The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://www.tandfonline.com/loi/uamm20

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G. T. Williams & W. B. Campbell

To cite this article: G. T. Williams & W. B. Campbell (1945) Numbers Generated By The Function e^{e^x-1}, The American Mathematical Monthly, 52:6, 323-327, DOI: 10.1080/00029890.1945.11991575

To link to this article: https://doi.org/10.1080/00029890.1945.11991575

Published online: 11 Apr 2018.

NUMBERS GENERATED BY THE FUNCTION e^{e^x-1}

G. T. WILLIAMS, Harvard University

1. Introduction. The integers which are the subject of this paper have been discussed cursorily by several writers* who contented themselves with discovering Lemma 1 and Corollary 1, and computing the first few. They occur in combinatory analysis, being, in fact, the sum of the horizontal entries in the table of p. 169 of Netto's *Lehrbuch der Combinatorik*. Their interest is primarily number-theoretic. Indeed, from Minetola's† work, it is evident that G_n (as defined below) is the number of ways in which a product of n distinct primes can be factored. Thus, $p_1p_2p_3=(p_1p_2)p_3=(p_1p_3)p_2=(p_2p_3)p_1=(p_1)(p_2)(p_3)$, and so $G_3=5$.

It will be convenient to give an algebraic definition of G_n.

2. Definition and algebraic properties. We determine the sequence of G's (we shall only be interested in the case where n is a non-negative integer) by the following

DEFINITION.

$$G_n e = \sum_{r=0}^{\infty} \frac{r^n}{r!} \qquad (n = 0, 1, \cdots).$$

It is plain from this that

$$G_0 = G_1 = 1.$$

More general summation are expressible in terms of the G's; in fact

THEOREM 1.

$$\sum_{r=0}^{\infty} \frac{(ar+b)^n}{r!} = (aG+b)^n e,$$

where the right-hand member means

$$e\sum_{r=0}^{n}\binom{n}{r}a^rG_rb^{n-r}.‡$$

For, expanding the first member, we find it equals

* Wohlsentolme gave just this as a problem, on the Tripos: to prove Lem. 1, Cor. 1, and find G_8. See, Bromwich, Infinite Series, p. 197.

† Silvio Minetola, Principii di Analisi Combinatoria, Jior. di Mat., vol. 45, 1907, pp. 333–366, vol. 47, 1909, pp. 173–200.

‡ We shall make considerable use of this standard symbolic convention: whenever we write a "product" of G's, say,

$$G^jG^kG^m \cdots,$$

we mean the sum of the exponents to be taken as a subscript

$$G_{j+k+m+\cdots}.$$

$$\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{s=0}^{n} \binom{n}{s} a^s r^s b^{n-s} = \sum_{s=0}^{n} \binom{n}{s} a^s b^{n-s} \sum_{r=0}^{\infty} \frac{r^s}{r!} \cdot$$

LEMMA 1. $$G_{n+1} = (G+1)^n \qquad (n = 0, 1, \cdots).$$

Write $a = b = 1$ in Theorem 1 and multiply numerator and denominator of the summand by $(r+1)$ to reduce it to the form of the definition. This proposition, together with the fact that the first two G's are integers, insures that all the G's are positive integers.

The lemma gives us a ready method of computing the numbers. We find

$G_0 = 1$	$G_5 = 52$	$G_{10} = 115975$
$G_1 = 1$	$G_6 = 203$	$G_{11} = 678570$
$G_2 = 2$	$G_7 = 877$	$G_{12} = 4213597$
$G_3 = 5$	$G_8 = 4140$	$G_{13} = 27644437$
$G_4 = 15$	$G_9 = 21147$	$G_{14} = 190899322$

and so on.

