Section 3-10 : Surface Area with Polar Coordinates

We will be looking at surface area in polar coordinates in this section. Note however that all we're going to do is give the formulas for the surface area since most of these integrals tend to be fairly difficult.

We want to find the surface area of the region found by rotating,

$$r = f\left(\theta\right) \qquad\qquad \alpha \le \theta \le \beta$$

about the *x* or *y*-axis.

As we did in the tangent and arc length sections we'll write the curve in terms of a set of parametric equations.

$$\begin{aligned} x &= r\cos\theta & \qquad\qquad y &= r\sin\theta \\ &= f\left(\theta\right)\cos\theta & &= f\left(\theta\right)\sin\theta \end{aligned}$$

If we now use the parametric formula for finding the surface area we'll get,

$$S = \int 2\pi y\, ds \qquad\qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x\, ds \qquad\qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad\qquad r = f\left(\theta\right), \quad \alpha \le \theta \le \beta$$

Note that because we will pick up a $d\theta$ from the *ds* we'll need to substitute one of the parametric equations in for *x* or *y* depending on the axis of rotation. This will often mean that the integrals will be somewhat unpleasant.

Section 3-9 : Arc Length with Polar Coordinates

We now need to move into the Calculus II applications of integrals and how we do them in terms of polar coordinates. In this section we'll look at the arc length of the curve given by,

$$r = f(\theta) \qquad \alpha \le \theta \le \beta$$

where we also assume that the curve is traced out exactly once. Just as we did with the tangent lines in polar coordinates we'll first write the curve in terms of a set of parametric equations,

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= f(\theta)\cos\theta & &= f(\theta)\sin\theta \end{aligned}$$

and we can now use the parametric formula for finding the arc length.

We'll need the following derivatives for these computations.

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta & \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta \\ &= \frac{dr}{d\theta}\cos\theta - r\sin\theta & &= \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{aligned}$$

We'll need the following for our *ds*.

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 \\ &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\qquad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2 + \left(\frac{dr}{d\theta}\right)^2 \end{aligned}$$

The arc length formula for polar coordinates is then,

$$L = \int ds$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Let's work a quick example of this.

Example 1 Determine the length of $r = \theta \quad 0 \le \theta \le 1$.

Solution

Okay, let's just jump straight into the formula since this is a fairly simple function.

$$L = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta$$

We'll need to use a trig substitution here.

$$\theta = \tan x \qquad\qquad d\theta = \sec^2 x\, dx$$

$$\theta = 0 \qquad 0 = \tan x \qquad x = 0$$

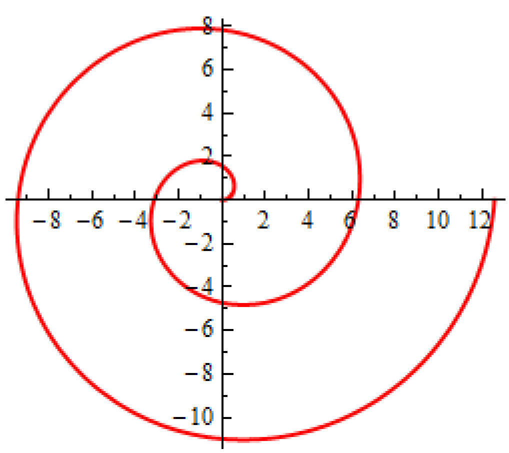
$$\theta = 1 \qquad 1 = \tan x \qquad x = \frac{\pi}{4}$$

$$\sqrt{\theta^2 + 1} = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = \left|\sec x\right| = \sec x$$

The arc length is then,

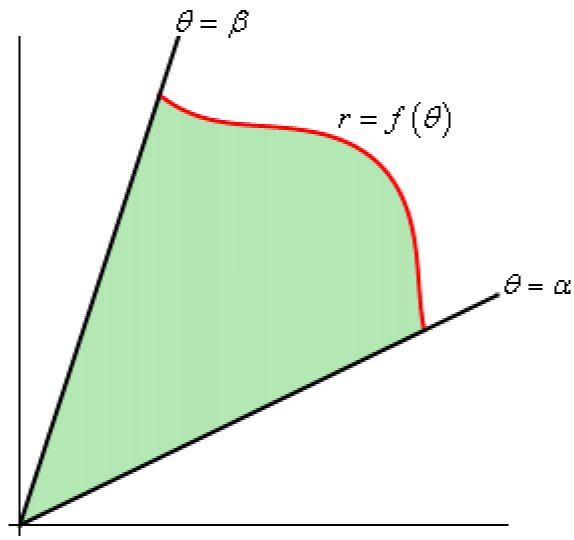
$$\begin{aligned} L &= \int_0^1 \sqrt{\theta^2 + 1}\, d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec^3 x\, dx \\ &= \frac{1}{2}\left(\sec x \tan x + \ln\left|\sec x + \tan x\right|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

Just as an aside before we leave this chapter. The polar equation $r = \theta$ is the equation of a spiral. Here is a quick sketch of $r = \theta$ for $0 \le \theta \le 4\pi$.



