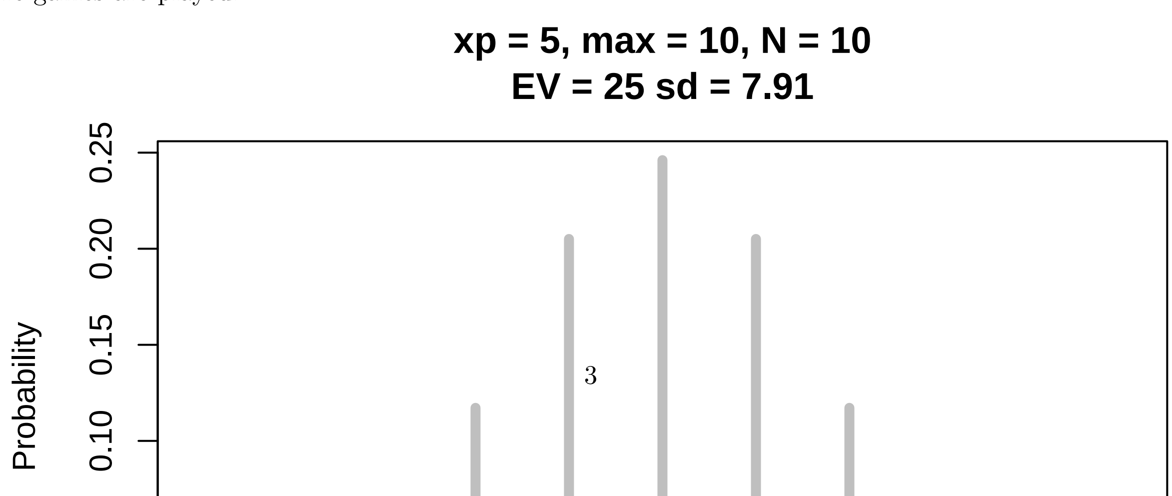
For example, consider a game with $N = 1$, $max = 10$, $x_p = 5$. Here, there are only two possible outcomes: a 50% chance of the balloon popping (for a reward of 0), and a 50% chance of saving the balloon (for a reward of 5)

Table 3: Outcome distribution for N = 1, max = 10, x.p = 5

points	prob
0	0.5
5	0.5

Ex: N = 10, max = 10, x.p = 5

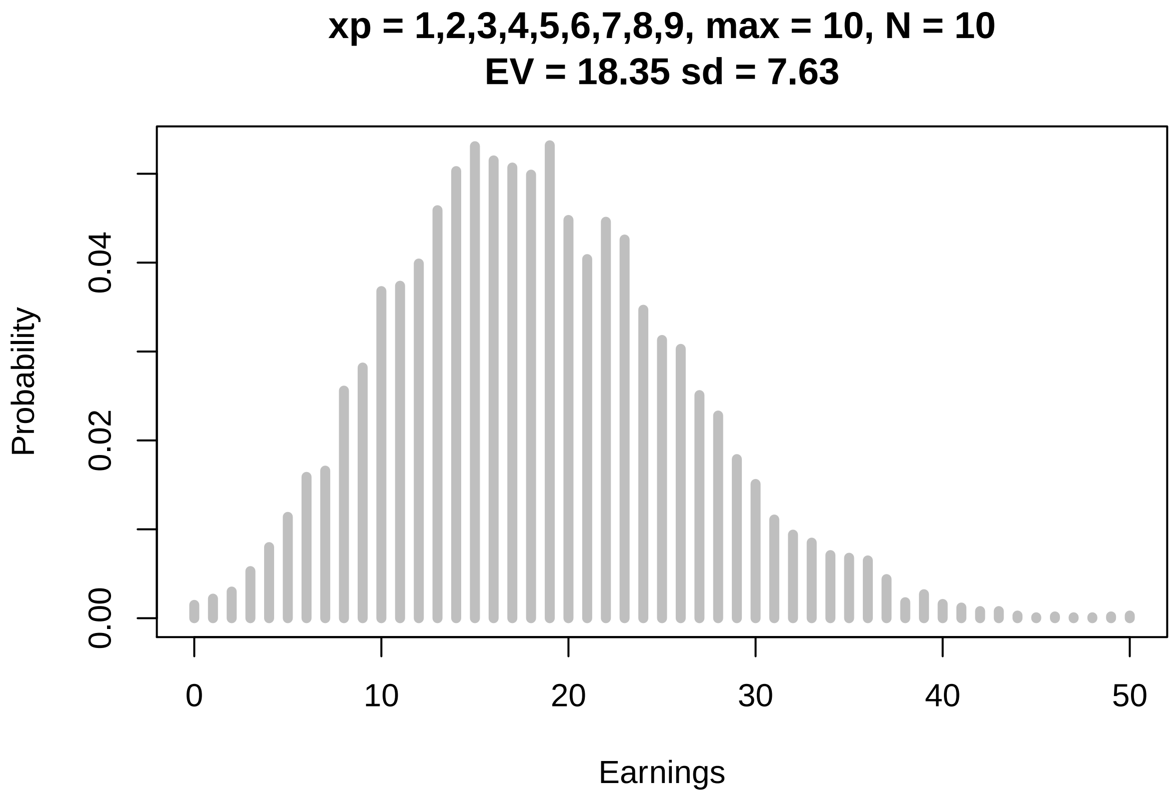
Now, let's increase the number of games to $N = 10$. Now there is a wider distribution of outcomes because more games are played.



