

# MATH-472: Exam 1 Makeup

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## Question 2 (a)

Let  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$  and  $g(x) = \frac{1}{\pi(1+x^2)}$ . Then  $\frac{f(x)}{g(x)} = \frac{\pi}{\sqrt{2\pi}} \cdot (1+x^2) \cdot e^{-x^2/2}$ .

We want to find the maximum of  $\frac{f(x)}{g(x)}$ . Taking the derivative, we have

$$\begin{aligned}\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] &= \frac{\pi}{\sqrt{2\pi}} \left( \frac{d}{dx} [e^{-x^2/2}] + \frac{d}{dx} [x^2 e^{-x^2/2}] \right) \\ &= \frac{\pi}{\sqrt{2\pi}} (-xe^{-x^2/2}) + \frac{\pi}{\sqrt{2\pi}} (2xe^{-x^2/2} - x^3 e^{-x^2/2}) \\ &= \frac{\pi}{\sqrt{2\pi}} (xe^{-x^2/2} - x^3 e^{-x^2/2}).\end{aligned}$$

Setting the derivative equal to zero, we see that

$$\begin{aligned}\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] &= 0 \\ \frac{\pi}{\sqrt{2\pi}} (xe^{-x^2/2} - x^3 e^{-x^2/2}) &= 0 \\ xe^{-x^2/2} - x^3 e^{-x^2/2} &= 0 \\ xe^{-x^2/2} &= x^3 e^{-x^2/2} \\ x &= x^3 \\ 1 &= x^2 \\ \text{thus, } x &= -1, 1.\end{aligned}$$

Evaluating  $\frac{f(x)}{g(x)}$  at the maximum to determine  $c$ , we have

$$c = \frac{f(-1)}{g(-1)} = \frac{f(1)}{g(1)} = \frac{\pi}{\sqrt{2\pi}} \cdot 2 \cdot e^{-1/2} \approx 1.5203.$$

Thus, the probability of acceptance is

$$Pr(\text{Accept}) = \frac{1}{c} \approx \frac{1}{1.5203} \approx 0.6577.$$

## Question 2 (b)

The probability of rejection for a single iteration is

$$Pr(\text{Reject}) = 1 - \frac{1}{c} \approx 1 - 0.6577 \approx 0.3423.$$

Thus, approximately 34 of 100 simulated numbers will be rejected, on average.

## Question 3

In determining the c.d.f. of  $X$ , I was too hasty and differentiated at key point instead of taking the anti-derivative. Here is a hopefully correct version of  $F_X(x)$  and its inverse:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^x f_X(t) dt \\ &= \int_0^x \theta t^{\theta-1} dt \\ &= \theta \cdot \frac{1}{\theta} \cdot t^{\theta} \Big|_{t=0}^{t=x} \\ &= (x^{\theta} - 0^{\theta}) \\ &= x^{\theta}, \text{ where } 0 \leq x \leq 1, 0 < \theta < \infty. \end{aligned}$$

$$\begin{aligned} F_X(x) = u &\Rightarrow u = x^{\theta} \\ &\Rightarrow \ln(u) = \theta \ln(x) \\ &\Rightarrow \frac{\ln(u)}{\theta} = \ln(x) \\ &\Rightarrow u \cdot e^{-\theta} = x \\ &\Rightarrow u \cdot e^{-\theta} = F_X^{-1}(u). \end{aligned}$$

## Question 4 (b) ii.

```
x <- c(0.9746, 0.8314, 0.9658, 0.8702, 0.8508, 0.6498, 0.7398, 0.9439)

neglog <- function(theta) -(8 * log(theta) + (theta - 1) * sum(log(x)))

fit <- stats4::mle(neglog, start = 5)
print(fit)
```

Call:

```
stats4::mle(minuslogl = neglog, start = 5)
```

Coefficients:

```
theta
5.988103
```

```
print(8 / 1.336)
```

```
[1] 5.988024
```