MATH 287 HOMEWORK 8

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Exercise 1. We need to find the determinants of these matrices:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}, \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$$

Answer. The determinants are

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1,$$

$$\det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = -1,$$

$$\det \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = 1,$$

$$\det \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix} = -1,$$

$$\det \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} = 1.$$

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Exercise 2. Proposition 9.7(ii): If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is surjective.

Proof. Assume that $f : A \to B$ and $g : B \to C$ are surjective functions. We intend to show that $g \circ f : A \to C$ is also surjective. That is

$$\exists c \in C, \forall a \in A : (g \circ f)(a) = c.$$

$g \circ f$ is a composition of f and g, defined as

$$g(f(a)) \text{ for all } a \in A.$$

By hypothesis, we know that f maps the entirety of A's elements (f is surjective). This means

$$\exists b \in B, \forall a \in A \text{ such that } f(a) = b.$$

We also know that g maps the entirety of B's elements (g is surjective). That is

$$\exists c \in C, \forall b \in B \text{ such that } g(b) = c.$$

Because $f \circ g$ is a composition, inputs (a's) are evaluated first by f, and the subsequent outputs (b's) are then fed as new inputs for g. This means, we can take any arbitrary element $b \in B$, and know that there is an $a \in A$ that satifies the following equation: $b = f(a)$. Thus we can write

$$\begin{aligned} g(b) &= c \\ g(f(a)) &= c \\ (g \circ f)(a) &= c. \end{aligned} \tag{1}$$

This means for every $a \in A$, there must be an $c \in C$ that corresponds to it. Therefore, $g \circ f$ must be surjective. This conclude the proof. $\square$

Exercise 3. Prove the claims:

For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each injective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is injective.

For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each surjective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is surjective.

For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each bijective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is bijective.

Claim 3.1. For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each injective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is injective.

Proof. We will prove the claim using induction. Let $P(n)$ denote the claim for a set of n functions. Using $P(2)$ as a base case case, we have f_1 and f_2 as two injective functions. With $f_1 \circ f_2$ being their composition, when applying Proposition 9.7 (i), we know that $f_1 \circ f_2$ is also injective. Therefore $P(2)$ is true.

Assuming $P(n)$ is true for some $n \in N$, we will use this fact to prove it is also true for $P(n+1)$. Ascending up to $P(n+1)$, we have an $n+1$ collection of functions,

$$f_1, f_2, \ldots, f_n, f_{n+1}$$

and we are assembling the following composition

$$f_1 \circ f_2 \circ \ldots \circ f_n \circ f_{n+1}.$$

Let g be a function, representing $f_1 \circ f_2 \circ \ldots \circ f_n$. We can rewrite our proposed composition as $g \circ f_{n+1}$. By hypothesis we know that g is injective. From our initial assumptions, we know that f_{n+1} is also injective. Then, applying Proposition 9.7 (i) to $g \circ f_{n+1}$ (a composition of two injective functions), we know the result will also be injective.

Thus, any composition of injective functions will also be injective, for all $n \in \mathbf{N}$, where $n \geq 2$. This concludes the induction, and the proof. $\square$

Claim 3.2. For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each surjective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is surjective.

Proof. We will prove the claim using induction. Let $P(n)$ denote the claim for a set of n functions. Using $P(2)$ as a base case case, we have f_1 and f_2 as two surjective functions. With $f_1 \circ f_2$ being their composition, when applying Proposition 9.7 (ii), we know that $f_1 \circ f_2$ is also surjective. Therefore $P(2)$ is true.

Assuming $P(n)$ is true for some $n \in N$, we will use this fact to prove it is also true for $P(n+1)$. Ascending up to $P(n+1)$, we have an $n+1$ collection of functions,

$$f_1, f_2, \ldots, f_n, f_{n+1}$$

and we are assembling the following composition

$$f_1 \circ f_2 \circ \ldots \circ f_n \circ f_{n+1}.$$

Let g be a function, representing $f_1 \circ f_2 \circ \ldots \circ f_n$. We can rewrite our proposed composition as $g \circ f_{n+1}$. By hypothesis we know that g is surjective. From our initial assumptions, we know that f_{n+1} is also surjective. Then, applying Proposition 9.7 (ii) to $g \circ f_{n+1}$ (a composition of two surjective functions), we know the result will also be surjective.

Thus, any composition of surjective functions will also be surjective, for all $n \in \mathbf{N}$, where $n \geq 2$. This concludes the induction, and the proof. $\square$

Claim 3.3. For any $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are each bijective, then $f_1 \circ f_2 \circ \cdots \circ f_n$ is bijective.

Proof. We will prove the claim using induction. Let $P(n)$ denote the claim for a set of n functions. Using $P(2)$ as a base case case, we have f_1 and f_2 as two bijective functions. With $f_1 \circ f_2$ being their composition, when applying Proposition 9.7 (iii), we know that $f_1 \circ f_2$ is also bijective. Therefore $P(2)$ is true.

