Section 3-9 : Undetermined Coefficients

In this section we will take a look at the first method that can be used to find a particular solution to a nonhomogeneous differential equation.

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

One of the main advantages of this method is that it reduces the problem down to an algebra problem. The algebra can get messy on occasion, but for most of the problems it will not be terribly difficult. Another nice thing about this method is that the complementary solution will not be explicitly required, although as we will see knowledge of the complementary solution will be needed in some cases and so we'll generally find that as well.

There are two disadvantages to this method. First, it will only work for a fairly small class of *g(t)*'s. The class of *g(t)*'s for which the method works, does include some of the more common functions, however, there are many functions out there for which undetermined coefficients simply won't work. Second, it is generally only useful for constant coefficient differential equations.

The method is quite simple. All that we need to do is look at *g(t)* and make a guess as to the form of $Y_P(t)$ leaving the coefficient(s) undetermined (and hence the name of the method). Plug the guess into the differential equation and see if we can determine values of the coefficients. If we can determine values for the coefficients then we guessed correctly, if we can't find values for the coefficients then we guessed incorrectly.

It's usually easier to see this method in action rather than to try and describe it, so let's jump into some examples.

Example 1 Determine a particular solution to

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t}$$

Solution

The point here is to find a particular solution, however the first thing that we're going to do is find the complementary solution to this differential equation. Recall that the complementary solution comes from solving,

$$y'' - 4y' - 12y = 0$$

The characteristic equation for this differential equation and its roots are.

$$r^2 - 4r - 12 = (r-6)(r+2) = 0 \qquad \Rightarrow \qquad r_1 = -2, \quad r_2 = 6$$

The complementary solution is then,

$$y_c(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t}$$

At this point the reason for doing this first will not be apparent, however we want you in the habit of finding it before we start the work to find a particular solution. Eventually, as we'll see, having the complementary solution in hand will be helpful and so it's best to be in the habit of finding it first prior to doing the work for undetermined coefficients.

Now, let's proceed with finding a particular solution. As mentioned prior to the start of this example we need to make a guess as to the form of a particular solution to this differential equation. Since *g(t)* is an exponential and we know that exponentials never just appear or disappear in the differentiation process it seems that a likely form of the particular solution would be

$$Y_P\left(t\right) = A\mathbf{e}^{5t}$$

Now, all that we need to do is do a couple of derivatives, plug this into the differential equation and see if we can determine what *A* needs to be.

Plugging into the differential equation gives

$$\begin{aligned} 25A\mathbf{e}^{5t} - 4\left(5A\mathbf{e}^{5t}\right) - 12\left(A\mathbf{e}^{5t}\right) &= 3\mathbf{e}^{5t} \\ -7A\mathbf{e}^{5t} &= 3\mathbf{e}^{5t} \end{aligned}$$

So, in order for our guess to be a solution we will need to choose *A* so that the coefficients of the exponentials on either side of the equal sign are the same. In other words we need to choose *A* so that,

$$-7A = 3 \qquad \Rightarrow \qquad A = -\frac{3}{7}$$

Okay, we found a value for the coefficient. This means that we guessed correctly. A particular solution to the differential equation is then,

$$Y_P\left(t\right) = -\frac{3}{7}\mathbf{e}^{5t}$$

Before proceeding any further let's again note that we started off the solution above by finding the complementary solution. This is not technically part the method of Undetermined Coefficients however, as we'll eventually see, having this in hand before we make our guess for the particular solution can save us a lot of work and/or headache. Finding the complementary solution first is simply a good habit to have so we'll try to get you in the habit over the course of the next few examples. At this point do not worry about why it is a good habit. We'll eventually see why it is a good habit.

Now, back to the work at hand. Notice in the last example that we kept saying "a" particular solution, not "the" particular solution. This is because there are other possibilities out there for the particular solution we've just managed to find one of them. Any of them will work when it comes to writing down the general solution to the differential equation.

Speaking of which… This section is devoted to finding particular solutions and most of the examples will be finding only the particular solution. However, we should do at least one full blown IVP to make sure that we can say that we've done one.

Example 2 Solve the following IVP

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t} \qquad\qquad y(0) = \frac{18}{7} \qquad\qquad y'(0) = -\frac{1}{7}$$

Solution

We know that the general solution will be of the form,

$$y(t) = y_c(t) + Y_P(t)$$

and we already have both the complementary and particular solution from the first example so we don't really need to do any extra work for this problem.

One of the more common mistakes in these problems is to find the complementary solution and then, because we're probably in the habit of doing it, apply the initial conditions to the complementary solution to find the constants. This however, is incorrect. The complementary solution is only the solution to the homogeneous differential equation and we are after a solution to the nonhomogeneous differential equation and the initial conditions must satisfy that solution instead of the complementary solution.

So, we need the general solution to the nonhomogeneous differential equation. Taking the complementary solution and the particular solution that we found in the previous example we get the following for a general solution and its derivative.

$$y(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t} - \frac{3}{7}\mathbf{e}^{5t}$$

$$y'(t) = -2c_1\mathbf{e}^{-2t} + 6c_2\mathbf{e}^{6t} - \frac{15}{7}\mathbf{e}^{5t}$$

Now, apply the initial conditions to these.

$$\frac{18}{7} = y(0) = c_1 + c_2 - \frac{3}{7}$$

$$-\frac{1}{7} = y'(0) = -2c_1 + 6c_2 - \frac{15}{7}$$

Solving this system gives $c_1 = 2$ and $c_2 = 1$. The actual solution is then.

$$y(t) = 2\mathbf{e}^{-2t} + \mathbf{e}^{6t} - \frac{3}{7}\mathbf{e}^{5t}$$

This will be the only IVP in this section so don't forget how these are done for nonhomogeneous differential equations!

Let's take a look at another example that will give the second type of *g(t)* for which undetermined coefficients will work.

Example 3 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = \sin(2t)$$

Solution

Again, let's note that we should probably find the complementary solution before we proceed onto the guess for a particular solution. However, because the homogeneous differential equation for this example is the same as that for the first example we won't bother with that here.

Now, let's take our experience from the first example and apply that here. The first example had an exponential function in the *g(t)* and our guess was an exponential. This differential equation has a sine so let's try the following guess for the particular solution.

$$Y_P(t) = A\sin(2t)$$

Differentiating and plugging into the differential equation gives,

$$-4A\sin(2t) - 4(2A\cos(2t)) - 12(A\sin(2t)) = \sin(2t)$$

Collecting like terms yields

$$-16A\sin(2t) - 8A\cos(2t) = \sin(2t)$$

We need to pick *A* so that we get the same function on both sides of the equal sign. This means that the coefficients of the sines and cosines must be equal. Or,

$$\begin{aligned} \cos(2t): \qquad & -8A = 0 \qquad \Rightarrow \qquad A = 0 \\ \sin(2t): \qquad & -16A = 1 \qquad \Rightarrow \qquad A = -\frac{1}{16} \end{aligned}$$

Notice two things. First, since there is no cosine on the right hand side this means that the coefficient must be zero on that side. More importantly we have a serious problem here. In order for the cosine to drop out, as it must in order for the guess to satisfy the differential equation, we need to set *A* = 0, but if *A* = 0, the sine will also drop out and that can't happen. Likewise, choosing *A* to keep the sine around will also keep the cosine around.

What this means is that our initial guess was wrong. If we get multiple values of the same constant or are unable to find the value of a constant then we have guessed wrong.

One of the nicer aspects of this method is that when we guess wrong our work will often suggest a fix. In this case the problem was the cosine that cropped up. So, to counter this let's add a cosine to our guess. Our new guess is

$$Y_P(t) = A\cos(2t) + B\sin(2t)$$

Plugging this into the differential equation and collecting like terms gives,

$$\begin{aligned} -4A\cos(2t) - 4B\sin(2t) - 4\left(-2A\sin(2t) + 2B\cos(2t)\right) - \\ 12\left(A\cos(2t) + B\sin(2t)\right) = \sin(2t) \\ (-4A - 8B - 12A)\cos(2t) + (-4B + 8A - 12B)\sin(2t) = \sin(2t) \\ (-16A - 8B)\cos(2t) + (8A - 16B)\sin(2t) = \sin(2t) \end{aligned}$$

Now, set the coefficients equal

$$\begin{aligned} \cos(2t): \qquad -16A - 8B &= 0 \\ \sin(2t): \qquad 8A - 16B &= 1 \end{aligned}$$

Solving this system gives us

$$A = \frac{1}{40} \qquad B = -\frac{1}{20}$$

We found constants and this time we guessed correctly. A particular solution to the differential equation is then,

$$Y_P(t) = \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t)$$

Notice that if we had had a cosine instead of a sine in the last example then our guess would have been the same. In fact, if both a sine and a cosine had shown up we will see that the same guess will also work.

Let's take a look at the third and final type of basic *g(t)* that we can have. There are other types of *g(t)* that we can have, but as we will see they will all come back to two types that we've already done as well as the next one.

Example 4 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = 2t^3 - t + 3$$

Solution

Once, again we will generally want the complementary solution in hand first, but again we're working with the same homogeneous differential equation (you'll eventually see why we keep working with the same homogeneous problem) so we'll again just refer to the first example.

For this example, *g(t)* is a cubic polynomial. For this we will need the following guess for the particular solution.

$$Y_P(t) = At^3 + Bt^2 + Ct + D$$

Notice that even though *g(t)* doesn't have a t^2 in it our guess will still need one! So, differentiate and plug into the differential equation.

$$6At + 2B - 4\left(3At^2 + 2Bt + C\right) - 12\left(At^3 + Bt^2 + Ct + D\right) = 2t^3 - t + 3$$
$$-12At^3 + \left(-12A - 12B\right)t^2 + \left(6A - 8B - 12C\right)t + 2B - 4C - 12D = 2t^3 - t + 3$$

Now, as we've done in the previous examples we will need the coefficients of the terms on both sides of the equal sign to be the same so set coefficients equal and solve.

$$\begin{aligned} t^3: &\qquad -12A = 2 &&\Rightarrow \quad A = -\frac{1}{6} \\ t^2: &\qquad -12A - 12B = 0 &&\Rightarrow \quad B = \frac{1}{6} \\ t^1: &\qquad 6A - 8B - 12C = -1 &&\Rightarrow \quad C = -\frac{1}{9} \\ t^0: &\qquad 2B - 4C - 12D = 3 &&\Rightarrow \quad D = -\frac{5}{27} \end{aligned}$$

Notice that in this case it was very easy to solve for the constants. The first equation gave *A*. Then once we knew *A* the second equation gave *B*, *etc*. A particular solution for this differential equation is then

$$Y_P(t) = -\frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$$

Now that we've gone over the three basic kinds of functions that we can use undetermined coefficients on let's summarize.

g(t)	$Y_P(t)$ guess
$a\mathbf{e}^{\beta t}$	$A\mathbf{e}^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots A_1 t + A_0$

Notice that there are really only three kinds of functions given above. If you think about it the single cosine and single sine functions are really special cases of the case where both the sine and cosine are present. Also, we have not yet justified the guess for the case where both a sine and a cosine show up. We will justify this later.

