MATH-472: Exam 1 Makeup

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Question 2 (a)

Let $f(x) = \frac{1}{\sqrt{2\pi}}\ e^{-x^2/2}$ and $g(x) = \frac{1}{\pi(1+x^2)}$. Then $\frac{f(x)}{g(x)} = \frac{\pi}{\sqrt{2\pi}} \cdot (1+x^2) \cdot e^{-x^2/2}$.

We want to find the maximum of $\frac{f(x)}{g(x)}$. Taking the derivative, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{f(x)}{g(x)}\Big] &= \frac{\pi}{\sqrt{2\pi}}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big[e^{-x^2/2}\Big] + \frac{\mathrm{d}}{\mathrm{d}x}\Big[x^2 e^{-x^2/2}\Big]\Big) \\
&= \frac{\pi}{\sqrt{2\pi}}(-xe^{-x^2/2}) + \frac{\pi}{\sqrt{2\pi}}(2xe^{-x^2/2} - x^3e^{-x^2/2}) \\
&= \frac{\pi}{\sqrt{2\pi}}(xe^{-x^2/2} - x^3e^{-x^2/2}).
\end{aligned}$$

Setting the derivative equal to zero, we see that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{f(x)}{g(x)}\Big] &= 0 \\
\frac{\pi}{\sqrt{2\pi}}(xe^{-x^2/2} - x^3e^{-x^2/2}) &= 0 \\
xe^{-x^2/2} - x^3e^{-x^2/2} &= 0 \\
xe^{-x^2/2} &= x^3e^{-x^2/2} \\
x &= x^3 \\
1 &= x^2 \\
\text{thus, } x &= -1, 1.
\end{aligned}$$

Evaluating $\frac{f(x)}{g(x)}$ at the maximum to determine c, we have

$$c = \frac{f(-1)}{g(-1)} = \frac{f(1)}{g(1)} = \frac{\pi}{\sqrt{2\pi}} \cdot 2 \cdot e^{-1/2} \approx 1.5203.$$

Thus, the probability of acceptance is

$$Pr(\text{Accept}) = \frac{1}{c} \approx \frac{1}{1.5203} \approx 0.6577.$$

Question 2 (b)

The probability of rejection for a single iteration is

$$Pr(\text{Reject}) = 1 - \frac{1}{c} \approx 1 - 0.6577 \approx 0.3423.$$

Thus, approximately 34 of 100 simulated numbers will be rejected, on average.

Question 3

In determining the c.d.f. of X, I was too hasty and differentiated at key point instead of taking the anti-derivative. Here is a hopefully correct version of $F_X(x)$ and its inverse:

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\ dt \\
&= \int_{0}^{x} \theta t^{\theta-1}\ dt \\
&= \theta \cdot \frac{1}{\theta} \cdot t^{\theta}\Big|_{t=0}^{t=x} \\
&= (x^{\theta} - 0^{\theta}) \\
&= x^{\theta},\ \text{where } 0 \leq x \leq 1, 0 < \theta < \infty.
\end{aligned}$$

$$\begin{aligned}
F_X(x) = u \Rightarrow\ & u = x^{\theta} \\
\Rightarrow\ & ln(u) = \theta ln(x) \\
\Rightarrow\ & \frac{ln(u)}{\theta} = ln(x) \\
\Rightarrow\ & u \cdot e^{-\theta} = x \\
\Rightarrow\ & u \cdot e^{-\theta} = F_X^{-1}(u).
\end{aligned}$$

Question 4 (b) ii.

```r
x <- c(0.9746, 0.8314, 0.9658, 0.8702, 0.8508, 0.6498, 0.7398, 0.9439)

neglog <- function(theta) -(8 * log(theta) + (theta - 1) * sum(log(x)))

fit <- stats4::mle(neglog, start = 5)
print(fit)
```

```
Call:
stats4::mle(minuslogl = neglog, start = 5)

Coefficients:
   theta
5.988103
```

```r
print(8 / 1.336)
```

```
[1] 5.988024
```