Chapter 1

Elliptic Functions

1.1 Theory

Introducing Elliptic Functions

[TBD]

The WeierstrassP function

The Weierstrass $\wp$ function is defined as follows:

$$\wp(z;\omega) = \frac{1}{z^2} + \sum_{\omega}{}'(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}),$$

is an even function and has a double pole at each $\omega \in \Omega$.

The Laurent expansion of the WeierstrassP function near the origin

We derive the Laurent expansion as follows:

$$\begin{aligned}
\wp(z;\omega) &= \frac{1}{z^2} + \sum_{\omega}{}'(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}) \\
&= \frac{1}{z^2} + \sum_{\omega}{}'(\frac{1}{\omega^2}\frac{1}{(1-\frac{z}{\omega})^2} - \frac{1}{\omega^2}) \\
&= \frac{1}{z^2} + \sum_{\omega}{}'(\frac{1}{\omega^2}(1 + \sum_{n=1}^{\infty}(n+1)(\frac{z}{\omega})^n) - \frac{1}{\omega^2}) \\
&= \frac{1}{z^2} + \sum_{\omega}{}'(\frac{1}{\omega^2}\sum_{n=1}^{\infty}(n+1)(\frac{z}{\omega})^n) \\
&= \frac{1}{z^2} + \sum_{\omega}{}'(\sum_{n=1}^{\infty}\frac{n+1}{\omega^{n+2}}z^n) \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)\sum_{\omega}{}'\frac{1}{\omega^{n+2}}z^n \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)G_{n+2}(\omega)z^n.
\end{aligned}$$

Because $\wp$ is an even function we can further simplify this as follows:

$$\begin{aligned}
\wp(z;\omega) &= \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2n+2}(\omega)z^{2n} \\
&= \frac{1}{z^2} + 3G_4(\omega)z^2 + 5G_6(\omega)z^4 + 7G_8(\omega)z^6 + 9G_{10}(\omega)z^8 + \cdots
\end{aligned}$$

Deriving the differential equation satisfied by the WeierstrassP function

Set $y = \wp(z;\omega)$ and consider the function:

$$f(z) = y'^2 - 4y^3 - 60G_4 y - 140G_6.$$

This function is elliptic and using the Laurent expansion of $\wp(z;\omega)$ we can show that $f(z)$ is holomorphic and vanishes at z = 0. Since it is elliptic it also vanishes on Ω and hence it has no poles because its only possible poles are those of y and y'. Therefore by Liouville's Theorem $f(z) = 0$ and can we conclude that $\wp(z;\omega)$ satisfies the following differential equation:

$$\wp'(z;\omega)^2 = 4\wp(z;\omega)^3 - 60G_4(\omega)\wp(z;\omega) - 140G_6(\omega).$$

The Eisenstein series and the invariants g2 and g3

[TBD]

The numbers e1, e2 and e3

[TBD]

Discrimant

[TBD]

Klein's modular J function

[TBD]

Invariance of J under unimodular transformations

[TBD]

Fourier expansion of g2, g3

[TBD]

Fourier expansion of D, J

[TBD]

1.2 Exercises

Question 1.

Given two pairs of complex numbers (ω_1, ω_2) and $(\omega_1{}', \omega_2{}')$ with non-real ratios ω_2/ω_1 and $\omega_2{}'/\omega_1{}'$. Prove that they generate the same set of periods iff there is a 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and determinant ± 1 such that

$$\begin{pmatrix} \omega_2{}' \\ \omega_1{}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}.$$

Answer

With 'the set of periods' is meant here the lattice generated by two complex numbers. We use vector notation for the complex numbers (ω_1, ω_2) and $(\omega_1{}', \omega_2{}')$. The condition that ω_2/ω_1 and $\omega_2{}'/\omega_1{}'$ have non-real ratios means, in vector terminology, that ω_1, ω_2, resp. $\omega_1{}', \omega_2{}'$ are both non-zero and independent. A 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and determinant ± 1 is called a unimodular matrix. The inverse of a unimodular matrix is also unimodular by Cramer's Rule for the inverse. The question we address here is how to determine if two given bases $\omega = (\omega_1, \omega_2)$ and $\omega' = (\omega_1{}', \omega_2{}')$ are equivalent, i.e. generate the same lattice $L(\omega) = L(\omega')$. We answer this by proving the following proposition.

