ICICLE

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Contents

1 Overview

This document describes the code ICICLE (Initial Conditions for Isolated CoLlisionless systEms) to generate stable initial conditions of isolated systems for N-body simulations. Section 2 describes how to use the code we developed. Section 3 gives a detailed explanation of the method used. Finally, Section 4 describes specific profiles included in the code. If you use our code, please cite the original paper [2].

2 The Code - User's Guide

To use the code, you simply alter the parameter file to specify the model and parameters you are using. Then, you run the program in the command line using the following syntax:

```
python ICICLE.py parameterfile.txt outputfile.txt
```

2.1 Files

The main part of the program is in the code "ICICLE.py". The content of this code is described thoroughly in Section 3, but, in short, given a model it returns positions and velocities of n particles within that distribution.

For each model there is an additional file named "ICs_*model*.py". These files contain information about the specific model (the density profile, cumulative mass distribution, gravitational potential and distribution function).

Finally, the parameter file has a key word followed by the value. Each value must be listed on a new line. It ignores any line that begins with the pound symbol, '#'. Key words are case sensitive. For more information on which each of the parameters mean, read the section for the specific model you are using.

2.2 Output

The output is written to the filename specified.

In the text file, the first line has the following information: the number of particles, the mass of each particle and the gravitational constant. The following lines display: particle number (indexed from 0), the x, y and z positions and the x, y, and z velocity components.

Note that if you choose to do a truncated NFW profile, the number of particles, n, will be less than the number requested. If you do an NFW profile, but choose to turn off the truncation, the resulting particles will NOT form a stable profile. See Section 4.1 for further information regarding this.

3 Method

In this section we outline the steps needed to select positions and velocities given an isotropic density profile, in a manner similar to [5]. First the radial distance is selected using the cumulative mass distribution, and then the position is selected assuming spherical symmetry at that radius. Next an energy is selected from the energy distribution. Once the position and energy of the particle have been determined, it is straightforward to calculate the velocity of the particle. Finally, the velocity direction is chosen isotropically.

3.1 Positions

The radius can be chosen from the mass distribution. Consider a density profile where the mass within a given radius is $M(r)$, and the largest radius is r_{max}. Then, the mass fraction at radius r is:

$$F_M = \frac{M(r)}{M(r_{max})} \quad . \tag{1}$$

Therefore, we choose an F_M uniformly from $(0, 1)$, and solve for r to find the radius.

Next, we need to choose a direction: since the distribution is isotropic, we will choose a point uniformly on the surface of a sphere with radius r.

Consider a sphere of radius r and recall that spherical coordinates are given by:

$$\begin{aligned} x &= rcos\theta sin\phi \\ y &= rsin\theta sin\phi \\ z &= rcos\phi \quad , \end{aligned} \tag{2}$$

where $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$.

Next, note that a volume area is given by $d\Omega = sin\phi d\theta d\phi$. It is clear that picking points uniformly from θ and ϕ will preferentially pick points near the poles. Therefore, we will introduce the substitution $u = -cos\phi$, where $u \in [-1, 1]$. Now, the volume element is $d\Omega = d\theta du$. Hence, we will pick points from u and θ to get an even distribution across the surface of the sphere. In terms of u and

θ, Equation (2) is:

$$
\begin{aligned}
x &= r cos\theta\sqrt{1-u^2} \\
y &= r sin\theta\sqrt{1-u^2} \\
z &= ru \ .
\end{aligned}
\tag{3}
$$

3.2 The Distribution Function

The distribution function, $f(\mathcal{E})$, is given by Eddington's formula. A complete derivation can be found in [1]:

$$
f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \frac{1}{\sqrt{\mathcal{E}-\Psi}}\frac{d^2\rho}{d\Psi^2}d\Psi + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{d\rho}{d\Psi}\right)_{\Psi=0}\right] \ , \tag{4}
$$

where $\mathcal{E}$, Ψ and ρ are the relative energy, the relative potential and the density profile, respectively. If the potential approaches zero as r approaches infinity, $\mathcal{E}$ is the binding energy, and $\Psi = -\Phi$, where Φ is the gravitational potential. The term $\left(\frac{d\rho}{d\Psi}\right)_{\Psi=0} = \left(\frac{d\rho}{dr}\frac{dr}{d\Psi}\right)_{\Psi=0}$ should evaluate to zero if the criteria $f(0) = 0$ holds.

