MATH-472: Homework 3

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Question 1

Do 6.1, 6.3, 6.6, 6.9, 6.10 in the exercises of Chapter 6.

6.1

Analytically:

$$\int_0^{\frac{\pi}{3}} sin(t)\ dt = -cos(t)\Big|_{t=0}^{t=\frac{\pi}{3}} = cos(0) - cos(\frac{\pi}{3}) = \frac{1}{2}$$

Monte-Carlo integration:

```
g <- function(t) sin(t)
u <- runif(10000, 0, pi/3)
pi/3 * mean(g(u))
```

```
[1] 0.5000456
```

6.3

We will compare two estimates for $\theta = \int_0^{\frac{1}{2}} e^{-x} \, dx$:

- θ, Simple Monte-Carlo
- θ^*, "Hit-and-Miss" method

```r
m <- 10000

# simple Monte-Carlo integration
g <- function(x) exp(-x)
u <- runif(m, 0, 1/2)
theta <- 1/2 * mean(g(u))
var_theta <- var(g(u)) / m

# hit-or-miss
e <- rexp(m, 1)
I <- e <= 1/2
theta_star <- mean(I)
var_theta_star <- theta_star * (1 - theta_star) / m

c("Variance - Simple MC" = var_theta, "Variance - Hit or Miss" = var_theta_star)
```

```
Variance - Simple MC Variance - Hit or Miss
        1.283763e-06           2.394938e-05
```

Based on $m = 10,000$, the we would say that θ is a more efficient estimator than θ^*.

6.6

```r
# empirical estimates of Cov(e^U, e^{1-U}) and Var(e^U + e^{1-U})
f <- function(x) exp(x)
u <- runif(500)
v <- 1 - u

cov_uv <- cov(f(u), f(v))
var_u_plus_v <- var(f(u)) + var(f(v)) + cov_uv
```

6.9

The Rayleigh Density is

$$f(x) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} \text{ where } x \geq 0, \sigma > 0.$$

Implement a function to generate samples from a *Rayleigh*(σ) distribution using antithetic variables. What is the percent reduction in variance of $\frac{X+X'}{2}$ compared with $\frac{X_1+X_2}{2}$ for independent X_1 and X_2?

```
rayleigh <- function(n, sigma, anti = TRUE) {
  u <- runif(n / 2)
  v <- if (anti) 1 - u else runif(n / 2)
  u <- c(u, v)
  sqrt(-2 * sigma^2 * log(u))
}

independent <- rayleigh(n = 3000, sigma = 2, anti = FALSE)
antithetic <- rayleigh(n = 3000, sigma = 2, anti = TRUE)

# f(x_1, x_2, ..., x_1500) ~ X_1
# f(x_1501, x_1502, ..., x_3000) ~ X_2
X1 <- independent[1:1500]
X2 <- independent[1501:3000]

# f(x_1, x_2, ..., x_1500) ~ X
# f(1 - x_1, 1 - x_2, ... 1 - x_1500) ~ X'
X <- antithetic[1:1500]
Xp <- antithetic[1501:3000]

# calculate variances
v1 <- 1/4 * var(X1) + 1/4 * var(X2) + 1/4 * 2 * cov(X1, X2)
v2 <- 1/4 * var(X) + 1/4 * var(Xp) + 1/4 * 2 * cov(X, Xp)

# percent reduction
p <- (v1 - v2) / v1
```

The reduction in variance is estimated as 94.56%, based on $\sigma = 2$ and $n = 3,000$.

6.10

Use Monte Carlo integration with antithetic variables to estimate

$$\int_0^1 \frac{e^{-x}}{1+x^2}\, dx,$$

and find the approximate reduction in variance as a percentage of the variance without variance reduction.

```r
f <- function(x) exp(-x) / (1 + x^2)
mc <- function(n, anti = TRUE) {
  u <- runif(n / 2)
  v <- if (anti) 1 - u else runif(n / 2)
  u <- c(u, v)
  mean(f(u))
}

n <- 3000
t1 <- t2 <- numeric(n)

for (i in 1:n) {
  t1[i] <- mc(1000, anti = FALSE)
  t2[i] <- mc(1000, anti = TRUE)
}

var_smc <- var(t1)
var_atv <- var(t2)

theta <- mean(t2)

p <- (var_smc - var_atv) / var_smc
```

From our simulation with $n = 3,000$, we have $\hat{\theta} = 0.525$. The estimated reduction in variance is 96.25%.