Proposition. *The lattices $L(\omega)$ and $L(\omega')$ are equivalent if and only if $\omega' = \omega \cdot U$.*

Proof. Assume $L(\omega) = L(\omega')$, then integer matrices exist, such that: $\omega' = \omega \cdot U$, and similarly $\omega = \omega' \cdot V$. Hence

$$\begin{aligned}
\omega' &= \omega' \cdot V \cdot U \\
{\omega'}^T \cdot \omega' &= (\omega' \cdot V \cdot U)^T \cdot (\omega' \cdot V \cdot U) && \text{transpose both sides} \\
{\omega'}^T \cdot \omega' &= (V \cdot U)^T \cdot ({\omega'}^T \cdot \omega') \cdot (V \cdot U) \\
\det({\omega'}^T \cdot \omega') &= \det((V \cdot U)^T \cdot ({\omega'}^T \cdot \omega') \cdot (V \cdot U)) && \text{taking determinants} \\
\det({\omega'}^T \cdot \omega') &= \det((V \cdot U))^2 \cdot \det({\omega'}^T \cdot \omega') \\
\det(V)\det(U) &= \pm 1
\end{aligned}$$

Since both U, V are integer matrices, we conclude that $\det(U) = \pm 1$ and that U is unimodular.

For the other direction, assume that $\omega' = \omega \cdot U$ for some unimodular matrix U. Therefore each column of ω' is contained in $L(\omega)$ and we get $L(\omega') \subseteq L(\omega)$. In addition, $L(\omega) = \omega' \cdot U^{-1}$, and since U^{-1} is unimodular we similarly get that $L(\omega) \subseteq L(\omega')$. We conclude that $L(\omega) = L(\omega')$. $\square$

Question 2.

Let $S(0)$ denote the sum of the zeros of an elliptic function f in a period parallelogram. Prove that $S(0) - S(\infty)$ is a period of f. (Hint: Integrate $\frac{z \cdot f'(z)}{f(z)}$).

Answer

So we need to prove that inside a fundamental parallelogram spanned by the complex numbers (ω_1, ω_2) the sum of the coordinates of the zeroes equals the sum of the coordinates of the poles modulo a period of f. We have to prove the following proposition.

Proposition.

$$S(0) - S(\infty) = m \cdot \omega_1 + n \cdot \omega_2 \ (\text{ for some } m, n \in \mathbb{Z} \ )$$

Proof.

$$\begin{aligned}
S(0) - S(\infty) &= \frac{1}{2\pi i} \oint_C z \frac{f'(z)}{f(z)} dz \\
&= \frac{1}{2\pi i} \int_0^{\omega_1} z \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_{\omega_1}^{\omega_1+\omega_2} z \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_{\omega_1+\omega_2}^{\omega_2} z \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_{\omega_2}^{0} z \frac{f'(z)}{f(z)} dz \\
&= \frac{1}{2\pi i} (\int_0^{\omega_1} z \frac{f'(z)}{f(z)} dz + \int_{\omega_1+\omega_2}^{\omega_2} z \frac{f'(z)}{f(z)} dz + \int_{\omega_1}^{\omega_1+\omega_2} z \frac{f'(z)}{f(z)} dz + \int_{\omega_2}^{0} z \frac{f'(z)}{f(z)} dz) \\
&= \frac{1}{2\pi i} (\int_0^{\omega_1} z \frac{f'(z)}{f(z)} dz - \int_{\omega_2}^{\omega_1+\omega_2} z \frac{f'(z)}{f(z)} dz + \int_{\omega_1}^{\omega_1+\omega_2} z \frac{f'(z)}{f(z)} dz - \int_0^{\omega_2} z \frac{f'(z)}{f(z)} dz) \\
&= \frac{1}{2\pi i} (\int_0^{\omega_1} z \frac{f'(z)}{f(z)} dz - \int_0^{\omega_1} (z+\omega_2) \frac{f'(z+\omega_2)}{f(z+\omega_2)} dz + \int_0^{\omega_2} (z+\omega_1) \frac{f'(z+\omega_1)}{f(z+\omega_1)} dz - \int_0^{\omega_2} z \frac{f'(z)}{f(z)} dz) \\
&= \frac{1}{2\pi i} (\int_0^{\omega_1} z \frac{f'(z)}{f(z)} dz - \int_0^{\omega_1} (z+\omega_2) \frac{f'(z)}{f(z)} dz + \int_0^{\omega_2} (z+\omega_1) \frac{f'(z)}{f(z)} dz - \int_0^{\omega_2} z \frac{f'(z)}{f(z)} dz) \\
&= \frac{1}{2\pi i} (\int_0^{\omega_1} (z - (z+\omega_2)) \frac{f'(z)}{f(z)} dz + \int_0^{\omega_2} ((z+\omega_1) - z) \frac{f'(z)}{f(z)} dz) \\
&= \frac{1}{2\pi i} (\omega_2 \int_0^{\omega_1} \frac{f'(z)}{f(z)} dz + \omega_1 \int_0^{\omega_2} \frac{f'(z)}{f(z)} dz) \\
&= \frac{1}{2\pi i} (\omega_2 \log f(z)|_0^{\omega_1} + \omega_1 \log f(z)|_0^{\omega_2}) \\
&= \frac{1}{2\pi i} (\omega_2 \log 1 + \omega_1 \log 1) \\
&= \frac{1}{2\pi i} (\omega_2 \cdot n \cdot 2\pi i + \omega_1 \cdot m \cdot 2\pi i) \\
&= m \cdot \omega_1 + n \cdot \omega_2
\end{aligned}$$

□

Question 3 a).

