# MATH 287 HOMEWORK 7

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*Exercise* 1. Type in the piecewise definition of the Fibonacci numbers.

*Answer.*

$$f_n = \begin{cases} 0, & \text{if n = 0} \\ 1, & \text{if n = 1} \\ f_{n-1} + f_{n-2} & \text{otherwise} \end{cases}$$

◇

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*Exercise* 2. Proposition 8.18: For all $m, n \in \mathbf{R}, (-m)(-n) = mn$.

*Proof.* Let $m, n \in \mathbf{R}$. First, we should note that $-m + m = 0$ and $-n + n = 0$ (Axiom 8.4). Multiplying the first equation by $n$ and the second equation by $-m$ gives us

$$n(-m + m) = 0 \text{ and } (-m)(-n + n) = 0.$$

From Prop. 8.15, we know that $-m \cdot 0 = 0$ and $n \cdot 0 = 0$, so the right-hand sides of each equation remain the same. After distributing and rearranging (Prop. 8.8 and 8.1 (iii)), we have

$$mn + (-m)n = 0 \text{ and } (-m)n + (-m)(-n) = 0.$$

Using Axiom 8.1 (i) to rearrange the equation on the left, we can show

$$(-m)n + mn = 0$$

$$(-m)n + (-m)(-n) = 0.$$

From Prop. 8.11, we know that the additive inverses of real numbers are unique. Therefore, $(-m)(-n)$ must be $mn$, i.e., $(-m)(-n) = mn$. $\square$

*Exercise* 3. Proposition 8.32: For all $w, x, y, z \in \mathbf{R}$:

(1) If $x < y$ then $x + z < y + z$.

(2) If $x < y$ and $z < w$ then $x + z < y + w$.

(3) If $0 < x < y$ and $0 < z \leq w$ then $xz < yw$.

(4) If $x < y$ and $z < 0$ then $yz < xz$.

*Claim* 3.1. If $x < y$ then $x + z < y + z$.

*Proof.* Let $x, y \in \mathbf{R}$. We are told that $x < y$, that is, $y - x \in \mathbf{R}_{>0}$. Then, suppose we take any arbitrary element $z \in \mathbf{R}$, and add $z$ to both $x$ and $y$:

$$\begin{aligned} x + z < y + z &\Rightarrow (y + z) - (x + z) \in \mathbf{R}_{>0} \text{ (definition of } <) \\ &\Rightarrow y + z - x - z \in \mathbf{R}_{>0} \text{ (distributing)} \\ &\Rightarrow y - x + 0 \in \mathbf{R}_{>0} \text{ (Axiom 8.4)} \\ &\Rightarrow x < y \text{ (definition of } <). \end{aligned} \tag{1}$$

We have shown that when adding an arbitrary constant value $z$ to both sides of the inequality, the quantity $z$ is zeroed out, preserving the inequality between $x$ and $y$. Thus we have shown that for all $z \in \mathbf{R}$, if $x < y$, $x + z < y + z$. $\square$

*Claim* 3.2. If $x < y$ and $z < w$ then $x + z < y + w$.

*Proof.* Let $w, x, y, z \in \mathbf{R}$. We are told that $x < y$ and $z < w$. That is, $y - x \in \mathbf{R}_{>0}$ and $w - z \in \mathbf{R}_{>0}$, by the definition of $<$. Adding $(y - x)$ and $(w - z)$ gives

$$\begin{aligned}(y - x) + (w - z) \in \mathbf{R}_{>0} &\Rightarrow y + w - x - z \in \mathbf{R}_{>0} \\ &\Rightarrow (y + w) - (x) - (z) \in \mathbf{R}_{>0} \text{ (Prop. 8.22)} \\ &\Rightarrow (y + w) - (x + z) \in \mathbf{R}_{>0} \text{ (Axiom 8.1 (ii))} \\ &\Rightarrow x + z < y + w.\end{aligned} \tag{2}$$

□

*Claim* 3.3. If $0 < x < y$ and $0 < z \leq w$ then $xz < yw$.