Section 3-8 : Area with Polar Coordinates

In this section we are going to look at areas enclosed by polar curves. Note as well that we said "enclosed by" instead of "under" as we typically have in these problems. These problems work a little differently in polar coordinates. Here is a sketch of what the area that we'll be finding in this section looks like.



We'll be looking for the shaded area in the sketch above. The formula for finding this area is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Notice that we use *r* in the integral instead of $f(\theta)$ so make sure and substitute accordingly when doing the integral.

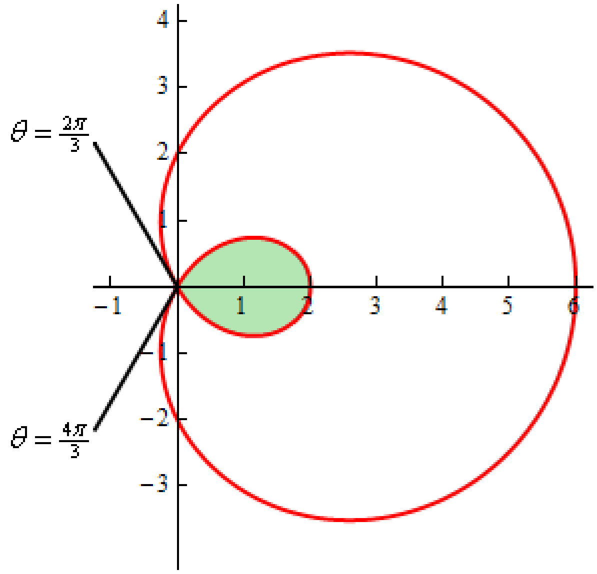
Let's take a look at an example.

Example 1 Determine the area of the inner loop of $r = 2 + 4\cos\theta$.

Solution
We graphed this function back when we first started looking at polar coordinates. For this problem we'll also need to know the values of θ where the curve goes through the origin. We can get these by setting the equation equal to zero and solving.

$$0 = 2 + 4\cos\theta$$
$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Here is the sketch of this curve with the inner loop shaded in.



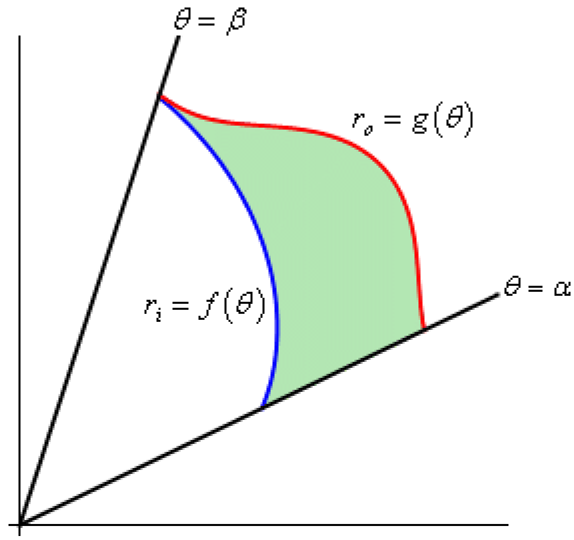
Can you see why we needed to know the values of θ where the curve goes through the origin? These points define where the inner loop starts and ends and hence are also the limits of integration in the formula.

So, the area is then,

$$\begin{aligned} A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(2+4\cos\theta\right)^2 \, d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4+16\cos\theta+16\cos^2\theta\right) d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2+8\cos\theta+4\left(1+\cos\left(2\theta\right)\right) d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6+8\cos\theta+4\cos\left(2\theta\right) d\theta \\ &= \left(6\theta+8\sin\theta+2\sin\left(2\theta\right)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \\ &= 4\pi-6\sqrt{3}=2.174 \end{aligned}$$

You did follow the work done in this integral didn't you? You'll run into quite a few integrals of trig functions in this section so if you need to you should go back to the Integrals Involving Trig Functions sections and do a quick review.

So, that's how we determine areas that are enclosed by a single curve, but what about situations like the following sketch where we want to find the area between two curves.