Prove that $\wp(u) = \wp(v)$, if and only if, $u - v$ or $u + v$ is a period of $\wp$.

Answer

So we have to prove the following proposition.

Proposition.

$$\wp(u) = \wp(v) \textit{ if and only if } u - v \textit{ or } u + v \in m \cdot \omega_1 + n \cdot \omega_2 \ (\textit{ for some } m, n \in \mathbb{Z}\ )$$

Proof. Assume $\wp(u) = \wp(v)$, then by **periodicity** of $\wp :\ u = \pm v + \omega \implies u \pm v \in \omega$. In the other direction assume $u \pm v \in \omega$, then $u = \pm v + \omega$ and by **periodicity** $\wp(u) = \wp(v)$. □

Question 3 b).

Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_m$ be complex numbers such that none of the numbers $\wp(a_i) - \wp(b_j)$ is zero. Let

$$f(z) = \prod_{i=1}^{n} [\wp(z) - \wp(a_i)] / \prod_{r=1}^{m} [\wp(z) - \wp(b_r)].$$

Prove that f is an even elliptic function with zeros at $a_1, \cdots, a_n$ and poles at $b_1, \cdots, b_m$.

Answer

We have to prove that f is elliptic, that f is even, that f has zeros at $a_1, \cdots, a_n$ and that f has poles at $b_1, \cdots, b_m$. We also need to explain the impact of any of the $\wp(a_i) - \wp(b_j)$ being zero.

Proof. The sum, difference, product and quotient of elliptic functions are also elliptic functions are also elliptic functions, since the set of all elliptic functions for a fixed lattice is a field. Hence, $f(z)$ is elliptic.
The function $f(z)$ is even because

$$f(z) = \prod_{i=1}^{n} [\wp(z) - \wp(a_i)] / \prod_{r=1}^{m} [\wp(z) - \wp(b_r)] = \prod_{i=1}^{n} [\wp(-z) - \wp(a_i)] / \prod_{r=1}^{m} [\wp(-z) - \wp(b_r)] = f(-z).$$

The function $f(z)$ has zeros at $a_1, \cdots, a_n$ since $\wp(a_i) - \wp(a_i) = 0$ and has poles at $b_1, \cdots, b_m$ since $\wp(b_j) - \wp(b_j) = 0$.
If $\wp(a_i) - \wp(b_j) = 0$ for some i, j then $f(z)$ might be undefined. □

Question 4.

Prove that every even elliptic function f is a rational function of $\wp$, where the periods of $\wp$ are a subset of the periods of f.

Answer

We answer this by proving the following two propositions.

Proposition (1)**.** *Let f be an even elliptic function, whose pole set is contained in L. Then f can be represented as a polynomial in $\wp$. The degree of this polynomial is half of the order of f.*

Proof. The Laurent series of f in 0 has only even coefficients, and is hence of the form

$$f(z) = a_{-2n} z^{-2n} + a_{-2(n-1)} z^{-2(n-1)} + \cdots \qquad n \geq 1, a_{-2n} \neq 0.$$

The Laurent series of $\wp$ is of the form

$$\wp(z) = z^{-2} + \cdots ,$$

and for the n-th power we obtain

$$\wp(z)^n = z^{-2n} + \cdots$$

Just as f and $\wp$ the function

$$g(z) = f(z) - a_{-2n}\wp(z)^n$$

is an even elliptic function whose pole set is contained in L. The order of g is strictly smaller than the order of f. The proof now is obtained by induction. □

Proposition (2)**.** *The field of all even elliptic functions for the lattice L is equal to $\mathbb{C}(\wp(z))$, as a subfield of $K(L)$, and is thus isomorphic to the field of rational functions.*

Proof. Let f be a non-constant even elliptic function. If a is a pole of f which is not contained in L, the function

$$z \mapsto (\wp(z) - \wp(a))^N \cdot f(z)$$

has in $z = a$ a removable singularity, if N is sufficiently large. Since f has only finitely many poles mod L, we find finitely many points $a_1, \cdots, a_m$ and natural numbers $N_1, \cdots, N_m$ such that

$$g(z) = f(z) \cdot \prod_{j=1}^{n} [\wp(z) - \wp(a_j)]^{N_j}$$

has no poles outside L. By proposition 1 $g(z)$ is a polynomial in $\wp(z)$. □

Question 5.

Prove that every elliptic function f can be expressed in the form

$$f(z) = R_1(\wp(z)) + \wp'(z)R_2(\wp(z)),$$

where R_1, R_2 are rational functions and $\wp$ has the same set of periods as f.