We now need move on to some more complicated functions. The more complicated functions arise by taking products and sums of the basic kinds of functions. Let's first look at products.

Example 5 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = t\mathbf{e}^{4t}$$

Solution

You're probably getting tired of the opening comment, but again finding the complementary solution first really a good idea but again we've already done the work in the first example so we won't do it again here. We promise that eventually you'll see why we keep using the same homogeneous problem and why we say it's a good idea to have the complementary solution in hand first. At this point all we're trying to do is reinforce the habit of finding the complementary solution first.

Okay, let's start off by writing down the guesses for the individual pieces of the function. The guess for the *t* would be

$$At + B$$

while the guess for the exponential would be

$$C\mathbf{e}^{4t}$$

Now, since we've got a product of two functions it seems like taking a product of the guesses for the individual pieces might work. Doing this would give

$$C\mathbf{e}^{4t}\left(At + B\right)$$

However, we will have problems with this. As we will see, when we plug our guess into the differential equation we will only get two equations out of this. The problem is that with this guess we've got three unknown constants. With only two equations we won't be able to solve for all the constants.

This is easy to fix however. Let's notice that we could do the following

$$C\mathbf{e}^{4t}\left(At + B\right) = \mathbf{e}^{4t}\left(ACt + BC\right)$$

If we multiply the *C* through, we can see that the guess can be written in such a way that there are really only two constants. So, we will use the following for our guess.

$$Y_P\left(t\right) = \mathbf{e}^{4t}\left(At + B\right)$$

Notice that this is nothing more than the guess for the *t* with an exponential tacked on for good measure.

Now that we've got our guess, let's differentiate, plug into the differential equation and collect like terms.

$$\mathbf{e}^{4t}\left(16At+16B+8A\right)-4\left(\mathbf{e}^{4t}\left(4At+4B+A\right)\right)-12\left(\mathbf{e}^{4t}\left(At+B\right)\right)=t\mathbf{e}^{4t}$$
$$\left(16A-16A-12A\right)t\mathbf{e}^{4t}+\left(16B+8A-16B-4A-12B\right)\mathbf{e}^{4t}=t\mathbf{e}^{4t}$$
$$-12At\mathbf{e}^{4t}+\left(4A-12B\right)\mathbf{e}^{4t}=t\mathbf{e}^{4t}$$

Note that when we're collecting like terms we want the coefficient of each term to have only constants in it. Following this rule we will get two terms when we collect like terms. Now, set coefficients equal.

$$t\mathbf{e}^{4t}: \qquad -12A=1 \qquad \Rightarrow \qquad A=-\frac{1}{12}$$
$$\mathbf{e}^{4t}: \qquad 4A-12B=0 \qquad \Rightarrow \qquad B=-\frac{1}{36}$$

A particular solution for this differential equation is then

$$Y_P\left(t\right)=\mathbf{e}^{4t}\left(-\frac{t}{12}-\frac{1}{36}\right)=-\frac{1}{36}\left(3t+1\right)\mathbf{e}^{4t}$$

This last example illustrated the general rule that we will follow when products involve an exponential. When a product involves an exponential we will first strip out the exponential and write down the guess for the portion of the function without the exponential, then we will go back and tack on the exponential without any leading coefficient.

Let's take a look at some more products. In the interest of brevity we will just write down the guess for a particular solution and not go through all the details of finding the constants. Also, because we aren't going to give an actual differential equation we can't deal with finding the complementary solution first.

Example 6 Write down the form of the particular solution to

$$y''+p\left(t\right)y'+q\left(t\right)y=g\left(t\right)$$

for the following *g(t)*'s.

(a) $g\left(t\right)=16\mathbf{e}^{7t}\sin\left(10t\right)$

(b) $g\left(t\right)=\left(9t^2-103t\right)\cos t$

(c) $g\left(t\right)=-\mathbf{e}^{-2t}\left(3-5t\right)\cos\left(9t\right)$

Solution

(a) $g\left(t\right)=16\mathbf{e}^{7t}\sin\left(10t\right)$

So, we have an exponential in the function. Remember the rule. We will ignore the exponential and write down a guess for $16\sin\left(10t\right)$ then put the exponential back in.

The guess for the sine is

$$A\cos\left(10t\right)+B\sin\left(10t\right)$$

Now, for the actual guess for the particular solution we'll take the above guess and tack an exponential onto it. This gives,

$$Y_P(t) = \mathbf{e}^{7t}\left(A\cos(10t) + B\sin(10t)\right)$$

One final note before we move onto the next part. The 16 in front of the function has absolutely no bearing on our guess. Any constants multiplying the whole function are ignored.

(b) $g(t) = \left(9t^2 - 103t\right)\cos t$

We will start this one the same way that we initially started the previous example. The guess for the polynomial is

$$At^2 + Bt + C$$

and the guess for the cosine is

$$D\cos t + E\sin t$$

If we multiply the two guesses we get.

$$\left(At^2 + Bt + C\right)\left(D\cos t + E\sin t\right)$$

Let's simplify things up a little. First multiply the polynomial through as follows.

$$\left(At^2 + Bt + C\right)\left(D\cos t\right) + \left(At^2 + Bt + C\right)\left(E\sin t\right)$$

$$\left(ADt^2 + BDt + CD\right)\cos t + \left(AEt^2 + BEt + CE\right)\sin t$$

Notice that everywhere one of the unknown constants occurs it is in a product of unknown constants. This means that if we went through and used this as our guess the system of equations that we would need to solve for the unknown constants would have products of the unknowns in them. These types of systems are generally very difficult to solve.

So, to avoid this we will do the same thing that we did in the previous example. Everywhere we see a product of constants we will rename it and call it a single constant. The guess that we'll use for this function will be.

$$Y_P(t) = \left(At^2 + Bt + C\right)\cos t + \left(Dt^2 + Et + F\right)\sin t$$

This is a general rule that we will use when faced with a product of a polynomial and a trig function. We write down the guess for the polynomial and then multiply that by a cosine. We then write down the guess for the polynomial again, using different coefficients, and multiply this by a sine.

(c) $g(t) = -\mathbf{e}^{-2t}(3 - 5t)\cos(9t)$

This final part has all three parts to it. First, we will ignore the exponential and write down a guess for.

$$-(3 - 5t)\cos(9t)$$

The minus sign can also be ignored. The guess for this is

$$(At+B)\cos(9t)+(Ct+D)\sin(9t)$$

Now, tack an exponential back on and we're done.

$$Y_P(t)=\mathbf{e}^{-2t}(At+B)\cos(9t)+\mathbf{e}^{-2t}(Ct+D)\sin(9t)$$

Notice that we put the exponential on both terms.

There a couple of general rules that you need to remember for products.

1. If *g(t)* contains an exponential, ignore it and write down the guess for the remainder. Then tack the exponential back on without any leading coefficient.

2. For products of polynomials and trig functions you first write down the guess for just the polynomial and multiply that by the appropriate cosine. Then add on a new guess for the polynomial with different coefficients and multiply that by the appropriate sine.

If you can remember these two rules you can't go wrong with products. Writing down the guesses for products is usually not that difficult. The difficulty arises when you need to actually find the constants.

Now, let's take a look at sums of the basic components and/or products of the basic components. To do this we'll need the following fact.

Fact

If $Y_{P1}(t)$ is a particular solution for

$$y''+p(t)y'+q(t)y=g_1(t)$$

and if $Y_{P2}(t)$ is a particular solution for

$$y''+p(t)y'+q(t)y=g_2(t)$$

then $Y_{P1}(t)+Y_{P2}(t)$ is a particular solution for

$$y''+p(t)y'+q(t)y=g_1(t)+g_2(t)$$

This fact can be used to both find particular solutions to differential equations that have sums in them and to write down guess for functions that have sums in them.

Example 7 Find a particular solution for the following differential equation.

$$y''-4y'-12y=3\mathbf{e}^{5t}+\sin(2t)+t\mathbf{e}^{4t}$$

Solution

This example is the reason that we've been using the same homogeneous differential equation for all the previous examples. There is nothing to do with this problem. All that we need to do it go back to the appropriate examples above and get the particular solution from that example and add them all together.

Doing this gives

$$Y_P(t) = -\frac{3}{7}\mathbf{e}^{5t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t) - \frac{1}{36}(3t+1)\mathbf{e}^{4t}$$

Let's take a look at a couple of other examples. As with the products we'll just get guesses here and not worry about actually finding the coefficients.

Example 8 Write down the form of the particular solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

for the following *g(t)*'s.

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$

(b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$

(c) $g(t) = \mathbf{e}^{7t} + 6$

(d) $g(t) = 6t^2 - 7\sin(3t) + 9$

(e) $g(t) = 10\mathbf{e}^{t} - 5t\mathbf{e}^{-8t} + 2\mathbf{e}^{-8t}$

(f) $g(t) = t^2\cos t - 5t\sin t$

(g) $g(t) = 5\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\cos(6t) - \sin(6t)$

Solution

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$

This first one we've actually already told you how to do. This is in the table of the basic functions. However, we wanted to justify the guess that we put down there. Using the fact on sums of function we would be tempted to write down a guess for the cosine and a guess for the sine. This would give.

$$\underbrace{A\cos(6t) + B\sin(6t)}_{\text{guess for the cosine}} + \underbrace{C\cos(6t) + D\sin(6t)}_{\text{guess for the sine}}$$

So, we would get a cosine from each guess and a sine from each guess. The problem with this as a guess is that we are only going to get two equations to solve after plugging into the differential equation and yet we have 4 unknowns. We will never be able to solve for each of the constants.

To fix this notice that we can combine some terms as follows.

$$(A+C)\cos(6t) + (B+D)\sin(6t)$$

Upon doing this we can see that we've really got a single cosine with a coefficient and a single sine with a coefficient and so we may as well just use

$$Y_P(t) = A\cos(6t) + B\sin(6t)$$

The general rule of thumb for writing down guesses for functions that involve sums is to always combine like terms into single terms with single coefficients. This will greatly simplify the work required to find the coefficients.