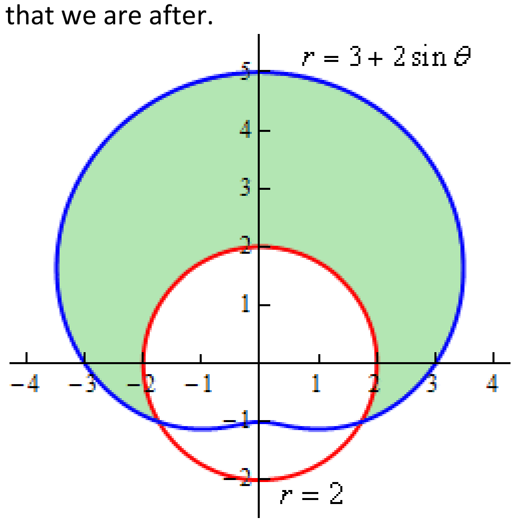
In this case we can use the above formula to find the area enclosed by both and then the actual area is the difference between the two. The formula for this is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_o^2 - r_i^2\right) d\theta$$

Let's take a look at an example of this.

Example 2 Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution
Here is a sketch of the region that we are after.

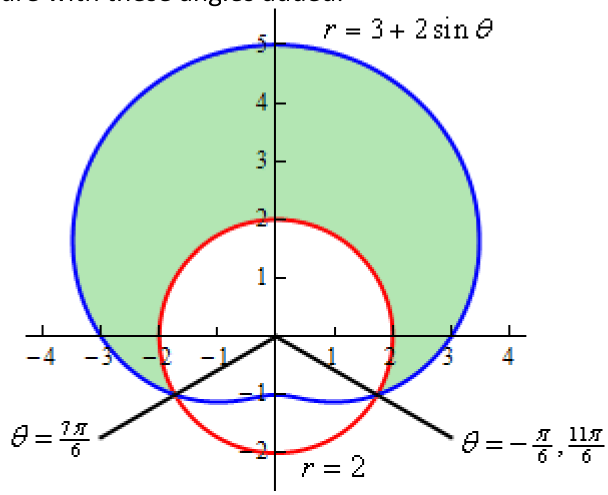


To determine this area, we'll need to know the values of θ for which the two curves intersect. We can determine these points by setting the two equations and solving.

$$3+2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well here that we also acknowledged that another representation for the angle $\frac{11\pi}{6}$ is $-\frac{\pi}{6}$. This is important for this problem. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However, if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area that we're after.

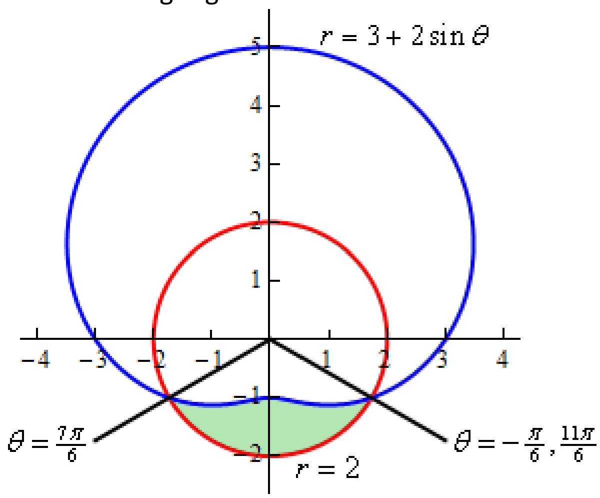
So, the area is then,

$$\begin{aligned}
A &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(\left(3+2\sin\theta\right)^2 - \left(2\right)^2\right) d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(5+12\sin\theta + 4\sin^2\theta\right) d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(7+12\sin\theta - 2\cos\left(2\theta\right)\right) d\theta \\
&= \frac{1}{2}\left(7\theta - 12\cos\theta - \sin\left(2\theta\right)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187
\end{aligned}$$

Let's work a slight modification of the previous example.

Example 3 Determine the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.

Solution
This time we're looking for the following region.



So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area for this region is,

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left((2)^2 - (3 + 2\sin\theta)^2\right) d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-5 - 12\sin\theta - 4\sin^2\theta\right) d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-7 - 12\sin\theta + 2\cos(2\theta)\right) d\theta$$

$$= \frac{1}{2}\left(-7\theta + 12\cos\theta + \sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

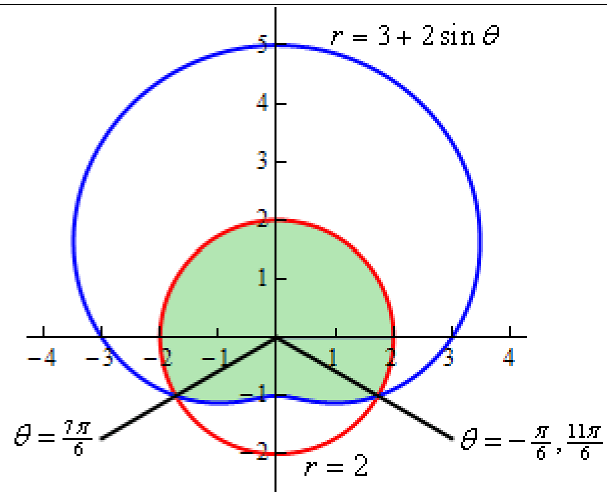
$$= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} = 2.196$$

Notice that for this area the "outer" and "inner" function were opposite!

