

MATH 287 HOMEWORK 5

ANDREW MOORE

Date: October 1, 2021.

Exercise 1. Show some of your favorite equations in *inline* and **display** mathematics.

Answer. Some people like small equations such as $e^{\pi i} + 1 = 0$ or $a^2 + b^2 = c^2$.

My favorite small equation is $A^T A \mathbf{x} = A^T \mathbf{b}$.

Other people like big equations like

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C.$$

My favorite big equation is

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

◇

Exercise 2. Project 5.16(ii). Prove or give a counterexample: For all sets A, B, C , $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Answer. I claim that this statement is true.

Proof. Taking an element on the left-hand side, called x , we can see that

$$\text{(because } x \in A \cap (B - C)) \quad x \in A$$

$$\text{(definition of } \cap) \quad \Rightarrow x \in (B - C)$$

$$\text{(definition of } -) \quad \Rightarrow (x \in B) \wedge (x \notin C)$$

$$\Rightarrow (x \in A \cap B) \wedge (x \notin A \cap C)$$

$$\Rightarrow x \in (A \cap B) - (A \cap C)$$

$$\Rightarrow A \cap (B - C) \subseteq (A \cap B) - (A \cap C).$$

Then, taking an element y on the right-hand side, we have

$$\text{(definition of } -) \quad (y \in A \cap B) \wedge (y \notin A \cap C)$$

$$\text{(definition of } \cap) \quad \Rightarrow (y \in A) \wedge (y \in B) \wedge (y \notin C)$$

$$\Rightarrow y \in (B - C)$$

$$\Rightarrow y \in A \cap (B - C)$$

$$\Rightarrow (A \cap B) - (A \cap C) \subseteq A \cap (B - C).$$

Thus, we have shown that

$$A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$$

$$(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$$

which means that $A \cap (B - C) = (A \cap B) - (A \cap C)$.

□

◇

Exercise 3. Proposition 5.20(ii). If A, B, C are any sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. Take an element x on the left-hand side. We'll also use i , and j to represent the ordered pair (i, j) that x constitutes. We can see that

(definition of \times) $i \in A$

(definition of \cap) $j \in B \cap C \Rightarrow (j \in B) \wedge (j \in C)$.

By the definition of \times , we know that

$$(i, j) \in A \times B \Rightarrow x \in A \times B$$

$$(i, j) \in A \times C \Rightarrow x \in A \times C.$$

This means we can say that $x \in (A \times B) \cap (A \times C)$ and therefore,

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C).$$

Now we will examine an element y from the right-hand side. We will use k , and l to represent the ordered pair (k, l) that y constitutes. From the

definition of \cap , we know that $(y \in A \times B) \wedge (y \in A \times C)$. We also know from the definition of \times that

$$(k, l) \in A \times B \Rightarrow (k \in A) \wedge (l \in B)$$

$$(k, l) \in A \times B \Rightarrow l \in C.$$

This means that $l \in B \cap C$. This means we can say

$$(k, l) \in A \times (B \cap C),$$

i.e., $y \in A \times (B \cap C)$. From this we can conclude that

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C).$$

To summarize, we have shown

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$$

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

and therefore

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

□

Exercise 4. Project 5.21(ii). $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Answer. I claim this statement is true.

Proof. Take an element x on the left-hand side. We will use i and j to represent the ordered pair (i, j) that x denotes. We can see that

$$x \in A \times B \Rightarrow (i, j) \in A \times B \Rightarrow (i \in A) \wedge (j \in B)$$

$$x \in C \times D \Rightarrow (i, j) \in C \times D \Rightarrow (i \in C) \wedge (j \in D).$$

This means that $i \in A \cap C$ and $j \in B \cap D$. This implies that the ordered pair (i, j) is an element of $(A \cap C) \times (B \cap D)$. So, $x \in (A \cap C) \times (B \cap D)$. We can then conclude

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D).$$

Now we will examine an element y on the right-hand side. We will use k and l to the ordered pair (k, l) that y denotes. We can see that

$$k \in A \cap C \Rightarrow (k \in A) \wedge (k \in C)$$

$$l \in B \cap D \Rightarrow (l \in B) \wedge (l \in D).$$

This means that $(k, l) \in A \times B$ and $(k, l) \in C \times D$ and thus, $(k, l) \in (A \times B) \cap (C \times D)$. That is,

$$y \in (A \times B) \cap (C \times D).$$

We can then conclude

$$(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D).$$

We have shown that

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$$

$$(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D),$$

and therefore

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

□

◇

Exercise 5. Find $\sum_{j=0}^k f_j^2 = (\text{your answer})$, where the are Fibonacci numbers as defined in the textbook. Prove your answer.

Your answer will have a clear statement: (your answer). Then, a proof of your answer.

For your proof, use induction.

For example, since the Fibonacci numbers start with , you will get values starting with 0, 1, 1, 2, 3, 5, 8, ..., you will get values starting with $0^2 = 0, 0^2 + 1^2 = 1, 1^2 + 1^2 = 2, 1^2 + 1^2 + 2^2 = 6$, and so on.

Hint: Work out some of the values and look for a pattern. It might be helpful to look at factorizations of the values.

Answer. After inspecting the series/summation, I propose that

$$\sum_{j=0}^k f_j^2 = f_k \cdot f_{k+1}$$

for $k \geq 1$.

Proof. Let $P(k)$ be defined as $\sum_{j=0}^k f_j^2 = f_k \cdot f_{k+1}$ for $k \geq 1$. As a base case, we will examine $P(1)$:

$$\sum_{j=0}^1 f_j^2 = 0^2 + 1^2 = 0 + 1 = 1$$

$$f_1 \cdot f_2 = 1 \cdot 1 = 1.$$

We know $P(k)$ is true for some k . Let us assume $P(k)$ is true, and use this to show that $P(k+1)$ is also true. That is, we intend to show that $P(k+1)$,

$$\sum_{j=0}^{k+1} f_j^2 = f_{k+1} \cdot f_{k+2}$$

is true.

□

◇