MATH 287 HOMEWORK 2

ANDREW MOORE

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Exercise 1. Proposition 2.7(iv). Let $m, n, p, q \in \mathbf{Z}$. If $m < n$ and $p < 0$ then $np < mp$.

Proof. We are told that $n - m \in \mathbf{N}$, and $0 - p \in \mathbf{N}$ (i.e., $-p \in \mathbf{N}$, based on Axiom 2.1.4). We will show that $mp - np \in \mathbf{N}$. Rearranging $mp - np$ we have:

$$\begin{aligned} mp - np &= p(m - n) \\ &= p(m(-1)(-1) + n(-1)) \\ &= p(-1)(m(-1) + n) \\ &= -p((-m) + n) \\ &= -p(n - m) \in \mathbf{N}. \end{aligned}$$

The series of steps above demonstrate that $mp - np$ can be expressed as two terms we know to be natural numbers (based on our initial assumptions): $-p$ and $(n - m)$. Axiom 2.1.2 tells us that the product of any two natural numbers is also in $\mathbf{N}$. Therefore, $(mp - np)$ must be in $\mathbf{N}$, i.e. $np < mp$. $\square$

Exercise 2. Proposition 2.12(iii). For all $m, n, p \in \mathbf{Z}$, if $p < 0$ and $mp < np$ then $n < m$.

Proof. We are told we can assume the following: $0 - p \in \mathbf{N}$, and $np - mp \in \mathbf{N}$. We must show that $n < m$, i.e. $m - n \in \mathbf{N}$. Rearranging $np - mp \in \mathbf{N}$, we have:

$$\begin{aligned} np - mp &= p(n - m) \\ &= p(n(-1)(-1) + m(-1)) \\ &= p(-1)(n(-1) + m) \\ &= -p(-n + m) \\ &= -p(m - n) \in \mathbf{N}. \end{aligned}$$

We can reexpress $0 - p \in \mathbf{N}$ as $-p \in \mathbf{N}$ (which we know from Axiom 2.1.4). From the series of equations above, we've seen that $np - mp \in \mathbf{N}$ can be expressed as $-p(m - n) \in \mathbf{N}$. Axiom 2.1.2 tells us that the product of any two numbers in $\mathbf{N}$ is also in $\mathbf{N}$. Therefore, $(m - n)$ must be in $\mathbf{N}$, i.e. $n < m$.

□

Exercise 3. Proposition 2.26. *For all integers* $k \geq -3$, $3k^2 + 21k + 37 >= 0$.

Proof. We are told that $k \in \mathbf{Z}$. We are asked to prove the claim that for *all* $k \geq -3$, the evaluation of $3k^2 + 21k + 37$ will be greater than or equal to 0. This claim can be proven using induction.

For induction, we test the statement using a *base case*. For the base case, we'll use the minimum value of k, that is, -3:

$$3k^2 + 21k + 37 \geq 0$$

$$3(-3)^2 + 21(-3) + 37 \geq 0$$

$$64 - 63 \geq 0$$

$$1 \geq 0.$$

Having shown the base case to be true, we know that at least for some values of $k \geq -3$, the statement $3k^2 + 21k + 37 \geq 0$ is true. Now let's imagine adding 1 to k, and evaluating the statement.

For $k = n + 1$ (where $n \geq -3$) we have:

$$3k^2 + 21k + 37 \geq 0$$

$$3(n+1)^2 + 21(n+1) + 37 \geq 0$$

$$(3n^2 + 6n + 3) + (21n + 21) + 37 \geq 0$$

$$3n^2 + 27n + 61 \geq 0.$$

This result is larger than what results when $k = n$, i.e. our inductive hypothesis:

$$(3n^2 + 27n + 61) - (3n^2 + 21n + 37) = 6n + 24$$

$$3n^2 + 27n + 61 \geq 6n + 24 \geq 0.$$

We know that $n \geq -3$, so $6n + 24 \geq 6 \geq 0$, and therefore $3(n+1)^2 + 21(n+1) + 37 \geq 0$. In essence, we have seen that if you pick an arbitrary $n \in \mathbf{N}$, such that $n \geq -3$, the value of any $n + 1$ applied to the expression will be greater than 0. This concludes our induction.

$\square$

Exercise 4. Project 2.28. Determine for which natural numbers $k^2 - 3k \geq 4$ and prove your answer.

Answer.

Claim 4.1. $k^2 - 3k \geq 4$ for $k \geq 4$.

Proof. We are given the inequality, $k^2 - 3k \geq 4$, and are asked to find (and justify) the values of k that make the inequality true. We will prove the claim $k^2 - 3k \geq 4$ for $k \geq 4$ using induction. First, note that $k \leq 3$ does not satisfy the inequality.

For $k = 3$:

$$k^2 - 3k \geq 4 \Rightarrow (3)^2 - 3(3) \geq 4 \Rightarrow 9 - 9 \geq 4 \Rightarrow 0 \geq 4$$

The statement $0 \geq 4$ is *false*, because $0 \neq 4$ and $0 - 4 \notin \mathbf{N}$. It is trivial to demonstrate the same result for $k = 1$ and $k = 2$, and this is left to the reader. Assuming that k can't be ≤ 3, let us redefine k to be $k = 3 + j$, such that $j \in \mathbf{N}$ and $j \geq 1$. We will now prove the base case ($j = 1$).

For $k = 3 + 1 = 4$:

$$k^2 - 3k \geq 4 \Rightarrow (4)^2 - 3(4) \geq 4 \Rightarrow 16 - 12 \geq 4 \Rightarrow 4 \geq 4$$

This last statement is true $(4 = 4)$. We will now show that the results apply for all j, by proving this is true for $k = 4 + (j + 1)$.

For $k = 4 + (j + 1) = 5 + j$:

$$(k+1)^2 - 3(k+1) \geq 4$$

$$(k^2 + 2k + 1) - 3k - 3 \geq 4$$

$$k^2 - k - 2 \geq 4$$

$$(5 + j)^2 - (5 + j) - 2 \geq 4$$

$$25 + 10j + j^2 - j - 7 \geq 4$$

$$j^2 + 9j + 18 \geq 4$$

The final statement $j^2 + 9j + 18 - 4 \in \mathbf{N}$ is true because there is no value of j in $\mathbf{N}$ that could cause the value of the statement to be ≤ 0. $\square$

$\diamond$

Exercise 5. The definition of "gcd" on page 22 of the textbook, in terms of smallest positive integer combination, might be unfamiliar. Soon we'll look again at gcd and relate this definition to the more familiar versions. This homework problem will answer a different question: What does this have to do with induction and the Well Ordering Principle? Why did gcd show up in Chapter 2?

a. As a thought experiment, let's try defining "gcd of rational numbers" like this. Suppose a and b are rational numbers, take the set of elements $ax + by$ where x and y are rational numbers, and $ax+by > 0$. Can we define "$gcd(a, b)$" to be the smallest element of that set? Why or why not? Explain.

b. Why does gcd of integers work? How does the Well Ordering Principle help?

Answer. a. It doesn't work. You can't compare two fractional numbers such as $\frac{2}{7}$ and $\frac{3}{8}$, and get a natural number as a combination of $\frac{2}{7}(x) + \frac{3}{8}(y)$. You'll always end up with a rational number.

b. $\diamond$