Answer

Proof. Let f be an elliptic function, E_i an even elliptic function and O_i an odd elliptic function, all for the lattice L, then

$$\begin{aligned}
f(z) &= \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2} \\
f(z) &= E_1(z) + O_1(z) \\
f(z) &= E_1(z) + \wp'(z)\frac{O_1(z)}{\wp'(z)} \\
f(z) &= E_1(z) + \wp'(z)E_2(z) \Rightarrow \\
f(z) &= R_1(\wp(z)) + \wp'(z)R_2(\wp(z))
\end{aligned}$$

□

Question 6.

Let f and g be two elliptic functions with the same set of periods. Prove that there exists a polynomial $P(x,y)$, not identically zero, such that

$$P(f(z), g(z)) = C$$

where C is a constant (depending on f and g but not on z).

Answer

We answer this by proving the following proposition.

Proposition. $1 = 1$

Proof. Assume that:

$$1 = 1$$

□

Question 7.

The discriminant of the polynomial $f(x) = 4(x - x_1)(x - x_2)(x - x_3)$ is the product $16((x_2 - x_1)(x_3 - x_2)(x_3 - x_1))^2$. Prove that the discriminant of $f(x) = 4x^3 - ax - b$ is $a^3 - 27b^2$.

Answer

Clearly, the definition of this discriminant is equal to the definition of the polynomial discriminant divided by 16, or $\Delta(p(x)) = \frac{1}{16}Disc(p(x))$. (By Mathematica) Discriminant$[4x^3 - ax - b, x] = 16(a^3 - 27b^2)$, so $\Delta(4x^3 - ax - b) = a^3 - 27b^2$.

Question 8.

The differential equation for $\wp(z)$ shows that $\wp'(z) = 0$ if $z = \omega_1/2$, $z = \omega_2/2$ or $z = (\omega_1 + \omega_2)/2$. Show that

$$\wp''(\frac{\omega_1}{2}) = 2(e_1 - e_2)(e_1 - e_3)$$

and obtain corresponding formulas for $\wp''(\frac{\omega_2}{2})$ and $\wp''(\frac{\omega_1+\omega_2}{2})$.

Answer

From 1.9(5) we know that $\wp''(z) = 6\wp(z)^2 - \frac{1}{2}g_2$ and from https://mathworld.wolfram.com/WeierstrassEllipticFunction.html (105) we know that $g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3)$. We apply this to $\frac{\omega_1}{2}$ and prove the following proposition.

Proposition. $\wp''(\frac{\omega_1}{2}) = 6e_1^2 - \frac{1}{2}(-4(e_1e_2 + e_1e_3 + e_2e_3)) = 2(e_1 - e_2)(e_1 - e_3)$.

Proof.

$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 - \frac{1}{2}(-4(e_1e_2 + e_1e_3 + e_2e_3))$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 + 2(e_1e_2 + e_1e_3 + e_2e_3)$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 + 2e_1e_2 + 2e_1e_3 + 2e_2e_3$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 + 2e_1e_3 + 2e_2(e_1 + e_3)$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 + 2e_1e_3 - 2{e_2}^2$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 - 2{e_1}^2 - 2e_1(-e_1 - e_3) - 2{e_2}^2$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 - 2{e_1}^2 - 2e_1e_2 - 2{e_2}^2$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 + 2e_1e_2 - 2{e_1}^2 - 4e_1e_2 - 2{e_2}^2$$
$$\wp''(\frac{\omega_1}{2}) = 6e_1^2 + 2e_1e_2 - 2(-e_1 - e_2)^2$$
$$\wp''(\frac{\omega_1}{2}) = 4e_1^2 + 2e_1^2 + 2e_1e_2 - 2(-e_1 - e_2)^2$$
$$\wp''(\frac{\omega_1}{2}) = 4e_1^2 - 2e_1(-e_1 - e_2) - 2{e_3}^2$$
$$\wp''(\frac{\omega_1}{2}) = 4e_1^2 - 2e_1e_3 - 2{e_3}^2$$
$$\wp''(\frac{\omega_1}{2}) = 2(2e_1 + e_3)(e_1 - e_3)$$
$$\wp''(\frac{\omega_1}{2}) = 2(e_1 - e_2)(e_1 - e_3)$$

□

Corresponding formulas for $\wp''(\frac{\omega_2}{2})$ and $\wp''(\frac{\omega_1+\omega_2}{2})$ are:

$$\wp''(\frac{\omega_1 + \omega_2}{2}) = 2(e_2 - e_1)(e_2 - e_3),$$

and

$$\wp''(\frac{\omega_2}{2}) = 2(e_3 - e_1)(e_3 - e_2).$$

Question 9.

According to Exercise 4, the function $\wp(2z)$ is a rational function of $\wp(z)$. Prove that, in fact,

$$\wp(2z) = \frac{(\wp(z)^2 + \frac{1}{4}g_2)^2 + 2g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3}.$$

Answer

We start from the 'Weierstrass Doubling Formula' $\wp(2z) = \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z)$.

