# MATH 287 HOMEWORK 9

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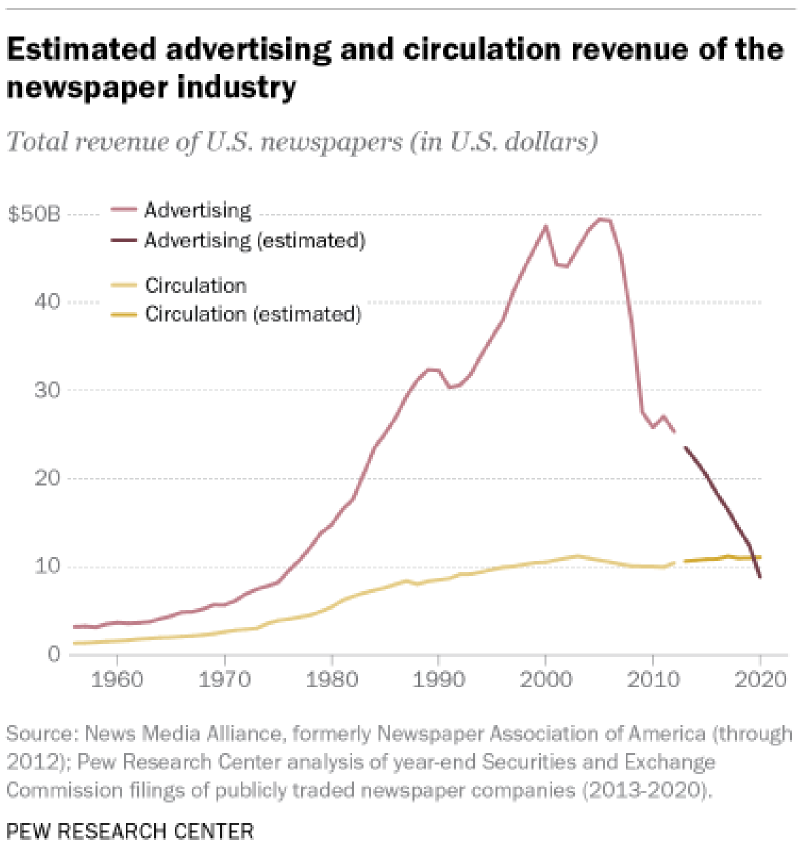
*Exercise* 1. Find or create a graphic to include, that you want to share.




*Date*: November 13, 2021.

*Exercise* 2. Proposition 10.8. For all $x, y \in \mathbf{R}$:

(ii) $|xy| = |x| \cdot |y|$.

(iii) $-|x| \leq x \leq |x|$.

*Claim* 2.1 (10.8 (ii)). $\forall x, y \in \mathbf{R} : |xy| = |x| \cdot |y|$

*Proof.* Let $x, y \in \mathbf{R}$. We will prove the claim using the following cases:

1. both $x, y > 0$.
2. both $x, y < 0$.
3. one of $x$ or $y$ is $< 0$.
4. one or both of $x$ and $y$ are equal to 0.

Case 1. Take an element $z \in \mathbf{R}$, which is the product of $x$ and $y$, i.e., $x \cdot y = z$. Because $x$ and $y$ are positive, we know that $z$ is also positive. Therefore, we can write

$$\begin{aligned} |x| \cdot |y| &= x \cdot y \text{ (definition of absolute value)} \\ &= z \\ &= |z| \\ &= |x \cdot y|. \end{aligned} \tag{1}$$

Case 2. Assume $z$ is the product of two negative real numbers denoted $-x$ and $-y$. We know from Prop. 8.18 that the product of two negative (real) numbers is a positive (real) number. Therefore, $z$ is still positive. Thus, we can write

$$
\begin{aligned}
|-x| \cdot |-y| &= x \cdot y \text{ (definition of absolute value)} \\
&= z \\
&= |z| \\
&= |-x \cdot -y|.
\end{aligned} \tag{2}
$$

Case 3. Assume we have two real numbers, $-x$ (negative) and $y$ (positive). From Prop. 8.22 (iii), we know that the product of a negative real number and a positive real number is a negative number. We will call this product $-z$. By the definition of absolute values, we know that $|-z| = z$. So, we can write

$$
\begin{aligned}
|-x| \cdot |y| &= x \cdot y \text{ (definition of absolute value)} \\
&= z \\
&= |-z| \\
&= |-x \cdot y|.
\end{aligned} \tag{3}
$$

Case 4. In an instance where either or both of $x$ and $y$ are 0, we can apply Proposition 10.8 (i), which lets us conclude $0 = |0| = |0 \cdot 0| = |x \cdot 0| = |0 \cdot y|$.

Thus we have shown that $|x| \cdot |y| = |x \cdot y|$ for all $x, y \in \mathbf{R}$. This concludes the proof. □

*Claim* 2.2 (10.8 (iii)). $\forall x \in \mathbf{R} : -|x| \leq x \leq |x|$

*Proof.* Let $x \in \mathbf{R}$. First, in the instance where $x = 0$, we have

$$-|0| \leq 0 \leq |0| \Rightarrow 0 = 0 = 0.$$

So, the statement holds. Now we will consider instances when $x \neq 0$. As a consequence, this means $-|x| < 0 < |x|$. Note that when $x > 0$, $|x| = x$, and also note that when $x < 0$, $|x| = -x$. Then, we can say

| if $x > 0$, then | if $x < 0$, then |
| --- | --- |
| $-\|x\| < x$ | $x < \|x\|$ |
| $-\|x\| < x \leq \|x\|$ | $-\|x\| \leq x < \|x\|.$ |

Thus, we have shown for all $x \in \mathbf{R}$, $-|x| \leq x \leq |x|$. This concludes the proof. $\square$

*Exercise* 3. Let $x, y \in \mathbf{R}$. Claim: $-y < x < y$ if and only if $|x| < y$.