Assuming $P(n)$ is true for some $n \in N$, we will use this fact to prove it is also true for $P(n+1)$. Ascending up to $P(n+1)$, we have an $n+1$ collection of functions,

$$f_1, f_2, \dots, f_n, f_{n+1}$$

and we are assembling the following composition

$$f_1 \circ f_2 \circ \dots \circ f_n \circ f_{n+1}.$$

Let g be a function, representing $f_1 \circ f_2 \circ \dots \circ f_n$. We can rewrite our proposed composition as $g \circ f_{n+1}$. By hypothesis we know that g is bijective. From our initial assumptions, we know that f_{n+1} is also bijective. Then, applying Proposition 9.7 (iii) to $g \circ f_{n+1}$ (a composition of two bijective functions), we know the result will also be bijective.

Thus, any composition of bijective functions will also be bijective, for all $n \in \mathbf{N}$, where $n \geq 2$. This concludes the induction, and the proof. $\square$

Exercise 4. Suppose $f : A \to B$ and $g : B \to C$, so $g \circ f$ is a function $A \to C$. Prove the following claims:

If $g \circ f$ is injective, then f is injective.

If $g \circ f$ is surjective, then g is surjective.

If $g \circ f$ is bijective, then f is injective and g is surjective. Explain why (use the previous claims) and give an example to show that f doesn't have to be surjective and g doesn't have to be injective.

Claim 4.1. If $g \circ f$ is injective, then f is injective.

Proof. Let $g \circ f : A \to C = g(f(a))$. We are told that $g \circ f$ is injective. We intend to show that this implies that f is also injective. Suppose we have $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$. We can feed this into g on each sides to see

$$\begin{aligned} f(a_1) &= f(a_2) \\ g(f(a_1)) &= g(f(a_2)) \qquad (2) \\ (g \circ f)(a_1) &= (g \circ f)(a_2). \end{aligned}$$

Because $g \circ f$ is injective, a_1 must be equal to a_2. Because a_1 and a_2 are arbitrary members of A, this is true for any $a_1, a_2 \in A$. Therefore f must be injective. This completes the proof. $\square$

Claim 4.2. If $g \circ f$ is surjective, then g is surjective.

Proof. Let $g \circ f : A \to C = g(f(a))$. We are told that $g \circ f$ is surjective. We intend to show that this implies g is also surjective, i.e.,

$$\exists c \in C, \forall b \in B \text{ such that } g(b) = c.$$

Assume we have some arbitrary $c \in C$. Because $g \circ f$ is surjective, this means there's an $a \in A$ which satisfies the following equation:

$$(g \circ f)(a) = c.$$

However, this is the same as writing

$$g(f(a)) = c.$$

We know from the claim that f maps $A \to B$. Therefore we could say, $f(a) = b$, where $b \in B$. This means we've found a b that satisfies $g(b) = c$, for any $c \in C$. Because b is arbitrary, this must be true for all $b \in B$, and therefore g must be surjective. This completes the proof. $\square$

Claim 4.3. If $g \circ f$ is bijective, then f is injective and g is surjective. Explain why (use the previous claims) and give an example to show that f doesn't have to be surjective and g doesn't have to be injective.

Proof. Let $g \circ f : A \to C = g(f(a))$. We are told that $g \circ f$ is bijective. We intend to show this implies f is injective, and g is surjective. If $g \circ f$ is bijective, this means it is both surjective and injective.

From the claim and associated proof for 4.1, we know that injectivity for $g \circ f$ is contingent on f's injectivity. This is because this property of f determines the domain of inputs that are fed into g (i.e., f's inputs are associated with unique outputs). So, g could be injective or not, but it doesn't affect the resulting composition, $g \circ f$.

Secondly, from the claim and associated proof of 4.2, we know that surjectivity for $g \circ f$ is contingent upon how g's surjectivity requires every element of C to have a corresponding b as an input to $g()$.

To show that it's not necessary for f to be surjective and g to be injective for $g \circ f$ to be bijective, take the following definitions for sets A, B, and C, and functions f, g and $g \circ f$:

$$
\begin{aligned}
A &= \{1, 2\} \\
B &= \{-2, 2, 4\} \\
C &= \{4, 16\} \\
f : A \to B &= f(a) = 2a \\
g : B \to C &= g(b) = b^2 \\
g \circ f : A \to C &= (g \circ f)(a) = g(f(a)) = (2a)^2
\end{aligned}
\tag{3}
$$

Having established our sets and functions, we can evaluate the functions on their related sets:

$$\begin{aligned}
f(a_1) &= 2(1) = 2 \\
f(a_2) &= 2(2) = 4 \\
g(b_1) &= (-2)^2 = 4 \\
g(b_2) &= (2)^2 = 4 \\
g(b_3) &= (4)^2 = 16 \\
(g \circ f)(a_1) &= (2(1))^2 = 4 \\
(g \circ f)(a_2) &= (2(2))^2 = 16
\end{aligned} \tag{4}$$

As demonstrated above, we have verified that

- f is injective but not surjective (because all a's map to a different b's, but not all elements of B have a corresponding a),
- that $g \circ f$ is bijective (both elements of A map to different elements of C, and all elements of A are mapped to elements in C), and
- that g is surjective but not injective (because all $b \in B$ have at least one value of c associated with them, but two elements of B map onto one element of C).

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