

## MATH 287 HOMEWORK 9

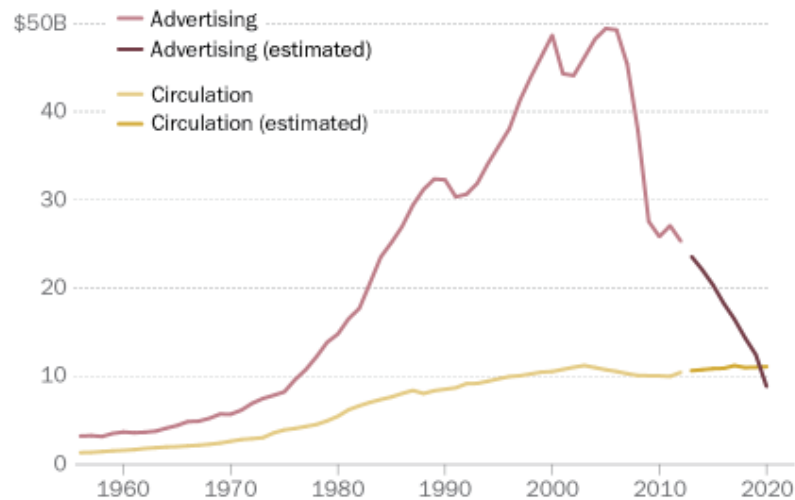
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*Exercise 1.* Find or create a graphic to include, that you want to share.

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### Estimated advertising and circulation revenue of the newspaper industry

*Total revenue of U.S. newspapers (in U.S. dollars)*



Source: News Media Alliance, formerly Newspaper Association of America (through 2012); Pew Research Center analysis of year-end Securities and Exchange Commission filings of publicly traded newspaper companies (2013-2020).

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*Exercise 2.* Proposition 10.8. For all  $x, y \in \mathbf{R}$ :

$$(ii) \quad |xy| = |x| \cdot |y|.$$

$$(iii) \quad -|x| \leq x \leq |x|.$$

*Claim 2.1* (10.8 (ii)).  $\forall x, y \in \mathbf{R} : |xy| = |x| \cdot |y|$

*Proof.* Let  $x, y \in \mathbf{R}$ . We will prove the claim using the following cases:

1. both  $x, y > 0$ .
2. both  $x, y < 0$ .
3. one of  $x$  or  $y$  is  $< 0$ .
4. one or both of  $x$  and  $y$  are equal to 0.

Case 1. Take an element  $z \in \mathbf{R}$ , which is the product of  $x$  and  $y$ , i.e.,  $x \cdot y = z$ . Because  $x$  and  $y$  are positive, we know that  $z$  is also positive.

Therefore, we can write

$$\begin{aligned}
 |x| \cdot |y| &= x \cdot y \text{ (definition of absolute value)} \\
 &= z \\
 (1) \qquad &= |z| \\
 &= |x \cdot y|.
 \end{aligned}$$

Case 2. Assume  $z$  is the product of two negative real numbers denoted  $-x$  and  $-y$ . We know from Prop. 8.18 that the product of two negative (real) numbers is a positive (real) number. Therefore,  $z$  is still positive. Thus, we can write

$$\begin{aligned} &|-x| \cdot |-y| = x \cdot y \text{ (definition of absolute value)} \\ &= z \\ (2) \quad &= |z| \\ &= |-x \cdot -y|. \end{aligned}$$

Case 3. Assume we have two real numbers,  $-x$  (negative) and  $y$  (positive). From Prop. 8.22 (iii), we know that the product of a negative real number and a positive real number is a negative number. We will call this product  $-z$ . By the definition of absolute values, we know that  $|-z| = z$ . So, we can

write

$$\begin{aligned}
 & | -x | \cdot |y| = x \cdot y \text{ (definition of absolute value)} \\
 & = z \\
 (3) \quad & = | -z | \\
 & = | -x \cdot y |.
 \end{aligned}$$

Case 4. In an instance where either or both of  $x$  and  $y$  are 0, we can apply Proposition 10.8 (i), which lets us conclude  $0 = |0| = |0 \cdot 0| = |x \cdot 0| = |0 \cdot y|$ .

Thus we have shown that  $|x| \cdot |y| = |x \cdot y|$  for all  $x, y \in \mathbf{R}$ . This concludes the proof.  $\square$

*Claim 2.2* (10.8 (iii)).  $\forall x \in \mathbf{R} : -|x| \leq x \leq |x|$

*Proof.* Let  $x \in \mathbf{R}$ . First, in the instance where  $x = 0$ , we have

$$-|0| \leq 0 \leq |0| \Rightarrow 0 = 0 = 0.$$

So, the statement holds. Now we will consider instances when  $x \neq 0$ . As a consequence, this means  $-|x| < 0 < |x|$ . Note that when  $x > 0$ ,  $|x| = x$ , and

also note that when  $x < 0$ ,  $|x| = -x$ . Then, we can say

if  $x > 0$ , then

if  $x < 0$ , then

$$-|x| < x$$

$$x < |x|$$

$$-|x| < x \leq |x|$$

$$-|x| \leq x < |x|.$$

Thus, we have shown for all  $x \in \mathbf{R}$ ,  $-|x| \leq x \leq |x|$ . This concludes the proof.  $\square$

*Exercise 3.* Let  $x, y \in \mathbf{R}$ . Claim:  $-y < x < y$  if and only if  $|x| < y$ .

