MATH 287 HOMEWORK 2

ANDREW MOORE

Date: September 10, 2021.

Exercise 1. Proposition 2.7(iv). Let $m, n, p, q \in \mathbf{Z}$. If $m < n$ and $p < 0$ then $np < mp$.

Proof. We are told that $n - m \in \mathbf{N}$, and $0 - p \in \mathbf{N}$ (i.e., $-p \in \mathbf{N}$, based on Axiom 2.1.4). We will show that $mp - np \in \mathbf{N}$. Rearranging $mp - np$ we have:

$$\begin{aligned} mp - np &= p(m - n) \\ &= p(m(-1)(-1) + n(-1)) \\ &= p(-1)(m(-1) + n) \\ &= -p((-m) + n) \\ &= -p(n - m) \in \mathbf{N}. \end{aligned}$$

The series of steps above demonstrate that $mp - np$ can be expressed as two terms we know to be natural numbers (based on our initial assumptions): $-p$ and $(n - m)$. Axiom 2.1.2 tells us that the product of any two natural numbers is also in $\mathbf{N}$. Therefore, $(mp - np)$ must be in $\mathbf{N}$, i.e. $np < mp$. $\square$

Exercise 2. Proposition 2.12(iii). For all $m, n, p \in \mathbf{Z}$, if $p < 0$ and $mp < np$ then $n < m$.

Proof. We are told we can assume the following: $0 - p \in \mathbf{N}$, and $np - mp \in \mathbf{N}$. We must show that $n < m$, i.e. $m - n \in \mathbf{N}$. Rearranging $np - mp \in \mathbf{N}$, we have:

$$\begin{aligned} np - mp &= p(n - m) \\ &= p(n(-1)(-1) + m(-1)) \\ &= p(-1)(n(-1) + m) \\ &= -p(-n + m) \\ &= -p(m - n) \in \mathbf{N}. \end{aligned}$$

We can reexpress $0 - p \in \mathbf{N}$ as $-p \in \mathbf{N}$ (which we know from Axiom 2.1.4). From the series of equations above, we've seen that $np - mp \in \mathbf{N}$ can be expressed as $-p(m - n) \in \mathbf{N}$. Axiom 2.1.2 tells us that the product of any two numbers in $\mathbf{N}$ is also in $\mathbf{N}$. Therefore, $(m - n)$ must be in $\mathbf{N}$, i.e. $n < m$.

$\square$

Exercise 3. Proposition 2.26. *For all integers* $k \geq -3$, $3k^2 + 21k + 37 >= 0$.

Proof. We are told that $k \in \mathbf{Z}$. We are asked to prove the claim that for *all* $k \geq -3$, the evaluation of $3k^2 + 21k + 37$ will be greater than or equal to 0. This claim can be proven using induction.

For induction, we test the statement using a *base case*. For the base case, we'll use the minimum value of k, that is, -3:

$$3k^2 + 21k + 37 \geq 0$$

$$3(-3)^2 + 21(-3) + 37 \geq 0$$

$$64 - 63 \geq 0$$

$$1 \geq 0.$$

Having shown the base case to be true, we know that at least for some values of $k \geq -3$, the statement $3k^2 + 21k + 37 \geq 0$ is true. Now let's imagine adding 1 to k, and evaluating the statement.

For $k = n + 1$ (where $n \geq -3$) we have:

$$3k^2 + 21k + 37 \geq 0$$

$$3(n+1)^2 + 21(n+1) + 37 \geq 0$$

$$(3n^2 + 6n + 3) + (21n + 21) + 37 \geq 0$$

$$3n^2 + 27n + 61 \geq 0.$$

This result is larger than what results when $k = n$, i.e. our inductive hypothesis:

$$(3n^2 + 27n + 61) - (3n^2 + 21n + 37) = 6n + 24$$

$$3n^2 + 27n + 61 \geq 6n + 24 \geq 0.$$

We know that $n \geq -3$, so $6n + 24 \geq 6 \geq 0$, and therefore $3(n+1)^2 + 21(n+1) + 37 \geq 0$. In essence, we have seen that if you pick an arbitrary $n \in \mathbf{N}$, such that $n \geq -3$, the value of any $n + 1$ applied to the expression will be greater than 0. This concludes our induction.

$\square$

Exercise 4. Project 2.28. Determine for which natural numbers $k^2 - 3k \geq 4$ and prove your answer.

Answer.

Claim 4.1. $k^2 - 3k \geq 4$ for $k \geq 4$.

Proof. We are given the inequality, $k^2 - 3k \geq 4$, and are asked to find (and justify) the values of k that make the inequality true. We will prove the claim $k^2 - 3k \geq 4$ for $k \geq 4$ using induction. First, note that $k \leq 3$ does not satisfy the inequality.

For $k = 3$:

$$k^2 - 3k \geq 4 \Rightarrow (3)^2 - 3(3) \geq 4 \Rightarrow 9 - 9 \geq 4 \Rightarrow 0 \geq 4$$

The statement $0 \geq 4$ is *false*, because $0 \neq 4$ and $0 - 4 \notin \mathbf{N}$. It is trivial to demonstrate the same result for $k = 1$ and $k = 2$, and this is left to the reader. Assuming that k can't be ≤ 3, let us redefine k to be $k = 3 + j$, such that $j \in \mathbf{N}$ and $j \geq 1$. We will now prove the base case ($j = 1$).

For $k = 3 + 1 = 4$:

$$k^2 - 3k \geq 4 \Rightarrow (4)^2 - 3(4) \geq 4 \Rightarrow 16 - 12 \geq 4 \Rightarrow 4 \geq 4$$

This last statement is true $(4 = 4)$. We will now show that the results apply for all j, by proving this is true for $k = 4 + (j + 1)$.

For $k = 4 + (j + 1) = 5 + j$:

$$(k + 1)^2 - 3(k + 1) \geq 4$$

$$(k^2 + 2k + 1) - 3k - 3 \geq 4$$

$$k^2 - k - 2 \geq 4$$

$$(5 + j)^2 - (5 + j) - 2 \geq 4$$

$$25 + 10j + j^2 - j - 7 \geq 4$$

$$j^2 + 9j + 18 \geq 4$$

The final statement $j^2 + 9j + 18 - 4 \in \mathbf{N}$ is true because there is no value of j in $\mathbf{N}$ that could cause the value of the statement to be ≤ 0. $\square$

$\diamond$