points	prob
40	0.044
45	0.010
50	0.001

Ex: N = 10, max = 10, x.p = {1, 2, 3, 4, 5, 6, 7, 8, 9}

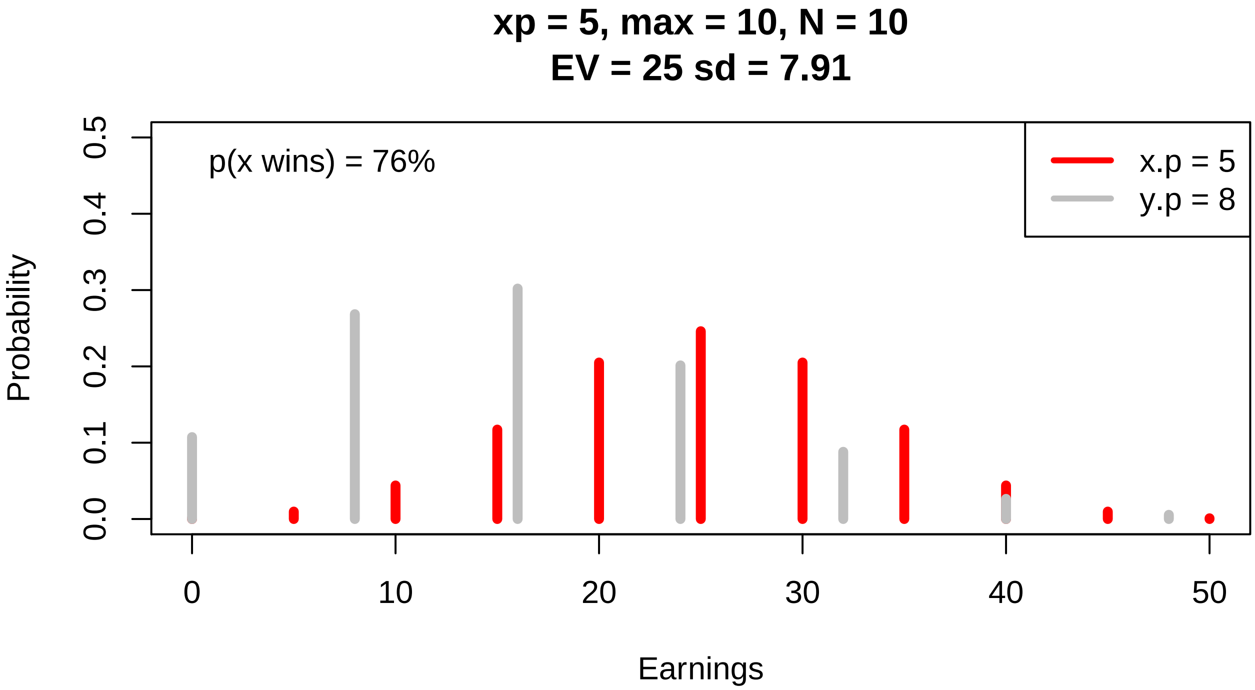
We can also do simulations where a player does not have a constant pumping rate, but rather a stochastic one. Here are the distributions of outcomes from a player who is equally likely to pump anywhere from 1 to 9 times. As we will see, this player has



Competition

Now, we calculate the probability of winning a game given two players x and y. At the start of the game, players x and y decide on their pumping values x_p, y_p. They then play N balloons. Players are not told how many points the other player earns over the course of the game (e.g.; as sequential balloons are played). When the game is finished, and all N balloons are completed, the player with the most points across all balloons wins an all-or-nothing bonus while the other player earns nothing.

We can calculate the probability that a player wins after N balloons by comparing each player's earning distributions after N balloons. For example, consider a player x who pumps 5 times ($x_p = 5$) and a player y who pumps 8 times ($y_{p} = 8) playing $N = 10$ balloons. Here are their expected earnings:



We can calculate the probability that x wins by comparing the two distributions directly (ie.; what is the summed probability of all outcomes where x wins?). Here, the probability that player x wins is 76%, and the probability that y wins is 24%.

Now, we calculate the probabiliy that x wins given all combinations of x_p and y_p for different numbers of balloons N.

Ex: N = 1, max = 10

- We'll start with a game with one balloon. That is, a one-shot game. Here, we see that a player's best response depends very much on his expectation of the other player. For example, if player y will pump just once, then player x should pump 2 times for a probability of winning of 80%. In contrast, if player y pumps 9 times, then player x should only pump 1 time for an 81% probability of winning. Finally, if player y pumps 5 times, then player y should pump just 1 time again.
- There is no Nash equilibrium in this short-horizon game as players should always adjust their strategy for every behavior of their opponent.

