MATH 287 HOMEWORK 3

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Date: September 17, 2021.

Exercise 1. Project 3.1. Express each of the following statements using quantifiers.

(iii) Every integer is the product of two integers.

(iv) The equation $x^2 - 2y^2 = 3$ has an integer solution.

Answer.

(iii) $\forall m \in \mathbf{Z}, \exists a \in \mathbf{Z}$ and $\exists b \in \mathbf{Z}$, such that $m = a \cdot b$.

(iv) $\exists x \in \mathbf{Z}$ and $\exists y \in \mathbf{Z}$, such that $x^2 - 2y^2 = 3$.

$\diamond$

Exercise 2. Project 3.2. In each of the following cases explain what is meant by the statement and decide whether it is true or false.

(iii) For each $x \in \mathbf{Z}$ there exists $y \in \mathbf{Z}$ such that $xy = x$.

(iv) There exists $y \in \mathbf{Z}$ such that for each $x \in \mathbf{Z}$, $xy = x$.

Answer.

(iii) This case states that for every integer m in the set of $\mathbf{Z}$, another intger y exists that satisfies the equation $xy = x$. I would say that it is true, but it is not specific enough. We know there exists only one y that satisfies the equation $xy = x$, which is $y = 1$ (Axiom 1.3).

(iv) This case states that there is an integer y, and for each integer m in $\mathbf{Z}$, y satisfies the equation $xy = x$. As above, I would say this statement is true, but should be made stronger by saying "there exists a unique integer $y = 1$, which satisfies the equation $xy = x$". Alternatively, the (strong) statement could be expressed as $\exists! y \in \mathbf{Z} \colon xy = x$.

$\diamond$

Exercise 3. Project 3.7. Negate the following statements.

(iv) The newspaper article was neither accurate nor entertaining. (Phrased differently: "The newspaper article was not accurate, and not entertaining.")

(v) If $\gcd(m, n)$ is odd, then m or n is odd.

(vi) H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G.

(vii) For each $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$.

Answer.

(iv) The newspaper article was either accurate, or entertaining.

(v) If $gcd(m, n)$ is even, then m and n are even.

(vi) H/N is a normal subgroup of G/N and H is not a normal subgroup of G, *or* H is a normal subgroup of G and H/N is not a normal subgroup of G/N.

(vii) There exists some $\varepsilon > 0$, for all N in $\mathbf{N}$, such that an $n \geq N$ exists where $|a_n - L|$ is greater than or equal to ε.

$\diamond$

Exercise 4. For all $k \in \mathbf{N}$, $5^k + 3$ is divisible by 4.

(1) Write what this statement says for $k = 1$. Is it true or false? Explain.

(2) Write what this statement says for $k = 2$. Is it true or false? Explain.

(3) Write what this statement says for $k = 3$. Is it true or false? Explain.

Now, prove the statement for all $k \in \mathbf{N}$, using induction.

(1) For $k = 1$, the statement says that $5^1 + 3$ is divisible by 4. It's true because $5 + 3 = 8$, and $8 = 4 \cdot 2$.

(2) For $k = 2$, the statement says that $5^2 + 3$ is divisible by 4. It's true because $25 + 3 = 28$, and $28 = 4 \cdot 7$.

(3) For $k = 3$ the statement says that $5^3 + 3$ is divisible by 4. It's true because $125 + 3 = 128$, and $128 = 4 \cdot 32$.

Proof. We will use induction on k. Let $P(k)$ denote the statement

$5^k + 3$ is divisible by 4.

The induction principle states we must check $P(1)$, i.e., the base case ($k = 1$):

$$\begin{aligned} 5^1 + 3 &= 5 + 3 \\ &= 8 \\ &= 4 \cdot 2. \end{aligned}$$

Next we will assume that $P(k)$ is true for some $k \in \mathbf{N}$, and show that $P(n+1)$ also holds. To assume that $5^k + 3$ is divisible by four is to state that there exists some $y \in \mathbf{Z}$ such that

$5^k + 3 = 4y.$

We now need to show that $5^{k+1} + 3 = 4z$ for some $z \in \mathbf{Z}$. That is

$5^{k+1} + 3 = (5^k \cdot 5^1) + 3 = 5 \cdot 5^k + 3 = 4z.$

We can rewrite the left-hand side as:

$5 \cdot (4y) = 4(5y) = 4z.$

By associativity, we can set $z = 5y$, which we know to be an integer (because y and z are integers). Thus we have shown that there exists a $z \in \mathbf{Z}$, such that $5^{k+1} + 3 = 4z$. This concludes our induction. $\square$