(b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$

For this one we will get two sets of sines and cosines. This will arise because we have two different arguments in them. We will get one set for the sine with just a *t* as its argument and we'll get another set for the sine and cosine with the 14*t* as their arguments.

The guess for this function is

$$Y_P(t) = A\cos t + B\sin t + C\cos(14t) + D\sin(14t)$$

(c) $g(t) = \mathbf{e}^{7t} + 6$

The main point of this problem is dealing with the constant. But that isn't too bad. We just wanted to make sure that an example of that is somewhere in the notes. If you recall that a constant is nothing more than a zeroth degree polynomial the guess becomes clear.

The guess for this function is

$$Y_p(t) = A\mathbf{e}^{7t} + B$$

(d) $g(t) = 6t^2 - 7\sin(3t) + 9$

This one can be a little tricky if you aren't paying attention. Let's first rewrite the function

$$g(t) = 6t^2 - 7\sin(3t) + 9 \qquad \text{as}$$
$$g(t) = 6t^2 + 9 - 7\sin(3t)$$

All we did was move the 9. However, upon doing that we see that the function is really a sum of a quadratic polynomial and a sine. The guess for this is then

$$Y_P(t) = At^2 + Bt + C + D\cos(3t) + E\sin(3t)$$

If we don't do this and treat the function as the sum of three terms we would get

$$At^2 + Bt + C + D\cos(3t) + E\sin(3t) + G$$

and as with the first part in this example we would end up with two terms that are essentially the same (the *C* and the *G*) and so would need to be combined. An added step that isn't really necessary if we first rewrite the function.

Look for problems where rearranging the function can simplify the initial guess.

(e) $g(t) = 10\mathbf{e}^{t} - 5t\mathbf{e}^{-8t} + 2\mathbf{e}^{-8t}$

So, this look like we've got a sum of three terms here. Let's write down a guess for that.

$$A\mathbf{e}^{t} + (Bt + C)\mathbf{e}^{-8t} + D\mathbf{e}^{-8t}$$

Notice however that if we were to multiply the exponential in the second term through we would end up with two terms that are essentially the same and would need to be combined. This is a case where the guess for one term is completely contained in the guess for a different term. When this happens we just drop the guess that's already included in the other term.

So, the guess here is actually.

$$Y_P(t) = A\mathbf{e}^{t} + (Bt + C)\mathbf{e}^{-8t}$$

Notice that this arose because we had two terms in our *g(t)* whose only difference was the polynomial that sat in front of them. When this happens we look at the term that contains the largest degree polynomial, write down the guess for that and don't bother writing down the guess for the other term as that guess will be completely contained in the first guess.

(f) $g(t) = t^2 \cos t - 5t \sin t$

In this case we've got two terms whose guess without the polynomials in front of them would be the same. Therefore, we will take the one with the largest degree polynomial in front of it and write down the guess for that one and ignore the other term. So, the guess for the function is

$$Y_P(t) = (At^2 + Bt + C)\cos t + (Dt^2 + Et + F)\sin t$$

(g) $g(t) = 5\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\cos(6t) - \sin(6t)$

This last part is designed to make sure you understand the general rule that we used in the last two parts. This time there really are three terms and we will need a guess for each term. The guess here is

$$Y_P(t) = A\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\left(B\cos(6t) + C\sin(6t)\right) + D\cos(6t) + E\sin(6t)$$

We can only combine guesses if they are identical up to the constant. So, we can't combine the first exponential with the second because the second is really multiplied by a cosine and a sine and so the two exponentials are in fact different functions. Likewise, the last sine and cosine can't be combined with those in the middle term because the sine and cosine in the middle term are in fact multiplied by an exponential and so are different.

So, when dealing with sums of functions make sure that you look for identical guesses that may or may not be contained in other guesses and combine them. This will simplify your work later on.

We have one last topic in this section that needs to be dealt with. In the first few examples we were constantly harping on the usefulness of having the complementary solution in hand before making the guess for a particular solution. We never gave any reason for this other that "trust us". It is now time to see why having the complementary solution in hand first is useful. This is best shown with an example so let's jump into one.

Example 9 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = \mathbf{e}^{6t}$$

Solution

This problem seems almost too simple to be given this late in the section. This is especially true given the ease of finding a particular solution for $g(t)$'s that are just exponential functions. Also, because the point of this example is to illustrate why it is generally a good idea to have the complementary solution in hand first we'll let's go ahead and recall the complementary solution first. Here it is,

$$y_c(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t}$$

Now, without worrying about the complementary solution for a couple more seconds let's go ahead and get to work on the particular solution. There is not much to the guess here. From our previous work we know that the guess for the particular solution should be,

$$Y_P(t) = A\mathbf{e}^{6t}$$

Plugging this into the differential equation gives,

$$\begin{aligned} 36A\mathbf{e}^{6t} - 24A\mathbf{e}^{6t} - 12A\mathbf{e}^{6t} &= \mathbf{e}^{6t} \\ 0 &= \mathbf{e}^{6t} \end{aligned}$$

Hmmmm…. Something seems wrong here. Clearly an exponential can't be zero. So, what went wrong? We finally need the complementary solution. Notice that the second term in the complementary solution (listed above) is exactly our guess for the form of the particular solution and now recall that both portions of the complementary solution are solutions to the homogeneous differential equation,

$$y'' - 4y' - 12y = 0$$

In other words, we had better have gotten zero by plugging our guess into the differential equation, it is a solution to the homogeneous differential equation!

So, how do we fix this? The way that we fix this is to add a t to our guess as follows.

$$Y_P(t) = At\mathbf{e}^{6t}$$

Plugging this into our differential equation gives,

$$\left(12A\mathbf{e}^{6t}+36At\mathbf{e}^{6t}\right)-4\left(A\mathbf{e}^{6t}+6At\mathbf{e}^{6t}\right)-12At\mathbf{e}^{6t}=\mathbf{e}^{6t}$$
$$\left(36A-24A-12A\right)t\mathbf{e}^{6t}+\left(12A-4A\right)\mathbf{e}^{6t}=\mathbf{e}^{6t}$$
$$8A\mathbf{e}^{6t}=\mathbf{e}^{6t}$$

Now, we can set coefficients equal.

$$8A=1 \qquad \Rightarrow \qquad A=\frac{1}{8}$$

So, the particular solution in this case is,

$$Y_P\left(t\right)=\frac{t}{8}\mathbf{e}^{6t}$$

So, what did we learn from this last example. While technically we don't need the complementary solution to do undetermined coefficients, you can go through a lot of work only to figure out at the end that you needed to add in a *t* to the guess because it appeared in the complementary solution. This work is avoidable if we first find the complementary solution and comparing our guess to the complementary solution and seeing if any portion of your guess shows up in the complementary solution.

If a portion of your guess does show up in the complementary solution then we'll need to modify that portion of the guess by adding in a *t* to the portion of the guess that is causing the problems. We do need to be a little careful and make sure that we add the *t* in the correct place however. The following set of examples will show you how to do this.

Example 10 Write down the guess for the particular solution to the given differential equation. Do not find the coefficients.

(a) $y''+3y'-28y=7t+\mathbf{e}^{-7t}-1$

(b) $y''-100y=9t^2\mathbf{e}^{10t}+\cos t-t\sin t$

(c) $4y''+y=\mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right)+6t\cos\left(\frac{t}{2}\right)$

(d) $4y''+16y'+17y=\mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right)+6t\cos\left(\frac{t}{2}\right)$

(e) $y''+8y'+16y=\mathbf{e}^{-4t}+\left(t^2+5\right)\mathbf{e}^{-4t}$

Solution

In these solutions we'll leave the details of checking the complementary solution to you.

(a) $y''+3y'-28y=7t+\mathbf{e}^{-7t}-1$

The complementary solution is

$$y_c\left(t\right)=c_1\mathbf{e}^{4t}+c_2\mathbf{e}^{-7t}$$

Remembering to put the "-1" with the 7*t* gives a first guess for the particular solution.

$$Y_P(t) = At + B + C\mathbf{e}^{-7t}$$

Notice that the last term in the guess is the last term in the complementary solution. The first two terms however aren't a problem and don't appear in the complementary solution. Therefore, we will only add a *t* onto the last term.

The correct guess for the form of the particular solution is.

$$Y_P(t) = At + B + Ct\mathbf{e}^{-7t}$$

(b) $y'' - 100y = 9t^2\mathbf{e}^{10t} + \cos t - t\sin t$

The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{10t} + c_2\mathbf{e}^{-10t}$$

A first guess for the particular solution is

$$Y_P(t) = \left(At^2 + Bt + C\right)\mathbf{e}^{10t} + \left(Et + F\right)\cos t + \left(Gt + H\right)\sin t$$

Notice that if we multiplied the exponential term through the parenthesis that we would end up getting part of the complementary solution showing up. Since the problem part arises from the first term the *whole* first term will get multiplied by *t*. The second and third terms are okay as they are.

The correct guess for the form of the particular solution in this case is.

$$Y_P(t) = t\left(At^2 + Bt + C\right)\mathbf{e}^{10t} + \left(Et + F\right)\cos t + \left(Gt + H\right)\sin t$$

So, in general, if you were to multiply out a guess and if any term in the result shows up in the complementary solution, then the whole term will get a *t* not just the problem portion of the term.

(c) $4y'' + y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

The complementary solution is

$$y_c(t) = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)$$

A first guess for the particular solution is

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + \left(Ct + D\right)\cos\left(\frac{t}{2}\right) + \left(Et + F\right)\sin\left(\frac{t}{2}\right)$$

In this case both the second and third terms contain portions of the complementary solution. The first term doesn't however, since upon multiplying out, both the sine and the cosine would have an exponential with them and that isn't part of the complementary solution. We only need to worry about terms showing up in the complementary solution if the only difference between the complementary solution term and the particular guess term is the constant in front of them.

So, in this case the second and third terms will get a *t* while the first won't

The correct guess for the form of the particular solution is.

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + t\left(Ct + D\right)\cos\left(\frac{t}{2}\right) + t\left(Et + F\right)\sin\left(\frac{t}{2}\right)$$

(d) $4y'' + 16y' + 17y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

To get this problem we changed the differential equation from the last example and left the *g(t)* alone. The complementary solution this time is

$$y_c(t) = c_1\mathbf{e}^{-2t}\cos\left(\frac{t}{2}\right) + c_2\mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right)$$

As with the last part, a first guess for the particular solution is

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + \left(Ct + D\right)\cos\left(\frac{t}{2}\right) + \left(Et + F\right)\sin\left(\frac{t}{2}\right)$$

This time however it is the first term that causes problems and not the second or third. In fact, the first term is exactly the complementary solution and so it will need a *t*. Recall that we will only have a problem with a term in our guess if it only differs from the complementary solution by a constant. The second and third terms in our guess don't have the exponential in them and so they don't differ from the complementary solution by only a constant.