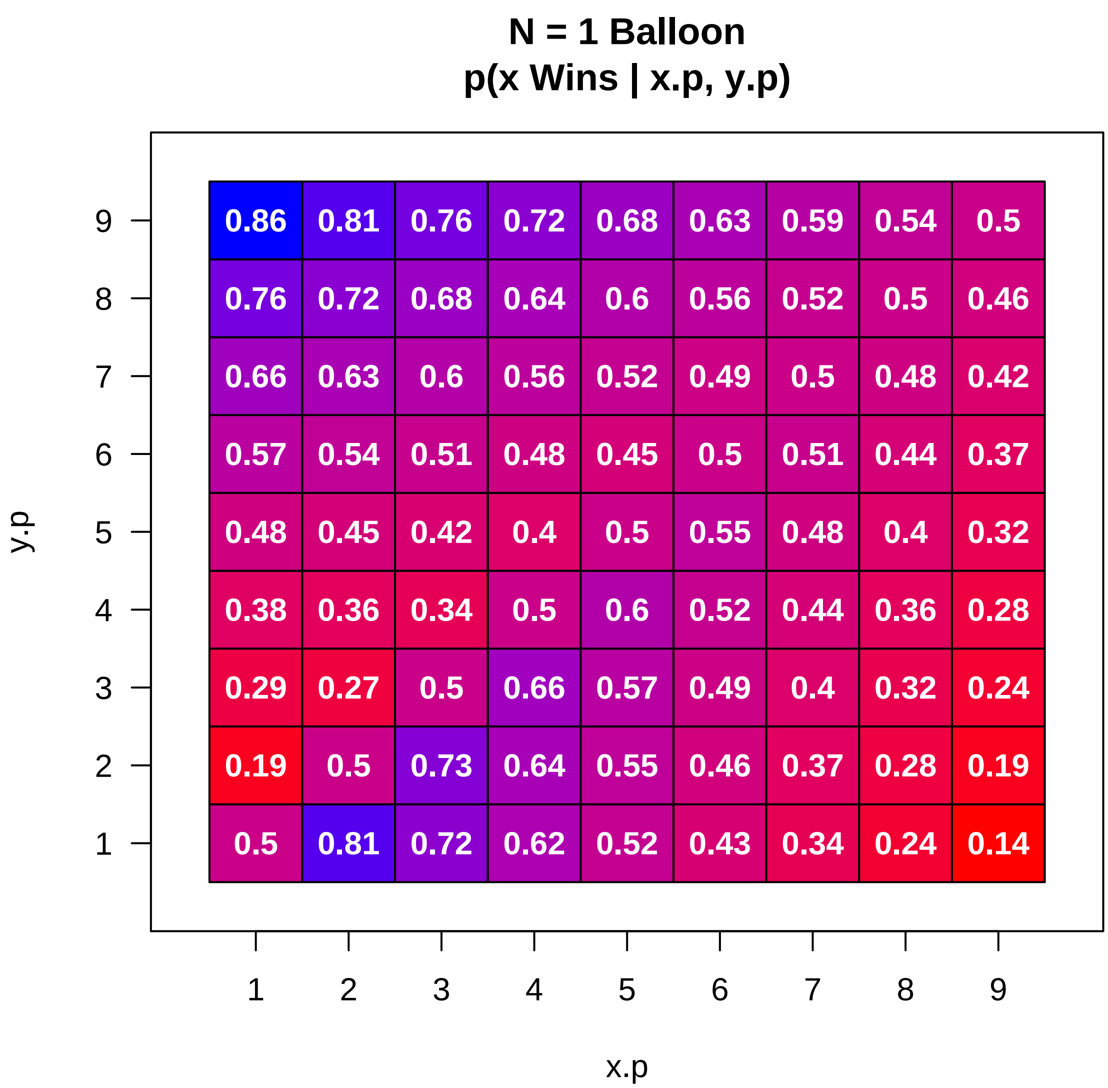
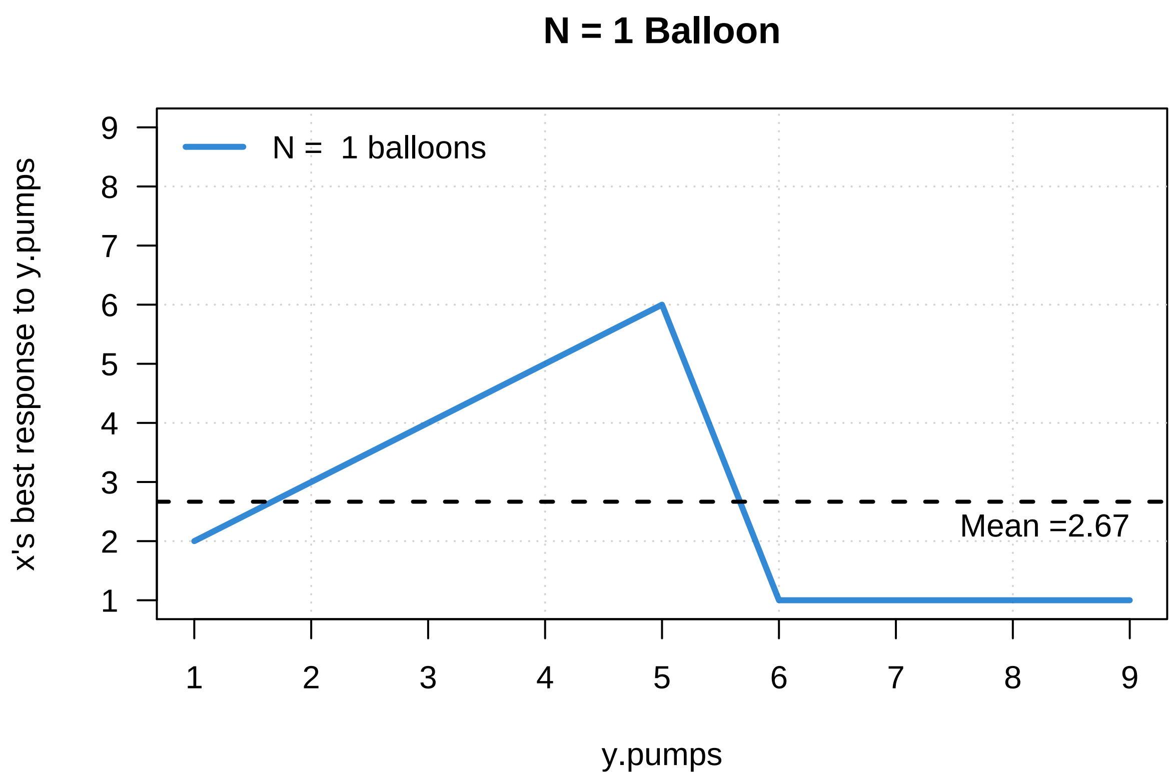
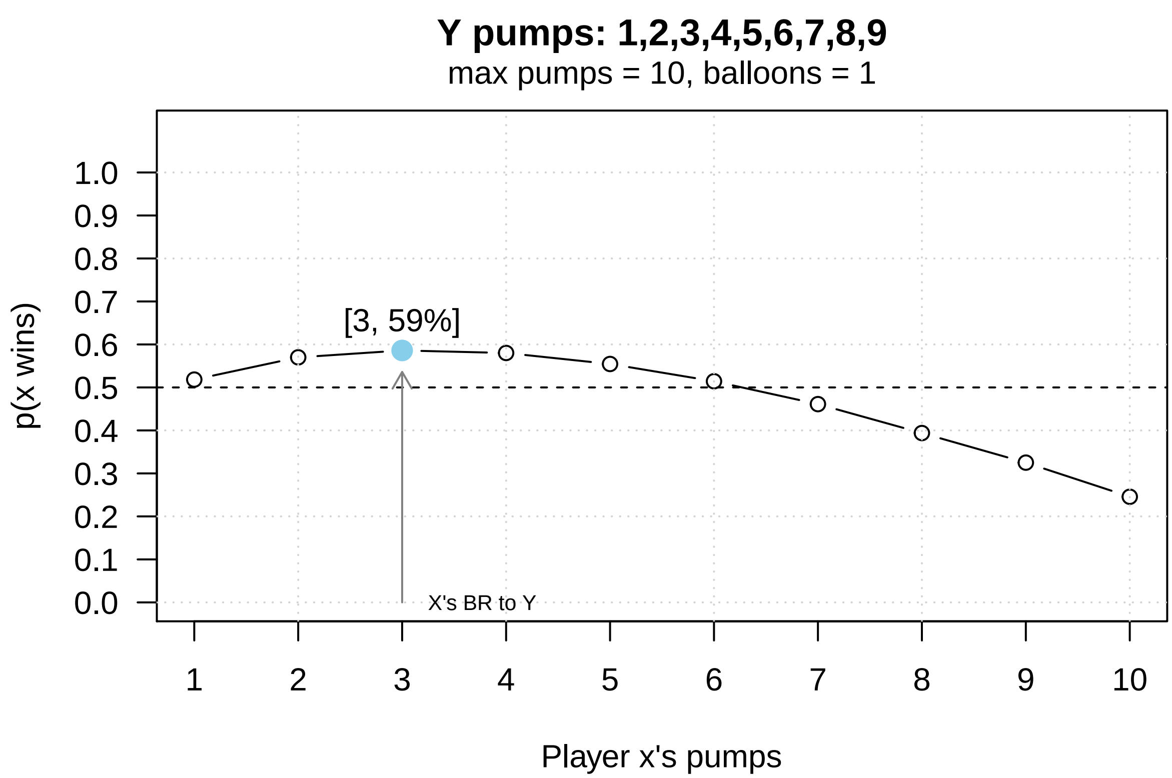