Proof.

$$\wp(2z) = \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z)$$
$$\wp(2z) = \frac{1}{4}\frac{(6\wp(z) - \frac{1}{2}g_2)^2}{4\wp(z)^3 - g_2\wp(z) - g_3} - 2\wp(z)$$
$$\wp(2z) = \frac{1}{4}\frac{(6\wp(z) - \frac{1}{2}g_2)^2 - 4\cdot 2\wp(z)(4\wp(z)^3 - g_2\wp(z) - g_3)}{4\wp(z)^3 - g_2\wp(z) - g_3}$$
$$\wp(2z) = \frac{1}{4}\frac{36\wp(z)^4 - 6g_2\wp(z)^2 + \frac{1}{4}g_2^2 - 32\wp(z)^4 + 8g_2\wp(z)^2 + 8g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3}$$
$$\wp(2z) = \frac{9\wp(z)^4 - \frac{3}{2}g_2\wp(z)^2 + \frac{1}{16}g_2^2 - 8\wp(z)^4 + 2g_2\wp(z)^2 + 2g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3}$$
$$\wp(2z) = \frac{\wp(z)^4 + \frac{1}{2}g_2\wp(z)^2 + \frac{1}{16}g_2^2 + 2g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3}$$
$$\wp(2z) = \frac{(\wp(z)^2 + \frac{1}{4}g_2)^2 + 2g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3}.$$

□

Question 10.

Let ω_1 and ω_2 be complex numbers with non-real ratio. Let $f(z)$ be an entire function and assume that there are constants a and b such that $f(z+\omega_1) = af(z), f(z+\omega_2) = bf(z)$, for all z. Show that $f(z) = Ae^{Bz}$, where A and B are constants.

Answer

We answer this by proving the following proposition.

Proposition. $1 = 1$

Proof. Assume that:

$$1 = 1$$

□

Question 11.

If $k \geq 2$ and $\tau \in H$ prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{m,n}{}'(m+n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n\tau}.$$

Answer

We answer this by starting from the known identity $\frac{1}{z} + \sum_m{}'(\frac{1}{z+m} - \frac{1}{m}) = \pi\cot(\pi z)$ and work our way from there.

Proof.

$$\frac{1}{z} + \sum_m{}'(\frac{1}{z+m} - \frac{1}{m}) = \pi\cot(\pi z) \tag{1}$$

$$\frac{1}{z} + \sum_m{}'(\frac{1}{z+m} - \frac{1}{m}) = \pi i - 2\pi i\sum_{n=1}^{\infty} e^{2\pi izn} \tag{2}$$

$$(-1)^j j!\sum_{m=-\infty}^{\infty}\frac{1}{(z+m)^{j+1}} = -(2\pi i)^{j+1}\sum_{n=1}^{\infty} n^j e^{2\pi izn} \tag{3}$$

$$(-1)^j j!\sum_{m=-\infty}^{\infty}\frac{1}{(m+N\tau)^{j+1}} = -(2\pi i)^{j+1}\sum_{n=1}^{\infty} n^j e^{(2\pi iN\tau)n} \tag{4}$$

$$(-1)^j j!\sum_{N=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+N\tau)^{j+1}} = -(2\pi i)^{j+1}\sum_{n=1}^{\infty} n^j \frac{e^{(2\pi i\tau)n}}{1-e^{(2\pi i\tau)n}} \tag{5}$$

$$\frac{1}{2}\sum_{m,N}{}'\frac{1}{(m+N\tau)^{2k}} - \sum_{m=1}^{\infty}\frac{1}{m^{2k}} = \frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} n^{2k-1}\frac{e^{(2\pi i\tau)n}}{1-e^{(2\pi i\tau)n}} \tag{6}$$

$$\sum_{m,N}{}'\frac{1}{(m+N\tau)^{2k}} = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} n^{2k-1}\frac{e^{(2\pi i\tau)n}}{1-e^{(2\pi i\tau)n}} \tag{7}$$

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi in\tau} \tag{8}$$

□

Remarks

(1): Starting point.
(2): Rework $\pi\cot(\pi z)$.
(3): Differentiate both sides over z, j times.
(4): Replace z by $N\tau$.
(5): Sum both sides over N from 1 to ∞.
(6): Now, assume j is odd, so $j = 2k-1$.
(7): Multiply both sides by 2 and replace $\sum_{m=1}^{\infty}\frac{1}{m^{2k}}$ by $\zeta(2k)$.
(8): Apply that $\sum_{n=1}^{\infty} n^{2k-1}\frac{e^{(2\pi i\tau)n}}{1-e^{(2\pi i\tau)n}} = \sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi in\tau}$.

Question 12.