Note that the distribution function is often described in terms of the probability density, ν, rather than the mass density ρ. The distribution function then gives the probability of a particle being located at a certain position in phase space, while the equation above gives the mass per phase space volume.

Consider the term $\frac{d^2\rho}{d\Psi^2}$. This can be expressed as:

$$
\begin{aligned}
\frac{d^2\rho}{d\Psi^2} &= \left(\frac{d\Psi}{dr}\right)^{-2}\left[\frac{d^2\rho}{dr^2} - \left(\frac{d\Psi}{dr}\right)^{-1}\frac{d^2\Psi}{dr^2}\frac{d\rho}{dr}\right] \\
\frac{d\Psi}{dr} &= -\frac{GM}{r^2} \\
\frac{d^2\Psi}{dr^2} &= \frac{2GM}{r^3} - 4\pi G\rho \\
\frac{d^2\rho}{d\Psi^2} &= \left(\frac{r^4}{G^2M^2}\right)\left[\frac{d^2\rho}{dr^2} + \left(\frac{r^2}{GM}\right)\left[\frac{2GM}{r^3} - 4\pi G\rho\right]\frac{d\rho}{dr}\right] \ .
\end{aligned}
\tag{5}
$$

Finally, we note that the density profile can be recovered from the distribution equation using the following equation:

$$
\rho(r) = 4\pi\int_0^{\Psi(r)} f(\mathcal{E})\sqrt{2(\Psi(r)-\mathcal{E})}d\mathcal{E} \ . \tag{6}
$$

3.3 Choosing From the Distribution Function

First, it is important to mention that the probability a particle located at radius r will have energy $\mathcal{E}$ is proportional to $f(\mathcal{E})\sqrt{\Psi - \mathcal{E}}$. Also, the maximum binding energy is $\mathcal{E}_{max} = \Psi(r)$.

Therefore, the cumulative distribution function (CDF), $F(\mathcal{E})$, for a particle at position r with relative potential energy $\Psi(r)$ is:

$$F(\mathcal{E}) = \frac{\int_0^{\mathcal{E}} f(\mathcal{E})\sqrt{\Psi - \mathcal{E}} d\mathcal{E}}{\int_0^{\Psi} f(\mathcal{E})\sqrt{\Psi - \mathcal{E}} d\mathcal{E}} \quad . \tag{7}$$

A random number is chosen uniformly between 0 and 1 for the value of the CDF, and the corresponding energy is found by a linear interpolation of the numerically determined CDF.

3.4 Velocities

At this point both a position and binding energy have been chosen for the particle. The velocity magnitude can then be calculated from a simple algebraic expression:

$$\mathcal{E} = \Psi - \frac{1}{2}v^2 \quad . \tag{8}$$

Finally, a direction for the velocity can be found in the same way as described for positions. That is, we select random numbers $\theta \in [0, 2\pi)$ and $u \in [-1, 1]$, and then the velocity components are:

$$\begin{aligned} v_x &= v \cos\theta \sqrt{1 - u^2} \\ v_y &= v \sin\theta \sqrt{1 - u^2} \\ v_z &= vu \quad . \end{aligned} \tag{9}$$

4 Profiles

4.1 NFW Profile

The Navarro-Frenk-White profile is used to describe cosmological dark matter halos [7, 8].

It is important to note that the enclosed mass of this profile diverges as the radius goes to infinity. Therefore, we necessarily only define particles within a finite radius, r_{cut}; however, this results in an unstable profile. Hence, to create a stable profile, we must modify the NFW profile in some way. In Section 4.1.5 we discuss one possible solution to this problem, which was first proposed in [2].

The NFW profile is uniquely determined by the mass and a scale radius, r_s. Additionally, we specify a finite radius, r_{cut}; particles will only be generated within this radius. In practice you may wish to set $r_{cut} = r_{vir}$. To perform the truncation outlined in Section 4.1.5, the user must set Truncate=True. Otherwise, the resulting initial conditions will not be stable.

4.1.1 Density Profile

The NFW profile is given by:

$$\rho(r) = \frac{\rho_0 r_s^3}{r(r_s + r)^2} \quad , \tag{10}$$

where $\rho(r)$ is the density at radius r. The constants ρ_0 and r_s are characteristic values of the specific system. We can also define the concentration parameter as $c = r_{vir}/r_s$, where r_{vir} is the virial radius.

