MATH 287 HOMEWORK 5

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Exercise 1. Show some of your favorite equations in *inline* and **display** mathematics.

Answer. Some people like small equations such as $e^{\pi i} + 1 = 0$ or $a^2 + b^2 = c^2$.

My favorite small equation is $A^T A\mathbf{x} = A^T\mathbf{b}$.

Other people like big equations like

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

My favorite big equation is

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

$\diamond$

Exercise 2. Project 5.16(ii). Prove or give a counterexample: For all sets A, B, C, $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Answer. I claim that this statement is true.

Proof. Take an element on the left-hand side, called x, we can see that

$$
\begin{aligned}
&\text{(because } x \in A \cap (B - C)\text{)} && x \in A \\
&\text{(definition of } \cap\text{)} && \Rightarrow x \in (B - C) \\
&\text{(definition of } -\text{)} && \Rightarrow (x \in B) \wedge (x \notin C) \\
& && \Rightarrow (x \in A \cap B) \wedge (x \notin A \cap C) \\
& && \Rightarrow x \in (A \cap B) - (A \cap C) \\
& && \Rightarrow A \cap (B - C) \subseteq (A \cap B) - (A \cap C).
\end{aligned}
$$

Then, taking an element y on the right-hand side, we have

$$
\begin{aligned}
&\text{(definition of } -) && (y \in A \cap B) \wedge (y \notin A \cap C) \\
&\text{(definition of } \cap) && \Rightarrow (y \in A) \wedge (y \in B) \wedge (y \notin C) \\
& && \Rightarrow y \in (B - C) \\
& && \Rightarrow y \in A \cap (B - C) \\
& && \Rightarrow (A \cap B) - (A \cap C) \subseteq A \cap (B - C).
\end{aligned}
$$

Thus we have shown that

$$A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$$

$$(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$$

which means that $A \cap (B - C) = (A \cap B) - (A \cap C)$. $\square$

$\diamond$

Exercise 3. Proposition 5.20(ii). If A, B, C are any sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. Take an element x on the left-hand side. We'll also use i, and j to represent the ordered pair (i, j) that x constitutes. We can see that

$$\text{(definition of } \times\text{)} \qquad i \in A$$

$$\text{(definition of } \cap\text{)} \qquad j \in B \cap C \Rightarrow (j \in B) \wedge (j \in C).$$

By the definition of $\times$, we know that

$$(i, j) \in A \times B \Rightarrow x \in A \times B$$

$$(i, j) \in A \times C \Rightarrow x \in A \times C.$$

This means we can say that $x \in (A \times B) \cap (A \times C)$ and therefore,

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C).$$

Now we will examine an element y from the right-hand side. We will use k, and l to represent the ordered pair (k, l) that y constitutes. From the definition of $\cap$, we know that $(y \in A \times B) \wedge (y \in A \times C)$. We also know from the definition of $\times$ that

$$(k, l) \in A \times B \Rightarrow (k \in A) \wedge (l \in B)$$

$$(k, l) \in A \times B \Rightarrow l \in C.$$

This means that $l \in B \cap C$. This means we can say

$$(k, l) \in A \times (B \cap C),$$

i.e., $y \in A \times (B \cap C)$. From this we can conclude that

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C).$$

To summarize, we have shown

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$$

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

and therefore

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

$\square$

Exercise 4. Project 5.21(ii).

(Prove or disprove. If you believe the equation is true, give a proof. If you believe it is not true for all sets, give a counterexample. You can try making up some random sets and working out each side of the equation to see if they match or not.)

Exercise 5. Find $\sum_{j=0}^{k} f_j^2$ = (your answer), where the are Fibonacci numbers as defined in the textbook. Prove your answer.

Your answer will have a clear statement: (your answer). Then, a proof of your answer.

For your proof, use induction.

For example, since the Fibonacci numbers start with , you will get values starting with 0, 1, 1, 2, 3, 5, 8, ..., you will get values starting with $0^2 = 0, 0^2 + 1^2 = 1, 1^2 + 1^2 = 2, 1^2 + 1^2 + 2^2 = 6$, and so on.

Hint: Work out some of the values and look for a pattern. It might be helpful to look at factorizations of the values.

Answer. After inspecting the series/summation, I propose that

$$\sum_{j=0}^{k} f_j^2 = f_k \cdot f_{k+1}$$

for $k \geq 1$.

Proof. Let $P(k)$ be defined as $\sum_{j=0}^{k} f_j^2 = f_k \cdot f_{k+1}$ for $k \geq 1$. As a base case, we will examine $P(1)$:

$$\sum_{j=0}^{1} f_j^2 = 0^2 + 1^2 = 0 + 1 = 1$$

$$f_1 \cdot f_2 = 1 \cdot 1 = 1.$$

We know $P(k)$ is true for some k. Let us assume $P(k)$ is true, and use this to show that $P(k+1)$ is also true. That is, we intend to show that $P(k+1)$,

$$\sum_{j=0}^{k+1} f_j^2 = f_{k+1} \cdot f_{k+2}$$

is true. □

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