MATH-472: Homework 2

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Question 1

The definition of the Gamma function where $n \in \mathbb{Z}^+$ is

$$\Gamma(n) = (n-1)!$$

This can be replicated in R using the `factorial()` function.

```r
posint_gamma <- function(n) {
  if (any(!is.integer(n) | n <= 0)) stop("n must be a positive integer.")

  factorial(n - 1)
}

all(posint_gamma(1:4) == gamma(1:4))
```

```
[1] TRUE
```

Question 2

We have

$$f(x;p) = \begin{cases} p(1-p)^x, x = 0, 1, 2, 3, \cdots \\ 0,\ \text{otherwise.} \end{cases}$$

We want the recursive formula for $f(x+1;p) = L \cdot f(x;p)$.

Solving for L, we see

$$\begin{aligned} f(x+1;p) &= L \cdot f(x;p) \\ p(1-p)^{(x+1)} &= L \cdot p(1-p)^x \\ p(1-p)^x(1-p)^1 &= L \cdot p(1-p)^x \\ (1-p) &= L. \end{aligned}$$

Question 3

Let X be a discrete R.V. where $x_1 = 10, x_2 = 30, x_3 = 50, x_4 = 70, x_5 = 90$, and having the following cdf:

X	10	30	50	70	90
$F(x) = P(X \leq x)$	0.27	0.41	0.64	0.92	1.00

```r
set.seed(20230208)

f <- function(print_check = FALSE) {
  u <- runif(1, 0, 1)
  x <- c(10, 30, 50, 70, 90)
  p <- c(0.27, 0.41, 0.64, 0.92, 1)
  keep <- dplyr::lag(p, default = 0) < u & u <= p

  if (print_check) print(data.frame(x, p, u, keep))

  x[keep]
}

f(print_check = TRUE) # run once as an example
```

```
   x    p         u  keep
1 10 0.27 0.7946032 FALSE
2 30 0.41 0.7946032 FALSE
3 50 0.64 0.7946032 FALSE
4 70 0.92 0.7946032  TRUE
5 90 1.00 0.7946032 FALSE
```

```
[1] 70
```

```r
x <- replicate(10000, f()) # examine the distribution, take 10k samples
x |> table() |> prop.table() |> cumsum() |> round(2)
```

```
  10   30   50   70   90 
0.27 0.41 0.64 0.92 1.00 
```

Question 4

(a) Calculate the inverse function of $u = F(x)$, i.e., $F_X^{-1}(u) = x$.

$$\begin{aligned} F(x) = u &\implies 1 - e^{-(0.25x^3)} = u \\ &\implies 1 - u = e^{-(0.25x^3)} \\ &\implies ln(1-u) = -\frac{1}{4}x^3 \\ &\implies -4ln(1-u) = x^3 \\ &\implies (-4ln(1-u))^{\frac{1}{3}} = x \\ &\implies F_X^{-1}(u) = x. \end{aligned}$$

(b) Calculate values of x for $u = 0.37, 0.45, 0.82$.

```r
f_inv <- function(u) (-4*log(1 - u))^(1/3)

f_inv(c(0.37, 0.45, 0.82)) |> round(digits = 5)
```

```
[1] 1.22719 1.33726 1.90002
```

Question 5

The Rayleigh density is

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, x \geq 0, \sigma > 0.$$

Develop an algorithm to generate random samples from a $Rayleigh(\sigma)$ distribution. Generate samples for several choices of $\sigma > 0$ and check that the mode of the generated samples is close to the theoretical mode σ (check the histogram).

We will use the inverse-transformation method. First, we need to find the cdf $F_X(x)$:

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f(t)\ dt \\
&= \int_{-\infty}^{0} 0\ dt + \int_{0}^{x} \frac{t}{\sigma^2} e^{-t^2/2\sigma^2}\ dt \\
&= \int_{0}^{x} \frac{t}{\sigma^2} e^{-t^2/2\sigma^2}\ dt \\
&= -\int_{0}^{x} e^u\ du && (\text{let } u = \tfrac{-t}{2\sigma^2} \text{ and } du = \tfrac{-t}{\sigma^2}\ dt) \\
&= -e^u \Bigg|_{t=0}^{t=x} \\
&= -e^{\frac{-t^2}{2\sigma^2}} \Bigg|_{t=0}^{t=x} = \left(-e^{\frac{-x^2}{2\sigma^2}} - (-1)\right) = 1 - e^{\frac{-x^2}{2\sigma^2}}, x \geq 0, \sigma > 0.
\end{aligned}$$

