MATH 287 HOMEWORK 1

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Exercise 1. Proposition 1.11(vi). If m, n, p, and q are integers, then

$(m(n+p))q = (mn)q + m(pq)$.

Proof. Let $m, n, p, q \in \mathbf{Z}$. We will use axioms related to multiplication to show that $(m(n+p))q = (mn)q + m(pq)$:

$$\begin{aligned} (m(n+p))q &= (m(n+p))q \\ \text{(Axiom 1.1.3)} \qquad &= (mn+mp)q \\ \text{(Axiom 1.1.3)} \qquad &= mnq + mpq \\ \text{(Axiom 1.1.5)} \qquad &= (mn)q + m(pq). \end{aligned}$$

□

Exercise 2. Proposition 1.22(i). For all $m \in \mathbf{Z}$, $-(-m) = m$.

Proof. Let $m \in Z$. We can start by re-expressing $-(-m)$ as $-1 \cdot (-1 \cdot m)$.

Rearranging the terms using our axioms, and simplifying, we get:

$$-(-m) = -1 \cdot (-1 \cdot m)$$

$$\text{(Axiom 1.1.4)} \qquad = (-1 \cdot -1) \cdot m$$

$$\text{(Corollary 1.21)} \qquad = 1 \cdot m$$

$$\text{(Axiom 1.3)} \qquad = m.$$

It must be for any integer m, $-(-m) = m$. □

Exercise 3. Proposition 1.22(ii). $-0 = 0$.

Proof. We can rewrite the equation as $0 = -1 \cdot 0$. Also observe, as defined in Axiom 1.4, that $1 + (-1) = 0$. With this in hand, we can see that

$$
\begin{aligned}
& & 0 &= -1 \cdot 0 \\
&\text{(Replacement)} & &= -1 \cdot (1 + (-1)) \\
&\text{(Axiom 1.1.3)} & &= -1 + (-1)(-1) \\
&\text{(Simplifying, using Corollary 1.21)} & &= -1 + 1 \\
& & &= 0.
\end{aligned}
$$

□

Exercise 4. Proposition 1.20. For all $m, n \in \mathbf{Z}, (-m)(-n) = mn$.

Proof. Let $m, n \in \mathbf{Z}$. From Axiom 1.4 we know the following:

$$m + (-m) = 0 \tag{1}$$

$$n + (-n) = 0 \tag{2}$$

When we multiply (1) and (2) by n and $(-m)$ respectively on both sides, we get:

$$(m + (-m))n = 0 \tag{3}$$

$$(n + (-n))(-m) = 0 \tag{4}$$

Proposition 1.14 shows that for any arbitrary integer $k \in \mathbf{Z}$, $k \cdot 0 = 0$. Because $-m$ and n are in $\mathbf{Z}$, the right-hand sides of (3) and (4) are each 0.

Next, we can distribute n and $-m$ through each equation, using Axiom 1.1.3 (and Proposition 1.6, which shows that the order of terms doesn't matter when distributing).

$$mn + (-m)n = 0 \tag{5}$$

$$(-m)n + (-m)(-n) = 0 \tag{6}$$

Applying Axiom 1.1.1 to (6) we can observe the following relationship between both equations:

$$mn + (-m)n = 0 \tag{7}$$

$$(-m)(-n) + (-m)n = 0 \tag{8}$$

Note that each equation has the same right-hand side, and that there are a total of 3 combinations of terms across both equations. From proposition 1.10, we know that if given three terms, across two equations each equal to the same value, that the unshared terms must be equivalent. That is, $mn = (-m)(-n)$, which concludes the proof.

$\square$

Exercise 5. Proposition 1.14. For all $m \in \mathbf{Z}, m \cdot 0 = 0 = 0 \cdot m$.

Hint: $0 + 0 = 0$.

Proof. Observe that, by Axiom 1.4: $1 + (-1) = 0$. Replacing this fact into the equation, and distributing we get

$$m \cdot 0 = m \cdot (1 + (-1))$$

(Axiom 1.1.3) $= m + m(-1)$

(Axiom 1.1.3) $= m + (-m)$

(Axiom 1.4) $= 0.$

It must be that for any integer m, $m \cdot 0 = 0$. $\square$