4.1.2 Cumulative Mass Distribution

It is straightforward to show that the mass within radius r, $M(r)$, is given by:

$$M(r) = \int_0^r 4\pi r'^2 \rho(r') dr' = 4\pi\rho_0 r_s^3 \left[\ln\left(\frac{r_s + r}{r_s}\right) - \frac{r}{r_s + r} \right] \quad . \tag{11}$$

The total mass diverges as $r \to \infty$.

4.1.3 Potential

Poisson's equation for a spherical system is:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho(r) \ . \tag{12}$$

With the boundary conditions $\left.\frac{d\Phi}{dr}\right|_{r=0} = 0$ and $\lim_{r\to\infty}\Phi(r) = 0$, we can solve for the potential energy:

$$\Phi(r) = -\frac{4\pi G\rho_s r_s^3}{r}\ln\left(\frac{r+r_s}{r_s}\right) \ . \tag{13}$$

This has a finite central potential. Evaluating the limit as $r \to 0$ yields

$$\Phi(0) = -4\pi G p_0 r_s^2 \ . \tag{14}$$

4.1.4 Distribution Function

First, consider Equation (5). Fom Equation (10), we determine:

$$\begin{aligned} \frac{d\rho}{dr} &= -\frac{p_0 r_s^3(r_s+3r)}{r^2(r_s+r)^3} \\ \frac{d^2\rho}{dr^2} &= \frac{2p_0 r_s^3(r_s^2+4rr_s+6r^2)}{r^3(r_s+r)^4} \ . \end{aligned} \tag{15}$$

Therefore:

$$\frac{d^2\rho}{d\Psi^2} = \frac{2p_0 r_s^3}{G^2M(r)^2(r+r_s)^4}\left[r(6r^2+4rr_s+r_s^2) - \left(1-\frac{2\pi r^3\rho(r)}{M(r)}\right)r(r+r_s)(3r+r_s)\right] \ . \tag{16}$$

Then, we can rewrite Equation (4) in terms of r. This gives the following equation for the distribution function:

$$f(\mathcal{E}) = \frac{1}{\sqrt{2}\pi^2 G}\int_{r_\mathcal{E}}^{\infty}\frac{\rho(r)}{M(r)^2\sqrt{\mathcal{E}-\Psi}}\left(\frac{r}{r+r_s}\right)^2\left[3M(r)+2\pi r(r+r_s)(3r+r_s)\rho(r)\right]dr \ , \tag{17}$$

where we have defined $r_\mathcal{E}$ as $\Psi(r_\mathcal{E}) = \mathcal{E}$.

4.1.5 Truncation

In the previous section we derived the distribution function for the NFW profile. However, as discussed previously, since we only are able to generate a finite number of particles, the generated particles will not actually be in equilibrium.

Our solution is to remove any particles that are not bound. That is, we iteratively remove any particles with $\mathcal{E} > \Psi r_{vir}$. For discussion on the stability of this profile, see [2].

4.2 Exponentially Truncated NFW Profile

In the previous section, we proposed a way to truncate the NFW profile, so that it has a finite mass. An alternative approach is to modify the NFW profile by introducing an exponential cutoff outside the virial radius. This profile was first proposed by [10], and is a mathematically convenient way to create a profile a profile that has finite mass and resembles the NFW profile within the virial radius.

This profile can be uniquely defined by the mass, scale radius, r_s, virial radius, r_{vir}, and a decay parameter. The specified decay parameter, d, is in units of scale radius: i.e. $d \equiv r_d/r_s$

4.2.1 Density Profile

The density profile is identical to the NFW profile within the virial radius, r_{vir}, but then is truncated exponentially. How fast this decay occurs is dependent on the parameter r_d.

Therefore, the profile, $\rho(r)$ is given by:

$$\rho(r) = \begin{cases} \rho_1(r) = \dfrac{\rho_0 r_s^3}{r(r_s+r)^2} & \text{if } r < r_{vir} \\ \rho_2(r) = \dfrac{\rho_0}{c(1+c)^2}\left(\dfrac{r}{r_{vir}}\right)^{\epsilon} \exp\left(-\dfrac{r-r_{vir}}{r_d}\right) & \text{if } r > r_{vir} \end{cases}, \tag{18}$$

where the constants ρ_0 and r_s are characteristic values of the specific system. We also define the concentration parameter as $c = r_{vir}/r_s$. Additionally, if we require that the logarithmic slope at

r_{vir} is continuous, we have the constraint:

$$\epsilon = -\frac{1+3c}{1+c} + \frac{r_{vir}}{r_d} \quad . \tag{19}$$

4.2.2 Cumulative Mass Distribution

The mass within radius r, $M(r)$, is given by:

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')dr' = \int_0^{\min(r,r_{vir})} 4\pi r'^2 \rho_1(r')dr' + \int_{\min(r,r_{vir})}^r 4\pi r'^2 \rho_2(r')dr' \quad . \tag{20}$$

The first integral can be calculated in the same way as for the regular NFW profile. We will define $m \equiv \min(r, r_{vir})$:

$$M_1(r) = \int_0^m 4\pi r'^2 \rho_1(r')dr' = 4\pi\rho_0 r_s^3 \left[\ln\left(\frac{r_s+m}{r_s}\right) - \frac{m}{r_s+m}\right] \quad . \tag{21}$$

The second integral is given by:

$$M_2(r) = \frac{4\pi p_0 r_s^3 r_d^2}{(r_s+r_{vir})^2} e^{r_{vir}/r_d} \left(\frac{r_d}{r_{vir}}\right)^{\epsilon+1} \left[\Gamma\left(\epsilon+3, \frac{m}{r_d}\right) - \Gamma\left(\epsilon+3, \frac{r}{r_d}\right)\right] \quad , \tag{22}$$

where Γ is the (upper) incomplete gamma function. To calculate the mass fraction at radius r, we need to determine the total mass, M_{tot}. Then, the mass fraction is simply $F_M = M(r)/M_{tot}$.

For this profile, the total mass is given by:

$$M_{tot} = \frac{4\pi p_0 r_s^3 r_d^2}{(r_s+r_{vir})^2} e^{r_{vir}/r_d} \left(\frac{r_d}{r_{vir}}\right)^{\epsilon+1} \Gamma\left(\epsilon+3, \frac{r_{vir}}{r_d}\right) + 4\pi\rho_0 r_s^3 \left[\ln\left(\frac{r_s+r_{vir}}{r_s}\right) - \frac{r_{vir}}{r_s+r_{vir}}\right] \quad . \tag{23}$$

4.2.3 Potential

First, we will find an expression for Ψ. We have:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho(r) \tag{24}$$

with the boundary conditions $\left.\frac{d\Phi}{dr}\right|_{r=0} = 0$ and $\lim_{r\to\infty}\Phi(r) = 0$.

Integrating once, we find:

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2} \tag{25}$$

At this point, we revert to numerical solutions, and find:

$$\Phi(r) = -\int_r^\infty \frac{GM(r')}{r'^2}dr' \ , \tag{26}$$

where the bounds of integration reflect that the potential goes to zero as r approaches infinity.

4.2.4 Distribution Function

As shown earlier,

$$\begin{aligned} \frac{d^2\rho}{d\Psi^2} &= \left(\frac{d\Psi}{dr}\right)^{-2}\left[\frac{d^2\rho}{dr^2} - \left(\frac{d\Psi}{dr}\right)^{-1}\frac{d^2\Psi}{dr^2}\frac{d\rho}{dr}\right] \\ \frac{d\Psi}{dr} &= -\frac{GM}{r^2} \\ \frac{d^2\Psi}{dr^2} &= \frac{2GM}{r^3} - 4\pi G\rho \ . \end{aligned} \tag{27}$$

When the radius is smaller than the virial radius, $\frac{d^2\rho}{d\Psi^2}$ is the same as calculated for the regular NFW profile. Outside the virial radius,

$$\begin{aligned} \frac{d\rho_2}{dr} &= \frac{\rho_0}{c(1+c)^2}\left(\frac{r}{r_{vir}}\right)^{\epsilon}\exp\left(-\frac{r-r_{vir}}{r_d}\right)\left(\frac{\epsilon r_d - r}{r r_d}\right) \\ \frac{d^2\rho_2}{dr^2} &= \frac{\rho_0}{c(1+c)^2}\left(\frac{r}{r_{vir}}\right)^{\epsilon}\exp\left(-\frac{r-r_{vir}}{r_d}\right)\left(\frac{r_d^2\epsilon^2 - r_d^2\epsilon - 2r_d\epsilon r + r^2}{r^2 r_d^2}\right) \ . \end{aligned} \tag{28}$$