Let's do one final modification of this example.

Example 4 Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

Solution
Here is the sketch for this example.



We are not going to be able to do this problem in the same fashion that we did the previous two. There is no set of limits that will allow us to enclose this area as we increase from one to the other. Remember that as we increase θ the area we're after must be enclosed. However, the only two ranges for θ that we can work with enclose the area from the previous two examples and not this region.

In this case however, that is not a major problem. There are two ways to do get the area in this problem. We'll take a look at both of them.

Solution 1
In this case let's notice that the circle is divided up into two portions and we're after the upper portion. Also notice that we found the area of the lower portion in Example 3. Therefore, the area is,

$$\begin{aligned}\text{Area} &= \text{Area of Circle} - \text{Area from Example 3} \\ &= \pi\left(2\right)^2 - 2.196 \\ &= 10.370\end{aligned}$$

Solution 2
In this case we do pretty much the same thing except this time we'll think of the area as the other portion of the limacon than the portion that we were dealing with in Example 2. We'll also need to actually compute the area of the limacon in this case.

So, the area using this approach is then,

$$\begin{aligned}
\text{Area} &= \text{Area of Limacon} - \text{Area from Example 2} \\
&= \int_0^{2\pi} \frac{1}{2}\left(3+2\sin\theta\right)^2 \, d\theta - 24.187 \\
&= \int_0^{2\pi} \frac{1}{2}\left(9+12\sin\theta+4\sin^2\theta\right) d\theta - 24.187 \\
&= \int_0^{2\pi} \frac{1}{2}\left(11+12\sin\theta-2\cos\left(2\theta\right)\right) d\theta - 24.187 \\
&= \frac{1}{2}\left(11\theta-12\cos\left(\theta\right)-\sin\left(2\theta\right)\right)\Big|_0^{2\pi} - 24.187 \\
&= 11\pi - 24.187 \\
&= 10.370
\end{aligned}$$

A slightly longer approach, but sometimes we are forced to take this longer approach.

As this last example has shown we will not be able to get all areas in polar coordinates straight from an integral.

Section 3-7 : Tangents with Polar Coordinates

We now need to discuss some calculus topics in terms of polar coordinates.

We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(\theta)$. With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we derived when we looked at tangent lines with parametric equations. To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.

From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(\theta)$. Substituting this into these equations gives the following set of parametric equations (with θ as the parameter) for the curve.

$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$

Now, we will need the following derivatives.

$$\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ &= \frac{dr}{d\theta}\cos\theta - r\sin\theta\end{aligned} \qquad\qquad \begin{aligned}\frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta \\ &= \frac{dr}{d\theta}\sin\theta + r\cos\theta\end{aligned}$$

The derivative $\frac{dy}{dx}$ is then,

Derivative with Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Note that rather than trying to remember this formula it would probably be easier to remember how we derived it and just remember the formula for parametric equations.

Let's work a quick example with this.

Example 1 Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \dfrac{\pi}{6}$.

Solution

We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos\theta$$

The formula for the derivative $\dfrac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + \left(3 + 8\sin\theta\right)\cos\theta}{8\cos^2\theta - \left(3 + 8\sin\theta\right)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

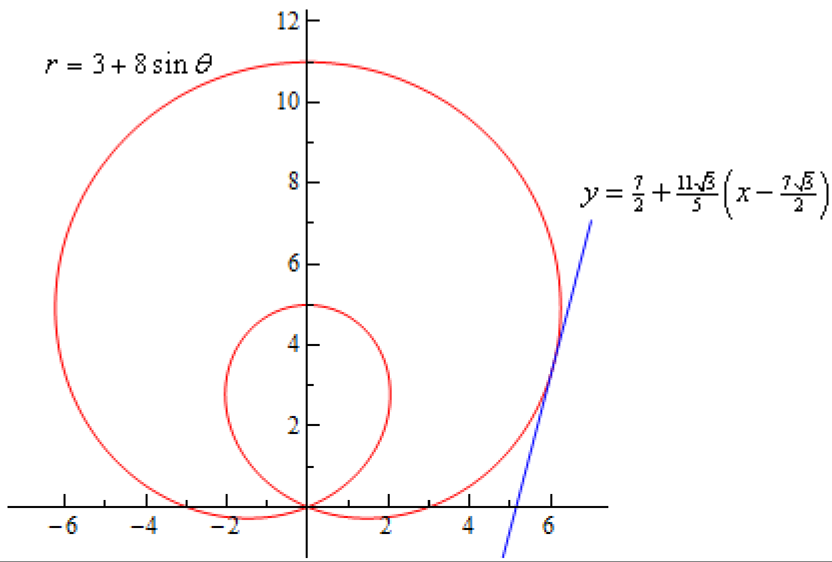
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$. We'll need to get the corresponding *x*-*y* coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad\qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

