MATH 287 HOMEWORK 4

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Exercise 1. Explain the proof of Proposition 4.6(ii). The textbook gives a proof of Proposition 4.6(ii). Rewrite the proof in more detail and with more explanation.

Proof. Let $P(k)$ denote the statement $b^m b^k = b^{m+k}$. We will prove $P(k)$ using induction, assuming $k, m \in \mathbf{Z}_{\geq 0}$, such that $k, m \geq 0$. We also assume $b \in \mathbf{Z}$.

We will first examine the base case $P(k = 0)$. By definition, we know that $b^0 = 1$. Below we will show that both sides of the base case can be made to equal b^m:

(by Axiom 1.3) $$b^m \cdot b^0 = b^m \cdot 1 = b^m$$

(combining the exponents, m + 0 = m) $$b^{m+0} = b^m.$$

We now know $P(k)$ is true for *some* k. We will now assume $P(k)$ is true, and use this to prove that $P(k+1)$ is also true. That is, we intend to conclude that $b^m b^{n+1} = b^{m+n+1}$. We will now use the definition of powers to rearrange the left and right-hand sides of the equation:

$$b^m b^{k+1} = b^m \cdot b^k \cdot b$$

$$b^{m+k+1} = b^{m+k} \cdot b.$$

We assumed $P(k)$ is true (our inductive hypothesis), that is, $b^m \cdot b^k = b^{m+k}$. With this assumption in place, it is clear the left and right-hand sides are equal. This concludes our proof. $\square$

Exercise 2. Proposition 4.7(iii). For all $k \in \mathbf{N}$, $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9.

Proof. Let $P(k)$ represent the statement $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9. We are asked to prove $P(k)$ is true for all $k \in \mathbf{N}$. We will prove $P(k)$ by induction. Beginning with 1 as a base case, we have $P(k = 1)$:

$$\begin{aligned}
10^k + 3 \cdot 4^{k+2} + 5 &= 10^1 + 3 \cdot 4^{1+2} + 5 \\
&= 10 + 3 \cdot 4^3 + 5 \\
&= 10 + 3 \cdot 64 + 5 \\
&= 10 + 192 + 5 \\
&= 207 \\
&= 23 \cdot 9.
\end{aligned}$$

Having seen $P(k)$ is true for some $k \in \mathbf{N}$, we will assume that $P(k)$ is true, and use this to prove $P(k+1)$. This means there exists some $y \in \mathbf{N}$ such that $10^k + 3 \cdot 4^{k+2} + 5 = 9y$. We intend to show that there exists some $z \in \mathbf{N}$, such that $10^{k+1} + 3 \cdot 4^{k+1+2} + 5 = 9z$. Re-arranging the left-hand side we have:

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\begin{aligned}
10^{k+1} + 3 \cdot 4^{k+1+2} + 5 &= 10 \cdot 10^k + 12 \cdot 4^{k+2} + 5 \\
&= (1+9) \cdot 10^k + (3+9) \cdot 4^{k+2} + 5 \\
&= 10^k + 9 \cdot 10^k + 3 \cdot 4^{k+2} + 9 \cdot 4^{k+2} + 5 \\
&= (10^k + 3 \cdot 4^{k+2} + 5) + (9 \cdot 10^k + 9 \cdot 4^{k+2}) \\
&= 9y + 9 \cdot 10^k + 9 \cdot 4^{k+2} \\
&= 9(y + 10^k + 4^{k+2}).
\end{aligned}
$$

Given that k and y are natural numbers, we know that $9(y + 10^k + 4^{k+2})$ is also a natural number (Axiom 2.1(ii)). This means that there exists some z that is equivalent to $y + 10^k + 4^{k+2}$, showing that the final statement is equivalent to $9z$. Because $9z$ is divisible by 9, we know that $P(k+1)$ is true. This concludes the proof. $\square$

Exercise 3. Project 4.9.

In this problem you will (1) determine for which natural numbers the statement is true, and (2) prove your answer. In your answer, you should state very clearly which natural numbers make the statement true: something like "For all natural numbers k such that (your answer here), $k^2 < 2^k$." Or you could phrase it differently, for example, as "If k is (your answer here), $k^2 < 2^k$." You will have to fill in what condition is needed for the k. Then, prove your statement.

To find the right condition, please try some k values. Try $k = 1, 2, 3, ...$ (we are talking about natural numbers so it makes sense to count up from 1). Which k values make $k^2 < 2^k$ true?