*Proof.* $\Rightarrow$ Let us assume $x \geq 0$. We are told that $-y < x < y$. This means that $y > 0$. Note that because $x \geq 0$, $x = |x|$. So, we can write

$$\begin{aligned} x < y &\Rightarrow y - x \in \mathbf{R}_{>0} \\ &\Rightarrow y - |x| \in \mathbf{R}_{>0} \\ &\Rightarrow |x| < y. \end{aligned} \tag{4}$$

Alternatively, assume $x < 0$. We are told that $-y < x$. This also indicates that $y > 0$. Note that since $x < 0$, $|x| = -x$. So, we have

$$\begin{aligned} -y < x &\Rightarrow y > -x \text{ (Proposition 8.37 (i))} \\ &\Rightarrow -x < y \text{ (Rearranging)} \\ &\Rightarrow |x| < y. \text{ (Because } |x| = -x) \end{aligned} \tag{5}$$

$\Leftarrow$ Assume $|x| < y$, and that $x \geq 0$. This means that $y > 0$. Note again that $x = |x|$. Then we can say

$$
\begin{aligned}
|x| < y &\Rightarrow y - |x| \in \mathbf{R}_{>0} \\
&\Rightarrow y - x \in \mathbf{R}_{>0} \\
&\Rightarrow x < y \\
&\Rightarrow x > -y \text{ (Proposition 8.37 (i))} \\
&\Rightarrow -y < x \text{ (Rearranging)} \\
&\Rightarrow -y < x < y.
\end{aligned} \tag{6}
$$

Alternatively, examine a case where $x < 0$. Note that $|x| = -x$. Then we have

$$
\begin{aligned}
|x| < y &\Rightarrow -x < y \\
&\Rightarrow x > -y \text{ (Proposition 8.37 (i))} \\
&\Rightarrow -y < x \text{ (Rearranging)} \\
&\Rightarrow -y < x < y.
\end{aligned} \tag{7}
$$

Thus, we have shown $-y < x < y$ if and only if $|x| < y$. This concludes the proof. $\square$

*Exercise* 4. Proposition 10.10. Let $x, y, z \in \mathbf{R}$.

(i) $|x - y| = 0$ if and only if $x = y$.

(ii) $|x - y| = |y - x|$.

(iii) $|x - z| \leq |x - y| + |y - z|$.

*Claim* 4.1. $|x - y| = 0$ if and only if $x = y$.

*Proof.* $\Rightarrow$ Let $j = (x - y)$. We are told that $|j| = 0$. From Proposition 10.8, we know that $j$ must be equal to 0. By Axiom 8.4 and Proposition 8.11, this means that $x$ must equal $y$.

$\Leftarrow$ Assume $x = y$. Via substitution we can say

$$\begin{aligned} |x - y| = 0 &\Rightarrow |x - x| = 0 \\ &\Rightarrow |0| = 0 \\ &\Rightarrow 0 = 0. \end{aligned} \tag{8}$$

Thus, we have shown that if $|x - y| = 0$, $x$ must equal $y$; and we have shown that if $x = y$, the absolute value of $(x - y)$ must be equal to 0. This concludes the proof. □

*Claim* 4.2. $|x - y| = |y - x|$

*Proof.* First, consider a case where either $x = 0$ or $y = 0$. For brevity, we will focus on $x = 0$, but the same conclusion follows if we were to use $y$, because $x$ and $y$ are arbitrary real numbers. If $x = 0$, then we have

$$|x - y| = |0 - y| \qquad |y - x| = |y - 0|$$

$$= |-y| \qquad = |y|$$

$$= y \qquad = y.$$

So, the statement holds. Now consider a case where $x, y \neq 0$. We have

$$|x - y| = |y - x|$$

$$= |(-1)(-y + x)|$$

$$= |-1| \cdot |-y + x| \text{ (Proposition 10.8 (ii))}$$

$$= 1 \cdot |-y + x|$$

$$= |-y + x|$$

$$= |x - y|. \text{ (Rearranging)}$$

Thus, we have shown that $|x - y| = |x - y|$. This concludes the proof. □

*Claim* 4.3. $|x - z| \leq |x - y| + |y - z|$

*Proof.* Let $x, y, z \in \mathbf{R}$. We know from Proposition 10.8 (iv), that $\forall j, k \in \mathbf{R}$, $|j + k| \leq |j| + |k|$. Let $j = x - y$, and $k = y - z$. This means we can say

$$|j + k| \leq |j| + |k| \text{ (Proposition 10.8 (iv))}$$

$$|(x - y) + (y - z)| \leq |x - y| + |y - z| \text{ (Substituting on the LHS)}$$

$$|x - y + y - z| \leq |x - y| + |y - z|$$

$$|x - z| \leq |x - y| + |y - z|.$$

Thus we have shown that $|x - z| \leq |x - y| + |y - z|$. This concludes the proof. $\square$