MATH 287 HOMEWORK 8

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Exercise 1. We need to find the determinants of these matrices:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}, \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$$

Answer. The determinants are

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1,$$

$$\det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = -1,$$

$$\det \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = 1,$$

$$\det \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix} = -1,$$

$$\det \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} = 1.$$

$\diamond$

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Exercise 2. Proposition 9.7(ii): If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is surjective.

Proof. Assume that $f : A \to B$ and $g : B \to C$ are surjective functions. We intend to show that $g \circ f$ is also surjective. That is

$$\exists c \in C, \forall a \in A : (g \circ f)(a) = c.$$

$g \circ f$ is a composition of f and g, defined as

$$g(f(a)) \text{ for all } a \in A.$$

By hypothesis, we know that f maps the entirety of B's elements (f is surjective). This means

$$\forall b \in B, \exists a \in A \text{ such that } f(a) = b.$$

We also know that g maps the entirety of C's elements (g is surjective). That is

$$\forall c \in C, \exists b \in B \text{ such that } g(b) = c.$$

Because $f \circ g$ is a composition, inputs (a's) are evaluated first by f, and the subsequent outputs (b's) are then fed as new inputs for g. Phrased differently, $g \circ f$ maps $A \to C$ by using the range of f as the domain of g. This means, under the composition, all inputs to g are outputs of f. Thus we can write

$$\begin{aligned} f(a) &= b \\ g(b) &= c \\ g(f(a)) &= c \\ (g \circ f)(a) &= c. \end{aligned} \tag{1}$$

This means for every $c \in C$ (i.e., each output of $g(b)$), there must be an $a \in A$ that corresponds to it. Therefore, $g \circ f$ must be surjective. This concludes the proof. $\square$

Exercise 3. Prove the claims:

For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each injective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is injective.

For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each surjective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is surjective.

For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each bijective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is bijective.

Claim 3.1. For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each injective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is injective.

Proof. We will prove the claim using induction. Let $P(n)$ denote the claim for a set of n functions. Using $P(2)$ as a base case case, we have f_1 and f_2 as two injective functions. With $f_1 \circ f_2$ being their composition, when applying Proposition 9.7 (i), we know that $f_1 \circ f_2$ is also injective. Therefore $P(2)$ is true.

Assuming $P(n)$ is true for some $n \in N$, we will use this fact to prove it is also true for $P(n+1)$. Ascending up to $P(n+1)$, we have an $n+1$ collection of functions,

$$f_1, f_2, \ldots, f_n, f_{n+1}$$

and we are assembling the following composition

$$f_1 \circ f_2 \circ \ldots \circ f_n \circ f_{n+1}.$$

Let g be a function, representing $f_1 \circ f_2 \circ \ldots \circ f_n$. We can rewrite our proposed composition as $g \circ f_{n+1}$. By hypothesis we know that g is injective. From our initial assumptions, we know that f_{n+1} is also injective. Then, applying Proposition 9.7 (i) to $g \circ f_{n+1}$ (a composition of two injective functions), we know the result will also be injective.

Thus, any composition of injective functions will also be injective, for all $n \in \mathbf{N}$, where $n \geq 2$. This concludes the induction, and the proof. $\square$

Claim 3.2. For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each surjective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is surjective.

Proof. We will prove the claim using induction. Let $P(n)$ denote the claim for a set of n functions. Using $P(2)$ as a base case case, we have f_1 and f_2 as two surjective functions. With $f_1 \circ f_2$ being their composition, when applying Proposition 9.7 (ii), we know that $f_1 \circ f_2$ is also surjective. Therefore $P(2)$ is true.

Assuming $P(n)$ is true for some $n \in N$, we will use this fact to prove it is also true for $P(n+1)$. Ascending up to $P(n+1)$, we have an $n+1$ collection of functions,

$$f_1, f_2, \ldots, f_n, f_{n+1}$$

and we are assembling the following composition

$$f_1 \circ f_2 \circ \ldots \circ f_n \circ f_{n+1}.$$

Let g be a function, representing $f_1 \circ f_2 \circ \ldots \circ f_n$. We can rewrite our proposed composition as $g \circ f_{n+1}$. By hypothesis we know that g is surjective. From our initial assumptions, we know that f_{n+1} is also surjective. Then, applying Proposition 9.7 (i) to $g \circ f_{n+1}$ (a composition of two surjective functions), we know the result will also be surjective.

Thus, any composition of surjective functions will also be surjective, for all $n \in \mathbf{N}$, where $n \geq 2$. This concludes the induction, and the proof. $\square$

Claim 3.3. For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each bijective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is bijective.

Proof. We will prove the claim using induction. Let $P(n)$ denote the claim for a set of n functions. Using $P(2)$ as a base case case, we have f_1 and f_2 as two bijective functions. With $f_1 \circ f_2$ being their composition, when applying Proposition 9.7 (iii), we know that $f_1 \circ f_2$ is also bijective. Therefore $P(2)$ is true.

Assuming $P(n)$ is true for some $n \in N$, we will use this fact to prove it is also true for $P(n+1)$. Ascending up to $P(n+1)$, we have an $n+1$ collection of functions,

$$f_1, f_2, \ldots, f_n, f_{n+1}$$

and we are assembling the following composition

$$f_1 \circ f_2 \circ \ldots \circ f_n \circ f_{n+1}.$$

Let g be a function, representing $f_1 \circ f_2 \circ \ldots \circ f_n$. We can rewrite our proposed composition as $g \circ f_{n+1}$. By hypothesis we know that g is bijective. From our initial assumptions, we know that f_{n+1} is also bijective. Then, applying Proposition 9.7 (iii) to $g \circ f_{n+1}$ (a composition of two bijective functions), we know the result will also be bijective.

Thus, any composition of bijective functions will also be bijective, for all $n \in \mathbf{N}$, where $n \geq 2$. This concludes the induction, and the proof. $\square$

Exercise 4. Suppose $f : A \to B$ and $g : B \to C$, so $g \circ f$ is a function $A \to C$. Prove the following claims:

If $g \circ f$ is injective, then f is injective.

If $g \circ f$ is surjective, then g is surjective.

If $g \circ f$ is bijective, then f is injective and g is surjective. Explain why (use the previous claims) and give an example to show that f doesn't have to be surjective and g doesn't have to be injective.

Claim 4.1. If $g \circ f$ is injective, then f is injective.

Proof. Suppose $g \circ f$ is injective. We intend to show that this implies f is also injective. Let $h = g \circ f = g(f(a))$. From its definition, h is injective. This means for all $a_1, a_2 \in A, a_1 \neq a_2$ implies that $h(a_1) \neq h(a_2)$. That is, each element in the codomain is associated with only one element in the domain. □

Claim 4.2. If $g \circ f$ is surjective, then g is surjective.

Proof. Suppose $g \circ f$ is surjective. We intend to show that this implies g is also surjective. Let $h = g \circ f = g(f(a))$. From its definition, h is surjective. This means that for each $c \in C, \exists a \in A$ such that $h(a) = c$. □

Claim 4.3. If $g \circ f$ is bijective, then f is injective and g is surjective. Explain why (use the previous claims) and give an example to show that f doesn't have to be surjective and g doesn't have to be injective.

Proof. □