Then, we can calculate

$$\frac{d^2\rho}{d\Psi^2} = \frac{\rho}{G^2M^2}\left(\frac{r}{r_d}\right)^2\left[r_d^2\epsilon^2 - r_d^2\epsilon - 2r_d\epsilon r + r^2 + r_d(\epsilon r_d - r)\left(2 - \frac{4\pi\rho r^3}{M}\right)\right] \tag{29}$$

We will define

$$\frac{d^2\rho}{d\Psi^2} = \begin{cases} D_1(r) \equiv \dfrac{\rho r^2}{G^2M^2}\left(\dfrac{r}{r+r_s}\right)^2\left[6+\dfrac{4\pi\rho}{M}r(r+r_s)(3r+r_s)\right] & \text{if } r < r_{vir} \\ D_2(r) \equiv \dfrac{\rho}{G^2M^2}\left(\dfrac{r}{r_d}\right)^2\left[r_d^2\epsilon^2 - r_d^2\epsilon - 2r_d\epsilon r + r^2 + r_d(\epsilon r_d - r)\left(2-\dfrac{4\pi\rho r^3}{M}\right)\right] & \text{if } r > r_{vir} \end{cases} . \tag{30}$$

The distribution function is now:

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_{r_\mathcal{E}}^{\max(r_\mathcal{E},r_v)} \frac{D_1}{\sqrt{\mathcal{E}-\Psi}}\frac{GM}{r^2}dr + \int_{\max(r_\mathcal{E},r_v)}^{\infty} \frac{D_2}{\sqrt{\mathcal{E}-\Psi}}\frac{GM}{r^2}dr\right] . \tag{31}$$

4.2.5 Stability Concerns

Finally, it is important to discuss whether the fact that this profile is not smooth at $r = r_{vir}$ affects the stability of this solution. Clearly, the term $\dfrac{d^2\rho}{d\Psi^2}$ is discontinuous at the virial radius. This it not necessarily a problem. However, the distribution function must be monotonically increasing to give a physical solution. Therefore, there is a further constraint that D_2 be positive. This is true if the decay parameter is chosen so that:

$$d \equiv \frac{r_d}{r_s} \geq \frac{(1+c)^2[(1+c)\ln(1+c)-c]}{(1+3c)[2(1+c)\ln(1+c)-c]} . \tag{32}$$

4.3 Hernquist Profile

In this section we discuss the Hernquist profile [4]. This profile was originally used to describe spherical galaxies, however it is also a good approximation of cosmological dark matter halo profiles. The advantage to this model is that it has simple analytic expressions for its density profile, mass profile, potential and distribution function. Derivations for all these can be found in [4], so we will simply list them here.

A Hernquist profile can be defined by its mass and a characteristic radius.

4.3.1 Density Profile

The density profile of the Hernquist profile is given by:

$$\rho(r) = \frac{M_{tot}}{2\pi}\frac{a}{r(r+a)^3} \ , \tag{33}$$

where M_{tot} is the total mass and a is a characteristic radius (the radius at which the enclosed mass is $M_{tot}/4$) .

4.3.2 Cumulative Mass Distribution

The mass of this profile is given by:

$$M(<r) = M_{tot}\frac{r^2}{(r+a)^2} \ . \tag{34}$$

4.3.3 Potential

The potential of the Hernquist Profile is:

$$\Phi(r) = -\frac{GM_{tot}}{r+a} \ . \tag{35}$$

4.3.4 Distribution Function

Finally, the distribution function is:

$$\begin{aligned} f(\mathcal{E}) &= \frac{M}{8\sqrt{2}\pi^3 a^3 v_g^3}\frac{1}{(1-q^2)^{5/2}}\left(3\sin^{-1} q + q(1-q^2)^{1/2}(1-2q^2)(8q^4-8q^2-3)\right) \\ q &= \sqrt{\frac{a\mathcal{E}}{GM}} \\ v_g &= \left(\frac{GM}{a}\right)^{1/2} \ . \end{aligned} \tag{36}$$

4.4 King Model

The King model is a lowered isothermal sphere model, altered so that it has a finite mass (within a well-defined tidal radius) [6].

There are many different possible parameterizations, but it can be uniquely defined by the total mass, tidal radius, r_t, and a dimensionless central potential, $P0 \equiv \Psi(0)/\sigma^2$, where σ is the velocity dispersion.

A demonstration of the relation between $\Psi(0)/\sigma^2$ and the concentration, $c \equiv \log(r_t/r_0)$, where r_0 is the King radius can be found in [1].