Figure 2: Probability of player x winning a game with 1 balloon given each combination of x.p and y.p



- If your opponent is equally likely to pump anywhere from 1 to 9 times, then you should pump 2, 3 or 4 times



Ex: N = 2, max = 10

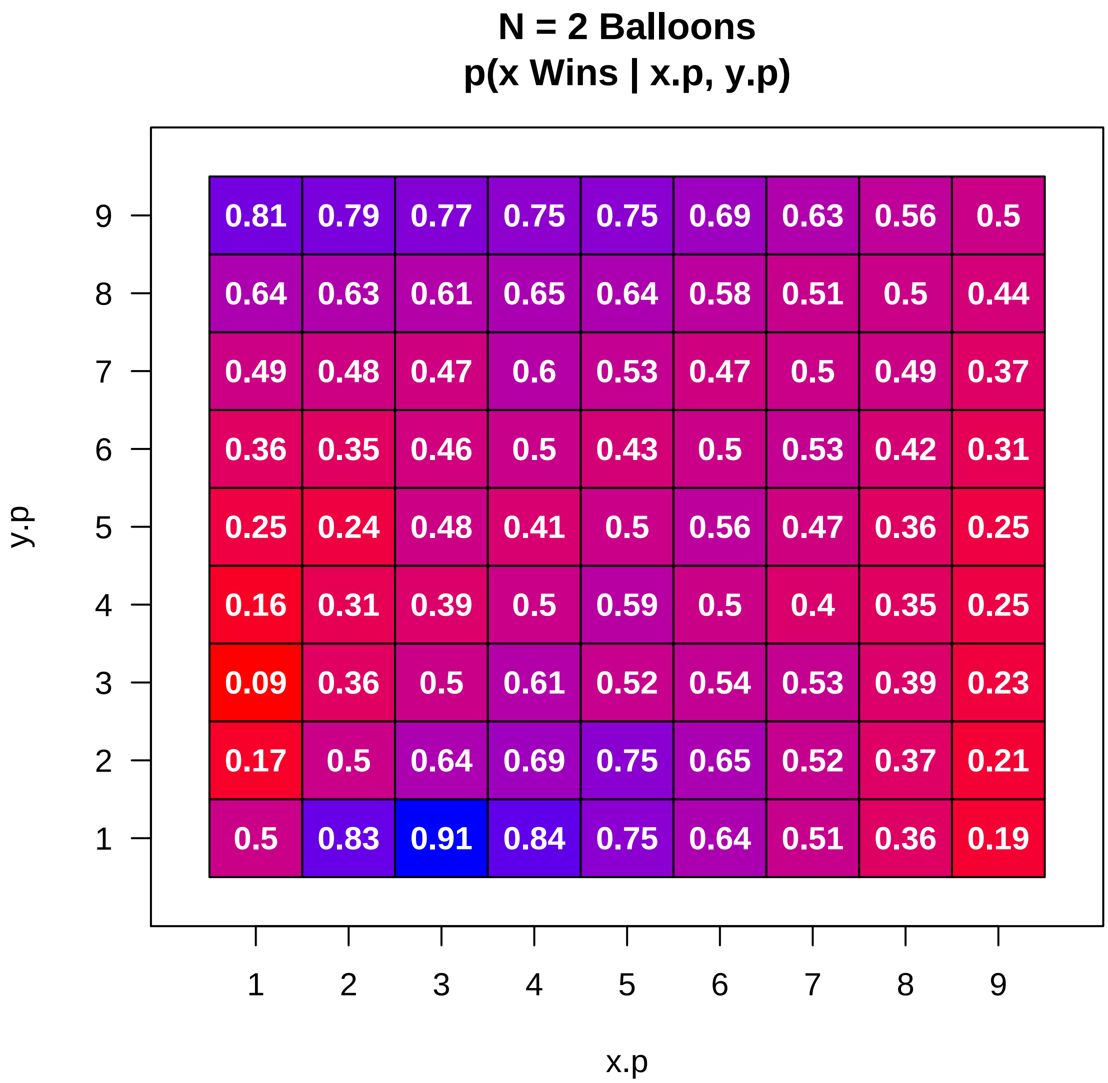
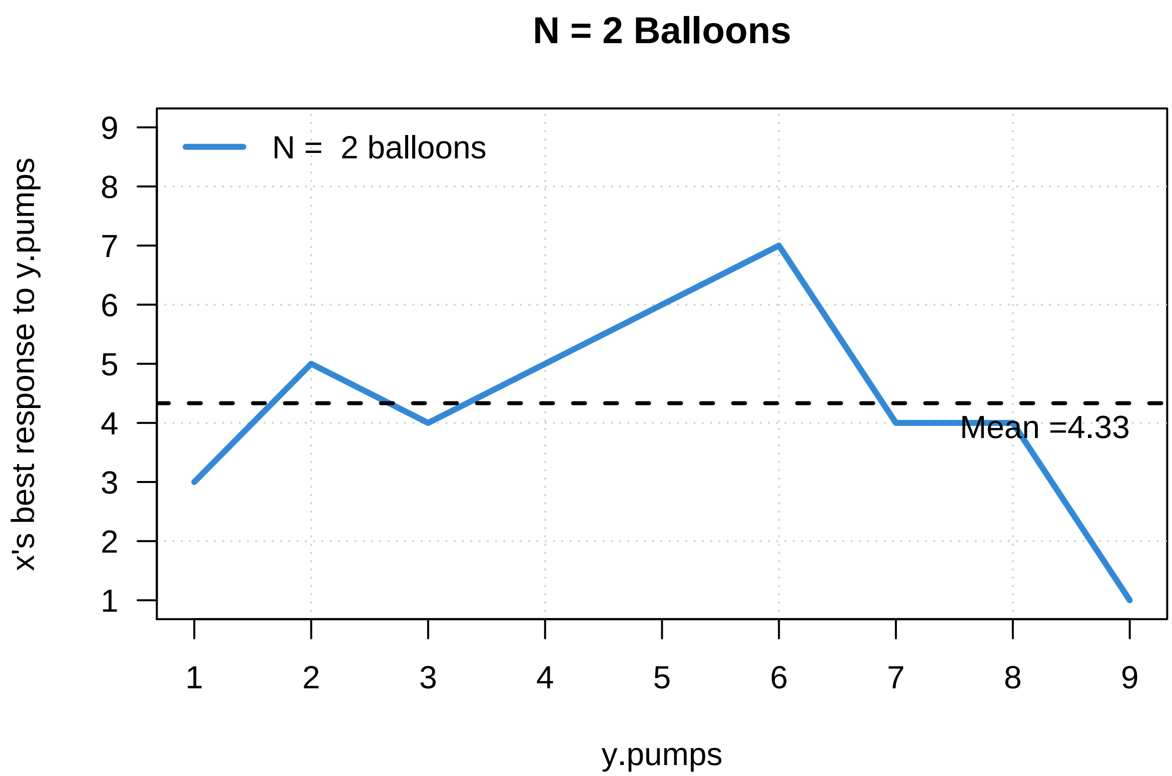
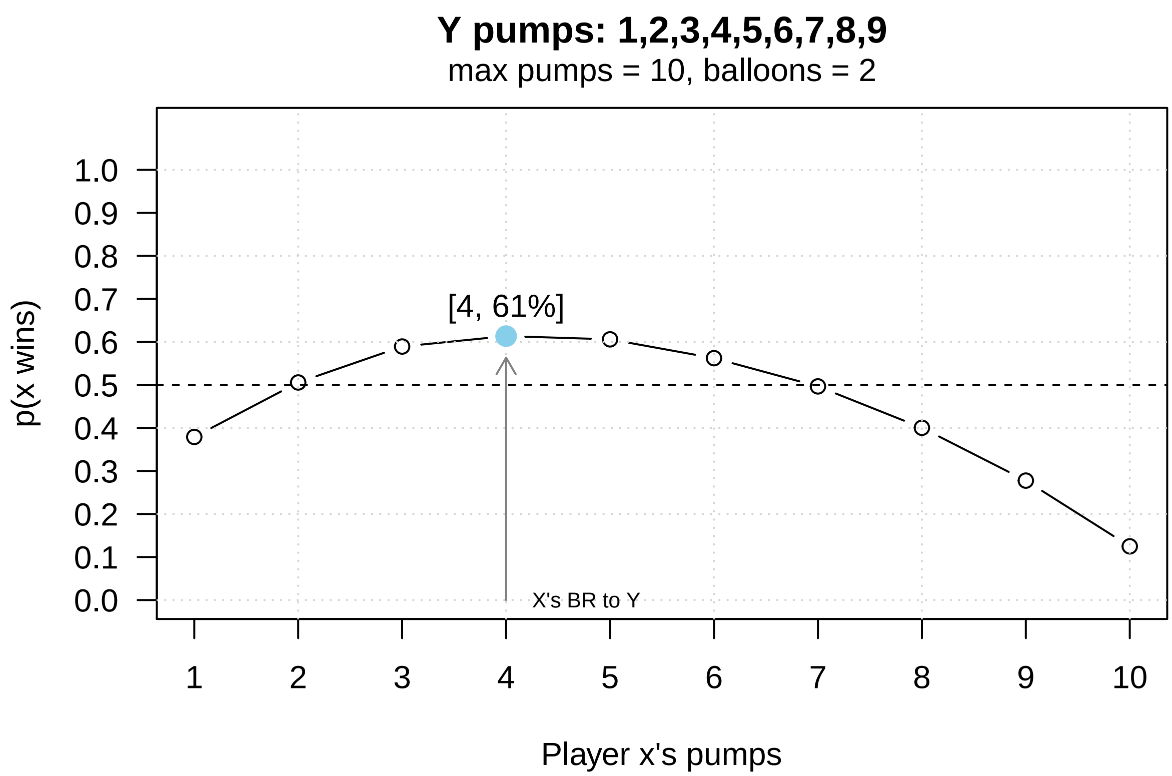


Figure 3: Probability of player x winning a game with 1 balloon given each combination of x.p and y.p

- When the number of balloons increases to 2, then the best response to a competitor with moderately high pumping values increases from 1. The biggest jump is when y.p = 5. Here, the best response is now x.p = 6.
- There is still no Nash equilibrium. Players should cycle between 4, 5 and 6 pumps.
- If your opponent is equally likely to pump anywhere from 1 to 9 times, then you should pump 3 or 4 times



- If your opponent is equally likely to pump anywhere from 1 to 9 times, then you should pump 4 times



