homologous to either zero or $\theta h_n$. In either case, when we apply $\theta$, we find that $\theta h_n - g_{\#}\theta c_n$ is homologous to zero. That is, $\theta h_n$ and $g_{\#}\theta c_n$ belong to the same homology class. Note that $\theta c_n$ is a cycle, because $\partial\theta c_n = \theta\partial c_n = \theta\theta c_{n-1} = 0$. Therefore, if $\beta$ is the homology class of $\theta c_n$, then $g_*(\beta)$ is the nonzero element of $\tilde{H}_n(S^n; \mathbb{Z}/2)$. It follows that $\beta$ is nonzero. Finally, the fact that $\theta\theta c_n = 0$ means that $v_{\#}\theta c_n = \theta c_n$, so $v_*(\beta) = \beta$.

**References**

1. M. K. Agoston, Algebraic Topology, Marcel Dekker, New York, 1976.
2. D. G. Bourgin, Modern Algebraic Topology, Macmillan, New York, 1963.

## ON THE NUMBER OF MULTIPLICATIVE PARTITIONS

JOHN F. HUGHES
*Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010*

J. O. SHALLIT
*Department of Mathematics, University of California, Berkeley, CA 94720*

**I. A Number-Theoretic Function.** In this note we show that if $f(n)$ is the number of essentially different factorizations of $n$, then

$$f(n) \leqslant 2n^{\sqrt{2}}.$$

In considering numbers that have exactly $k$ divisors, one is led to examine this function $f(n)$, the number of ways to write $n$ as the product of integers $\geqslant 2$, where we consider factorizations that differ only in the order of the factors to be the same. We call these representations of $n$ **multiplicative partitions**. For example, $f(12) = 4$, since

$$12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$$

are the four multiplicative partitions of 12. From these four representations, we can conclude that a number has exactly 12 divisors if and only if its prime factorization is one of the following:

$$p^{11}, p^5q, p^3q^2, p^2qr.$$

This follows from the expression for $\tau(n)$, the number of divisors of $n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$.

$$\tau(n) = \prod_{j=1}^{k} (1 + a_j).$$

For example, see [1].

The behavior of $f(n)$ is quite erratic, and apparently has not been previously studied in this form. We observe that if $q$ is prime, then $f(q^k) = p(k)$, the number of additive partitions of $k$. Also, if $q_1, q_2, \ldots, q_k$ are distinct primes, then $f(q_1q_2 \cdots q_k) = B(k)$, the $k$th Bell number. See [2].

More generally, $f(q_1^{a_1} \cdots q_k^{a_k})$ is the number of additive partitions of the "multi-partite number" $(a_1, a_2, \ldots, a_k)$, where addition is defined component-wise. See [3] for further details.

We will show that

$$f(n) \leqslant 2n^{\sqrt{2}}. \tag{1}$$

For a table of $f(n)$ for $1 \leqslant n \leqslant 100$, see the Appendix.

**II. Proof of the Main Result.** To prove (1) we first define an auxiliary function:

$g(m, n) =$ the number of multiplicative partitions of $n$ with all elements $\leqslant m$.

Clearly $f(n) = g(n, n)$. We have the following

THEOREM 1.

$$g(m, n) = \sum_{\substack{d \mid n \\ d \leqslant m}} g(d, n/d). \tag{2}$$

*Proof*. We define $g(m, 1) = 1$ and $g(1, n) = 0$ for $n \neq 1$. Let $n = a_1 a_2 \cdots a_k$ be a multiplicative partition of $n$ with all factors $\leqslant m$. Then we may assume the factors are arranged in decreasing order, so $a_1$ is the largest factor in the product. The number of ways to choose $a_2 \cdots a_k$ is therefore $g(a_1, n/a_1)$. But $a_1$ was unspecified, and therefore could be any divisor $d$ of $n$ such that $d \leqslant m$. Summing over all such $d$ gives the result. □

From Theorem 1 we can obtain a simple estimate for $g(m, n)$.

THEOREM 2.

$$g(m, n) \leqslant mn.$$

*Proof*. The theorem is clearly true for $m = 1$ or $n = 1$. We will show it is true by induction on the product $mn$. Assume true for all $m, n$ such that $mn < MN$, where $M \geqslant 2$. Then from Theorem 1 we have

$$g(M, N) = \sum_{\substack{d \mid N \\ d \leqslant M}} g(d, N/d).$$

Since $d \cdot N/d = N < MN$, we may apply the induction hypothesis to the terms inside the summation. We find

$$\begin{aligned} g(M, N) &\leqslant \sum_{\substack{d \mid N \\ d \leqslant M}} d \cdot N/d \\ &\leqslant \sum_{d \leqslant M} N \\ &= MN, \end{aligned}$$

and the theorem is true by induction. □

Theorem 2 gives the estimate $f(n) = g(n, n) \leqslant n^2$. It is possible to improve this estimate, which we do in a moment. First we need three easy lemmas.

