MATH-472: Homework 3

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Question 1

Do 6.1, 6.3, 6.6, 6.9, 6.10 in the exercises of Chapter 6.

6.1

Analytically:

$$\int_0^{\frac{\pi}{3}} sin(t)\ dt = -cos(t)\Big|_{t=0}^{t=\frac{\pi}{3}} = cos(0) - cos(\frac{\pi}{3}) = \frac{1}{2}$$

Monte-Carlo integration:

```
g <- function(t) sin(t)
u <- runif(10000, 0, pi/3)
mean(g(u))
```

```
[1] 0.4775084
```

6.3

```
g <- function(x) exp(-x)
f <- function(x) sqrt(x)

gen_thetas <- function(N = 1000, n = 200) {
  t <- numeric(N)
  tstar <- numeric(N)
```

```
  for (i in 1:N) {
    u <- runif(n, 0, 0.5)
    e <- rexp(n, rate = 1)
    t[i] <- mean(g(u))
    tstar[i] <- mean(e <= 0.5)
  }

  c(theta = var(t), theta_star = var(tstar))
}

gen_thetas()
```

```
      theta   theta_star
6.638171e-05 1.128808e-03
```

6.6

```
mc <- function(f, n, r, anti = TRUE) {
  t <- numeric(n)
  for (i in 1:n) {
    m <- runif(r / 2)
    n <- if (anti) 1 - m else runif(r / 2)
    u <- c(m, n)
    t[i] <- mean(f(u))
  }
  return(t)
}

f <- function(x) exp(x)

smc <- mc(f, 1000, 100, anti = FALSE)
ava <- mc(f, 1000, 100)
```

6.9

The Rayleigh Density is

$$f(x) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} \text{ where } x \geq 0, \sigma > 0.$$

Implement a function to generate samples from a $Rayleigh(\sigma)$ distribution using antithetic variables. What is the percent reduction in variance of $\frac{X+X'}{2}$ compared with $\frac{X_1+X_2}{2}$ for independent X_1 and X_2?

6.10

Use Monte Carlo integration with antithetic variables to estimate

$$\int_0^1 \frac{e^{-x}}{1+x^2}\, dx,$$

and find the approximate reduction in variance as a percentage of the variance without variance reduction.

Question 2

Suppose you use the importance sampling method to obtain a Monte Carlo estimate of

$$\theta = \int_0^\infty g(x)\, dx,$$

where

$$g(x) = \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2}.$$

(a) A possible importance function for the purpose could be

$$f(x) = \frac{1}{\Gamma(3/2)} 2^{3/2} x^{3/2-1} e^{-2x}, 1 < x < \infty.$$

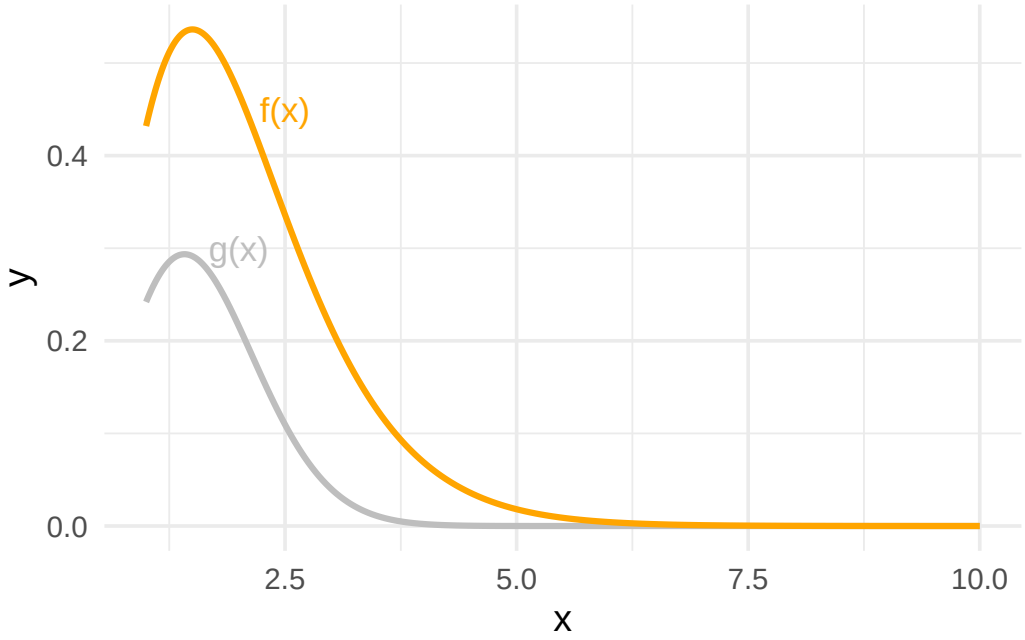
Note that $t = x - 1$ has a gamma distribution with shape $3/2$ and rate 2. Draw two functions $y = g(x)$ and $y = f(x)$ on the xy-plane for the following values: `x <- seq(1, 10, 0.01)`.

(b) Estimate θ using the importance function in (a).

```r
g <- function(x) x^2 / sqrt(2 * pi) * exp(-x^2 / 2)
f <- function(x) 1 / gamma(3 / 2) * 2^(3/2) * x^(3) * exp(-2 * x)
x <- seq(1, 10, 0.01)
d <- data.frame(x = x, y0 = g(x), y1 = f(x))

library(ggplot2)

ggplot(d, aes(x = x)) +
  geom_line(aes(y = y0), size = 1.1, color = "grey") +
  geom_line(aes(y = y1), size = 1.1, color = "orange") +
  annotate("text", x = 2.0, y = 0.3, label = "g(x)", color = "grey", size = 4.5) +
  annotate("text", x = 2.5, y = 0.45, label = "f(x)", color = "orange", size = 4.5) +
  theme_minimal(base_size = 14) +
  labs(x = "x", y = "y")
```


f(x)
g(x)
0.4
0.2
0.0
y
2.5
5.0
7.5
10.0
x