The correct guess for the form of the particular solution is.

$$Y_P(t) = t\mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + \left(Ct + D\right)\cos\left(\frac{t}{2}\right) + \left(Et + F\right)\sin\left(\frac{t}{2}\right)$$

(e) $y'' + 8y' + 16y = \mathbf{e}^{-4t} + \left(t^2 + 5\right)\mathbf{e}^{-4t}$

The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{-4t} + c_2 t\mathbf{e}^{-4t}$$

The two terms in *g(t)* are identical with the exception of a polynomial in front of them. So this means that we only need to look at the term with the highest degree polynomial in front of it. A first guess for the particular solution is

$$Y_P(t) = \left(At^2 + Bt + C\right)\mathbf{e}^{-4t}$$

Notice that if we multiplied the exponential term through the parenthesis the last two terms would be the complementary solution. Therefore, we will need to multiply this whole thing by a *t*.

The next guess for the particular solution is then.

$$Y_P\left(t\right)=t\left(At^2+Bt+C\right)\mathbf{e}^{-4t}$$

This still causes problems however. If we multiplied the *t* and the exponential through, the last term will still be in the complementary solution. In this case, unlike the previous ones, a *t* wasn't sufficient to fix the problem. So, we will add in another *t* to our guess.

The correct guess for the form of the particular solution is.

$$Y_P\left(t\right)=t^2\left(At^2+Bt+C\right)\mathbf{e}^{-4t}$$

Upon multiplying this out none of the terms are in the complementary solution and so it will be okay.

As this last set of examples has shown, we really should have the complementary solution in hand before even writing down the first guess for the particular solution. By doing this we can compare our guess to the complementary solution and if any of the terms from your particular solution show up we will know that we'll have problems. Once the problem is identified we can add a *t* to the problem term(s) and compare our new guess to the complementary solution. If there are no problems we can proceed with the problem, if there are problems add in another *t* and compare again.

Can you see a general rule as to when a *t* will be needed and when a t^2 will be needed for second order differential equations?

to get the value of y_1. We could use this tangent line as an approximation for the solution on the interval [t_0, t_1]. Likewise, we used the tangent line

$$y = y_1 + f\left(t_1, y_1\right)\left(t - t_1\right)$$

to get the value of y_2. We could use this tangent line as an approximation for the solution on the interval [t_1, t_2]. Continuing in this manner we would get a set of lines that, when strung together, should be an approximation to the solution as a whole.

In practice you would need to write a computer program to do these computations for you. In most cases the function *f(t,y)* would be too large and/or complicated to use by hand and in most serious uses of Euler's Method you would want to use hundreds of steps which would make doing this by hand prohibitive. So, here is a bit of *pseudo-code* that you can use to write a program for Euler's Method that uses a uniform step size, *h*.

1. **define** $f\left(t, y\right)$.
2. **input** t_0 and y_0.
3. **input** step size, *h* and the number of steps, *n*.
4. **for** *j* from 1 to *n* **do**
    a. $m = f\,(t_0, y_0)$
    b. $y_1 = y_0 + h*m$
    c. $t_1 = t_0 + h$
    d. Print t_1 and y_1
    e. $t_0 = t_1$
    f. $y_0 = y_1$
5. **end**

The *pseudo-code* for a non-uniform step size would be a little more complicated, but it would essentially be the same.

So, let's take a look at a couple of examples. We'll use Euler's Method to approximate solutions to a couple of first order differential equations. The differential equations that we'll be using are linear first order differential equations that can be easily solved for an exact solution. Of course, in practice we wouldn't use Euler's Method on these kinds of differential equations, but by using easily solvable differential equations we will be able to check the accuracy of the method. Knowing the accuracy of any approximation method is a good thing. It is important to know if the method is liable to give a good approximation or not.

Example 1 For the IVP

$$y' + 2y = 2 - \mathbf{e}^{-4t} \qquad y\left(0\right) = 1$$

Use Euler's Method with a step size of $h = 0.1$ to find approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$, and 0.5. Compare them to the exact values of the solution at these points.

Solution

This is a fairly simple linear differential equation so we'll leave it to you to check that the solution is

$$y(t) = 1 + \frac{1}{2}\mathbf{e}^{-4t} - \frac{1}{2}\mathbf{e}^{-2t}$$

In order to use Euler's Method we first need to rewrite the differential equation into the form given in (1).

$$y' = 2 - \mathbf{e}^{-4t} - 2y$$

From this we can see that $f(t, y) = 2 - \mathbf{e}^{-4t} - 2y$. Also note that $t_0 = 0$ and $y_0 = 1$. We can now start doing some computations.

$$f_0 = f(0,1) = 2 - \mathbf{e}^{-4(0)} - 2(1) = -1$$
$$y_1 = y_0 + h\, f_0 = 1 + (0.1)(-1) = 0.9$$

So, the approximation to the solution at $t_1 = 0.1$ is $y_1 = 0.9$.

At the next step we have

$$f_1 = f(0.1, 0.9) = 2 - \mathbf{e}^{-4(0.1)} - 2(0.9) = -0.470320046$$
$$y_2 = y_1 + h\, f_1 = 0.9 + (0.1)(-0.470320046) = 0.852967995$$

Therefore, the approximation to the solution at $t_2 = 0.2$ is $y_2 = 0.852967995$.

I'll leave it to you to check the remainder of these computations.

$$f_2 = -0.155264954 \qquad y_3 = 0.837441500$$
$$f_3 = 0.023922788 \qquad y_4 = 0.839833779$$
$$f_4 = 0.1184359245 \qquad y_5 = 0.851677371$$

Here's a quick table that gives the approximations as well as the exact value of the solutions at the given points.

Time, t_n	**Approximation**	**Exact**	**Error**
$t_0 = 0$	$y_0 = 1$	$y(0) = 1$	0 %
$t_1 = 0.1$	$y_1 = 0.9$	$y(0.1) = 0.925794646$	2.79 %
$t_2 = 0.2$	$y_2 = 0.852967995$	$y(0.2) = 0.889504459$	4.11 %
$t_3 = 0.3$	$y_3 = 0.837441500$	$y(0.3) = 0.876191288$	4.42 %
$t_4 = 0.4$	$y_4 = 0.839833779$	$y(0.4) = 0.876283777$	4.16 %
$t_5 = 0.5$	$y_5 = 0.851677371$	$y(0.5) = 0.883727921$	3.63 %

We've also included the error as a percentage. It's often easier to see how well an approximation does if you look at percentages. The formula for this is,

$$percent\ error = \frac{|exact - approximate|}{exact} \times 100$$

We used absolute value in the numerator because we really don't care at this point if the approximation is larger or smaller than the exact. We're only interested in how close the two are.

The maximum error in the approximations from the last example was 4.42%, which isn't too bad, but also isn't all that great of an approximation. So, provided we aren't after very accurate approximations this didn't do too badly. This kind of error is generally unacceptable in almost all real applications however. So, how can we get better approximations?

Recall that we are getting the approximations by using a tangent line to approximate the value of the solution and that we are moving forward in time by steps of h. So, if we want a more accurate approximation, then it seems like one way to get a better approximation is to not move forward as much with each step. In other words, take smaller h's.

Example 2 Repeat the previous example only this time give the approximations at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

Solution

Below are two tables, one gives approximations to the solution and the other gives the errors for each approximation. We'll leave the computational details to you to check.

Approximations

Time	**Exact**	***h* = 0.1**	***h* = 0.05**	***h* = 0.01**	***h* = 0.005**	***h* = 0.001**
$t = 1$	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
$t = 2$	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
$t = 3$	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
$t = 4$	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
$t = 5$	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

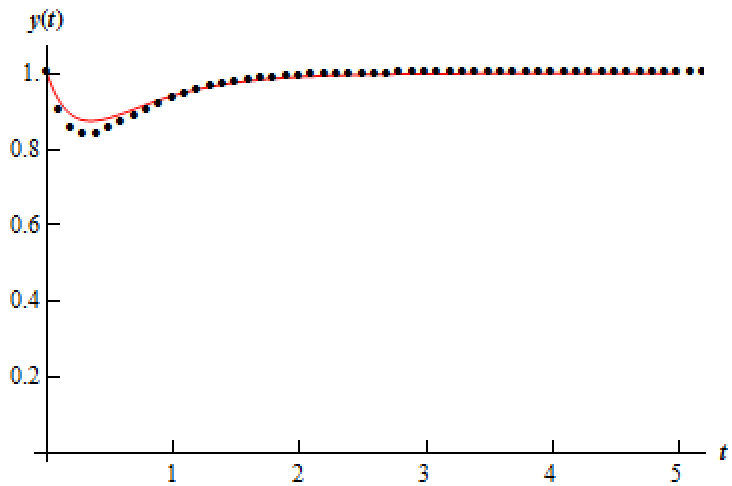
Percentage Errors

Time	***h* = 0.1**	***h* = 0.05**	***h* = 0.01**	***h* = 0.005**	***h* = 0.001**
$t = 1$	1.08 %	0.53 %	0.105 %	0.053 %	0.0105 %
$t = 2$	0.036 %	0.010 %	0.00094 %	0.00041 %	0.0000703 %
$t = 3$	0.029 %	0.013 %	0.0025 %	0.0013 %	0.00025 %
$t = 4$	0.0065 %	0.0033 %	0.00067 %	0.00034 %	0.000067 %
$t = 5$	0.0012 %	0.00064 %	0.00013 %	0.000068 %	0.000014 %

We can see from these tables that decreasing h does in fact improve the accuracy of the approximation as we expected.

There are a couple of other interesting things to note from the data. First, notice that in general, decreasing the step size, *h*, by a factor of 10 also decreased the error by about a factor of 10 as well.

Also, notice that as *t* increases the approximation actually tends to get better. This isn't the case completely as we can see that in all but the first case the *t* = 3 error is worse than the error at *t* = 2, but after that point, it only gets better. This should not be expected in general. In this case this is more a function of the shape of the solution. Below is a graph of the solution (the line) as well as the approximations (the dots) for *h* = 0.1.



Notice that the approximation is worst where the function is changing rapidly. This should not be too surprising. Recall that we're using tangent lines to get the approximations and so the value of the tangent line at a given *t* will often be significantly different than the function due to the rapidly changing function at that point.

