MATH 287 HOMEWORK 5

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Exercise 1. Show some of your favorite equations in *inline* and **display** mathematics.

Answer. Some people like small equations such as $e^{\pi i} + 1 = 0$ or $a^2 + b^2 = c^2$.

My favorite small equation is $A^T A\mathbf{x} = A^T\mathbf{b}$.

Other people like big equations like

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

My favorite big equation is

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

$\diamond$

Exercise 2. Project 5.16(ii). Prove or give a counterexample: For all sets A, B, C, $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Answer. I claim that this statement is true.

Proof. Taking an element on the left-hand side, called x, we can see that

(because $x \in A \cap (B - C)$)	$x \in A$
(definition of $\cap$)	$\Rightarrow x \in (B - C)$
(definition of $-$)	$\Rightarrow (x \in B) \wedge (x \notin C)$
	$\Rightarrow (x \in A \cap B) \wedge (x \notin A \cap C)$
	$\Rightarrow x \in (A \cap B) - (A \cap C)$
	$\Rightarrow A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$.

Then, taking an element y on the right-hand side, we have

$$
\begin{aligned}
&\text{(definition of } -) && (y \in A \cap B) \wedge (y \notin A \cap C) \\
&\text{(definition of } \cap) && \Rightarrow (y \in A) \wedge (y \in B) \wedge (y \notin C) \\
& && \Rightarrow y \in (B - C) \\
& && \Rightarrow y \in A \cap (B - C) \\
& && \Rightarrow (A \cap B) - (A \cap C) \subseteq A \cap (B - C).
\end{aligned}
$$

Thus, we have shown that

$$A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$$

$$(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$$

which means that $A \cap (B - C) = (A \cap B) - (A \cap C)$. $\square$

$\diamond$

Exercise 3. Proposition 5.20(ii). If A, B, C are any sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. Take an element x on the left-hand side. We'll also use i, and j to represent the ordered pair (i, j) that x constitutes. We can see that

$$\text{(definition of } \times) \qquad i \in A$$

$$\text{(definition of } \cap) \qquad j \in B \cap C \Rightarrow (j \in B) \wedge (j \in C).$$

By the definition of $\times$, we know that

$$(i, j) \in A \times B \Rightarrow x \in A \times B$$

$$(i, j) \in A \times C \Rightarrow x \in A \times C.$$

This means we can say that $x \in (A \times B) \cap (A \times C)$ and therefore,

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C).$$

Now we will examine an element y from the right-hand side. We will use k, and l to represent the ordered pair (k, l) that y constitutes. From the definition of $\cap$, we know that $(y \in A \times B) \wedge (y \in A \times C)$. We also know from the definition of $\times$ that

$$(k, l) \in A \times B \Rightarrow (k \in A) \wedge (l \in B)$$

$$(k, l) \in A \times B \Rightarrow l \in C.$$

This means that $l \in B \cap C$. This means we can say

$$(k, l) \in A \times (B \cap C),$$

i.e., $y \in A \times (B \cap C)$. From this we can conclude that

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C).$$

To summarize, we have shown

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$$

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

and therefore

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

$\square$

Exercise 4. Project 5.21(ii). $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Answer. I claim this statement is true.

Proof. Take an element x on the left-hand side. We will use i and j to represent the ordered pair (i, j) that x denotes. We can see that

$$x \in A \times B \Rightarrow (i, j) \in A \times B \Rightarrow (i \in A) \wedge (j \in B)$$

$$x \in C \times D \Rightarrow (i, j) \in C \times D \Rightarrow (i \in C) \wedge (j \in D).$$

This means that $i \in A \cap C$ and $j \in B \cap D$. This implies that the ordered pair (i, j) is an element of $(A \cap C) \times (B \cap D)$. So, $x \in (A \cap C) \times (B \cap D)$. We can then conclude

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D).$$

Now we will examine an element y on the right-hand side. We will use k and l to the ordered pair (k, l) that y denotes. We can see that

$$k \in A \cap C \Rightarrow (k \in A) \wedge (k \in C)$$

$$l \in B \cap D \Rightarrow (l \in B) \wedge (l \in D).$$

This means that $(k,l) \in A \times B$ and $(k,l) \in C \times D$ and thus, $(k,l) \in (A \times B) \cap (C \times D)$. That is,

$$y \in (A \times B) \cap (C \times D).$$

We can then conclude

$$(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D).$$

We have shown that

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$$

$$(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D),$$

and therefore

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

$\square$

$\diamond$

Exercise 5. Find $\sum_{j=0}^{k} f_j^2 =$ (your answer), where the are Fibonacci numbers as defined in the textbook. Prove your answer.

Answer. After inspecting the series/summation, I propose that

$$\sum_{j=0}^{k} f_j^2 = f_k \cdot f_{k+1}$$

for all $k \geq 1$ ($k \in \mathbf{N}$).

Proof. Let $P(k)$ be defined as $\sum_{j=0}^{k} f_j^2 = f_k \cdot f_{k+1}$ for all $k \in \mathbf{N}$ such that $k \geq 1$. As a base case, we will examine $P(1)$:

$$\sum_{j=0}^{1} f_j^2 = 0^2 + 1^2 = 0 + 1 = 1$$

$$f_1 \cdot f_2 = 1 \cdot 1 = 1.$$

We know $P(k)$ is true for some k. Let us assume $P(k)$ is true, and use this to show that $P(k+1)$ is also true. That is, we intend to show that $P(k+1)$,

$$\sum_{j=0}^{k+1} f_j^2 = f_{k+1} \cdot f_{k+2}$$

is true. Re-expressing the sum, we can show

$$\begin{aligned}
&\text{(sum to k, plus result of } f_{k+1}^2) & \sum_{j=0}^{k+1} f_j^2 &= \sum_{j=0}^{k} f_j^2 + f_{k+1}^2 \\
&\text{(the inductive hypothesis, plus 1)} & &= f_k \cdot f_{k+1} + f_{k+1}^2 \\
&\text{(factor out } f_{k+1}) & &= f_{k+1}(f_k \cdot 1 + f_{k+1}) \\
& & &= f_{k+1}(f_k + f_{k+1}) \\
&\text{(adding two Fibonacci numbers together)} & &= f_{k+1}(f_{k+2}) \\
& & &= f_{k+1} \cdot f_{k+2}.
\end{aligned}$$

We have shown $P(k+1)$ is true, which concludes our induction, and the proof. $\square$

$\diamond$