We now need to find $F_X^{-1}(u) = x$:

$$\begin{aligned}
F_X(x) = u &\implies 1 - e^{-x^2} 2\sigma^2 = u \\
&\implies 1 - u = e^{-x^2} 2\sigma^2 \\
&\implies -2\sigma^2 ln(1-u) = x^2 \\
&\implies \sqrt{-2\sigma^2 ln(1-u)} = x \\
&\implies F_X^{-1}(u) = x.
\end{aligned}$$

We can then apply the algorithm:

1. Generate $u \sim U(0,1)$
2. Compute $X'(u) = F_X^{-1}(u)$ to get realizations from X.

Demonstrations with several values of σ, using R.

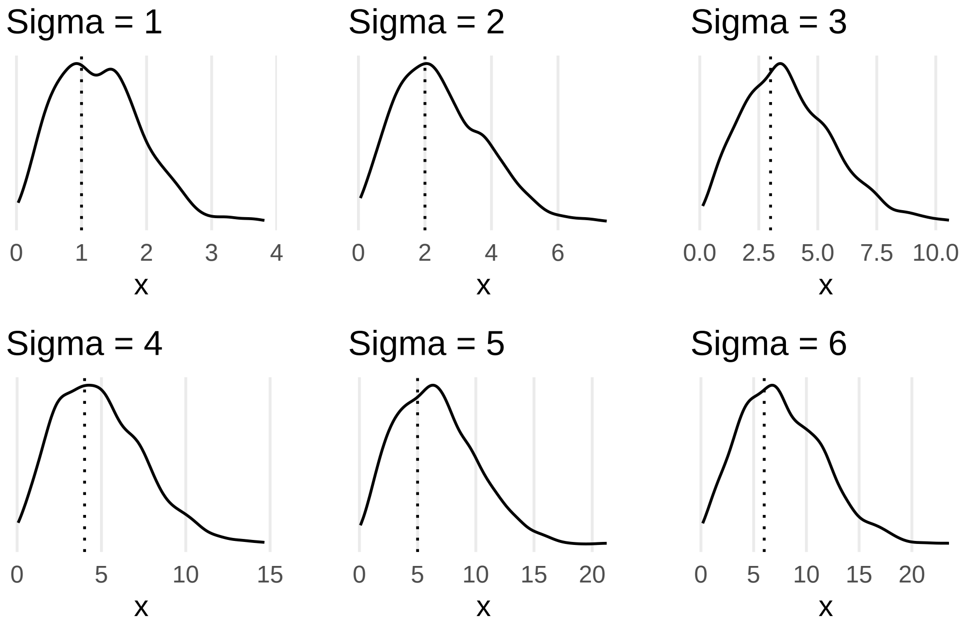
```r
library(tidyverse)
library(patchwork)

finv <- function(u, sigma) sqrt(-2*sigma^2 * log(1- u))

gen_density <- function(n = 1000, sigma = 1) {
  dat <- tibble(u = runif(n, 0, 1), x = finv(u, sigma))

  ggplot(dat, aes(x = x)) +
    geom_vline(xintercept = sigma, lty = "dotted") +
    geom_density() +
    labs(title = str_glue("Sigma = {sigma}"), x = "x", y = "") +
    theme_minimal(base_size = 11) +
    theme(
      axis.text.y = element_blank(),
      panel.grid.minor = element_blank(), panel.grid.major.y = element_blank()
    )
}

(gen_density(sigma = 1) | gen_density(sigma = 2) | gen_density(sigma = 3)) /
  (gen_density(sigma = 4) | gen_density(sigma = 5) | gen_density(sigma = 6))
```