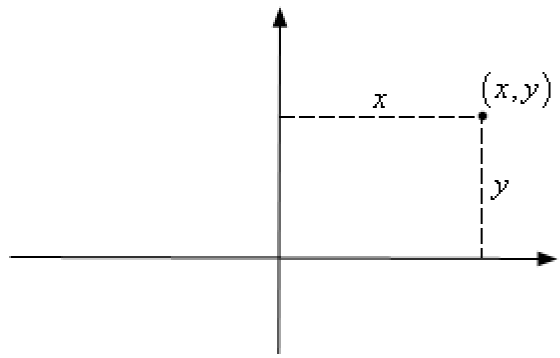
For the sake of completeness here is a graph of the curve and the tangent line.



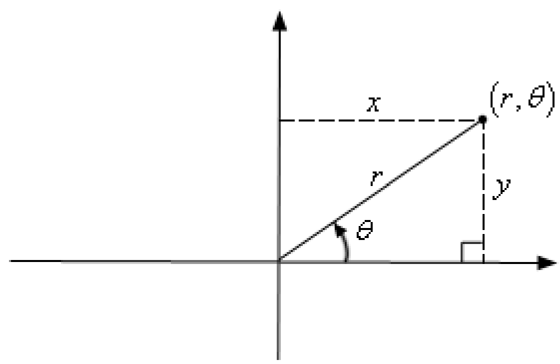
Section 3-6 : Polar Coordinates

Up to this point we've dealt exclusively with the Cartesian (or Rectangular, or *x*-*y*) coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. So, in this section we will start looking at the polar coordinate system.

Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates $\left(x, y\right)$ and we use this to define the point by starting at the origin and then moving *x* units horizontally followed by *y* units vertically. This is shown in the sketch below.

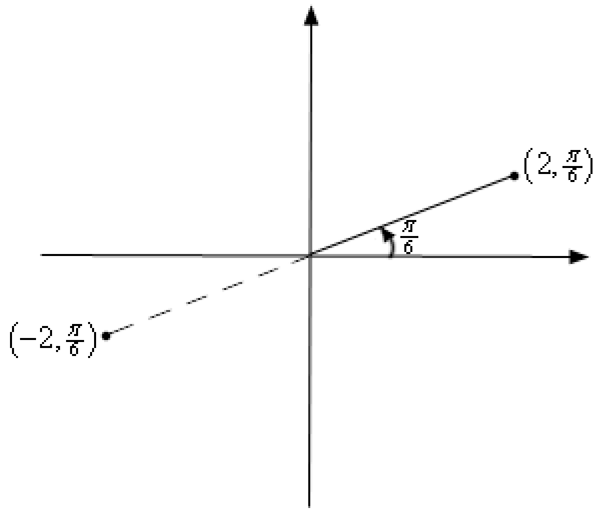


This is not, however, the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive *x*-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive *x*-axis as the coordinates of the point. This is shown in the sketch below.



Coordinates in this form are called **polar coordinates**.

The above discussion may lead one to think that *r* must be a positive number. However, we also allow *r* to be negative. Below is a sketch of the two points $\left(2, \frac{\pi}{6}\right)$ and $\left(-2, \frac{\pi}{6}\right)$.