Answer. Trying a few values of k, such as $k = 1, 2, 3, 4, 5, 6, 7$, we have

(k = 1, true)	$1^2 < 2^1 \Rightarrow 1 < 2$
(k = 2, false)	$2^2 < 2^2 \Rightarrow 4 < 4$
(k = 3, false)	$3^2 < 2^3 \Rightarrow 9 < 8$
(k = 4, false)	$4^2 < 2^4 \Rightarrow 16 < 16$
(k = 5, true)	$5^2 < 2^5 \Rightarrow 25 < 32$
(k = 6, true)	$6^2 < 2^6 \Rightarrow 36 < 64$
(k = 7, true)	$7^2 < 2^7 \Rightarrow 49 < 128$

Based on these initial observations, I would claim that $k^2 < 2^k$ is true if $k \in \mathbf{N}$ is greater than or equal to 5. We'll set aside the trivial exception of $k = 1$.

Proof. Let $P(k)$ denote the statement $k^2 < 2^k$. We will use induction to prove that $P(k)$ is true for all $k \in \mathbf{N}$, such that $k \geq 5$. First, as a base case, we will show that $P(k = 5)$ is true:

$$5^2 < 2^5 \Rightarrow 25 < 32.$$

Assuming $P(k)$ is true for some $k \geq 5$, we will now show that $P(k+1)$ is true.

Starting on the left-hand side, we have:

$$\begin{aligned} & & (k+1)^2 &= k^2 + 2k + 1 \\ &(\because 2k+1 < k^2, \forall k \in \mathbf{N} : k \geq 3; \text{ see sub-proof below}) & &< k^2 + k^2 \\ &(\text{re-expressing addition as multiplication}) & &= 2(k^2) \\ &(\text{the inductive hypothesis, } k^2 < 2^k) & &< 2(2^k) \\ &(\text{via Proposition 4.6(ii)}) & &= 2^{k+1}. \end{aligned}$$

Sub-proof. Above we claimed that $2k+1 < k^2$ for all $k \in \mathbf{N}$ such that $k \geq 3$. We will refer to this claim as $Q(k)$, and will prove it using induction. Starting with the base case $Q(3)$, we have: $2(3) + 1 < (3)^2 \Rightarrow 7 < 9$. Assuming $Q(k)$ is true for some $k \geq 3$, we will show $Q(k+1)$ is also true:

$$2(k+1)+1 < (k+1)^2$$

(distributing on both sides) $$2k+3 < k^2+2k+1$$

(subtracting out common terms) $$2 < k^2.$$

We know that k is at least 3, and that $2 < 3^2$. As k increases infinitely, this statement will remain true. Therefore $2k+1 < k^2$ for $k \geq 3$. ■

Having shown this underlying claim as true, we are able to conclude that $(k+1)^2 < 2^{k+1}$, thus finishing our induction. □

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Exercise 4. Find $\sum_{j=0}^{k} f_j$, where the f_j are Fibonacci numbers as defined in the textbook. Prove your answer.

Your answer will have a clear statement: $\sum_{j=0}^{k} f_j =$ (your answer). Then, a proof of your answer.

Hint: Try $\sum_{j=0}^{k} f_j$ for several values of k (e.g., $k = 1, 2, 3, ..., 6, ...$). Look for a pattern. This is "experimental mathematics", where you try some things, gather data, and look for a pattern!

For your proof, use induction.

Answer. A formula for the sum is $\sum_{j=0}^{k} f_j = f_{k+2} - 1$ for all $k \geq 2$.

Proof. Let $P(k)$ represent the claim $\sum_{j=0}^{k} f_j = f_{k+2} - 1$, such that $k \geq 2$. We will prove this claim using induction. For the base case, $P(k = 2)$, we have:

$$f_0 = 0$$

$$f_1 = 1$$

$$f_2 = 1$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3$$

$$\sum_{j=0}^{2} f_j = f_{k+2} - 1 = f_4 - 1 = 3 - 1 = 2$$

$$f_0 + f_1 + f_2 = 0 + 1 + 1 = 2.$$

We know that $P(k)$ is true for some k. We will now assume $P(k)$ and use this to prove $P(k+1)$ is true. That is, we intend to show $\sum_{j=0}^{k+1} f_j = f_{k+3} - 1$, for all $k \geq 2$.

We can re-express the sum to see:

$$\begin{aligned}
&\text{(sum to k, plus result of } f_{k+1}) && \sum_{j=0}^{k+1} f_j = \sum_{j=0}^{k} f_j + f_{k+1} \\
&\text{(the inductive hypothesis, plus 1)} && = f_{k+2} - 1 + f_{k+1} \\
& && = f_{k+1} + f_{k+2} - 1 \\
&\text{(sum of two sequential Fibonacci \#s.)} && = f_{k+3} - 1.
\end{aligned}$$

This concludes the proof.

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