LEMMA 3.

$$g(a, b) \leqslant g(b, b).$$

*Proof*. This follows immediately, since if $a \geqslant b$, we have strict equality, while if $a < b$, we have summing over fewer terms of equation (2). □

LEMMA 4. *Let* $0 < c < 1$. *Then*

$$f(n) \leqslant g(n^c, n) + \sum_{d=1}^{n^{1-c}} f(d).$$

*Proof*.

$$\begin{aligned} f(n) = g(n, n) &= \sum_{d \mid n} g(d, n/d) \\ &= \sum_{\substack{d \mid n \\ d \leqslant n^c}} g(d, n/d) + \sum_{\substack{d \mid n \\ d > n^c}} g(d, n/d) \end{aligned}$$

$$\leqslant g(n^c, n) + \sum_{\substack{d|n \\ d>n^c}} g(n/d, n/d) \text{ (by Theorem 1 and Lemma 3)}$$

$$= g(n^c, n) + \sum_{\substack{d|n \\ d<n^{1-c}}} g(d, d)$$

$$\leqslant g(n^c, n) + \sum_{d=1}^{n^{1-c}} f(d),$$

which is the desired result. □

LEMMA 5. *Let* $a \geqslant 0$. *Then*

$$\sum_{d=1}^{k} d^a \leqslant \frac{k^{a+1}}{a+1} + k^a.$$

*Proof*. This is easily proved by comparison with the integral $\int_1^k t^a\, dt$.

We are now in a position to prove our main result.

THEOREM 6.

$$f(n) \leqslant 2n^{\sqrt{2}}.$$

*Proof*. The table in the Appendix shows the theorem is true for $n \leqslant 69$. We will prove the theorem by induction on $n$. Assume $f(d) \leqslant kd^{c+1}$ for $d < n$, where $n \geqslant 70$ and $c$ and $k$ are constants to be specified later. Then from Lemma 4 we have

$$f(n) \leqslant g(n^c, n) + \sum_{d=1}^{n^{1-c}} f(d)$$

$$\leqslant n^{c+1} + \sum_{d=1}^{n^{1-c}} f(d) \text{ (by Theorem 2)}$$

$$\leqslant n^{c+1} + k\sum_{d=1}^{n^{1-c}} d^{c+1} \text{ (by induction)}$$

$$\leqslant n^{c+1} + k\left(\frac{(n^{1-c})^{c+2}}{c+2} + (n^{1-c})^{c+1}\right) \text{ (by Lemma 5)}.$$

Now put $k = 2$ and $c = \sqrt{2} - 1$ to get

$$f(n) \leqslant n^{\sqrt{2}} + \frac{2}{\sqrt{2}+1}n^{\sqrt{2}} + 2n^{2(\sqrt{2}-1)}$$

$$\leqslant 2n^{\sqrt{2}}$$

since $2/\sqrt{2} + 1 < 5/6$ and $2n^{2(\sqrt{2}-1)} \leqslant 1/6n^{\sqrt{2}}$ for $n \geqslant 70$.

Our theorem is now proved by induction. □

**III. Two Conjectures.** Numerical evidence seems to indicate that the exponent $\sqrt{2}$ in Theorem 6 is too large. We make two conjectures; the second is more doubtful.

CONJECTURE 1.

$$f(n) \leqslant n.$$

CONJECTURE 2.

$$f(n) \leqslant \frac{n}{\log n} \text{ for } n \neq 144.$$

Both these conjectures have been verified by computer for $n \leqslant 10{,}000$.

**Appendix**

| $n$ | $f(n)$ | $n$ | $f(n)$ | $n$ | $f(n)$ | $n$ | $f(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 26 | 2 | 51 | 2 | 76 | 4 |
| 2 | 1 | 27 | 3 | 52 | 4 | 77 | 2 |
| 3 | 1 | 28 | 4 | 53 | 1 | 78 | 5 |
| 4 | 2 | 29 | 1 | 54 | 7 | 79 | 1 |
| 5 | 1 | 30 | 5 | 55 | 2 | 80 | 12 |
| 6 | 2 | 31 | 1 | 56 | 7 | 81 | 5 |
| 7 | 1 | 32 | 7 | 57 | 2 | 82 | 2 |
| 8 | 3 | 33 | 2 | 58 | 2 | 83 | 1 |
| 9 | 2 | 34 | 2 | 59 | 1 | 84 | 11 |
| 10 | 2 | 35 | 2 | 60 | 11 | 85 | 2 |
| 11 | 1 | 36 | 9 | 61 | 1 | 86 | 2 |
| 12 | 4 | 37 | 1 | 62 | 2 | 87 | 2 |
| 13 | 1 | 38 | 2 | 63 | 4 | 88 | 7 |
| 14 | 2 | 39 | 2 | 64 | 11 | 89 | 1 |
| 15 | 2 | 40 | 7 | 65 | 2 | 90 | 11 |
| 16 | 5 | 41 | 1 | 66 | 5 | 91 | 2 |
| 17 | 1 | 42 | 5 | 67 | 1 | 92 | 4 |
| 18 | 4 | 43 | 1 | 68 | 4 | 93 | 2 |
| 19 | 1 | 44 | 4 | 69 | 2 | 94 | 2 |
| 20 | 4 | 45 | 4 | 70 | 5 | 95 | 2 |
| 21 | 2 | 46 | 2 | 71 | 1 | 96 | 19 |
| 22 | 2 | 47 | 1 | 72 | 16 | 97 | 1 |
| 23 | 1 | 48 | 12 | 73 | 1 | 98 | 4 |
| 24 | 7 | 49 | 2 | 74 | 2 | 99 | 4 |
| 25 | 2 | 50 | 4 | 75 | 4 | 100 | 9 |

**References**


**1.** G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, Clarendon Press, 1971, p. 239.

**2.** G. T. Williams, Numbers generated by the function $e^{e^{x}-1}$, this MONTHLY, 52 (1945) 323–327.

**3.** George Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications 2, Gian-Carlo Rota, Editor, Addison-Wesley, Reading, Mass. 1976.


## ANSWERS TO PHOTOS ON PAGE 437

No, they are partial. They are two of the most famous partial differential equators in the world. Top: Lars Hörmander of Lund; bottom: Olga Ladyženskaja of Leningrad.