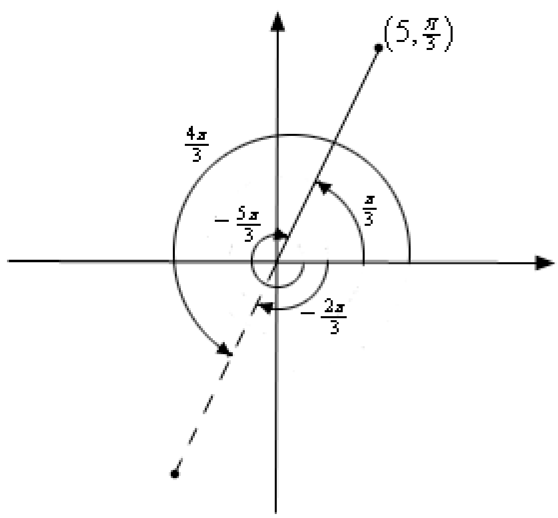


From this sketch we can see that if *r* is positive the point will be in the same quadrant as θ. On the other hand if *r* is negative the point will end up in the quadrant exactly opposite θ. Notice as well that the coordinates $\left(-2,\frac{\pi}{6}\right)$ describe the same point as the coordinates $\left(2,\frac{7\pi}{6}\right)$ do. The coordinates $\left(2,\frac{7\pi}{6}\right)$ tells us to rotate an angle of $\frac{7\pi}{6}$ from the positive *x*-axis, this would put us on the dashed line in the sketch above, and then move out a distance of 2.

This leads to an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$\left(5,\frac{\pi}{3}\right)=\left(5,-\frac{5\pi}{3}\right)=\left(-5,\frac{4\pi}{3}\right)=\left(-5,-\frac{2\pi}{3}\right)$$

Here is a sketch of the angles used in these four sets of coordinates.



In the second coordinate pair we rotated in a clock-wise direction to get to the point. We shouldn't forget about rotating in the clock-wise direction. Sometimes it's what we have to do.

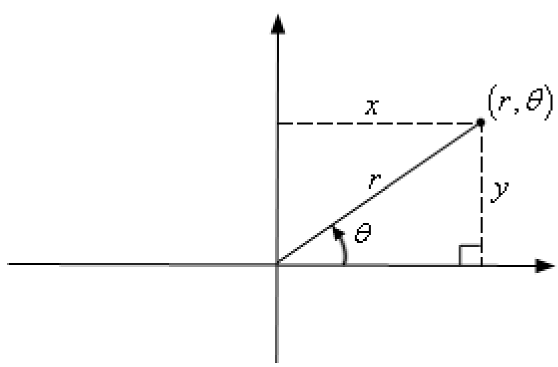
The last two coordinate pairs use the fact that if we end up in the opposite quadrant from the point we can use a negative *r* to get back to the point and of course there is both a counter clock-wise and a clock-wise rotation to get to the angle.

These four points only represent the coordinates of the point without rotating around the system more than once. If we allow the angle to make as many complete rotations about the axis system as we want then there are an infinite number of coordinates for the same point. In fact, the point (r,θ) can be represented by any of the following coordinate pairs.

$$(r,\theta+2\pi n) \qquad\qquad (-r,\theta+(2n+1)\pi), \qquad \text{where } n \text{ is any integer.}$$

Next, we should talk about the origin of the coordinate system. In polar coordinates the origin is often called the **pole**. Because we aren't actually moving away from the origin/pole we know that $r=0$. However, we can still rotate around the system by any angle we want and so the coordinates of the origin/pole are $(0,\theta)$.

Now that we've got a grasp on polar coordinates we need to think about converting between the two coordinate systems. Well start out with the following sketch reminding us how both coordinate systems work.



Note that we've got a right triangle above and with that we can get the following equations that will convert polar coordinates into Cartesian coordinates.

Polar to Cartesian Conversion Formulas

$x = r\cos\theta$	$y = r\sin\theta$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$\begin{aligned} x^2+y^2 &= (r\cos\theta)^2+(r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta+\sin^2\theta\right) = r^2 \end{aligned}$$

This is a very useful formula that we should remember, however we are after an equation for *r* so let's take the square root of both sides. This gives,

$$r = \sqrt{x^2 + y^2}$$

Note that technically we should have a plus or minus in front of the root since we know that *r* can be either positive or negative. We will run with the convention of positive *r* here.

Getting an equation for θ is almost as simple. We'll start with,

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

Taking the inverse tangent of both sides gives,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

We will need to be careful with this because inverse tangents only return values in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Recall that there is a second possible angle and that the second angle is given by $\theta + \pi$.

Summarizing then gives the following formulas for converting from Cartesian coordinates to polar coordinates.

Cartesian to Polar Conversion Formulas

$$r^2 = x^2 + y^2 \qquad\qquad r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Let's work a quick example.

Example 1 Convert each of the following points into the given coordinate system.

(a) $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

(b) $\left(-1, -1\right)$ into polar coordinates.

Solution

(a) Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

This conversion is easy enough. All we need to do is plug the points into the formulas.

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$.

(b) Convert $\left(-1,-1\right)$ into polar coordinates.

Let's first get *r*.

$$r = \sqrt{\left(-1\right)^2 + \left(-1\right)^2} = \sqrt{2}$$

Now, let's get θ.