Question 6

We have $f(x) = \frac{1}{2}e^{-|x|}$. This function is symmetric around 0. So, the resulting cdf $F_X(x)$ will be a piecewise function. If $x < 0$, we have

$$\begin{aligned}
\int_{-\infty}^{x} f(t)\ dt &= \int_{-\infty}^{x} \frac{1}{2}e^{-|t|}\ dt \\
&= \frac{1}{2}\int_{-\infty}^{x} e^{t}\ dt && \text{(because } -\infty < t < 0\text{)} \\
&= \frac{1}{2}e^{t}\Big|_{-\infty}^{x} \\
&= \lim_{n\to-\infty} \frac{1}{2}e^{t}\Big|_{n}^{x} \\
&= \lim_{n\to-\infty} \frac{1}{2}(e^{x} - e^{n}) \\
&= \lim_{n\to-\infty} \frac{1}{2}e^{x} - \cancelto{0}{\frac{1}{2}e^{n}} \\
&= \frac{1}{2}e^{x}.
\end{aligned}$$

Note that $\int_{-\infty}^{0} f(x)\ dx = \frac{1}{2}$ because

$$\begin{aligned}
\int_{-\infty}^{0} f(x)\ dx &= \int_{-\infty}^{0} \frac{1}{2}e^{-|x|}\ dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{x}\ dx && \text{(because } -\infty < x < 0\text{)} \\
&= \frac{1}{2}e^{x}\Big|_{-\infty}^{0} \\
&= \lim_{n\to-\infty} \frac{1}{2}e^{x}\Big|_{n}^{0} \\
&= \lim_{n\to-\infty} \frac{1}{2}(e^{0} - e^{n}) \\
&= \lim_{n\to-\infty} \frac{1}{2} - \cancelto{0}{\frac{1}{2}e^{n}} \\
&= \frac{1}{2}.
\end{aligned}$$

Now, if $x \geq 0$, we have

$$\begin{aligned}
\int_{-\infty}^{0} f(t)\ dt + \int_{0}^{x} f(t)\ dt &= \frac{1}{2} + \frac{1}{2}\int_{0}^{x} e^{-|t|}\ dt \\
&= \frac{1}{2} + \frac{1}{2}\int_{0}^{x} e^{-t}\ dt && \text{(because } 0 \leq t \leq x\text{)} \\
&= \frac{1}{2} - \frac{1}{2}e^{-t}\Big|_{0}^{x} \\
&= \frac{1}{2} - \frac{1}{2}(e^{-x} - e^{0}) \\
&= \frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-x} \\
&= 1 - \frac{1}{2}e^{-x}.
\end{aligned}$$

Thus, $F_X(x)$ is defined as

$$F_X(x) = \begin{cases} \frac{1}{2}e^{x}, & x < 0 \\ 1 - \frac{1}{2}e^{-x}, & x \geq 0 \end{cases}$$

and given some x, we have $F_X(x) = u$. We now need an inverse $F_X^{-1}(u)$ satisfying $F_X^{-1}(u) = x$. This function will also be piecewise.

If $x < 0$

$$\begin{aligned}
F_X(x) = \frac{1}{2}e^{x} &= u \\
\implies e^{x} &= 2u \\
\implies x &= ln(2u.)
\end{aligned}$$

If $x \geq 0$

$$\begin{aligned}
F_X(x) = 1 - \frac{1}{2}e^{-x} &= u \\
\implies \frac{1}{2}e^{-x} &= 1 - u \\
\implies e^{-x} &= 2 - 2u \\
\implies -x &= ln(2 - 2u) \\
\implies x &= -ln(2 - 2u).
\end{aligned}$$

So

$$F_X^{-1}(u) = \begin{cases} ln(2u) & u < 0.5 \\ -ln(2 - 2u) & u \geq 0.5. \end{cases}$$