Question 2

Suppose you use the importance sampling method to obtain a Monte Carlo estimate of

$$\theta = \int_1^\infty g(x)\ dx,$$

where

$$g(x) = \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2}.$$

(a) A possible importance function for the purpose could be

$$f(x) = \frac{1}{\Gamma(3/2)} 2^{3/2} x^{3/2-1} e^{-2x}, 1 < x < \infty.$$

Note that $t = x - 1$ has a gamma distribution with shape 3/2 and rate 2. Draw two functions $y = g(x)$ and $y = f(x)$ on the xy-plane for the following values: `x <- seq(1, 10, 0.01)`.

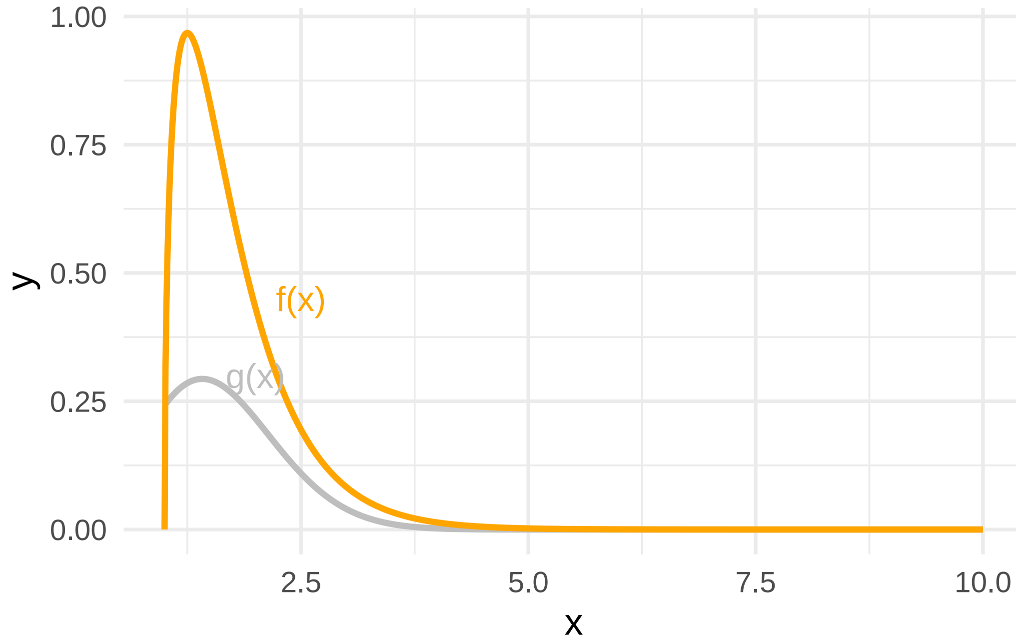
(b) Estimate θ using the importance function in (a).

```
g <- function(x) x^2 / sqrt(2 * pi) * exp(-x^2 / 2)
f <- function(x) 1 / gamma(3 / 2) * 2^(3/2) * (x - 1)^(3/2 - 1) * exp(-2 * (x - 1))
x <- seq(1, 10, 0.01)
d <- data.frame(x = x, y0 = g(x), y1 = f(x))

library(ggplot2)

theme_set(theme_minimal(base_size = 14))

ggplot(d, aes(x = x)) +
  geom_line(aes(y = y0), size = 1.1, color = "grey") +
  geom_line(aes(y = y1), size = 1.1, color = "orange") +
  annotate("text", x = 2.0, y = 0.3, label = "g(x)", color = "grey", size = 4.5) +
  annotate("text", x = 2.5, y = 0.45, label = "f(x)", color = "orange", size = 4.5) +
  labs(x = "x", y = "y")
```



To estimate θ using importance sampling, we use the following procedure.

1. Generate $x_1, x_2, \cdots, x_n \sim f_X(x)$
2. Estimate $\hat{\theta}$ as $\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{g(x_i)}{f(x_i)}$.

```
x <- rgamma(10000, 3/2, 2) + 1
theta <- mean(g(x) / f(x))
theta
```

```
[1] 0.3997343
```

```
integrate(g, 1, Inf)
```

```
0.400626 with absolute error < 5.7e-07
```