Ex: N = 5, max = 10

- When the number of balloons increases to 5, the best response for all y.p values greater than 2 is now

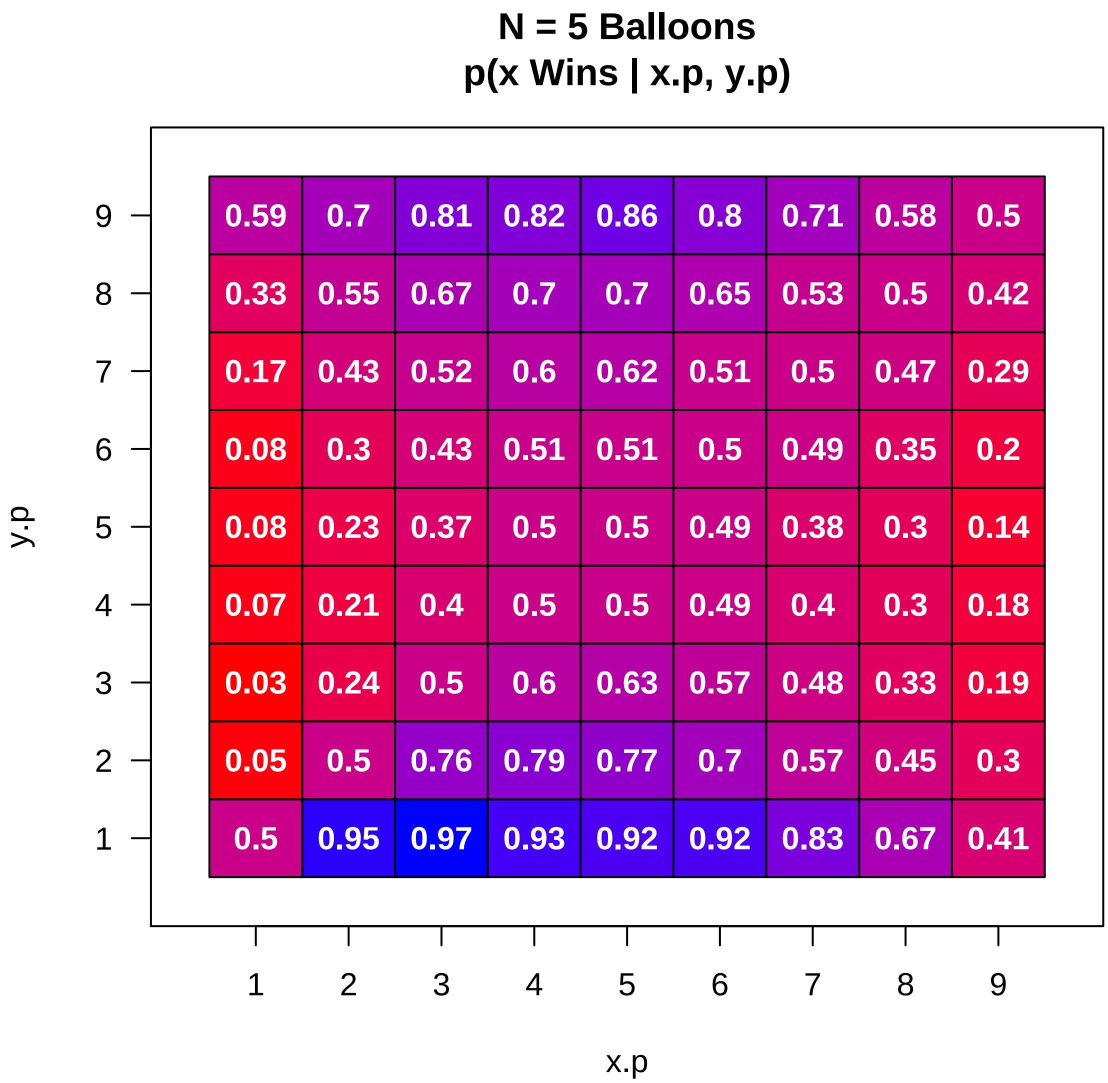
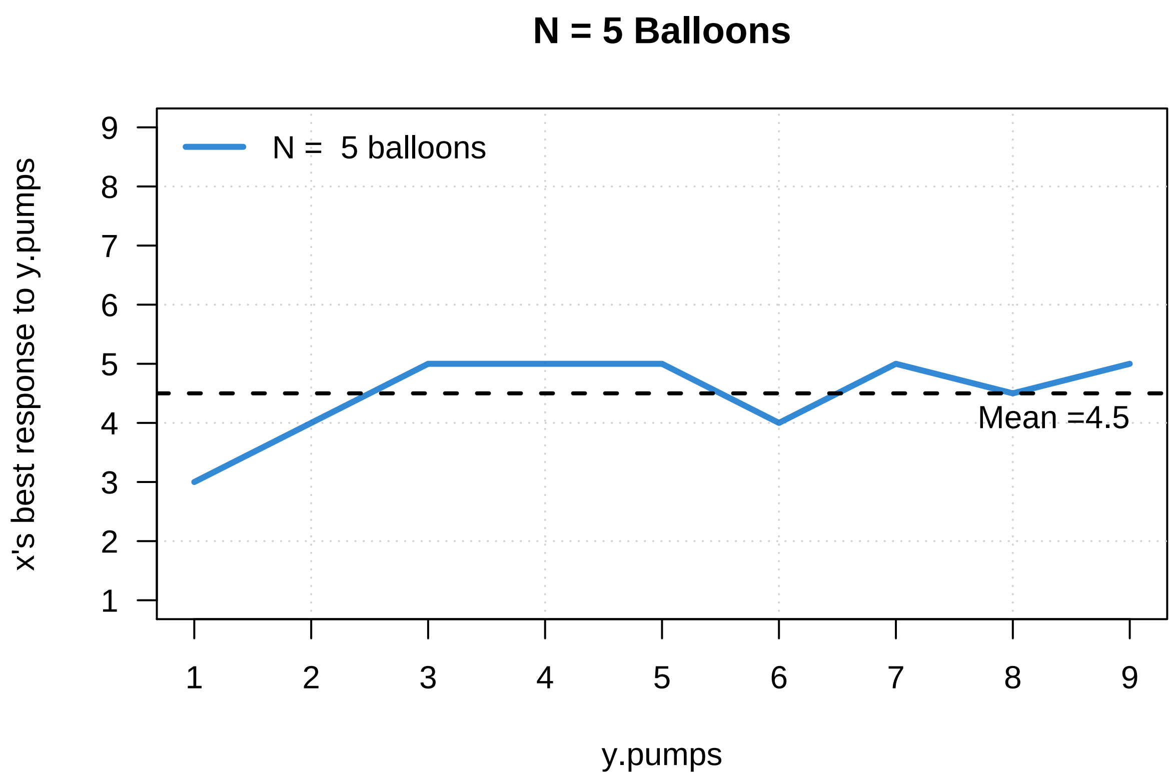


