

MATH-472: Homework 2

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Question 1

The definition of the Gamma function where $n \in \mathbb{Z}^+$ is

$$\Gamma(n) = (n-1)!$$

This can be replicated in R using the `factorial()` function.

```
posint_gamma <- function(n) {  
  if (any(!is.integer(n) | n <= 0)) stop("n must be a positive integer.")  
  
  factorial(n - 1)  
}  
  
all(posint_gamma(1:4) == gamma(1:4))
```

```
[1] TRUE
```

Question 2

We have

$$f(x; p) = \begin{cases} p(1-p)^x, & x = 0, 1, 2, 3, \dots \\ 0, & \text{otherwise.} \end{cases}$$

We want the recursive formula for $f(x+1; p) = L \cdot f(x; p)$.

Solving for L , we see

$$\begin{aligned} f(x+1; p) &= L \cdot f(x; p) \\ p(1-p)^{(x+1)} &= L \cdot p(1-p)^x \\ p(1-p)^x(1-p)^1 &= L \cdot p(1-p)^x \\ (1-p) &= L. \end{aligned}$$

Question 3

Let X be a discrete R.V. where $x_1 = 10, x_2 = 30, x_3 = 50, x_4 = 70, x_5 = 90$, and having the following cdf:

X	10	30	50	70	90
$F(x) = P(X \leq x)$	0.27	0.41	0.64	0.92	1.00

```
set.seed(20230208)

f <- function(print_check = FALSE) {
  u <- runif(1, 0, 1)
  x <- c(10, 30, 50, 70, 90)
  p <- c(0.27, 0.41, 0.64, 0.92, 1)
  keep <- dplyr::lag(p, default = 0) < u & u <= p

  if (print_check) print(data.frame(x, p, u, keep))

  x[keep]
}

f(print_check = TRUE) # run once as an example
```

```
      x      p      u keep
1 10 0.27 0.7946032 FALSE
2 30 0.41 0.7946032 FALSE
3 50 0.64 0.7946032 FALSE
4 70 0.92 0.7946032  TRUE
5 90 1.00 0.7946032 FALSE
```

```
[1] 70
```

```
x <- replicate(10000, f()) # examine the distribution, take 10k samples
x |> table() |> prop.table() |> cumsum() |> round(2)
```

```
      10      30      50      70      90
0.27 0.41 0.64 0.92 1.00
```

Question 4

- (a) Calculate the inverse function of $u = F(x)$, i.e., $F_X^{-1}(u) = x$.

$$\begin{aligned} F(x) = u &\implies 1 - e^{-(0.25x^3)} = u \\ &\implies 1 - u = e^{-(0.25x^3)} \\ &\implies \ln(1 - u) = -\frac{1}{4}x^3 \\ &\implies -4\ln(1 - u) = x^3 \\ &\implies (-4\ln(1 - u))^{\frac{1}{3}} = x \\ &\implies F_X^{-1}(u) = x. \end{aligned}$$

- (b) Calculate values of x for $u = 0.37, 0.45, 0.82$.

```
f_inv <- function(u) (-4*log(1 - u))^(1/3)
f_inv(c(0.37, 0.45, 0.82)) |> round(digits = 5)
```

```
[1] 1.22719 1.33726 1.90002
```

Question 5

The Rayleigh density is

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, x \geq 0, \sigma > 0.$$

Develop an algorithm to generate random samples from a *Rayleigh*(σ) distribution. Generate samples for several choices of $\sigma > 0$ and check that the mode of the generated samples is close to the theoretical mode σ (check the histogram).

