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MATH-471, Homework 4

1a.

We have a sample of data $\underline{x}$ = 6, 4, 8, 4, 4, 4, 2, 5, 5, 7. The sample size is 10.

We are told that the data represent the nightly count of animals caught in Alex's trap. Based on the description, we will assume the data comes from a *Poisson* distribution. The probability mass function for the Poisson distribution is

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, 2, ... \text{ and } \lambda > 0.$$

1b.

For the Poisson distribution, both the mean and variance are controlled by the same parameter, λ. To estimate the mean and variance, we will maximize the likelihood function, defined as

$$L(\lambda|x_1, x_2, ..., x_{10}) = \prod_{i=1}^{10} f(x_i|\lambda) = \prod_{i=1}^{10} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}.$$

Taking the natural log of L, we have

$$\begin{aligned} lnL(\lambda|x_1, x_2, ..., x_{10}) &= \sum_{i=1}^{10} ln\frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\ &= \sum_{i=1}^{10}\big[-\lambda + x_i ln\lambda - ln(x_i!)\big] \\ &= -\sum_{i=1}^{10}\lambda + \sum_{i=1}^{10} x_i ln\lambda - ln(x_i!) \\ &= -10\lambda + \sum_{i=1}^{10} x_i ln\lambda - ln(x_i!). \end{aligned}$$

Differentiating lnL results in

$$\frac{\partial}{\partial\lambda}\big[lnL(\lambda|x_1, x_2, ..., x_{10})\big] = \frac{\partial}{\partial\lambda}\big[-10\lambda + \sum_{i=1}^{10} x_i ln\lambda - ln(x_i!)\big] = -10 + \frac{\sum_{i=1}^{10} x_i}{\lambda}.$$

Setting the equation to 0, and solving for $\hat{\lambda}$ we have

$$
\begin{aligned}
0 &= -10 + \frac{\sum_{i=1}^{10} x_i}{\hat{\lambda}} \\
10 &= \frac{\sum_{i=1}^{10} x_i}{\hat{\lambda}} \\
10\hat{\lambda} &= \sum_{i=1}^{10} x_i \\
\hat{\lambda} &= \frac{\sum_{i=1}^{10} x_i}{10} \\
\hat{\lambda} &= \frac{49}{10} = 4.9.
\end{aligned}
$$

Thus, our estimate for the parameter is $\hat{\lambda} = 4.9$, i.e., $\mu = 4.9$ and $\sigma^2 = 4.9$.

We can test whether this point is the global maximum by using the 2nd-derivative test. Taking the second derivative of lnL we have

$$
\frac{\partial^2}{\partial^2 \lambda}\big[ln(L(\lambda|x_1, x_2, ..., x_{10}))\big] = \frac{\partial}{\partial \lambda}\big[-10 + \frac{\sum_{i=1}^{10} x_i}{\lambda}\big] = \frac{-\sum_{i=1}^{10} x_i}{\lambda^2}
$$

This result is negative, and will be negative for all λ (because, by definition, $\lambda > 0$). This means that we have found the maximum.

5.1

5.5

5.6

5.8

5.10

5.14

5.15