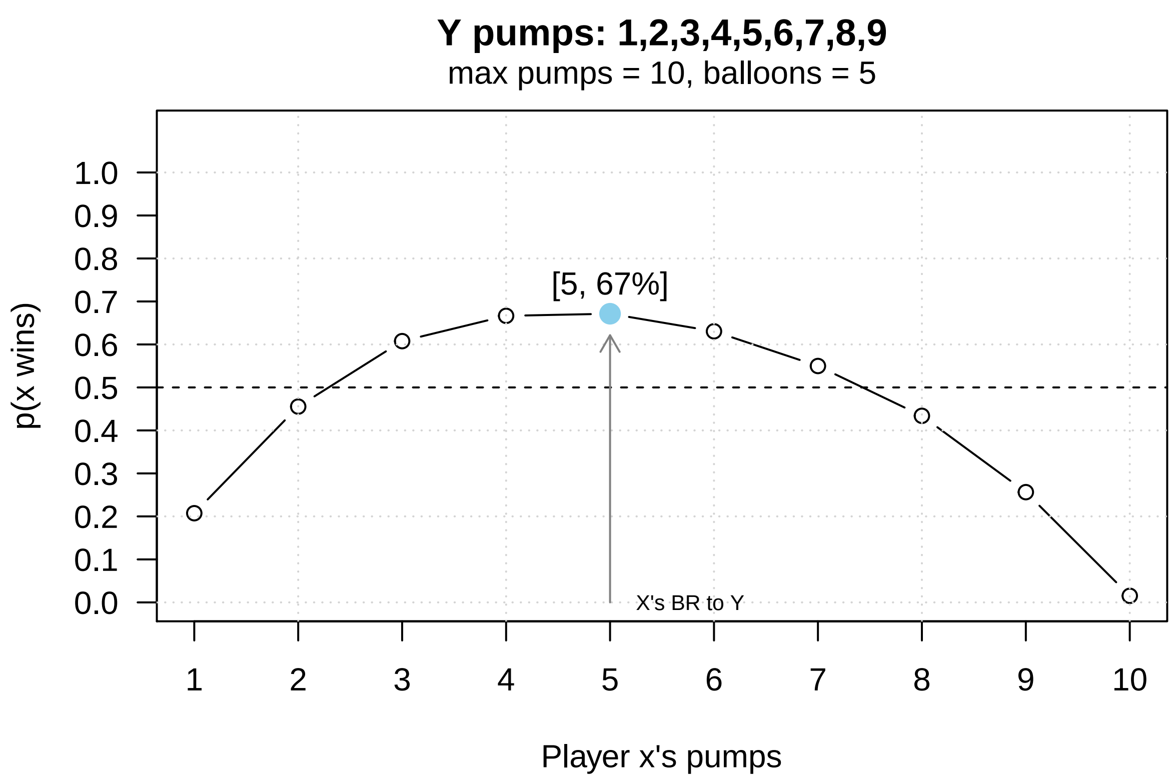
Figure 4: Probability of player x winning a game with 1 balloon given each combination of x.p and y.p

either 5 or 6.

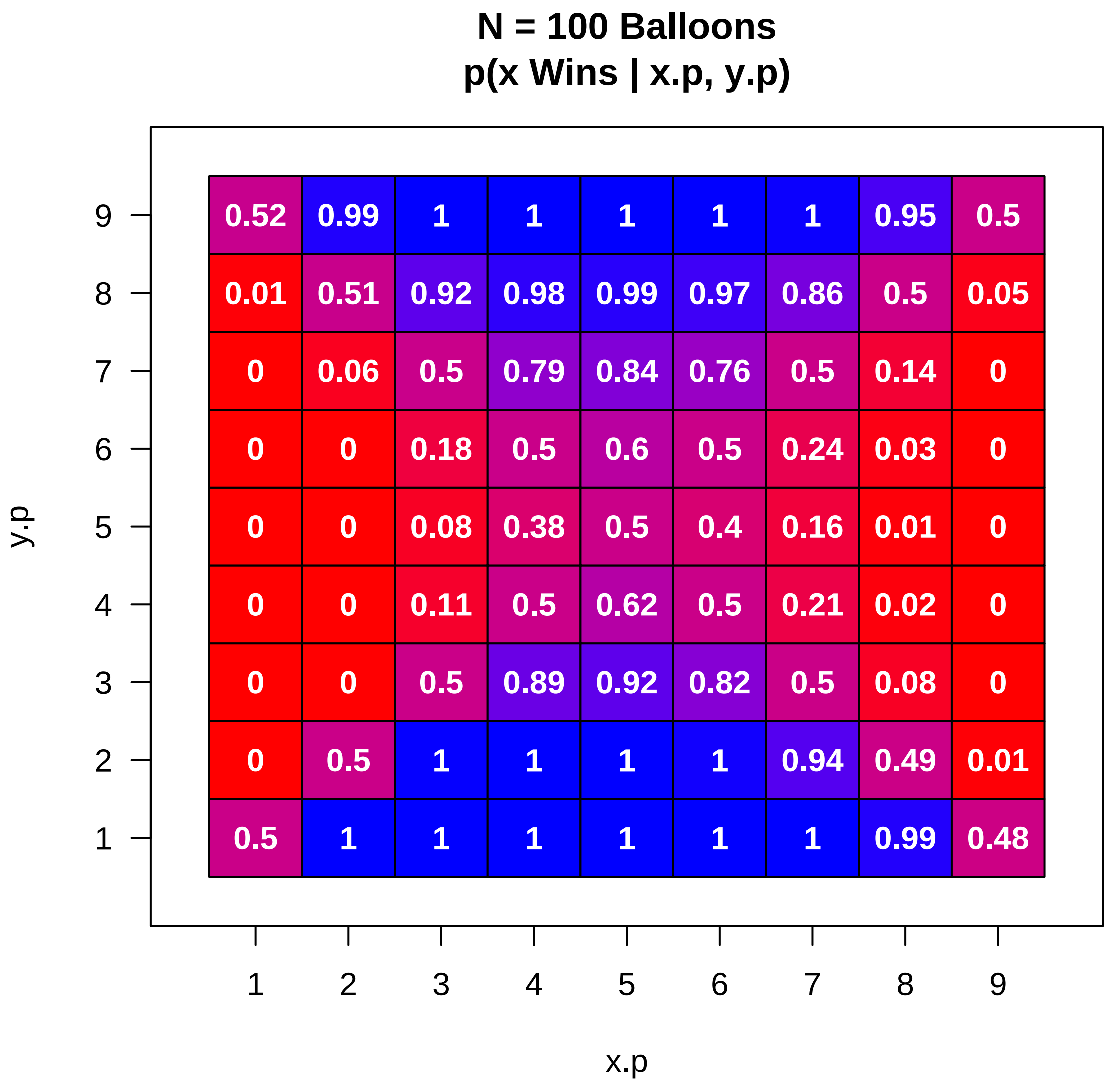
- Still no equilibrium
- If your opponent is equally likely to pump anywhere from 1 to 9 times, then you should pump 5 times.