4.4.1 Distribution Function

The King model has the distribution function:

$$f(\mathcal{E}) = \rho_1 (2\pi\sigma^2)^{-\frac{3}{2}} (e^{\mathcal{E}/\sigma^2} - 1) \ \ , \mathcal{E} > 0 \ \ . \tag{37}$$

where σ is the velocity dispersion, and ρ_1 is a characteristic density.

The King radius, r_0 is defined as:

$$r_0 \equiv \sqrt{\frac{9\sigma^2}{4\pi G\rho_0}} \ \ , \tag{38}$$

where ρ_0 is the central density. The concentration is then defined as:

$$c \equiv \log_{10}(r_t/r_0) \ \ . \tag{39}$$

We note that the King model has many parameters: c, r_0, r_t, ρ_1, ρ_0, σ, Ψ_0/σ^2 and M_{tot}. However, we can uniquely determine the profile with only three parameters. For this code we will use the parameterization Ψ_0/σ^2, M_{tot} and r_t.

4.4.2 Density Profile

We know that the density profile is related to the distribution function through the equation $\rho(\Psi) = 4\pi \int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi - \mathcal{E})}d\mathcal{E}$. Therefore, the density profile is:

$$\rho(\Psi) = \rho_1 \left[e^{\Psi/\sigma^2} \text{erf}\left(\frac{\sqrt{\Psi}}{\sigma}\right) - \sqrt{\frac{4\Psi}{\pi\sigma^2}}\left(1 + \frac{2\Psi}{3\sigma^2}\right)\right] \ . \tag{40}$$

However, the density profile is defined in terms of the relative potential Ψ. In order to determine the density as a function of radius, we need a relation between Ψ and r

4.4.3 Potential

To relate Ψ to r, we will use Poisson's equation.

$$\nabla^2\Psi = -4\pi G\rho = -4\pi G\rho_1 \left[e^{\Psi/\sigma^2} \text{erf}\left(\frac{\sqrt{\Psi}}{\sigma}\right) - \sqrt{\frac{4\Psi}{\pi\sigma^2}}\left(1 + \frac{2\Psi}{3\sigma^2}\right)\right] \ . \tag{41}$$

Additionally, we use the boundary conditions $d\Psi/dr = 0$ at $r = 0$, and $\Psi(r_t) = 0$, where r_t is the tidal radius. This can be solved numerically.

Additionally, we note that Ψ is related to the potential as $\Psi(r) = -\Phi(r)+\Phi(r_t) = -\Phi(r)-GM_{tot}/r_t$, where, we have used the fact that the total mass is enclosed within the tidal radius, r_t.

Consider Poisson's equation, and introduce the dimensionless parameters $P = \Psi/\sigma^2$, $p = \rho/\rho_1$, $R = r/r_0$ and $a = \rho_1/\rho_0$:

$$\begin{aligned}
\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) &= -4\pi G\rho(\Psi)r^2 \\
\frac{d^2\Psi}{dr^2} + \frac{2}{r}\frac{d\Psi}{dr} &= -4\pi G\rho(\Psi) \\
\frac{d^2P}{dR^2} + \frac{2}{R}\frac{dP}{dR} &= -\frac{4\pi G r_0^2}{\sigma^2}\rho(\Psi) \\
\frac{d^2P}{dR^2} + \frac{2}{R}\frac{dP}{dR} &= -9ap(P) \\
\frac{d^2P}{dR^2} + \frac{2}{R}\frac{dP}{dR} &= -9a\left[e^P\text{erf}\left(\sqrt{P}\right) - \sqrt{\frac{4P}{\pi}}\left(1 + \frac{2}{3}P\right)\right] \ .
\end{aligned} \tag{42}$$

The parameter a can be expressed in terms of the central potential, $a = 1/p(P_0)$. The second ODE can then be solved with the conditions $P(0) = P_0$ and $\frac{dP}{dR}(0) = 0$. We note that specifying the central potential determines the tidal radius by the condition $P(R_t) = 0$. A higher central potential results in a larger tidal radius. In the limit that $P_0 \to \infty$, the King model approaches an isothermal sphere.

4.4.4 Cumulative Mass Distribution

Finally, to determine the mass within radius r, the equation $M(r) = \int_0^r 4\pi\rho(r)r^2 dr$ can be solved numerically.