Also, in this case, because the function ends up fairly flat as *t* increases, the tangents start looking like the function itself and so the approximations are very accurate. This won't always be the case of course.

Let's take a look at one more example.

Example 3 For the IVP

$$y' - y = -\frac{1}{2}\mathbf{e}^{\frac{t}{2}}\sin(5t) + 5\mathbf{e}^{\frac{t}{2}}\cos(5t) \qquad\qquad y(0) = 0$$

Use Euler's Method to find the approximation to the solution at *t* = 1, *t* = 2, *t* = 3, *t* = 4, and *t* = 5. Use *h* = 0.1, *h* = 0.05, *h* = 0.01, *h* = 0.005, and *h* = 0.001 for the approximations.

Solution

We'll leave it to you to check the details of the solution process. The solution to this linear first order differential equation is.

$$y(t) = \mathbf{e}^{\frac{t}{2}}\sin(5t)$$

Here are two tables giving the approximations and the percentage error for each approximation.

Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
$t = 2$	-1.47880	0.65270	-0.34327	-1.23907	-1.35810	-1.45453
$t = 3$	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
$t = 4$	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
$t = 5$	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

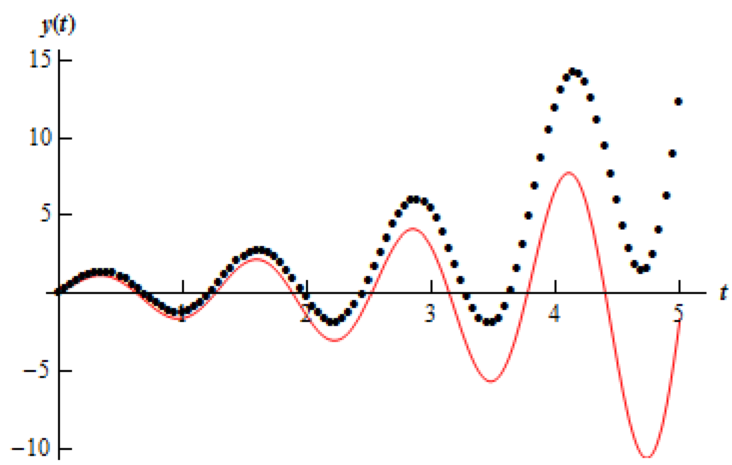
Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	38.54 %	19.98 %	4.12 %	2.07 %	0.42 %
$t = 2$	144.14 %	76.79 %	16.21 %	8.16 %	1.64 %
$t = 3$	150.55 %	83.46 %	18.20 %	9.20 %	1.86 %
$t = 4$	130.68 %	75.64 %	17.08 %	8.67 %	1.76 %
$t = 5$	1461.63 %	859.14 %	196.79 %	100.12 %	20.30 %

So, with this example Euler's Method does not do nearly as well as it did on the first IVP. Some of the observations we made in Example 2 are still true however. Decreasing the size of h decreases the error as we saw with the last example and would expect to happen. Also, as we saw in the last example, decreasing h by a factor of 10 also decreases the error by about a factor of 10.

However, unlike the last example increasing t sees an increasing error. This behavior is fairly common in the approximations. We shouldn't expect the error to decrease as t increases as we saw in the last example. Each successive approximation is found using a previous approximation. Therefore, at each step we introduce error and so approximations should, in general, get worse as t increases.

Below is a graph of the solution (the line) as well as the approximations (the dots) for $h = 0.05$.



As we can see the approximations do follow the general shape of the solution, however, the error is clearly getting much worse as t increases.

So, Euler's method is a nice method for approximating fairly nice solutions that don't change rapidly. However, not all solutions will be this nicely behaved. There are other approximation methods that do a much better job of approximating solutions. These are not the focus of this course however, so I'll leave it to you to look further into this field if you are interested.

Also notice that we don't generally have the actual solution around to check the accuracy of the approximation. We generally try to find bounds on the error for each method that will tell us how well an approximation should do. These error bounds are again not really the focus of this course, so I'll leave these to you as well if you're interested in looking into them.

$$\int N(y)\,dy = \int M(x)\,dx \qquad (3)$$

So, if we compare (2) and (3) we can see that the only difference is on the left side and even then the only real difference is (2) has the integral in terms of *u* and (3) has the integral in terms of *y*. Outside of that there is no real difference. The integral on the left is exactly the same integral in each equation. The only difference is the letter used in the integral. If we integrate (2) and then back substitute in for *u* we would arrive at the same thing as if we'd just integrated (3) from the start.

Therefore, to make the work go a little easier, we'll just use (3) to find the solution to the differential equation. Also, after doing the integrations, we will have an implicit solution that we can hopefully solve for the explicit solution, $y(x)$. Note that it won't always be possible to solve for an explicit solution.

Recall from the Definitions section that an implicit solution is a solution that is not in the form $y = y(x)$ while an explicit solution has been written in that form.

We will also have to worry about the interval of validity for many of these solutions. Recall that the interval of validity was the range of the independent variable, x in this case, on which the solution is valid. In other words, we need to avoid division by zero, complex numbers, logarithms of negative numbers or zero, *etc.* Most of the solutions that we will get from separable differential equations will not be valid for all values of x.

Let's start things off with a fairly simple example so we can see the process without getting lost in details of the other issues that often arise with these problems.

Example 1 Solve the following differential equation and determine the interval of validity for the solution.

$$\frac{dy}{dx} = 6y^2x \qquad y(1) = \frac{1}{25}$$

Solution

It is clear, hopefully, that this differential equation is separable. So, let's separate the differential equation and integrate both sides. As with the linear first order officially we will pick up a constant of integration on both sides from the integrals on each side of the equal sign. The two can be moved to the same side and absorbed into each other. We will use the convention that puts the single constant on the side with the *x*'s given that we will eventually be solving for *y* and so the constant would end up on that side anyway.

$$\begin{aligned} y^{-2}dy &= 6x\,dx \\ \int y^{-2}dy &= \int 6x\,dx \\ -\frac{1}{y} &= 3x^2 + c \end{aligned}$$

So, we now have an implicit solution. This solution is easy enough to get an explicit solution, however before getting that it is usually easier to find the value of the constant at this point. So apply the initial condition and find the value of *c*.

$$-\frac{1}{1/25} = 3(1)^2 + c \qquad c = -28$$

Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 - 28$$

$$y(x) = \frac{1}{28 - 3x^2}$$

Now, as far as solutions go we've got the solution. We do need to start worrying about intervals of validity however.

Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, $x = 1$ in this case.

So, for our case we've got to avoid two values of *x*. Namely, $x \neq \pm\sqrt{\frac{28}{3}} \approx \pm 3.05505$ since these will give us division by zero. This gives us three possible intervals of validity.

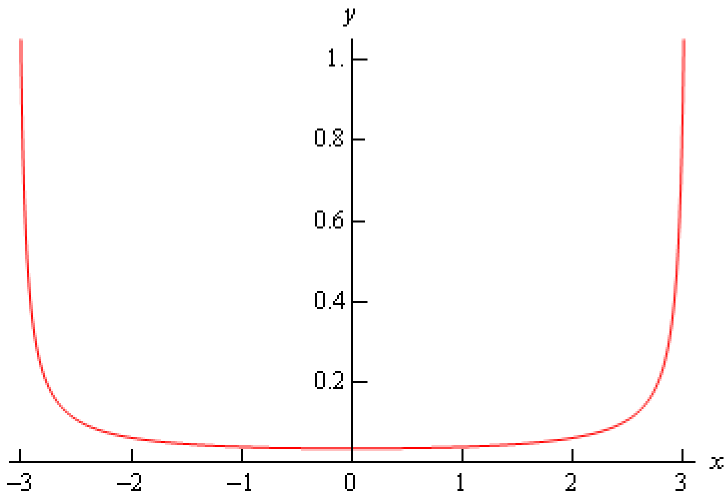
$$-\infty < x < -\sqrt{\frac{28}{3}} \qquad -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}} \qquad \sqrt{\frac{28}{3}} < x < \infty$$

However, only one of these will contain the value of *x* from the initial condition and so we can see that

$$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

must be the interval of validity for this solution.

Here is a graph of the solution.



Note that this does not say that either of the other two intervals listed above can't be the interval of validity for any solution to the differential equation. With the proper initial condition either of these could have been the interval of validity.

We'll leave it to you to verify the details of the following claims. If we use an initial condition of

$$y(-4) = -\frac{1}{20}$$

we will get exactly the same solution however in this case the interval of validity would be the first one.

$$-\infty < x < -\sqrt{\frac{28}{3}}$$

Likewise, if we use

$$y(6) = -\frac{1}{80}$$

as the initial condition we again get exactly the same solution and, in this case, the third interval becomes the interval of validity.

$$\sqrt{\frac{28}{3}} < x < \infty$$

So, simply changing the initial condition a little can give any of the possible intervals.

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \qquad y(1) = 3$$

Solution

This differential equation is clearly separable, so let's put it in the proper form and then integrate both sides.

$$(2y-4)\,dy = \left(3x^2+4x-4\right)dx$$
$$\int (2y-4)\,dy = \int \left(3x^2+4x-4\right)dx$$
$$y^2-4y = x^3+2x^2-4x+c$$

We now have our implicit solution, so as with the first example let's apply the initial condition at this point to determine the value of *c*.

$$(3)^2-4(3) = (1)^3+2(1)^2-4(1)+c \qquad c=-2$$

The implicit solution is then

$$y^2-4y = x^3+2x^2-4x-2$$

We now need to find the explicit solution. This is actually easier than it might look and you already know how to do it. First, we need to rewrite the solution a little

$$y^2-4y-\left(x^3+2x^2-4x-2\right)=0$$

To solve this all we need to recognize is that this is quadratic in *y* and so we can use the quadratic formula to solve it. However, unlike quadratics you are used to, at least some of the "constants" will not actually be constant but will in fact involve *x*'s.

So, upon using the quadratic formula on this we get.

$$\begin{aligned} y(x) &= \frac{4\pm\sqrt{16-4(1)\left(-\left(x^3+2x^2-4x-2\right)\right)}}{2} \\ &= \frac{4\pm\sqrt{16+4\left(x^3+2x^2-4x-2\right)}}{2} \end{aligned}$$

Next, notice that we can factor a 4 out from under the square root (it will come out as a 2…) and then simplify a little.

$$\begin{aligned} y(x) &= \frac{4\pm2\sqrt{4+\left(x^3+2x^2-4x-2\right)}}{2} \\ &= 2\pm\sqrt{x^3+2x^2-4x+2} \end{aligned}$$

We are almost there. Notice that we've actually got two solutions here (the "$\pm$") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x=1$ into the solution gives.

$$3 = y(1) = 2\pm\sqrt{1+2-4+2} = 2\pm1 = 3,1$$

In this case it looks like the "+" is the correct sign for our solution. Note that it is completely possible that the "–" could be the solution (*i.e.* using an initial condition of $y(1)=1$) so don't always expect it to be one or the other.