Refer to exercise 11. If $\tau \in H$ prove that

$$G_{2k}(-\frac{1}{\tau}) = \tau^{2k}G_{2k}(\tau).$$

Answer

Proof.

$$
\begin{aligned}
G_{2k}(-\frac{1}{\tau}) &= \sum_{m,n}{}' \frac{1}{(m+n(\frac{-1}{\tau}))^{2k}} \\
G_{2k}(-\frac{1}{\tau}) &= \sum_{m,n}{}' \frac{1}{(m-\frac{n}{\tau})^{2k}} \\
G_{2k}(-\frac{1}{\tau}) &= \sum_{m,n}{}' \tau^{2k} \frac{1}{(m\tau-n)^{2k}} \\
G_{2k}(-\frac{1}{\tau}) &= \tau^{2k} \sum_{m,n}{}' \frac{1}{(m\tau-n)^{2k}} \\
G_{2k}(-\frac{1}{\tau}) &= \tau^{2k} \sum_{m,n}{}' \frac{1}{(-n+m\tau)^{2k}} \\
G_{2k}(-\frac{1}{\tau}) &= \tau^{2k} \sum_{m,n}{}' \frac{1}{(n+m\tau)^{2k}} \\
G_{2k}(-\frac{1}{\tau}) &= \tau^{2k} \sum_{m,n}{}' \frac{1}{(m+n\tau)^{2k}} \\
G_{2k}(-\frac{1}{\tau}) &= \tau^{2k} \sum_{m,n}{}' G_{2k}(\tau)
\end{aligned}
$$

□

Question 12 a).

Refer to exercise 11. If $\tau \in H$ deduce that

$$G_{2k}(\frac{i}{2}) = (-4)^k G_{2k}(2i).$$

Answer

We answer this by using $G_{2k}(-\frac{1}{\tau}) = \tau^{2k} \sum_{m,n}{}' G_{2k}(\tau)$ and set $\tau = 2i$.

Proof.

$$
\begin{aligned}
G_{2k}(\frac{i}{2}) &= (2i)^{2k} G_{2k}(2i) \\
&= (-4)^k G_{2k}(2i)
\end{aligned}
$$

□

Question 12 b).

Refer to exercise 11. If $\tau \in H$ deduce that

$$G_{2k}(i) = 0, \text{if k is odd.}$$

Answer

We answer this by using $G_{2k}(-\frac{1}{\tau}) = \tau^{2k}\sum_{m,n}{}' G_{2k}(\tau)$, set $\tau = i$, k odd. Note that $\frac{-1}{i} = i$.

Proof.

$$
\begin{aligned}
G_{2k}(i) &= (i)^{2k} G_{2k}(i) \\
G_{2k}(i) &= (-1)^{k} G_{2k}(i) \\
G_{2k}(i) &= 0
\end{aligned}
$$

□

Question 12 c).

Refer to exercise 11. If $\tau \in H$ deduce that

$$G_{2k}(e^{2\pi i/3}) = 0, \text{if } k \not\equiv 0 \pmod 3.$$

Answer

We answer this by using $G_{2k}(-\frac{1}{\tau}) = \tau^{2k}\sum_{m,n}{}' G_{2k}(\tau)$ and set $\tau = e^{\frac{1}{3}\pi i}$. Note that $\frac{-1}{e^{\frac{1}{3}\pi i}} = e^{\frac{2}{3}\pi i}$.

Proof.

$$
\begin{aligned}
G_{2k}(e^{\frac{2}{3}\pi i}) &= (e^{\frac{1}{3}\pi i})^{2k} G_{2k}(e^{\frac{1}{3}\pi i}) \\
G_{2k}(e^{\frac{2}{3}\pi i}+1) &= (e^{\frac{1}{3}\pi i})^{2k} G_{2k}(e^{\frac{1}{3}\pi i}) \\
G_{2k}(e^{\frac{1}{3}\pi i}) &= (e^{\frac{1}{3}\pi i})^{2k} G_{2k}(e^{\frac{1}{3}\pi i}) \\
G_{2k}(e^{\frac{1}{3}\pi i}) &= (e^{\frac{2}{3}\pi i})^{k} G_{2k}(e^{\frac{1}{3}\pi i}) \\
G_{2k}(e^{\frac{1}{3}\pi i}) &= \begin{cases} (e^{\frac{2}{3}\pi i}) G_{2k}(e^{\frac{1}{3}\pi i}) & \text{if } k \equiv 1 \mod 3, \\ (e^{\frac{4}{3}\pi i}) G_{2k}(e^{\frac{1}{3}\pi i}) & \text{if } k \equiv 2 \mod 3, \\ G_{2k}(e^{\frac{1}{3}\pi i}) & \text{if } k \equiv 0 \mod 3 \end{cases}
\end{aligned}
$$

The cases $k = 1, 2$ can only occur if $G_{2k}(e^{\frac{1}{3}\pi i}) = 0$. □

Question 13.