It is evident that the G's increase very rapidly; this is reflected in

THEOREM 2. $$G_n \geqq k^n/k! \qquad (k = 0, 1, \cdots).$$

For $k = 0$ this is trivial; if it is true for some k, we have

$$G_n = \sum \binom{n-1}{r} G_r \geqq \sum \binom{n-1}{r} \frac{k^r}{k!} = \frac{(k+1)^n}{(k+1)!} \cdot$$

Unfortunately G_{n+1} is not a very good upper bound for the function $x^n/\Gamma(x)$; *e.g.*, the greatest value attained by the function $x^3/\Gamma(x)$ is approximately 13.56.

Lemma 1 generalizes inductively to

THEOREM 3.

$$\sum_{r=0}^{k} (-)^r \binom{k}{r} G_{n-r+1} = \sum_{r=0}^{n-k} \binom{n-k}{r} G_{n-r} \qquad (0 \leqq k \leqq n).$$

This is certainly valid for $k = 0$. Now, let $F(n, k)$ denote either member of the equation which we assume true for this particular k. Then, by the well known properties of the binomial coefficients, we have

$$F(n, k) - F(n-1, k) = F(n, k+1),$$

which establishes the theorem, by induction on k.

The case $k = n$ is of some interest.

COROLLARY 1. $$G_n = G(G-1)^n.$$

We justify the title of the paper by

THEOREM 4. $$e^{e^x-1}=e^{Gx},$$

where, again, e^{Gx} is the symbolic representation of

$$\sum_{r=0}^{\infty} G_r x^r/r!.$$

For,

$$e^{e^x} = \sum_r \frac{1}{r!}\sum_s \frac{r^s x^s}{s!} = \sum_s \frac{x^s}{s!}\sum_r \frac{r^s}{r!}.$$

This theorem gives rise to a recurrence relation for the G's which is quite different from any we have yet stated.

THEOREM 5. $$G(G-1)\cdots(G-n+1)=1 \qquad (n=0, 1, \cdots).$$

We shall adopt the notation S_j^i to stand for the sum of all possible products of the numbers $1, 2, \cdots, (i-1)$, taken j at a time. Then, by a change of variable ($x=\log(1+u)$) in Theorem 4,

$$e^x = \sum \frac{G_r}{r!}\log^r(1+x) = \sum_r G_r \sum_n (-)^{n-r} S_{n-r}^n \frac{x^n}{n!}$$

$$= \sum_n \frac{x^n}{n!}\sum_r (-)^{n-r} S_{n-r}^n G_r,$$

whence,

$$\sum_{r=0}^{n} (-)^{n-r} S_{n-r}^n G_r = 1.$$

There is also an elegant symbolic proof of this.

$$e^x = e^{G\log(1+x)} = (1+x)^G = \sum_{n=0}^{\infty} G\cdots(G-n+1)\frac{x^n}{n!}$$

3. Number-theoretic properties. We are now in a position to discuss the curious arithmetic properties which these numbers possess.

LEMMA 2. $$G_p \equiv 2 \pmod{p},$$

where, as in all the following theorems, p is a prime number.

Write $n=p$ in Theorem 5. By Lagrange's proof of Fermat's Theorem, all the intermediate coefficients are congruent to zero, modulo p; the first to $+1$; and the last to -1.

This proposition is the starting point for a series of inductions, which culminates in Theorem 6.

LEMMA 3. $$G_{p+n} \equiv G_n + G_{n+1} \pmod{p}.$$

When $n=0$ this reduces to Lemma 2. Write $p+n-1$ for n, and p for k, in Theorem 3; then

$$G_{p+n} - G_n \equiv \sum \binom{n-1}{r} G_{p+n-r-1} \pmod{p}.$$

Now, assume that, for all integers $<n$ the statement is true. We then have

$$G_{p+n} - G_n \equiv \sum \binom{n-1}{r} G_{n-r-1} + \sum \binom{n-1}{r} G_{n-r} \pmod{p}$$
$$= \sum \binom{n}{r} G_{n-r} = G_{n+1}.$$