We can recover the central density of the profile from the total mass. Consider the non-dimensionless form of M:

$$L_{tot} \equiv \frac{M_{tot}}{4\pi r_0^3 \rho_1} = \int_0^{R_t} p(R)R^2 dR \ . \tag{43}$$

L_{tot} can be calculated from the integrand. Then, it is straightforward to solve for ρ_1. We have already shown that $a = \rho_1/\rho_0$ can be determined from P_0, and thus ρ_0 can also be calculated.

4.5 Einasto Profile

The Einasto profile was originally proposed to fit star counts in the Milky Way [3], however it may also be a very accurate description of dark matter halos [9].

This profile is defined by its mass, shape parameter, α, and a characteristic radius, r_{-2}.

4.5.1 Density Profile

The density profile is expressed as:

$$\begin{aligned} \rho(r) &= \rho_{-2} \exp\left(-\frac{2}{\alpha}\left[\left(\frac{r}{r_{-2}}\right)^{\alpha} - 1\right]\right) \\ &= \rho_0 \exp\left(-\frac{2}{\alpha}\left(\frac{r}{r_{-2}}\right)^{\alpha}\right) \ . \end{aligned} \tag{44}$$

where r_{-2} is the radius at which $d\log\rho/d\log r = -2$ and $\rho_{-2} = \rho(r_{-2})$.

4.5.2 Cumulative Mass Distribution

To solve the mass profile, we start with:

$$M(r) = \int_0^r 4\pi r'^2 \rho(r') dr' \tag{45}$$

Then, we define x as:

$$\begin{aligned} x &= \frac{2}{\alpha}\left(\frac{r}{r_{-2}}\right)^{\alpha} \\ r &= r_{-2}\left(\frac{\alpha x}{2}\right)^{1/\alpha} \quad . \end{aligned} \tag{46}$$

Therefore,

$$\begin{aligned} dx &= \frac{2}{r_{-}2}\left(\frac{r}{r_{-2}}\right)^{1-\alpha} dr \\ dr &= \frac{r}{\alpha x} dx \quad . \end{aligned} \tag{47}$$

Substituting these in gives:

$$\begin{aligned} M(r) &= \frac{4\pi\rho_0 r_{-2}^3}{\alpha}\left(\frac{\alpha}{2}\right)^{3/\alpha} \int_0^x x'^{3/\alpha-1} e^{x'} dx' \\ &= \frac{4\pi\rho_0 r_{-2}^3}{\alpha}\left(\frac{\alpha}{2}\right)^{3/\alpha} \gamma(3/\alpha, x) \\ &= \frac{4\pi\rho_0 r_{-2}^3}{\alpha}\left(\frac{\alpha}{2}\right)^{3/\alpha} \gamma\left(\frac{3}{\alpha}, \frac{2}{\alpha}\left(\frac{r}{r_{-2}}\right)^{\alpha}\right) \quad , \end{aligned} \tag{48}$$

and the total mass is then:

$$M_{tot} = \frac{4\pi\rho_0 r_{-2}^3}{\alpha}\left(\frac{\alpha}{2}\right)^{3/\alpha} \Gamma\left(\frac{3}{\alpha}, 0\right) \quad , \tag{49}$$

where γ is the lower and Γ is the upper incomplete gamma function.

4.5.3 Potential

The potential is solved as followed:

$$\begin{aligned} \Phi(r) &= -\int_r^{\infty} \frac{GM(r')}{r'^2} dr' \\ &= -G\frac{4\pi\rho_0 r_{-2}^3}{\alpha}\left(\frac{\alpha}{2}\right)^{3/\alpha} \int_r^{\infty} \frac{1}{r'^2}\gamma\left(\frac{3}{\alpha}, \frac{2}{\alpha}\left(\frac{r'}{r_{-2}}\right)^{\alpha}\right) dr' \quad . \end{aligned} \tag{50}$$

