MATH-472: Homework 1

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1/25/23

Question 1

Let U be a random variable with support $\mathbb{R}_U = (0, 1)$.

Define the pdf and cdf of U respectively as

$$f_U(u) = \begin{cases} \frac{1}{1} & 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_U(u) = \begin{cases} 0 & 0 \leq 0 \\ u & 0 < u < 1 \\ 1 & u \geq 1 \end{cases}$$

Let $T = e^U$. Then, $g(u) = e^u$ is strictly increasing on $(0, 1)$. So, we then have $g^{-1}(t) = ln(t)$.

Define $F_T(t)$ and $f_T(t)$ as

$$F_T(t) = \begin{cases} 0 & t \leq 1 \\ F_U(g^{-1}(t)) = ln(t) & 1 < t < e \\ 1 & t \geq e. \end{cases}$$

$$f_T(t) = \frac{d}{dt} F_T(t) = \begin{cases} ln(t) \cdot \frac{1}{t} & t \in (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
E[T] &= \int_{-\infty}^{\infty} tf(t)\ dt \\
&= \int_{1}^{e} t\frac{ln(t)}{t}\ dt \\
&= \int_{1}^{e} ln(t)\ dt \\
&= tln(t)\Big|_{1}^{e} - \int_{1}^{e}\ dt \\
&= (eln(e) - ln(1)) - t\Big|_{1}^{e} \\
&= e - e + 1 \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
E[T^2] &= \int_{-\infty}^{\infty} t^2 f(t)\ dt \\
&= \int_{1}^{e} t^2\frac{ln(t)}{t}\ dt \\
&= \int_{1}^{e} tln(t)\ dt \\
&\int_{1}^{e} u\ dv = uv\Big|_{1}^{e} - \int_{1}^{e} v\ du \\
&\text{let } u = ln(t), du = \frac{1}{t}\ dt, v = \frac{1}{2}t^2, dv = t\ dt \\
&= \frac{1}{2}t^2 ln(t)\Big|_{1}^{e} - \frac{1}{2}\int_{1}^{e} t\ dt \\
&= \frac{1}{2}e^2 - \frac{1}{4}t^2\Big|_{1}^{e} \\
&= \frac{1}{2}e^2 - \frac{1}{4}(e^2 - 1) \\
&= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} \\
&= \frac{1}{4}e^2 + \frac{1}{4} \\
&= \frac{1}{4}(e^2 + 1).
\end{aligned}$$

$$Var(T) = E[T^2] - (E[T])^2 = \frac{1}{4}(e^2+1) - 1 = \frac{1}{4}e^2 - \frac{3}{4}.$$

Let $l(u) = u$.

$$E[l(u)] = \int_{-\infty}^{\infty} ul(u)\ du = \int_0^1 u\ du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}$$

$$E[l(u)^2] = \int_{-\infty}^{\infty} u^2 l(u)\ du = \int_0^1 u^2\ du = \frac{1}{3}u^3\Big|_0^1 = \frac{1}{3}$$

$$Var(l(u)) = E[l(u)^2] - (E[l(u)])^2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Question 2

Let $I(x) = \begin{cases} 1 & 0 < X < 0.5 \\ 0 & \text{otherwise} \end{cases}$, where $X \sim Uniform(0, 1)$.

$$\begin{aligned} P(X \leq 0.5) = F(0.5) &= \int_0^{0.5} \frac{1}{1-0}\ dx = x\Big|_0^{0.5} \\ &= (0.5 - 0) \\ &= 0.5 \\ &= 1 \cdot (0.5) + 0 \cdot (0.5) \\ &= P(I(x) = 1) + P(I(x) = 0) \\ &= E[I(x)]. \end{aligned}$$

$$\int_0^c 1\ dx = x\Big|_0^c = (c - 0) = c(1 - 0) = c\Big[y\Big]_0^1 = \int_0^1 c\ dy$$

Question 3

Let $x_1, x_2, \cdots, x_{10} \overset{i.i.d.}{\sim} f(x_i;\sigma^2) = \begin{cases} \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} & x > 0, \sigma^2 > 0 \\ 0 & \text{otherwise} \end{cases}$.

First, we find the likelihood function $L(\sigma^2)$ and the log-likelihood function $\ell(\sigma^2)$.

$$\begin{aligned} L(\sigma^2) = \prod_{i=1}^{10} f(x_i;\sigma^2) &= \prod_{i=1}^{10} \frac{x_i}{\sigma^2} e^{\frac{-x_i^2}{2\sigma^2}} \\ &= (\prod_{i=1}^{10} \frac{x_i}{\sigma^2})(\prod_{i=1}^{10} e^{\frac{-x_i^2}{2\sigma^2}}) \\ &= (\prod_{i=1}^{10} \frac{x_i}{\sigma^2}) e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{10} x_i^2}. \end{aligned}$$

$$\begin{aligned} \ell(\sigma^2) = ln[L(\sigma^2)] &= ln\Big[(\prod_{i=1}^{10} \frac{x_i}{\sigma^2}) e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{10} x_i^2}\Big] \\ &= ln(\prod_{i=1}^{10} \frac{x_i}{\sigma^2}) + ln(e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{10} x_i^2}) \\ &= \sum_{i=1}^{10} ln(\frac{x_i}{\sigma^2}) - \frac{1}{2\sigma^2}\sum_{i=1}^{10} x_i^2 \\ &= \sum_{i=1}^{10}(ln(\frac{1}{\sigma^2}) + ln(x_i)) - \frac{1}{2\sigma^2}\sum_{i=1}^{10} x_i^2 \\ &= 10ln(\frac{1}{\sigma^2}) + \sum_{i=1}^{10} ln(x_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{10} x_i^2 \end{aligned}$$

Taking the derivative and setting it equal to zero, we solve for $\hat{\sigma^2}$:

$$
\begin{aligned}
\frac{d}{d\sigma}[\ell(\sigma^2)] &= \frac{-20}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{10} x_i^2 \\
0 &\overset{set}{=} \frac{-20}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{10} x_i^2 \\
\implies \frac{20}{\sigma} &= \frac{1}{\sigma^3}\sum_{i=1}^{10} x_i^2 \\
\implies \hat{\sigma^2} &= \frac{\sum_{i=1}^{10} x_i^2}{20} \\
\implies \hat{\sigma^2} &\approx 74.50549.
\end{aligned}
$$

Evaluating the second derivative at $\hat{\sigma^2}$ we see

$$
\begin{aligned}
\frac{d^2}{d\sigma^2}[\ell(\sigma^2)]_{\sigma^2=74.50549} &= \frac{d}{d\sigma}\Big[\frac{-20}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{10} x_i^2\Big]_{\sigma^2=74.50549} \\
&= \frac{20}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{10} x_i^2\Big|_{\sigma^2=74.50549} \\
&\approx 0.2684366 - 0.8053076 \\
&< 0 \\
&\therefore \sigma^2 \text{ is a local maximum.}
\end{aligned}
$$

We now confirm these results numerically.

```
x <- c(16.88, 10.23, 4.59, 6.66, 13.68, 14.23, 19.87, 9.40, 6.51, 10.95)

ll <- function(sigma) {
  -(10 * log(1 / sigma^2) + sum(log(x)) - (1 / (2*sigma^2)) * sum(x^2))
}

stats4::mle(ll, 10)
```

```
Call:
stats4::mle(minuslogl = ll, start = 10)
```

```
Coefficients:
   sigma
8.631656
```

Squaring the coefficient (approximately) matches our analytical calculation for $\hat{\sigma^2}$: $8.631656^2 = 74.5054853$.