MATH 287 HOMEWORK 2

ANDREW MOORE

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Exercise 1. Proposition 2.7(iv). Let $m, n, p, q \in \mathbf{Z}$. If $m < n$ and $p < 0$ then $np < mp$.

Proof. We are told that $n - m \in \mathbf{N}$, and $0 - p \in \mathbf{N}$ (i.e., $-p \in \mathbf{N}$, based on Axiom 2.1.4). We will show that $mp - np \in \mathbf{N}$. Rearranging $mp - np$ we have:

$$\begin{aligned} mp - np &= p(m - n) \\ &= p(m(-1)(-1) + n(-1)) \\ &= p(-1)(m(-1) + n) \\ &= -p((-m) + n) \\ &= -p(n - m) \in \mathbf{N}. \end{aligned}$$

The series of steps above demonstrate that $mp - np$ can be expressed as two terms we know to be natural numbers (based on our initial assumptions): $-p$ and $(n - m)$. Axiom 2.1.2 tells us that the product of any two natural numbers is also in $\mathbf{N}$. Therefore, $(mp - np)$ must be in $\mathbf{N}$, i.e. $np < mp$. $\square$

Exercise 2. Proposition 2.12(iii). For all $m, n, p \in \mathbf{Z}$, if $p < 0$ and $mp < np$ then $n < m$.

Proof. We are told we can assume the following: $0 - p \in \mathbf{N}$, and $np - mp \in \mathbf{N}$. We must show that $n < m$, i.e. $m - n \in \mathbf{N}$. Rearranging $np - mp \in \mathbf{N}$, we have:

$$\begin{aligned}
np - mp &= p(n - m) \\
&= p(n(-1)(-1) + m(-1)) \\
&= p(-1)(n(-1) + m) \\
&= -p(-n + m) \\
&= -p(m - n) \in \mathbf{N}.
\end{aligned}$$

We can reexpress $0 - p \in \mathbf{N}$ as $-p \in \mathbf{N}$ (which we know from Axiom 2.1.4). From the series of equations above, we've seen that $np - mp \in \mathbf{N}$ can be expressed as $-p(m - n) \in \mathbf{N}$. Axiom 2.1.2 tells us that the product of any two numbers in $\mathbf{N}$ is also in $\mathbf{N}$. Therefore, $(m - n)$ must be in $\mathbf{N}$, i.e. $n < m$.

$\square$

Exercise 3. Proposition 2.26. In this problem, the textbook gives a proof. Your homework is to rewrite the proof in more detail.

Imagine a student in the class is confused by the proof. Rewrite the proof in a way that would make sense and be clear for a confused student.

Exercise 4. Project 2.28. Determine for which natural numbers $k^2 - 3k \geq 4$ and prove your answer.

Answer.

Claim 4.1. $k^2 - 3k \geq 4$ for $k \geq 4$.

Proof. We are given the inequality, $k^2 - 3k \geq 4$, and are asked to find (and justify) the values of k that make the inequality true. We will prove the claim $k^2 - 3k \geq 4$ for $k \geq 4$ using induction. First, note that $k \leq 3$ does not satisfy the inequality.

For $k = 3$:

$$k^2 - 3k \geq 4 \Rightarrow (3)^2 - 3(3) \geq 4 \Rightarrow 9 - 9 \geq 4 \Rightarrow 0 \geq 4$$

The statement $0 \geq 4$ is *false*, because $0 \neq 4$ and $0 - 4 \notin \mathbf{N}$. It is trivial to demonstrate the same result for $k = 1$ and $k = 2$, and this is left to the reader. Assuming that k can't be ≤ 3, let us redefine k to be $k = 3 + j$, such that $j \in \mathbf{N}$ and $j \geq 1$. We will now prove the base case ($j = 1$).

For $k = 3 + 1 = 4$:

$$k^2 - 3k \geq 4 \Rightarrow (4)^2 - 3(4) \geq 4 \Rightarrow 16 - 12 \geq 4 \Rightarrow 4 \geq 4$$

This last statement is true $(4 = 4)$. We will now show that the results apply for all j, by proving this is true for $k = 4 + (j + 1)$.

For $k = 4 + (j + 1) = 5 + j$:

$$(k + 1)^2 - 3(k + 1) \geq 4$$

$$(k^2 + 2k + 1) - 3k - 3 \geq 4$$

$$k^2 - k - 2 \geq 4$$

$$(5 + j)^2 - (5 + j) - 2 \geq 4$$

$$25 + 10j + j^2 - j - 7 \geq 4$$

$$j^2 + 9j + 18 \geq 4$$

The final statement $j^2 + 9j + 18 - 4 \in \mathbf{N}$ is true because there is no value of j in $\mathbf{N}$ that could cause the value of the statement to be ≤ 0. $\square$

$\diamond$