The explicit solution for our differential equation is.

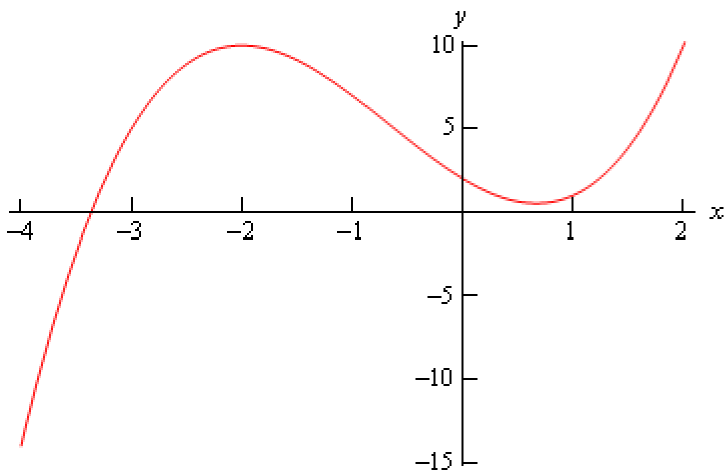
$$y(x)=2+\sqrt{x^3+2x^2-4x+2}$$

To finish the example out we need to determine the interval of validity for the solution. If we were to put a large negative value of x in the solution we would end up with complex values in our solution and we want to avoid complex numbers in our solutions here. So, we will need to determine which values of x will give real solutions. To do this we will need to solve the following inequality.

$$x^3+2x^2-4x+2\geq 0$$

In other words, we need to make sure that the quantity under the radical stays positive.

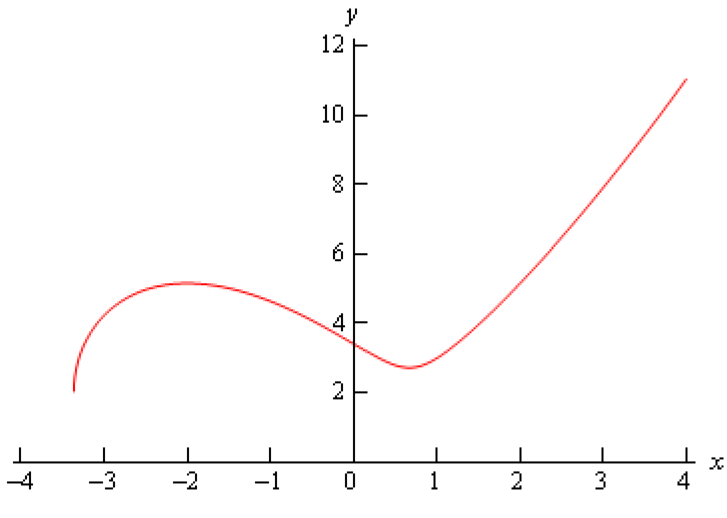
Using a computer algebra system like Maple or Mathematica we see that the left side is zero at $x = -3.36523$ as well as two complex values, but we can ignore complex values for interval of validity computations. Finally, a graph of the quantity under the radical is shown below.



So, in order to get real solutions we will need to require $x\geq -3.36523$ because this is the range of x's for which the quantity is positive. Notice as well that this interval also contains the value of x that is in the initial condition as it should.

Therefore, the interval of validity of the solution is $x\geq -3.36523$.

Here is graph of the solution.



Example 3 Solve the following IVP and find the interval of validity of the solution.

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad\qquad y(0) = -1$$

Solution

First separate and then integrate both sides.

$$y^{-3}dy = x\left(1+x^2\right)^{-\frac{1}{2}} dx$$

$$\int y^{-3}dy = \int x\left(1+x^2\right)^{-\frac{1}{2}} dx$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the initial condition to get the value of *c*.

$$-\frac{1}{2} = \sqrt{1} + c \qquad\qquad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for *y(x)*.

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y(x) = \pm\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Reapplying the initial condition shows us that the "–" is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Let's get the interval of validity. That's easier than it might look for this problem. First, since $1+x^2 \geq 0$ the "inner" root will not be a problem. Therefore, all we need to worry about is division by zero and negatives under the "outer" root. We can take care of both by requiring

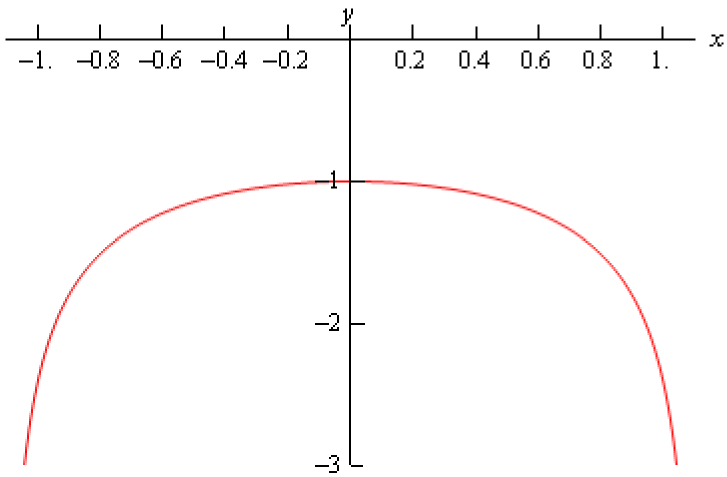
$$\begin{aligned} 3 - 2\sqrt{1+x^2} &> 0 \\ 3 &> 2\sqrt{1+x^2} \\ 9 &> 4\left(1+x^2\right) \\ \frac{9}{4} &> 1+x^2 \\ \frac{5}{4} &> x^2 \end{aligned}$$

Note that we were able to square both sides of the inequality because both sides of the inequality are guaranteed to be positive in this case. Finally solving for *x* we see that the only possible range of *x*'s that will not give division by zero or square roots of negative numbers will be,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and nicely enough this also contains the initial condition *x=0*. This interval is therefore our interval of validity.

Here is a graph of the solution.



Example 4 Solve the following IVP and find the interval of validity of the solution.

$$y' = \mathbf{e}^{-y}\left(2x-4\right) \qquad y\left(5\right) = 0$$

Solution

This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$\mathbf{e}^{y}\,dy = \left(2x-4\right)dx$$

$$\int \mathbf{e}^{y}\,dy = \int \left(2x-4\right)dx$$

$$\mathbf{e}^{y} = x^2 - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad c = -4$$

This then gives an implicit solution of.

$$\mathbf{e}^{y} = x^2 - 4x - 4$$

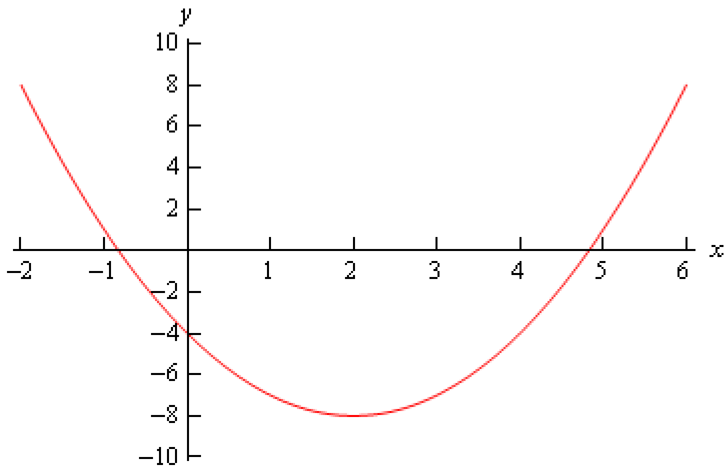
We can easily find the explicit solution to this differential equation by simply taking the natural log of both sides.

$$y\left(x\right) = \ln\left(x^2 - 4x - 4\right)$$

Finding the interval of validity is the last step that we need to take. Recall that we can't plug negative values or zero into a logarithm, so we need to solve the following inequality

$$x^2 - 4x - 4 > 0$$

The quadratic will be zero at the two points $x = 2 \pm 2\sqrt{2}$. A graph of the quadratic (shown below) shows that there are in fact two intervals in which we will get positive values of the polynomial and hence can be possible intervals of validity.



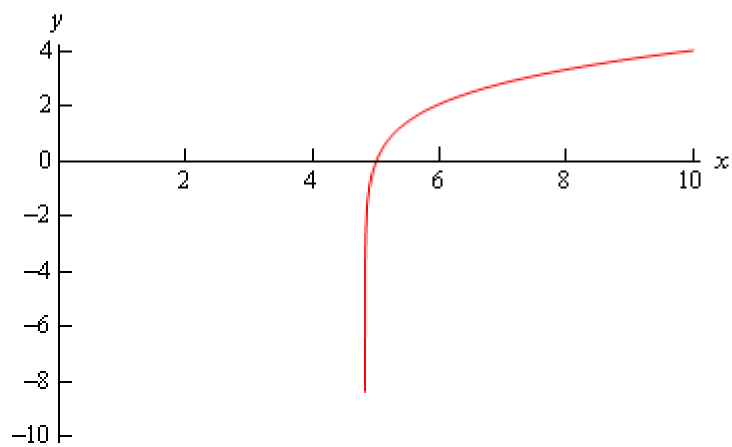
So, possible intervals of validity are

$$-\infty < x < 2 - 2\sqrt{2}$$
$$2 + 2\sqrt{2} < x < \infty$$

From the graph of the quadratic we can see that the second one contains *x* = 5, the value of the independent variable from the initial condition. Therefore, the interval of validity for this solution is.

$$2 + 2\sqrt{2} < x < \infty$$

Here is a graph of the solution.



Example 5 Solve the following IVP and find the interval of validity for the solution.

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} \qquad r(1) = 2$$

Solution

This is actually a fairly simple differential equation to solve. We're doing this one mostly because of the interval of validity.

So, get things separated out and then integrate.

$$\frac{1}{r^2}dr = \frac{1}{\theta}d\theta$$
$$\int \frac{1}{r^2}dr = \int \frac{1}{\theta}d\theta$$
$$-\frac{1}{r} = \ln|\theta| + c$$

Now, apply the initial condition to find *c*.

$$-\frac{1}{2} = \ln(1) + c \qquad\qquad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for *r* gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

Now, there are two problems for our solution here. First, we need to avoid $\theta = 0$ because of the natural log. Notice that because of the absolute value on the θ we don't need to worry about θ being negative. We will also need to avoid division by zero. In other words, we need to avoid the following points.