*Proof.* Let $w, x, y, z \in \mathbf{R}$. We know that $y - x \in \mathbf{R}_{>0}$, $x - 0 \in \mathbf{R}_{>0}$, and $z - 0 \in \mathbf{R}_{>0}$. We also know that $w - z \in \mathbf{R}_{>0}$ or $w = z$. If $w = z$ then, we have

$$\begin{aligned}z(y - x) \in \mathbf{R}_{>0} &\Rightarrow yz - xz \in \mathbf{R}_{>0} \\ &\Rightarrow xz < yz \\ &\Rightarrow (xz < yz) \equiv (xw < yw) \equiv (xw < yz) \equiv (xz < yw).\end{aligned} \tag{3}$$

If $z < w$, then $w - z \in \mathbf{R}_{>0}$. Multiplying $(y - x)$ and $(w - z)$, we have

$$(y - x) \cdot (w - z) \in \mathbf{R}_{>0} \Rightarrow yw - yz - xw - xz \in \mathbf{R}_{>0}.$$

Letting $j = -(yz + xw + xz)$, we can see that $yw - j \in \mathbf{R}_{>0} \Rightarrow j < yw$. Given that $xz$ is part of $j$'s sum, and each of $x, w, y, z > 0$, we can conclude that $yw - xz \in \mathbf{R}_{>0}$, i.e., $xz < yw$. □

*Claim* 3.4. If $x < y$ and $z < 0$ then $yz < xz$.

*Proof.* Let $x, y, z \in \mathbf{R}$. We are told that $x < y$ and $z < 0$. This means that $y - x \in \mathbf{R}_{>0}$ and $0 - z \in \mathbf{R}_{>0}$ (i.e., $-z \in \mathbf{R}_{>0}$). We know that $\mathbf{R}$ is closed under multiplication, so we can multiply $-z$ and $y - x$ to show that

$$\begin{aligned}
(-z) \cdot (y - x) \in \mathbf{R}_{>0} &\Rightarrow -yz \cdot (-z)(-x) \in \mathbf{R}_{>0} \\
&\Rightarrow xz - yz \in \mathbf{R}_{>0} \text{ (Prop. 8.18 \& Axiom 8.1 (i))} \qquad (4) \\
&\Rightarrow yz < xz \text{ (by the definition of } <\text{).}
\end{aligned}$$

□

*Exercise* 4. Proposition 8.53: Every nonempty subset of $\mathbf{R}$ that is bounded below has a greatest lower bound.

*Proof.* Let $A \subseteq \mathbf{R}$ where $A \neq \emptyset$. Let us also assume that $A$ is bounded below. We intend to show that $A$ has a greatest lower bound. That is we will prove that $\exists c \in \mathbf{R}$ such that

(1) $\forall a \in A, c \leq a$ and

(2) $\forall r \in \mathbf{R}$, if $r$ is a lower bound of $A$, $r \leq c$.

First, let $M$ be a lower bound for $A$, and let $B = \{-a | a \in A\}$. If $M$ is a lower bound for $A$, this means that $\forall a \in A, M \leq a$. Then, let $b$ be an arbitrary member of $B$:

$$
\begin{aligned}
& b \in B \\
& \Rightarrow -b \in A \text{ (negation of b is in A)} \\
& \Rightarrow -b \geq M \text{ (all elements of A are bounded below by M)} \\
& \Rightarrow b \leq -M.
\end{aligned} \tag{5}
$$

Because $b$ is arbitrary, this means that $-M$ is an upper bound of B. By Axiom 8.52, we know that every non-empty subset of $\mathbf{R}$ has a least upper bound. $B$ is a non-empty subset of $\mathbf{R}$, so $\exists k \in \mathbf{R}$ such that:

(1) $\forall b \in B, b \leq k$ and

(2) $\forall r \in \mathbf{R}$, if $r$ is an upper bound of $B$, $k \leq r$.

Given that we know some $k$ exists, we can say

$$k \in \mathbf{R}$$

$$\begin{aligned} &\forall b \in B, b \leq k \text{ (definition of an upper bound)} \\ &\forall b \in B, -b \geq -k \text{ (multiplying both sides by -1)} \\ &\forall a \in A, a \geq -k \text{ (because an arbitrary } a = -b\text{).} \end{aligned} \tag{6}$$

This means that $-k$ is a lower bound of $A$. Given $k$'s relationship with $B$, we can infer that $\forall r \in \mathbf{R}$, if $r$ is a lower bound of $A$, then $-k \geq r$. This meets the criteria we established at the beginning of the proof, meaning that $-k$ must be equal to $c$, the greatest lower bound of $A$. $\square$