MATH 287 HOMEWORK 11

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Exercise 1. Suppose $f : A \to B$ is surjective. Then, for any set C and functions $g_1, g_2 : B \to C$, if $g_1 \circ f = g_2 \circ f$, $g_1 = g_2$.

Proof. Let $f : A \to B$ and $g_1, g_2 : B \to C$ be functions. We are examining compositions between f and g_1, g_2. From our hypothesis, we have $g_1 \circ f = g_2 \circ f$. From the definition of a composition, we know $g_1 \circ f : A \to C$ is defined by $(g_1 \circ f)(a) = c$ for all $a \in A$, and $g_2 \circ f : A \to C$ is defined by $(g_2 \circ f)(a) = c$ for all $a \in A$. Additionally, we know that f is surjective. This means that for

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Date: December 5, 2021.

every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. Thus, we can write

$$
\begin{aligned}
&\text{(starting assumption)} && g_1 \circ f = g_2 \circ f \\
& && (g_1 \circ f)(a) = (g_2 \circ f)(a) \\
&\text{(definition of a composition)} && g_1(f(a)) = g_2(f(a)) \\
& && g_1(b) = g_2(b) \\
& && g_1 = g_2.
\end{aligned}
$$

Because for all $b \in B, \exists a \in A$ such that $f(a) = b$ (surjectivity of f), and $g_1 \circ f = g_2 \circ f$, we know that the a being fed to f is the same element of A. This means the result of $f(a)$, is the same b on both sides of the equality. Thus we have shown $g_1 = g_2$. This concludes the proof. □

Exercise 2. a. Find and prove a formula for $2+5+8+11+\cdots+(3n-1)$.

b. Prove: for all positive odd integers $n, 5^n - n^2$ is divisible by 4.

Claim 2.1. I claim that

$$2+5+8+11+\cdots+(3n-1)=\sum_{i=1}^{n}(3n-1)=\frac{1}{2}(3n^2+n).$$

Proof. We will show that $\sum_{i=1}^{n}(3i-1)=\frac{1}{2}(3n^2+n)$, for $n \geq 1$ using induction.

As a base case, $n=1$, we have

$$\sum_{i=1}^{1}(3i-1)=2=\frac{1}{2}(3(1)^2+1).$$

We will now assume the formula holds for all natural numbers up to n. We will then use this to show that it also holds for $n+1$. That is, we intend to demonstrate $\sum_{i=1}^{n+1}(3i-1)=\frac{1}{2}(3(n+1)^2+(n+1))$. Re-expressing the sum, we see

$$\sum_{i=1}^{n+1}(3i-1) = \sum_{i=1}^{n}(3i-1) + (3n+2)$$

(by hypothesis)
$$= \frac{1}{2}(3n^2+n) + (3n+2)$$

(ensuring a common denominator)
$$= \frac{(3n^2+n)}{2} + \frac{2(3n+2)}{2}$$

$$= \frac{(3n^2+n)+(6n+4)}{2}$$

(simplifying)
$$= \frac{3n^2+7n+4}{2}$$

$$= \frac{1}{2}(3n^2+7n+4)$$

$$= \frac{1}{2}(3(n^2+2n+1)+(n+1))$$

$$= \frac{1}{2}(3(n+1)^2+(n+1)).$$

This concludes the induction, and the proof. □

Claim 2.2. For all positive odd integers $n, 5^n - n^2$ is divisible by 4.

Proof. By induction on n. As a base case, $n = 1$, we see $5^1 - 1^2 = 4$. Assuming the statement holds for all odd integers up to n, we will show that it also holds for $n + 2$. To state that $5^n - n^2$ is divisible by 4 is to assert $\exists y \in \mathbf{Z}$ such that $4y = 5^n - n^2$. We will demonstrate that there also $\exists z \in \mathbf{Z}$ such that $4z = 5^{n+2} - (n+2)^2$. First, note that $5^n = 4y + n^2$. Then, we can see

$$4z = 5^{n+2} - (n+2)^2$$

$$4z = 5^n \cdot 5^2 - (n+2)^2$$

(by hypothesis) $$4z = (4y + n^2) \cdot 5^2 - (n+2)^2$$

(distributing and expanding) $$4z = 100y + 25n^2 - (n^2 + 4n + 4)$$

(simplifying) $$4z = 100y + 24n^2 - 4n - 4$$

$$4z = 4(25y + 6n^2 - n - 1)$$

$$z = 25y + 6n^2 - n - 1.$$

We have shown that there exists an integer z, such that $4z = 5^{n+2} - (n+2)^2$, meaning that the result of the statement is divisible by 4. We can thus conclude the statement is true for all odd integers. This concludes the induction and the proof. $\square$

Exercise 3. Proposition 11.25. Let $b, c, p, q \in \mathbf{R}$. If $x^2 - bx - c = 0$ has two solutions s and t, and if we define a sequence $(a_k)_{k=1}^{\infty}$ by $a_k := ps^k + qt^k$, then this sequence satisfies a recurrence relation $a_n = ba_{n-1} + ca_{n-2}$ for all $n \geq 3$.

Proof. $\square$

Exercise 4. Re-do a problem: Homework 5, problem 4: "$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$."

Answer. I was marked down for not being adequately thorough in my answer. Here is my revised proof.

Proof. Take an element x on the left-hand side, and let $x = (i, j)$. This element is in the intersection of $A \times B$ and $C \times D$, that is, $x \in A \times B$ *and* $x \in C \times D$. This means that $i \in A$ and $j \in B$ and that $i \in C$ and $j \in D$. From these observations, we can determine that $i \in A \cap C$ and $j \in B \cap D$. So, we can conclude that $x = (i, j) \in (A \cap C) \times (B \cap D)$, i.e.,

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D).$$

Now, we will examine an element y on the right-hand side, letting $y = (k, l)$. The element y is in the cross product of the two intersections of $A \cap C$ and $B \cap D$. By definition, this means that $k \in A \cap C$ and $l \in B \cap D$. Stated explicitly, this means $k \in A, l \in B, k \in C,$ and $l \in D$. If we take the cross product of A and B, (k, l) is a member of $A \times B$. Similarly, if we take the cross product of

C and D, (k, l) is a member of $C \times D$. Thus, $y = (k, l) \in (A \times B) \cap (C \times D)$.

So, we can say

$$(A \times B) \cap (C \times D) \supseteq (A \cap C) \times (B \cap D).$$

We have shown that

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$$

and

$$(A \times B) \cap (C \times D) \supseteq (A \cap C) \times (B \cap D).$$

Thus, we can conclude

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

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