*Proof.*  $\Rightarrow$  Let us assume  $x \geq 0$ . We are told that  $-y < x < y$ . This means that  $y > 0$ . Note that because  $x \geq 0$ ,  $x = |x|$ . So, we can write

$$\begin{aligned} x < y &\Rightarrow y - x \in \mathbf{R}_{>0} \\ (4) \qquad \qquad \qquad &\Rightarrow y - |x| \in \mathbf{R}_{>0} \\ &\Rightarrow |x| < y. \end{aligned}$$

Alternatively, assume  $x < 0$ . We are told that  $-y < x$ . This also indicates that  $y > 0$ . Note that since  $x < 0$ ,  $|x| = -x$ . So, we have

$$\begin{aligned} -y < x &\Rightarrow y > -x \text{ (Proposition 8.37 (i))} \\ (5) \qquad \qquad \qquad &\Rightarrow -x < y \text{ (Rearranging)} \\ &\Rightarrow |x| < y. \text{ (Because } |x| = -x) \end{aligned}$$

$\Leftarrow$  Assume  $|x| < y$ , and that  $x \geq 0$ . This means that  $y > 0$ . Note again that  $x = |x|$ . Then we can say

$$\begin{aligned} |x| < y &\Rightarrow y - |x| \in \mathbf{R}_{>0} \\ &\Rightarrow y - x \in \mathbf{R}_{>0} \\ &\Rightarrow x < y \\ (6) \quad &\Rightarrow x > -y \text{ (Proposition 8.37 (i))} \\ &\Rightarrow -y < x \text{ (Rearranging)} \\ &\Rightarrow -y < x < y. \end{aligned}$$

Alternatively, examine a case where  $x < 0$ . Note that  $|x| = -x$ . Then we have

$$\begin{aligned} |x| < y &\Rightarrow -x < y \\ &\Rightarrow x > -y \text{ (Proposition 8.37 (i))} \\ (7) \quad &\Rightarrow -y < x \text{ (Rearranging)} \\ &\Rightarrow -y < x < y. \end{aligned}$$

Thus, we have shown  $-y < x < y$  if and only if  $|x| < y$ . This concludes the proof. □



*Exercise 4.* Proposition 10.10. Let  $x, y, z \in \mathbf{R}$ .

- (i)  $|x - y| = 0$  if and only if  $x = y$ .
- (ii)  $|x - y| = |y - x|$ .
- (iii)  $|x - z| \leq |x - y| + |y - z|$ .

*Claim 4.1.*  $|x - y| = 0$  if and only if  $x = y$ .

*Proof.*  $\Rightarrow$  Let  $j = (x - y)$ . We are told that  $|j| = 0$ . From Proposition 10.8, we know that  $j$  must be equal to 0. By Axiom 8.4 and Proposition 8.11, this means that  $x$  must equal  $y$ .

$\Leftarrow$  Assume  $x = y$ . Via substitution we can say

$$|x - y| = 0 \Rightarrow |x - x| = 0$$

$$(8) \qquad \qquad \qquad \Rightarrow |0| = 0$$

$$\Rightarrow 0 = 0.$$

Thus, we have shown that if  $|x - y| = 0$ ,  $x$  must equal  $y$ ; and we have shown that if  $x = y$ , the absolute value of  $(x - y)$  must be equal to 0. This concludes the proof.  $\square$

*Claim 4.2.*  $|x - y| = |y - x|$

*Proof.* First, consider a case where either  $x = 0$  or  $y = 0$ . For brevity, we will focus on  $x = 0$ , but the same conclusion follows if we were to use  $y$ , because  $x$  and  $y$  are arbitrary real numbers. If  $x = 0$ , then we have

$$\begin{aligned} |x - y| &= |0 - y| & |y - x| &= |y - 0| \\ &= |-y| & &= |y| \\ &= y & &= y. \end{aligned}$$

Now consider a case where  $x > y$ , and let  $j = x - y$  and  $h = y - x$ . So,  $x - y \in \mathbf{R}_{>0}$ . This means that  $j \in \mathbf{R}_{>0}$ . By the definition of absolute value,  $|j| \in \mathbf{R}_{>0}$ , because  $|j| = j$ . □

*Claim 4.3.*  $|x - z| \leq |x - y| + |y - z|$