Ramanujan's tau function $\tau(n)$ is defined by the Fourier expansion

$$\Delta(\tau) = (2\pi)^{12}\sum_{n=1}^{\infty} \tau(n) e^{2\pi i n\tau},$$

derived in Theorem 2.19. Prove that

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n),$$

where $f \circ g$ denotes the Cauchy product of two sequences,

$$(f \circ g)(n) = \sum_{k=0}^{n} f(k)g(n-k),$$

and

$$\sigma_\alpha(n) = \sum_{d/n} d^\alpha \text{ for } n \geq 1, \text{ with } \sigma_3(0) = \frac{1}{240}, \sigma_5(0) = -\frac{1}{504}.$$

(Hint: Theorem 1.18.)

Answer

We answer this by using the following identities:

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

$$g_2(\tau) = \frac{4}{3}\pi^4 \cdot 240 \cdot \sum_{n=0}^{\infty} \sigma_3(n)e^{2\pi i n\tau}$$

$$g_3(\tau) = -\frac{8}{27}\pi^6 \cdot 504 \cdot \sum_{n=0}^{\infty} \sigma_5(n)e^{2\pi i n\tau}$$

Proof.

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

$$\Delta(\tau) = \left(\frac{4}{3}\pi^4 \cdot 240 \cdot \sum_{n=0}^{\infty} \sigma_3(n)e^{2\pi i n\tau}\right)^3 - 27 \cdot \left(-\frac{8}{27}\pi^6 \cdot 504 \cdot \sum_{n=0}^{\infty} \sigma_5(n)e^{2\pi i n\tau}\right)^2$$

$$\Delta(\tau) = 32768000\pi^{12} \sum_{n=0}^{\infty} \sigma_3 \circ \sigma_3 \circ \sigma_3(n)e^{2\pi i n\tau} - 602112\pi^{12} \sum_{n=0}^{\infty} \sigma_5 \circ \sigma_5(n)e^{2\pi i n\tau}$$

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=0}^{\infty} (8000\sigma_3 \circ \sigma_3 \circ \sigma_3(n) - 147\sigma_5 \circ \sigma_5(n))e^{2\pi i n\tau}$$

□

Question 14.

A series of the form $\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}$ is called a Lambert series. Assuming absolute convergence, prove that:

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n)x^n,$$

where $F(n) = \sum_{d/n} f(d)$.

Answer

We answer this by using the following identity:

$$\frac{x^n}{1-x^n} = \frac{1}{1-x^n} - 1 = x^n + x^{2n} + x^{3n} + \cdots$$

Proof.

$$\begin{aligned}\textstyle\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} = f(1)x &+ f(1)x^2 + f(1)x^3 + f(1)x^4 + f(1)x^5 + f(1)x^6 + f(1)x^7 + f(1)x^8 + \cdots \\ &+ f(2)x^2 \qquad\qquad + f(2)x^4 \qquad\qquad + f(2)x^6 \qquad\qquad + f(2)x^8 + \cdots \\ &\qquad\qquad + f(3)x^3 \qquad\qquad\qquad\qquad + f(3)x^6 \qquad\qquad\qquad\qquad + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + f(4)x^8 + \cdots\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} &= f(1)x + (f(1)+f(2))x^2 + (f(1)+f(3))x^3 + (f(1)+f(2)+f(4))x^4 + \cdots \\ &= \sum_{n=1}^{\infty}\sum_{d/n} f(d)x^n \\ &= \sum_{n=1}^{\infty} F(n)x^n\end{aligned}$$

□

Question 14 a).

Apply the result of exercise 14 to obtain the following formula valid for $|x| < 1$.

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} = x.$$

Answer

We answer this by using the following identity:

$$\sum_{d/n} \mu(d) = I(n) = \begin{cases} 1 \text{ if } n = 1, \\ 0 \text{ if } n > 1. \end{cases}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} \mu(n)\frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty}\sum_{d/n} \mu(d)x^n \\ &= \sum_{n=1}^{\infty} I(n)x^n \\ &= x\end{aligned}$$

□

Question 14 b).

Apply the result of exercise 14 to obtain the following formula valid for $|x| < 1$.

$$\sum_{n=1}^{\infty} \frac{\varphi(n)x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

Answer

We answer this by using the following identities:

$$\sum_{d/n} \varphi(d) = n$$

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} \varphi(n)\frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty}\sum_{d/n} \varphi(d)x^n \\ &= \sum_{n=1}^{\infty} nx^n \\ &= \frac{x}{(1-x)^2}\end{aligned}$$

□

Question 14 c).

Apply the result of exercise 14 to obtain the following formula valid for $|x| < 1$.

$$\sum_{n=1}^{\infty} \frac{n^\alpha x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_\alpha(n)x^n.$$

Answer

We answer this by using the following identities:

$$\sum_{d/n} d^\alpha = \sigma_\alpha(n)$$

Proof.

$$\sum_{n=1}^{\infty} n^\alpha \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty}\sum_{d/n} \sigma_\alpha(d)x^n$$

$$= \sum_{n=1}^{\infty} \sigma_\alpha(n) x^n$$

□

Question 14 d).