Noting that $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$, and using integration by parts shows:

$$
\begin{aligned}
&\int_r^\infty \frac{\gamma(a,x)}{r'^2}dr' \\
&= \int_x^\infty \gamma(a,x)\frac{1}{r_{-2}}\left(\frac{2}{\alpha x}\right)^{1/\alpha}\frac{dx}{\alpha x} \\
&= \frac{1}{\alpha r_{-2}}\left(\frac{2}{\alpha}\right)^{1/\alpha}\int_x^\infty \frac{\gamma(a,x)}{x^{1/\alpha+1}}dx \\
&= \frac{1}{r_{-2}}\left(\frac{2}{\alpha}\right)^{1/\alpha}\left[-\frac{\gamma(a,x)}{x^{1/\alpha}}\bigg|_x^\infty + \int_x^\infty \frac{x^{a-1}e^x}{x^{1/\alpha}}dx\right] \\
&= \frac{1}{r_{-2}}\left(\frac{2}{\alpha}\right)^{1/\alpha}\left[\frac{\gamma(a,x)}{x^{1/\alpha}} + \int_x^\infty x^{a-1/\alpha-1}e^x dx\right] \\
&= \frac{1}{r_{-2}}\left(\frac{2}{\alpha}\right)^{1/\alpha}\left[\frac{\gamma(a,x)}{x^{1/\alpha}} + \Gamma(a-1/\alpha,x)\right] \\
&= \frac{1}{r}\left[\gamma\left(\frac{3}{\alpha},\frac{2}{\alpha}\left(\frac{r'}{r_{-2}}\right)^\alpha\right) + \left(\frac{2}{\alpha}\right)^{1/\alpha}\left(\frac{r}{r_{-2}}\right)\Gamma\left(\frac{2}{\alpha},\frac{2}{\alpha}\left(\frac{r}{r_{-2}}\right)^\alpha\right)\right] \quad .
\end{aligned}
\tag{51}
$$

Therefore:

$$
\Phi(r) = -G\frac{4\pi\rho_0 r_{-2}^3}{\alpha r}\left(\frac{\alpha}{2}\right)^{3/\alpha}\left[\gamma\left(\frac{3}{\alpha},\frac{2}{\alpha}\left(\frac{r}{r_{-2}}\right)^\alpha\right) + \left(\frac{2}{\alpha}\right)^{1/\alpha}\left(\frac{r}{r_{-2}}\right)\Gamma\left(\frac{2}{\alpha},\frac{2}{\alpha}\left(\frac{r}{r_{-2}}\right)^\alpha\right)\right] \quad , \tag{52}
$$

where the central potential is $\Phi_0 \equiv \Phi(0) = -G\frac{4\pi\rho_0 r_{-2}^2}{\alpha}\left(\frac{\alpha}{2}\right)^{2/\alpha}\Gamma\left(\frac{2}{\alpha},0\right)$.

4.5.4 Distribution Function

To determine the distribution function, we first calculate $d^2\rho/d\Psi^2$:

$$
\begin{aligned}
\frac{d\rho(r)}{dr} &= -\frac{2}{r}\left(\frac{r}{r_{-2}}\right)^\alpha \rho(r) \\
\frac{d^2\rho(r)}{dr^2} &= -\frac{2}{r^2}\left(\frac{r}{r_{-2}}\right)^\alpha\left(\alpha - 1 - 2\left(\frac{r}{r_{-2}}\right)^\alpha\right)\rho(r) \\
\frac{d^2\rho}{d\Psi^2} &= 2\left(\frac{r^2\rho}{G^2M^2}\right)\left(\frac{r}{r_{-2}}\right)^\alpha\left[2\left(\frac{r}{r_{-2}}\right)^\alpha + \frac{4\pi r^3\rho}{M} - \alpha - 1\right] \quad .
\end{aligned}
\tag{53}
$$

Therefore, the distribution function is given by:

$$
\begin{aligned}
f(\mathcal{E}) &= \frac{1}{\sqrt{8}\pi^2}\int_0^{\mathcal{E}} \frac{2}{\sqrt{\mathcal{E}-\Psi}}\left(\frac{r^2\rho}{G^2M^2}\right)\left(\frac{r}{r_{-2}}\right)^{\alpha}\left[2\left(\frac{r}{r_{-2}}\right)^{\alpha}+\frac{4\pi r^3\rho}{M}-\alpha-1\right]d\Psi \\
&= \frac{1}{\sqrt{8}\pi^2}\int_{r_{\mathcal{E}}}^{\infty} \frac{2}{\sqrt{\mathcal{E}-\Psi}}\left(\frac{\rho}{GM}\right)\left(\frac{r}{r_{-2}}\right)^{\alpha}\left[2\left(\frac{r}{r_{-2}}\right)^{\alpha}+\frac{4\pi r^3\rho}{M}-\alpha-1\right]dr \ .
\end{aligned}
\tag{54}
$$

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