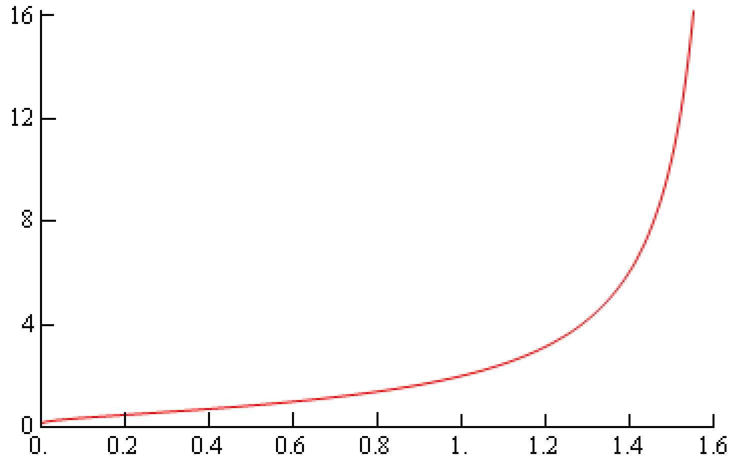
$$\begin{aligned} \frac{1}{2} - \ln|\theta| &= 0 \\ \ln|\theta| &= \frac{1}{2} \qquad \text{exponentiate both sides} \\ |\theta| &= \mathbf{e}^{\frac{1}{2}} \\ \theta &= \pm\sqrt{\mathbf{e}} \end{aligned}$$

So, these three points break the number line up into four portions, each of which could be an interval of validity.

$$-\infty < \theta < -\sqrt{\mathbf{e}}$$
$$-\sqrt{\mathbf{e}} < \theta < 0$$
$$0 < \theta < \sqrt{\mathbf{e}}$$
$$\sqrt{\mathbf{e}} < \theta < \infty$$

The interval that will be the actual interval of validity is the one that contains $\theta = 1$. Therefore, the interval of validity is $0 < \theta < \sqrt{\mathbf{e}}$.

Here is a graph of the solution.



Example 6 Solve the following IVP.

$$\frac{dy}{dt} = \mathbf{e}^{y-t}\sec(y)\left(1+t^2\right) \qquad y(0) = 0$$

Solution

This problem will require a little work to get it separated and in a form that we can integrate, so let's do that first.

$$\frac{dy}{dt} = \frac{\mathbf{e}^{y}\mathbf{e}^{-t}}{\cos(y)}\left(1+t^2\right)$$

$$\mathbf{e}^{-y}\cos(y)\,dy = \mathbf{e}^{-t}\left(1+t^2\right)dt$$

Now, with a little integration by parts on both sides we can get an implicit solution.

$$\int \mathbf{e}^{-y}\cos(y)\,dy = \int \mathbf{e}^{-t}\left(1+t^2\right)dt$$

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+c$$

Applying the initial condition gives.

$$\frac{1}{2}(-1) = -(3)+c \qquad c = \frac{5}{2}$$

Therefore, the implicit solution is.

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y)-\cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+\frac{5}{2}$$

Example 1 Find the solution to the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v$$

Solution

First, we need to get the differential equation in the correct form.

$$\frac{dv}{dt} + 0.196v = 9.8$$

From this we can see that $p(t)=0.196$ and so $\mu(t)$ is then.

$$\mu(t) = \mathbf{e}^{\int 0.196dt} = \mathbf{e}^{0.196t}$$

Note that officially there should be a constant of integration in the exponent from the integration. However, we can drop that for exactly the same reason that we dropped the *k* from (8).

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$\mathbf{e}^{0.196t}\frac{dv}{dt} + 0.196\mathbf{e}^{0.196t}v = 9.8\mathbf{e}^{0.196t}$$

$$\left(\mathbf{e}^{0.196t}v\right)' = 9.8\mathbf{e}^{0.196t}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\int \left(\mathbf{e}^{0.196t}v\right)' dt = \int 9.8\mathbf{e}^{0.196t}\, dt$$

$$\mathbf{e}^{0.196t}v + k = 50\mathbf{e}^{0.196t} + c$$

Okay. It's time to play with constants again. We can subtract *k* from both sides to get.

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c - k$$

Both *c* and *k* are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as *c*. So, we now have

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c$$

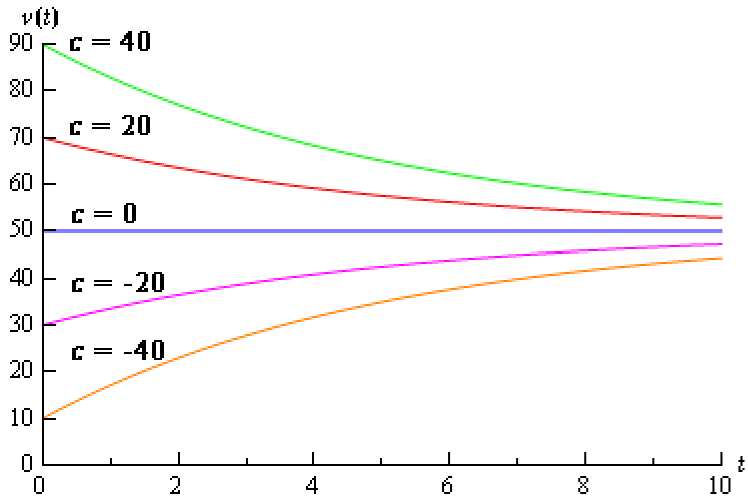
From this point on we will only put one constant of integration down when we integrate both sides knowing that if we had written down one for each integral, as we should, the two would just end up getting absorbed into each other.

The final step in the solution process is then to divide both sides by $\mathbf{e}^{0.196t}$ or to multiply both sides by $\mathbf{e}^{-0.196t}$. Either will work, but we usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t) = 50 + c\mathbf{e}^{-0.196t}$$

From the solution to this example we can now see why the constant of integration is so important in this process. Without it, in this case, we would get a single, constant solution, $v(t)=50$. With the constant of integration we get infinitely many solutions, one for each value of c.

Back in the direction field section where we first derived the differential equation used in the last example we used the direction field to help us sketch some solutions. Let's see if we got them correct. To sketch some solutions all we need to do is to pick different values of c to get a solution. Several of these are shown in the graph below.



So, it looks like we did pretty good sketching the graphs back in the direction field section.

Now, recall from the Definitions section that the Initial Condition(s) will allow us to zero in on a particular solution. Solutions to first order differential equations (not just linear as we will see) will have a single unknown constant in them and so we will need exactly one initial condition to find the value of that constant and hence find the solution that we were after. The initial condition for first order differential equations will be of the form

$$y(t_0) = y_0$$

Recall as well that a differential equation along with a sufficient number of initial conditions is called an Initial Value Problem (IVP).

Example 2 Solve the following IVP.

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad\qquad v(0) = 48$$

Solution

To find the solution to an IVP we must first find the general solution to the differential equation and then use the initial condition to identify the exact solution that we are after. So, since this is the same differential equation as we looked at in Example 1**,** we already have its general solution.

$$v = 50 + c\mathbf{e}^{-0.196t}$$

Now, to find the solution we are after we need to identify the value of c that will give us the solution we are after. To do this we simply plug in the initial condition which will give us an equation we can solve for c. So, let's do this

$$48 = v(0) = 50 + c \qquad \Rightarrow \qquad c = -2$$

So, the actual solution to the IVP is.

$$v = 50 - 2\mathbf{e}^{-0.196t}$$

A graph of this solution can be seen in the figure above.

Let's do a couple of examples that are a little more involved.

Example 3 Solve the following IVP.

$$\cos(x)\,y' + \sin(x)\,y = 2\cos^3(x)\sin(x) - 1 \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \qquad 0 \le x < \frac{\pi}{2}$$

Solution :

Rewrite the differential equation to get the coefficient of the derivative a one.

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)\,y = 2\cos^2(x)\sin(x) - \sec(x)$$

Now find the integrating factor.

$$\mu(t) = \mathbf{e}^{\int \tan(x)\,dx} = \mathbf{e}^{\ln|\sec(x)|} = \mathbf{e}^{\ln \sec(x)} = \sec(x)$$

Can you do the integral? If not rewrite tangent back into sines and cosines and then use a simple substitution. Note that we could drop the absolute value bars on the secant because of the limits on *x*. In fact, this is the reason for the limits on *x*. Note as well that there are two forms of the answer to this integral. They are equivalent as shown below. Which you use is really a matter of preference.

$$\int \tan(x)\,dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1} = \ln|\sec(x)|$$

Also note that we made use of the following fact.

$$\mathbf{e}^{\ln f(x)} = f(x) \tag{11}$$

This is an important fact that you should always remember for these problems. We will want to simplify the integrating factor as much as possible in all cases and this fact will help with that simplification.

Now back to the example. Multiply the integrating factor through the differential equation and verify the left side is a product rule. Note as well that we multiply the integrating factor through the rewritten differential equation and NOT the original differential equation. Make sure that you do this. If you multiply the integrating factor through the original differential equation you will get the wrong solution!

$$\sec(x)\,y' + \sec(x)\tan(x)\,y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$\left(\sec(x)\,y\right)' = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrate both sides.

$$\begin{aligned}\int \left(\sec(x)\,y(x)\right)'\,dx &= \int 2\cos(x)\sin(x) - \sec^2(x)\,dx\\ \sec(x)\,y(x) &= \int \sin(2x) - \sec^2(x)\,dx\\ \sec(x)\,y(x) &= -\frac{1}{2}\cos(2x) - \tan(x) + c\end{aligned}$$

Note the use of the trig formula $\sin(2\theta) = 2\sin\theta\cos\theta$ that made the integral easier. Next, solve for the solution.

$$\begin{aligned}y(x) &= -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x)\\ &= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x)\end{aligned}$$

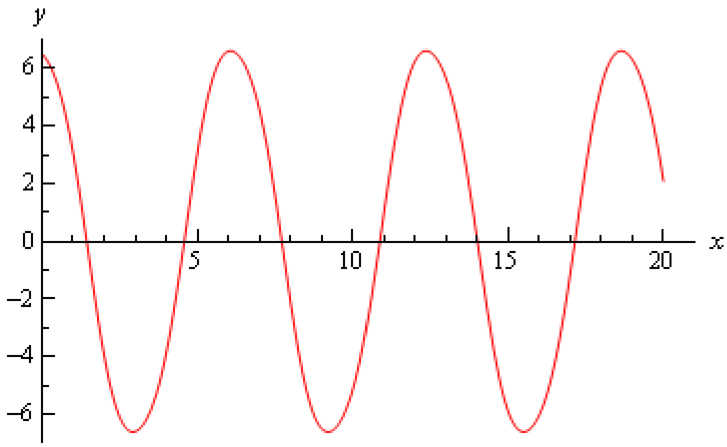
Finally, apply the initial condition to find the value of *c*.

$$\begin{aligned}3\sqrt{2} = y\left(\frac{\pi}{4}\right) &= -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)\\ 3\sqrt{2} &= -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2}\\ c &= 7\end{aligned}$$

The solution is then.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Below is a plot of the solution.