Apply the result of exercise 14 to obtain the following formula valid for $|x| < 1$.

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

Answer

We answer this by using the following identity:

$$\sum_{d/n} \lambda(d) = \begin{cases} 1 \text{ if n is a square} \\ 0 \text{ otherwise.} \end{cases}$$

Proof.

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sum_{d/n} \lambda(d) x^n$$
$$= \sum_{n=1}^{\infty} x^{n^2}$$

□

Question 14 e).

Apply the result of exercise 14 c) to express $g_2(\tau)$ and $g_3(\tau)$ in terms of Lambert series in $x = e^{2\pi i \tau}$.

Answer

We answer this by using the following identities:

$$g_2(\tau) = \frac{4}{3}\pi^4 \cdot (1 + 240 \cdot \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n \tau})$$

$$g_3(\tau) = \frac{8}{27}\pi^6 (1 - \cdot 504 \cdot \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n \tau})$$

Proof. Let $x = e^{2\pi i \tau}$ and using 14 c) we get the following expressions for $g2(\tau)$, and $g3(\tau)$:

$$g2(\tau) = \frac{4}{3}\pi^4 \cdot (1 + 240 \cdot \sum_{n=1}^{\infty} n^3 \frac{x^n}{1 - x^n})$$

$$g3(\tau) = \frac{8}{27}\pi^6 \cdot (1 - 504 \cdot \sum_{n=1}^{\infty} n^5 \frac{x^n}{1-x^n})$$

$\square$

Question 15 a).

Let

$$G(x) = \sum_{n=1}^{\infty} \frac{n^5 x^n}{1-x^n},$$

and

$$F(x) = \sum_{\substack{n=1 \\ \text{n odd}}}^{\infty} \frac{n^5 x^n}{1-x^n}.$$

Show that

$$F(x) = G(x) - 34G(x^2) + 64G(x^4).$$

Answer

We will use the following identities:

$$G(x) = \sigma_5(n)x^n$$
$$G(x^2) = \sigma_5(n)x^{2n}$$
$$G(x^4) = \sigma_5(n)x^{4n}$$
$$G(x) - 32G(x^2) + 64G(x^4) = (\sigma_5(n) - 32\sigma_5(\frac{n}{2}) + 64\sigma_5(\frac{n}{4}))x^n$$
$$F(x) = \sigma'_5(n)(-1)^{n+1}x^n$$

where $\sigma'_\alpha(n) = \sum_{\substack{d/n \\ d \text{ odd}}} d^\alpha$, and answer the question by proving the following proposition.

Proposition. $\sigma'_5(n)(-1)^{n+1} = (\sigma_5(n) - 34\sigma_5(\frac{n}{2}) + 64\sigma_5(\frac{n}{4}))$

Proof. We distinguish 3 cases: $n \mod 4 = 1, 3$, $n \mod 4 = 2$, and $n \mod 4 = 0$.

If $n \mod 4 = 1, 3$ then n is odd, and thus $(-1)^{n+1} = 1$ and $\sigma'_5(n) = \sigma_5(n)$ since all the divisors of n are odd, and clearly $\sigma_5(\frac{n}{2})$ and $\sigma_5(\frac{n}{4})$ are zero, since not defined.

If $n \mod 4 = 2$, then n is divible by 2, but not divisible by 4, $(-1)^{n+1} = -1$, so in this case we need the sum of the odd divisors multiplied by -1. Since there is only one factor of 2 in n $(\sigma_5(n) - 2^5\sigma_5(\frac{n}{2})$ is equal to the sum of the odd divisors. In fact, $\sigma_5(\frac{n}{2}$ is also equal to the sum of the odd divisors, in order to switch the sign we can substract it an additional two times, making the expression $(\sigma_5(n) - 34\sigma_5(\frac{n}{2}))$ and clearly $\sigma_5(\frac{n}{4})$ is zero, since not defined.

If $n \mod 4 = 0$, then n is divible by 4 and $(-1)^{n+1} = -1$. Now, $(\sigma_5(n) - 34\sigma_5(\frac{n}{2}))$ is not what we want, because additional even divisors have been subtracted. This can be corrected by adding $2 \cdot 2^5\sigma_5(\frac{n}{4})$, making the expression $(\sigma_5(n) - 34\sigma_5(\frac{n}{2}) + 64\sigma_5(\frac{n}{4}))$. $\square$

Question 15 b).

Show that

$$F(x) = \sum_{\substack{n=1 \\ \text{n odd}}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{504}.$$

Answer

We answer this by proving the following proposition.

Proposition. $1 = 1$

Proof. Assume that:

$$1 = 1$$

□

1.3 Title

Subtitle

Plain text.
And another sentence.

Another subtitle

Een subsection tekstje.