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}\left(1\right) = \frac{\pi}{4}$$

This is not the correct angle however. This value of θ is in the first quadrant and the point we've been given is in the third quadrant. As noted above we can get the correct angle by adding π onto this. Therefore, the actual angle is,

$$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

So, in polar coordinates the point is $\left(\sqrt{2}, \frac{5\pi}{4}\right)$. Note as well that we could have used the first θ that we got by using a negative *r*. In this case the point could also be written in polar coordinates as $\left(-\sqrt{2}, \frac{\pi}{4}\right)$.

We can also use the above formulas to convert equations from one coordinate system to the other.

Example 2 Convert each of the following into an equation in the given coordinate system.

(a) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.

(b) Convert $r = -8\cos\theta$ into Cartesian coordinates.

Solution

(a) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.

In this case there really isn't much to do other than plugging in the formulas for *x* and *y* (*i.e.* the Cartesian coordinates) in terms of *r* and θ (*i.e.* the polar coordinates).

$$2\left(r\cos\theta\right) - 5\left(r\cos\theta\right)^3 = 1 + \left(r\cos\theta\right)\left(r\sin\theta\right)$$

$$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

(b) Convert $r = -8\cos\theta$ into Cartesian coordinates.

This one is a little trickier, but not by much. First notice that we could substitute straight for the *r*. However, there is no straight substitution for the cosine that will give us only Cartesian coordinates. If we had an *r* on the right along with the cosine then we could do a direct substitution. So, if an *r* on the right side would be convenient let's put one there, just don't forget to put one on the left side as well.

$$r^2 = -8r\cos\theta$$

We can now make some substitutions that will convert this into Cartesian coordinates.

$$x^2 + y^2 = -8x$$

Before moving on to the next subject let's do a little more work on the second part of the previous example.

The equation given in the second part is actually a fairly well known graph; it just isn't in a form that most people will quickly recognize. To identify it let's take the Cartesian coordinate equation and do a little rearranging.

$$x^2 + 8x + y^2 = 0$$

Now, complete the square on the *x* portion of the equation.

$$x^2 + 8x + 16 + y^2 = 16$$
$$(x+4)^2 + y^2 = 16$$

So, this was a circle of radius 4 and center $(-4, 0)$.

This leads us into the final topic of this section.

Common Polar Coordinate Graphs

Let's identify a few of the more common graphs in polar coordinates. We'll also take a look at a couple of special polar graphs.

Lines

Some lines have fairly simple equations in polar coordinates.

1. $\theta = \beta$.

   We can see that this is a line by converting to Cartesian coordinates as follows

$$\theta = \beta$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \beta$$
$$\frac{y}{x} = \tan\beta$$
$$y = (\tan\beta)\,x$$

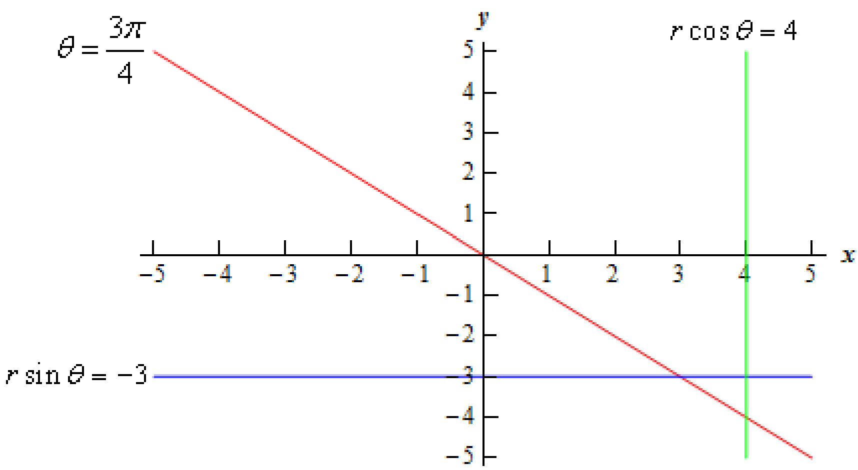
This is a line that goes through the origin and makes an angle of β with the positive *x*-axis. Or, in other words it is a line through the origin with slope of $\tan\beta$.

2. $r\cos\theta = a$
   This is easy enough to convert to Cartesian coordinates to $x = a$. So, this is a vertical line.

3. $r\sin\theta = b$
   Likewise, this converts to $y = b$ and so is a horizontal line.

Example 3 Graph $\theta = \frac{3\pi}{4}$, $r\cos\theta = 4$ and $r\sin\theta = -3$ on the same axis system.

Solution
There really isn't too much to this one other than doing the graph so here it is.



