

MATH 287 HOMEWORK 7

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Exercise 1. Type in the piecewise definition of the Fibonacci numbers.

Answer.

$$f_n = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{otherwise} \end{cases}$$

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Exercise 2. Proposition 8.18: For all $m, n \in \mathbf{R}$, $(-m)(-n) = mn$.

Proof. Let $m, n \in \mathbf{R}$. First, we should note that $-m + m = 0$ and $-n + n = 0$ (Axiom 8.4). Multiplying the first equation by n and the second equation by $-m$ gives us

$$n(-m + m) = 0 \text{ and } (-m)(-n + n) = 0.$$

From Prop. 8.15, we know that $-m \cdot 0 = 0$ and $n \cdot 0 = 0$, so the right-hand sides of each equation remain the same. After distributing and rearranging (Prop. 8.8 and 8.1 (iii)), we have

$$mn + (-m)n = 0 \text{ and } (-m)n + (-m)(-n) = 0.$$

Using Axiom 8.1 (i) to rearrange the equation on the left, we can show

$$(-m)n + mn = 0$$

$$(-m)n + (-m)(-n) = 0.$$

From Prop. 8.11, we know that the additive inverses of real numbers are unique. Therefore, $(-m)(-n)$ must be mn , i.e., $(-m)(-n) = mn$. \square

Exercise 3. Proposition 8.32: For all $w, x, y, z \in \mathbf{R}$:

- (1) If $x < y$ then $x + z < y + z$.
- (2) If $x < y$ and $z < w$ then $x + z < y + w$.
- (3) If $0 < x < y$ and $0 < z \leq w$ then $xz < yw$.
- (4) If $x < y$ and $z < 0$ then $yz < xz$.

Claim 3.1. If $x < y$ then $x + z < y + z$.

Proof. Let $x, y \in \mathbf{R}$. We are told that $x < y$, that is, $y - x \in \mathbf{R}_{>0}$. Then, suppose we take any arbitrary element $z \in \mathbf{R}$, and add z to both x and y :

$$\begin{aligned}
 x + z < y + z &\Rightarrow (y + z) - (x + z) \in \mathbf{R}_{>0} \text{ (definition of } <) \\
 &\Rightarrow y + z - x - z \in \mathbf{R}_{>0} \text{ (distributing)} \\
 (1) \quad &\Rightarrow y - x + 0 \in \mathbf{R}_{>0} \text{ (Axiom 8.4)} \\
 &\Rightarrow x < y \text{ (definition of } <).
 \end{aligned}$$

We have shown that when adding an arbitrary constant value z to both sides of the inequality, the quantity z is zeroed out, preserving the inequality between x and y . Thus we have shown that for all $z \in \mathbf{R}$, if $x < y$, $x + z < y + z$. \square

Claim 3.2. If $x < y$ and $z < w$ then $x + z < y + w$.

Proof. Let $w, x, y, z \in \mathbf{R}$. We are told that $x < y$ and $z < w$. That is, $y - x \in \mathbf{R}_{>0}$ and $w - z \in \mathbf{R}_{>0}$, by the definition of $<$. Adding $(y - x)$ and $(w - z)$ gives

$$\begin{aligned}
 (y - x) + (w - z) &\in \mathbf{R}_{>0} \Rightarrow y + w - x - z \in \mathbf{R}_{>0} \\
 &\Rightarrow (y + w) - (x + z) \in \mathbf{R}_{>0} \text{ (Prop. 8.22)} \\
 (2) \quad &\Rightarrow (y + w) - (x + z) \in \mathbf{R}_{>0} \text{ (Axiom 8.1 (ii))} \\
 &\Rightarrow x + z < y + w.
 \end{aligned}$$

□

Claim 3.3. If $0 < x < y$ and $0 < z \leq w$ then $xz < yw$.

Proof. Let $w, x, y, z \in \mathbf{R}$. We know that $y - x \in \mathbf{R}_{>0}$, $x - 0 \in \mathbf{R}_{>0}$, and $z - 0 \in \mathbf{R}_{>0}$. We also know that $w - z \in \mathbf{R}_{>0}$ or $w = z$. If $w = z$ then, we have

$$\begin{aligned}
 z(y - x) &\in \mathbf{R}_{>0} \Rightarrow yz - xz \in \mathbf{R}_{>0} \\
 (3) \quad &\Rightarrow xz < yz \\
 &\Rightarrow (xz < yz) \equiv (xw < yw) \equiv (xw < yz) \equiv (xz < yw).
 \end{aligned}$$

If $z < w$, then $w - z \in \mathbf{R}_{>0}$. Multiplying $(y - x)$ and $(w - z)$, we have

$$(y - x) \cdot (w - z) \in \mathbf{R}_{>0} \Rightarrow yw - yz - xw - xz \in \mathbf{R}_{>0}.$$

Letting $j = -(yz + xw + xz)$, we can see that $yw - j \in \mathbf{R}_{>0} \Rightarrow j < yw$.

Given that xz is part of j 's sum, and each of $x, w, y, z > 0$, we can conclude

that $yw - xz \in \mathbf{R}_{>0}$, i.e., $xz < yw$. □

Claim 3.4. If $x < y$ and $z < 0$ then $yz < xz$.

Proof. Let $x, y, z \in \mathbf{R}$. We are told that $x < y$ and $z < 0$. This means that $y - x \in \mathbf{R}_{>0}$ and $0 - z \in \mathbf{R}_{>0}$ (i.e., $-z \in \mathbf{R}_{>0}$). We know that \mathbf{R} is closed under multiplication, so we can multiply $-z$ and $y - x$ to show that

$$(-z) \cdot (y - x) \in \mathbf{R}_{>0} \Rightarrow -yz \cdot (-z)(-x) \in \mathbf{R}_{>0}$$

$$(4) \qquad \qquad \qquad \Rightarrow xz - yz \in \mathbf{R}_{>0} \text{ (Prop. 8.18 \& Axiom 8.1 (i))}$$

$$\Rightarrow yz < xz \text{ (by the definition of } < \text{).}$$

□

Exercise 4. Proposition 8.53: Every nonempty subset of \mathbf{R} that is bounded below has a greatest lower bound.

Proof. Let $A \subset \mathbf{R}$ where $A \neq \emptyset$, and let M be a lower bound of A . \square