MATH 287 HOMEWORK 10

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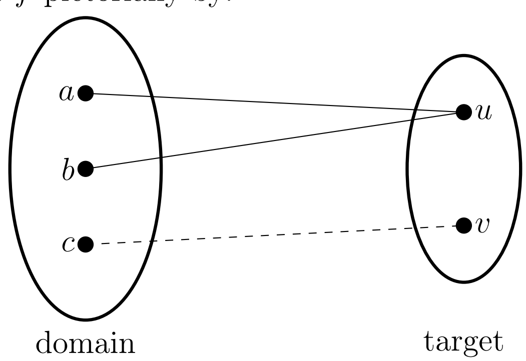
Exercise 1. Suppose f is a function $f : \{a, b, c\} \to \{u, v\}$ and we have $f(a) = u$ and $f(b) = u$. How should we define $f(c)$ if we want f to be surjective (onto)?

Answer. We should define $f(c) = v$. Then f is given by the table:

x	a	b	c
$f(x)$	u	u	v

We can represent f pictorially by:



We used a dashed line to show $f(c) = v$ that was the answer to the question. $\diamond$

Date: November 28, 2021.

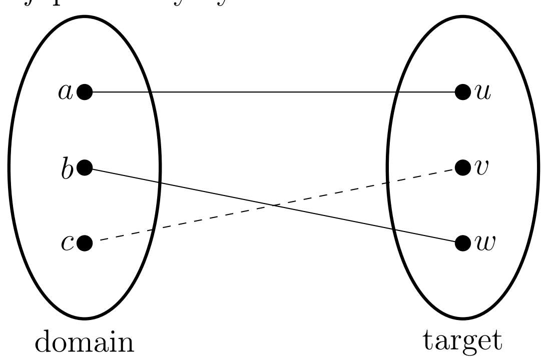
Exercise 2. Suppose f is a function $f : \{a, b, c\} \to \{u, v, w\}$ and we have $f(a) = u, f(b) = w$. How should we define $f(c)$ if we want f to be injective (one-to-one)?

Answer. We should define $f(c) = v$.

Then f is given by the table:

x	a	b	c
$f(x)$	u	w	v

We can represent f pictorially by:



We used a dashed line to show $f(c) = v$ that was the answer to the question. $\diamond$

Exercise 3. Finally, redo one problem from a previous homework assignment.

Problem: "Prove that for all $k \in \mathbf{N}, 5^k + 3$ is divisible by 4.

Answer. This is question 4 from Homework 3. I've omitted the 3 initial items that ask to show the statement is true for $k = 1, 2, 3$, which I hope is okay. I was docked points for arithmetic mistakes during the proof's inductive step. Here is my full and (hopefully) corrected version.

Proof. We will use induction on k. Let $P(k)$ denote the statement $5^k + 3$ is divisible by 4. As a base case $P(k = 1)$ we see

$$5^1 + 3 = 5 + 3 = 8 = 4 \cdot 2.$$

We will now assume $P(k)$ is true, and use this to show that $P(k + 1)$ is also true. To say $P(k)$ is true means $\exists y \in \mathbf{Z}$ such that $4y = 5^k + 3$. We need to show that $P(k + 1)$ is true, i.e., that $4z = 5^{k+1} + 3$ for some $z \in \mathbf{Z}$. First, note that $5^k = 4y - 3$. Then, rearranging the right-hand side, we have

$$4z = 5^{k+1} + 3$$

$$4z = 5^k \cdot 5^1 + 3$$

$$4z = (4y - 3) \cdot 5 + 3$$

$$4z = 20y - 15 + 3$$

$$4z = 20y - 12$$

$$z = 5y - 3.$$

We have shown that $\exists z \in \mathbf{Z}$, such that $4 \cdot z = 5^{k+1} + 3$. By our inductive hypothesis, we know that $y \in \mathbf{Z}$, and from our axioms, we know that the multiplication and subtraction of integers (i.e., $5y - 3$) results in another integer. So, we know the statement is true for all $k \in \mathbf{N}$. This concludes our induction, and the proof. $\square$

From the units that covered induction and the natural numbers, I've developed an appreciation for how powerful induction is. When initially exposed to induction as a method, I wasn't really able to understand the sort of "domino"-esque fulfillment of such proofs. I think this was because I was fixated on actually trying to conceptualize performing a given operation an infinite number of times.

However, as I worked through more problems requiring induction, I realize their power and elegance comes from the ability to choose an arbitrary k (after evaluating the statement P at k) and show that the statement P holds for the immediately following k (i.e., $k + 1$). The method actually allows for a smooth and familiar writing style that can actually result in a compact and well-organized proof, compared to other approaches on the same problem. However, as my stumbling on the problem initially shows, it requires careful handling of the variables being used for induction. I guess one can always benefit from more practice working with exponents!

$\diamond$