Example 4 Find the solution to the following IVP.

$$t\,y' + 2y = t^2 - t + 1 \qquad\qquad y(1) = \frac{1}{2}$$

Solution

First, divide through by the *t* to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int \frac{2}{t}dt} = \mathbf{e}^{2\ln|t|}$$

Now, we need to simplify $\mu(t)$. However, we can't use (11) yet as that requires a coefficient of one in front of the logarithm. So, recall that

$$\ln x^r = r\ln x$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = \mathbf{e}^{2\ln|t|} = \mathbf{e}^{\ln|t|^2} = |t|^2 = t^2$$

We were able to drop the absolute value bars here because we were squaring the *t*, but often they can't be dropped so be careful with them and don't drop them unless you know that you can. Often the absolute value bars must remain.

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here…) by the integrating factor.

$$\left(t^2 y\right)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$t^2 y = \int t^3 - t^2 + t\, dt$$
$$= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c$$
$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$$

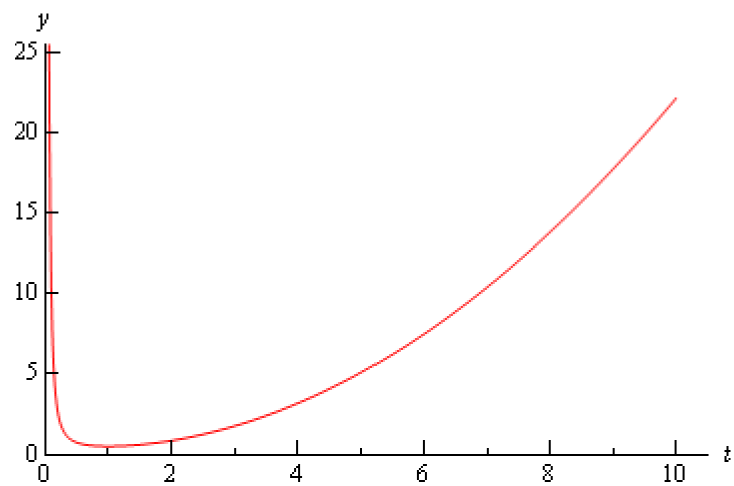
Finally, apply the initial condition to get the value of *c*.

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \qquad \Rightarrow \qquad c = \frac{1}{12}$$

The solution is then,

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

Here is a plot of the solution.



Example 5 Find the solution to the following IVP.

$$t\,y' - 2y = t^5 \sin(2t) - t^3 + 4t^4 \qquad y(\pi) = \frac{3}{2}\pi^4$$

Solution

First, divide through by *t* to get the differential equation in the correct form.

$$y' - \frac{2}{t}y = t^4 \sin(2t) - t^2 + 4t^3$$

Now that we have done this we can find the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{2}{t}dt} = \mathbf{e}^{-2\ln|t|}$$

Do not forget that the "-" is part of *p(t)*. Forgetting this minus sign can take a problem that is very easy to do and turn it into a very difficult, if not impossible problem so be careful!

Now, we just need to simplify this as we did in the previous example.

$$\mu(t) = \mathbf{e}^{-2\ln|t|} = \mathbf{e}^{\ln|t|^{-2}} = |t|^{-2} = t^{-2}$$

Again, we can drop the absolute value bars since we are squaring the term.

Now multiply the differential equation by the integrating factor (again, make sure it's the rewritten one and not the original differential equation).

$$\left(t^{-2}y\right)' = t^2\sin(2t) - 1 + 4t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^{-2}y(t) &= \int t^2\sin(2t)\,dt + \int -1 + 4t\,dt \\ t^{-2}y(t) &= -\frac{1}{2}t^2\cos(2t) + \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t) - t + 2t^2 + c \\ y(t) &= -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + ct^2 \end{aligned}$$

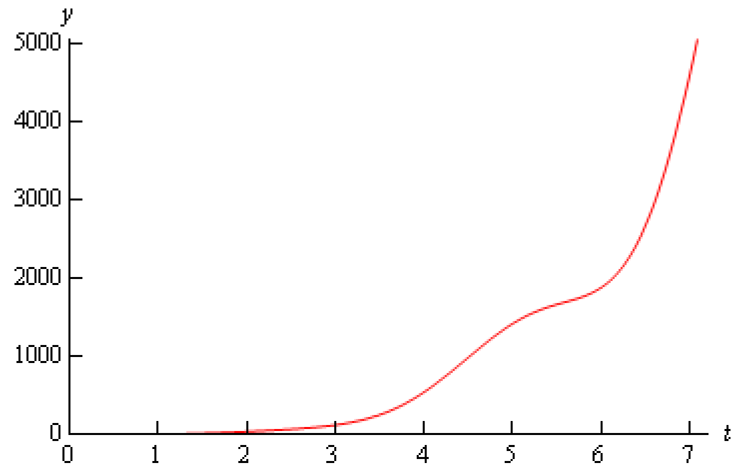
Apply the initial condition to find the value of *c*.

$$\begin{aligned} \frac{3}{2}\pi^4 = y(\pi) &= -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + c\pi^2 = \frac{3}{2}\pi^4 - \pi^3 + \frac{1}{4}\pi^2 + c\pi^2 \\ \pi^3 - \frac{1}{4}\pi^2 &= c\pi^2 \\ c &= \pi - \frac{1}{4} \end{aligned}$$

The solution is then

$$y(t) = -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + \left(\pi - \frac{1}{4}\right)t^2$$

Below is a plot of the solution.



Let's work one final example that looks more at interpreting a solution rather than finding a solution.

Example 6 Find the solution to the following IVP and determine all possible behaviors of the solution as $t \to \infty$. If this behavior depends on the value of y_0 give this dependence.

$$2y' - y = 4\sin(3t) \qquad y(0) = y_0$$

Solution

First, divide through by a 2 to get the differential equation in the correct form.

$$y' - \frac{1}{2}y = 2\sin(3t)$$

Now find $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{1}{2}dt} = \mathbf{e}^{-\frac{t}{2}}$$

Multiply $\mu(t)$ through the differential equation and rewrite the left side as a product rule.

$$\left(\mathbf{e}^{-\frac{t}{2}}y\right)' = 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)$$

Integrate both sides (the right side requires integration by parts – you can do that right?) and solve for the solution.

$$\mathbf{e}^{-\frac{t}{2}}y = \int 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)\,dt + c$$

$$\mathbf{e}^{-\frac{t}{2}}y = -\frac{24}{37}\mathbf{e}^{-\frac{t}{2}}\cos(3t) - \frac{4}{37}\mathbf{e}^{-\frac{t}{2}}\sin(3t) + c$$

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + c\mathbf{e}^{\frac{t}{2}}$$

Apply the initial condition to find the value of *c* and note that it will contain y_0 as we don't have a value for that.

$$y_0 = y(0) = -\frac{24}{37} + c \qquad \Rightarrow \qquad c = y_0 + \frac{24}{37}$$

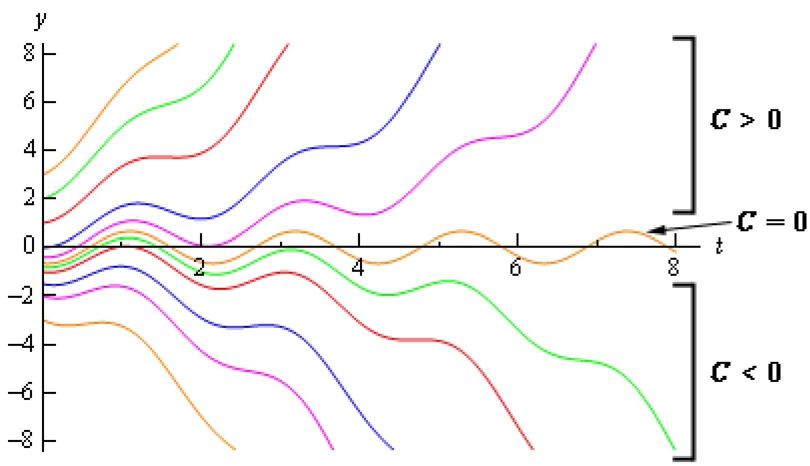
So, the solution is

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + \left(y_0 + \frac{24}{37}\right)\mathbf{e}^{\frac{t}{2}}$$

Now that we have the solution, let's look at the long term behavior (*i.e.* $t \to \infty$) of the solution. The first two terms of the solution will remain finite for all values of t. It is the last term that will determine the behavior of the solution. The exponential will always go to infinity as $t \to \infty$, however depending on the sign of the coefficient *c* (yes we've already found it, but for ease of this discussion we'll continue to call it *c*). The following table gives the long term behavior of the solution for all values of c.

Range of *c*	Behavior of solution as $t \to \infty$
$c < 0$	$y(t) \to -\infty$
$c = 0$	$y(t)$ remains finite
$c > 0$	$y(t) \to \infty$

This behavior can also be seen in the following graph of several of the solutions.



Now, because we know how *c* relates to y_0 we can relate the behavior of the solution to y_0. The following table give the behavior of the solution in terms of y_0 instead of *c*.

Range of y_0	Behavior of solution as $t \to \infty$
$y_0 < -\dfrac{24}{37}$	$y(t) \to -\infty$
$y_0 = -\dfrac{24}{37}$	$y(t)$ remains finite
$y_0 > -\dfrac{24}{37}$	$y(t) \to \infty$

Note that for $y_0 = -\frac{24}{37}$ the solution will remain finite. That will not always happen.

Investigating the long term behavior of solutions is sometimes more important than the solution itself. Suppose that the solution above gave the temperature in a bar of metal. In this case we would want the solution(s) that remains finite in the long term. With this investigation we would now have the value of the initial condition that will give us that solution and more importantly values of the initial condition that we would need to avoid so that we didn't melt the bar.