We will use the inverse-transformation method. First, we need to find the cdf $F_X(x)$:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^x f(t) dt \\ &= \int_{-\infty}^0 0 dt + \int_0^x \frac{t}{\sigma^2} e^{-t^2/2\sigma^2} dt \\ &= \int_0^x \frac{t}{\sigma^2} e^{-t^2/2\sigma^2} dt \\ &= - \int_0^x e^u du \quad (\text{let } u = \frac{-t^2}{2\sigma^2} \text{ and } du = \frac{-t}{\sigma^2} dt) \\ &= -e^u \Big|_{t=0}^{t=x} \\ &= -e^{\frac{-x^2}{2\sigma^2}} \Big|_{t=0}^{t=x} = \left(-e^{\frac{-x^2}{2\sigma^2}} - (-1) \right) = 1 - e^{\frac{-x^2}{2\sigma^2}}, x \geq 0, \sigma > 0. \end{aligned}$$

We now need to find $F_X^{-1}(u) = x$:

$$\begin{aligned} F_X(x) = u &\implies 1 - e^{-x^2/2\sigma^2} = u \\ &\implies 1 - u = e^{-x^2/2\sigma^2} \\ &\implies -2\sigma^2 \ln(1 - u) = x^2 \\ &\implies \sqrt{-2\sigma^2 \ln(1 - u)} = x \\ &\implies F_X^{-1}(u) = x. \end{aligned}$$

We can then apply the algorithm:

1. Generate $u \sim U(0, 1)$
2. Compute $X'(u) = F_X^{-1}(u)$ to get realizations from X .

Demonstrations with several values of σ , using R.

```

library(tidyverse)
library(patchwork)

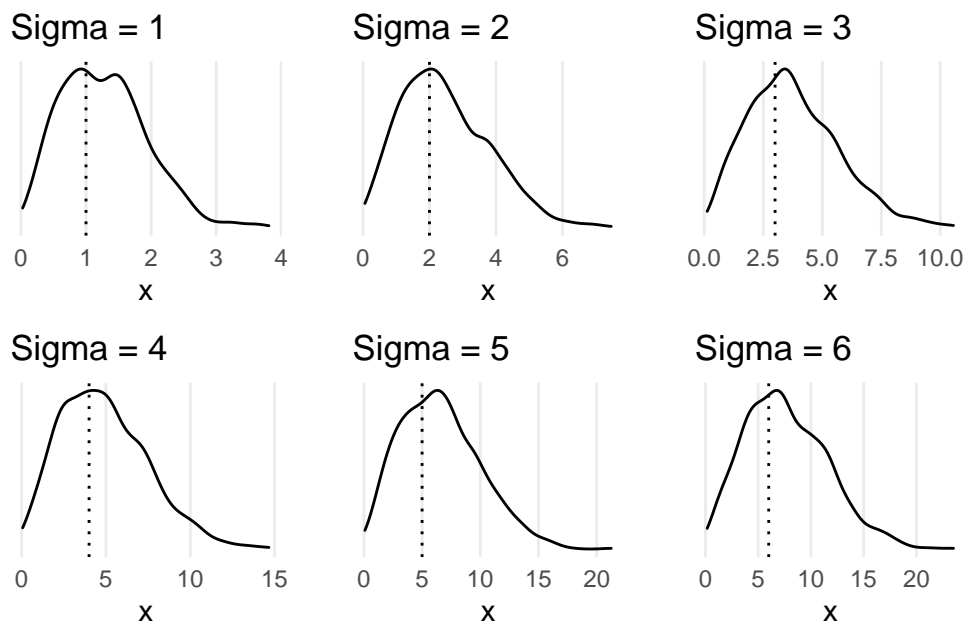
finv <- function(u, sigma) sqrt(-2*sigma^2 * log(1- u))

gen_density <- function(n = 1000, sigma = 1) {
  dat <- tibble(u = runif(n, 0, 1), x = finv(u, sigma))

  ggplot(dat, aes(x = x)) +
    geom_vline(xintercept = sigma, lty = "dotted") +
    geom_density() +
    labs(title = str_glue("Sigma = {sigma}"), x = "x", y = "") +
    theme_minimal(base_size = 11) +
    theme(
      axis.text.y = element_blank(),
      panel.grid.minor = element_blank(), panel.grid.major.y = element_blank()
    )
}

(gen_density(sigma = 1) | gen_density(sigma = 2) | gen_density(sigma = 3)) /
  (gen_density(sigma = 4) | gen_density(sigma = 5) | gen_density(sigma = 6))

```



Question 6

$$f(x) = \frac{1}{2}e^{-|x|}.$$

Question 7