- If your opponent is equally likely to pump anywhere from 1 to 9 times, then you should pump 4 or 5 times



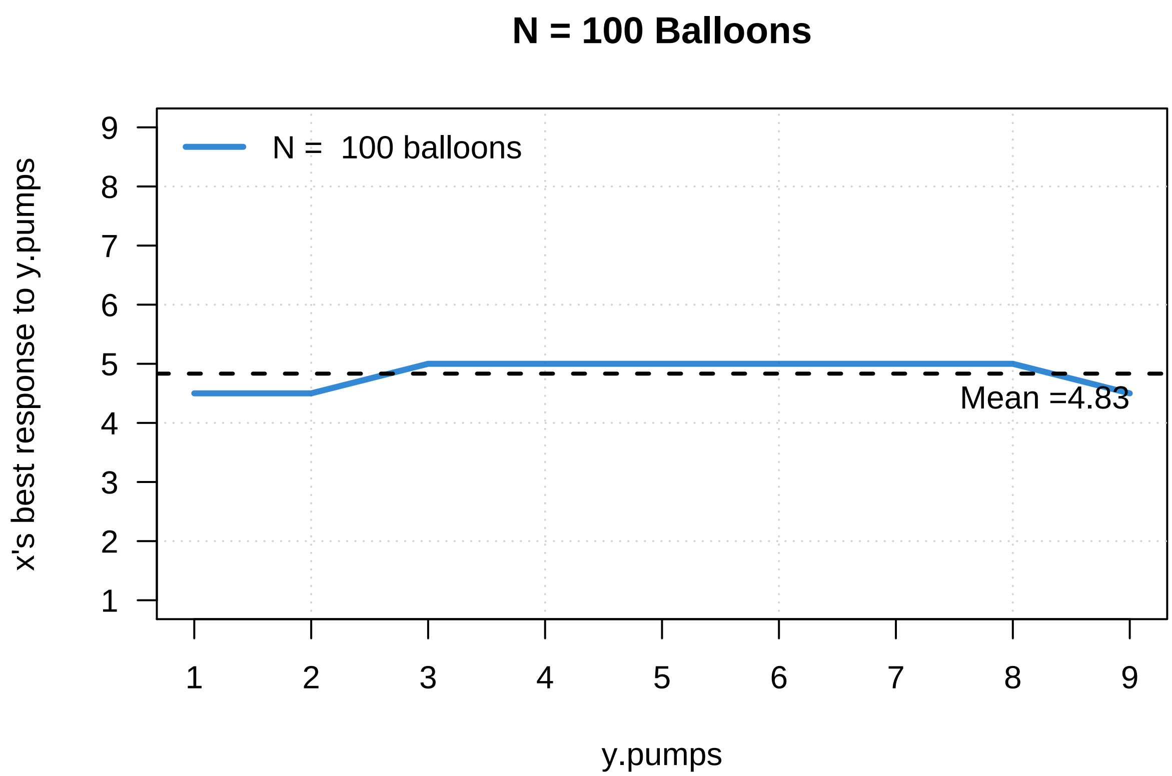
Ex: N = 100, max = 10



y.p \ x.p	1	2	3	4	5	6	7	8	9
9	0.52	0.99	1	1	1	1	1	0.95	0.5
8	0.01	0.51	0.92	0.98	0.99	0.97	0.86	0.5	0.05
7	0	0.06	0.5	0.79	0.84	0.76	0.5	0.14	0
6	0	0	0.18	0.5	0.6	0.5	0.24	0.03	0
5	0	0	0.08	0.38	0.5	0.4	0.16	0.01	0
4	0	0	0.11	0.5	0.62	0.5	0.21	0.02	0
3	0	0	0.5	0.89	0.92	0.82	0.5	0.08	0
2	0	0.5	1	1	1	1	0.94	0.49	0.01
1	0.5	1	1	1	1	1	1	0.99	0.48

Figure 5: Probability of player x winning a game with 1 balloon given each combination of x.p and y.p

- Finally, when the number of balloons increases to 100, the best response is always 5.
- In a long-horizon game, the Nash equilibrum is the individual maxization strategy.
- If your opponent is equally likely to pump anywhere from 1 to 9 times, then you should pump 5 times.



- If your opponent is equally likely to pump anywhere from 1 to 9 times, then you should pump 4 or 5 times