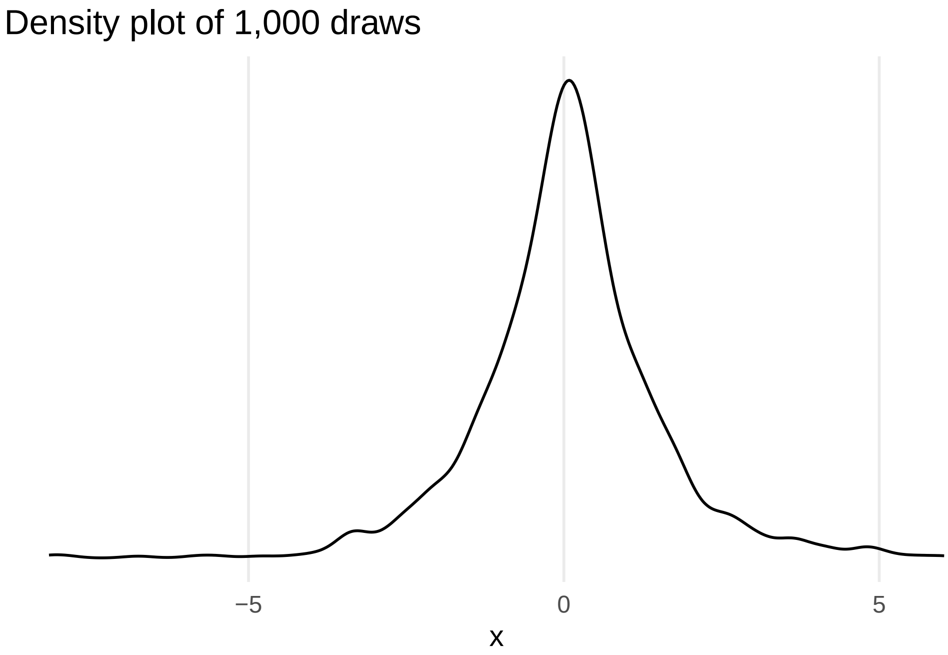
Now we can apply the algorithm

1. Generate $u \sim U(0, 1)$
2. Compute $X'(u) = F_X^{-1}(u)$ to get realizations from X.

Implemented in R code, to generate 1,000 realizations.

```r
finv <- function(u) ifelse(u < 0.5, log(2*u), -log(2 - 2*u))
lapl <- function(n = 1000) finv(runif(n, 0, 1))

ggplot(tibble(x = lapl())) +
  geom_density(aes(x = x)) +
  theme_minimal(base_size = 11) +
  theme(
    axis.text.y = element_blank(),
    panel.grid.minor = element_blank(), panel.grid.major.y = element_blank()
  ) +
  labs(y = "", title = "Density plot of 1,000 draws")
```



Question 7

We will use the Box-Mueller Transform to generate samples from standard normal distributions.

```
bmt <- function(n) {
  u1 <- runif(n, 0, 1)
  u2 <- runif(n, 0, 1)

  r <- sqrt(-2*log(u1))
  theta <- 2*pi*u2
  x <- r*cos(theta)
  y <- r*sin(theta)

  data.frame(x, y)
}
```

(a)

10 values from $\chi^2(5)$.

```
dat <- bmt(25)

indices <- seq(1, 25, 5)
chi <- numeric()
for (element in indices) {
  x2 <- c(sum(dat$x[element:element + 4]^2), sum(dat$y[element:element + 4]^2))
  chi <- c(chi, x2)
}

print(chi)
```

```
 [1] 0.768997559 0.380225654 3.472371432 1.704080166 0.008472789 0.255175259
 [7] 0.002790484 0.320066156 1.301886762 0.842165637
```

(b)

10 values from $t(3)$.

```
dat <- bmt(30)

z <- dat$x[1:10]
v <- numeric()
q <- 3

indices <- seq(1, q * 10, q)
for (element in indices) {
  x2 <- sum(dat$y[element:element + (q - 1)]^2)
  v <- c(v, x2)
}

z / sqrt(v / q)
```

```
 [1] -16.35420125  -1.80444116   1.41025368   4.11248280  -2.87063397
 [6]   3.13739856   0.55614796   0.55255069   0.02208829  -0.09158468
```

(c)

10 values from $F(6, 10)$.

```
dat <- bmt(100)

v <- numeric()
m <- 6
vind <- seq(1, m * 10, m)
for (element in vind) {
  x2 <- sum(dat$x[element:element + (m - 1)]^2)
  v <- c(v, x2)
}

w <- numeric()
n <- 10
wind <- seq(1, n * 10, n)
for (element in wind) {
  x2 <- sum(dat$y[element:element + (n - 1)]^2)
  w <- c(w, x2)
}

(v / m) / (w / n)
```

```
[1] 2.057290e+03 5.044342e-01 3.006081e+00 3.935956e-01 9.975536e+01
[6] 3.793622e+00 5.699611e-03 3.505360e+00 2.946466e+03 1.257410e+01
```