Letting $p=2$ we obtain the significant

COROLLARY 2. $\qquad G_n+G_{n+1}+G_{n+2}\equiv 0 \pmod{2}.$

LEMMA 4. $\qquad G_{kp^s+n}\equiv G^n(G+s)^k \pmod{p}.$

We shall show that its truth for fixed s, all n, and $k=1$, implies its truth for the same s, all n, and all k. For, assume it for some s and k, and all n; then

$$G_{(k+1)p^s+n} = G_{kp^s+(p^s+n)} \equiv G^{p^s+n}(G+s)^k \pmod{p}$$
$$\equiv G^n(G+s)(G+s)^k = G^n(G+s)^{k+1}.$$

Now, assume the theorem for all k, all n, and some s. Writing $k=p$,

$$G_{p^{s+1}+n} \equiv G_{p+n} + s^p G_n \equiv G_{n+1} + (s+1)G_n \pmod{p},$$

which, by our preliminary remark, implies its validity for all k, all n, and $s+1$. Since the statement is obviously correct for $s=0$, it is universally true by induction on s.

If the subscript of G is not of the form of the lemma, but is a polynomial in p, it can be reduced by considering everything after the leading term as n. Repeated application of the proposition yields

THEOREM 6. $\qquad G_{\Sigma k_r p^r}\equiv \Pi(G+r)^{k_r} \pmod{p},$

where the limits of the summation and product are the same.

THEOREM 7. *The G's have a "congruence-period" of* $(p^p-1)/(p-1)$ *places; i.e.,*

$$G_{n+(p^p-1)/(p-1)} \equiv G_n \pmod{p} \qquad (n=0, 1, \cdots).$$

For, by Theorem 6, we have

$$G_{p^{p-1}+\cdots+p+1+n} \equiv G^{n+1}(G+1)\cdots(G+p-1) \pmod{p}$$
$$\equiv G_{p+n} - G_{n+1} \equiv G_n.$$

We have shown therefore that the smallest period of the least residues of the G's is a divisor of $(p^p-1)/(p-1)$. Indeed, when $p=2$, 3, or 5, it is precisely this, although for $p=5$, either of the factors of 781, 11 and 71 might be a priori candidates. The author has succeeded in computing the least residue pattern in these three cases, but since the latter is rather excessive, we give only the first two. For 2, it is

$$1,\ 1,\ 0$$

and for 3

$$1,\ 1,\ 2,\ 2,\ 0,\ 1,\ 2,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1.$$

For primes >5 the situation is unknown.

We close with what is, in view of Theorem 7, a natural generalization of Corollary 2. We are indebted to Dr. Irving Kaplansky for the proof.

THEOREM 8. *The sum of $(p^p-1)/(p-1)$ consecutive G's is a multiple of p.*

We pass to the Galois field of p elements, so that congruence modulo p becomes equality. Solving the difference equation

$$G_{n+p} - G_{n+1} - G_n = 0$$

by standard methods, we find

$$G_n = a_1x_1^n + a_2x_2^n + \cdots + a_px_p^n$$

where $x_1, \cdots, x_p$ are the (distinct) roots of

$$x^p - x - 1 = 0.$$

Now, precisely as in the case of the G's, we find

$$x^{p^n} = x + n,$$

so that

$$x^{(p^p-1)/(p-1)} = x(x+1)\cdots(x+p-1) = x^p - x = 1,$$

whence

$$1 + x + \cdots + x^{(p^p-1)/(p-1)-1} = \frac{1-1}{x-1} = 0$$

which proves the theorem.

DESCRIPTIVE GEOMETRY AS USED IN THE SLAUGHTER HOUSE

(As reported in the Kansas City Star to have been quoted from a 24-page booklet from the Office of Price Administration)

"Then all fat shall be removed which extends above a flat plane using the following two lines as guides for each edge of the plane: an imaginary line parallel with the full length of the protruding edge of the lumbar section of the chine bone which line extends one inch directly above such protruding edge; a line on the inside of the loin two inches from the flank edge, and running parallel with such edge for the full length of the loin."

Shades of Taurus—can it be that our classes soon will be filled with embryo butchers and meat-cutters, learning the rudiments of their trade via courses in descriptive geometry?

W. B. Campbell.