Circles
Let's take a look at the equations of circles in polar coordinates.

1. $r = a$.
   This equation is saying that no matter what angle we've got the distance from the origin must be *a*. If you think about it that is exactly the definition of a circle of radius *a* centered at the origin.

   So, this is a circle of radius *a* centered at the origin. This is also one of the reasons why we might want to work in polar coordinates. The equation of a circle centered at the origin has a very nice equation, unlike the corresponding equation in Cartesian coordinates.

2. $r = 2a\cos\theta$.
We looked at a specific example of one of these when we were converting equations to Cartesian coordinates.

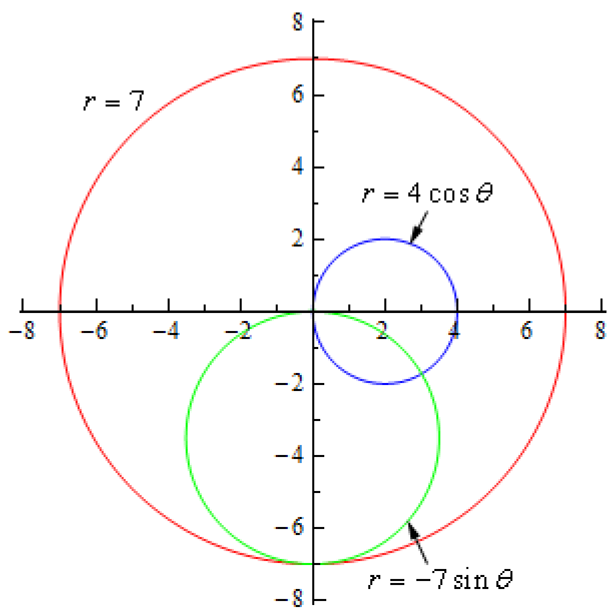
This is a circle of radius $|a|$ and center $(a,0)$. Note that *a* might be negative (as it was in our example above) and so the absolute value bars are required on the radius. They should not be used however on the center.

3. $r = 2b\sin\theta$.
This is similar to the previous one. It is a circle of radius $|b|$ and center $(0,b)$.

4. $r = 2a\cos\theta + 2b\sin\theta$.
This is a combination of the previous two and by completing the square twice it can be shown that this is a circle of radius $\sqrt{a^2+b^2}$ and center (a,b). In other words, this is the general equation of a circle that isn't centered at the origin.

Example 4 Graph $r = 7$, $r = 4\cos\theta$, and $r = -7\sin\theta$ on the same axis system.

Solution
The first one is a circle of radius 7 centered at the origin. The second is a circle of radius 2 centered at $(2,0)$. The third is a circle of radius $\frac{7}{2}$ centered at $\left(0,-\frac{7}{2}\right)$. Here is the graph of the three equations.



Note that it takes a range of $0 \le \theta \le 2\pi$ for a complete graph of $r = a$ and it only takes a range of $0 \le \theta \le \pi$ to graph the other circles given here. You can verify this with a quick table of values if you'd like to.

Cardioids and Limacons
These can be broken up into the following three cases.

1. Cardioids : $r = a \pm a\cos\theta$ and $r = a \pm a\sin\theta$.
   These have a graph that is vaguely heart shaped and always contain the origin.

2. Limacons with an inner loop : $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a < b$.
   These will have an inner loop and will always contain the origin.

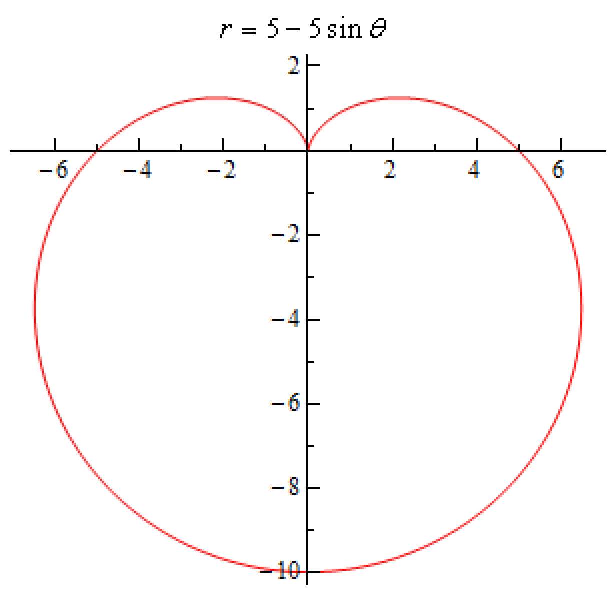
3. Limacons without an inner loop : $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a > b$.
   These do not have an inner loop and do not contain the origin.

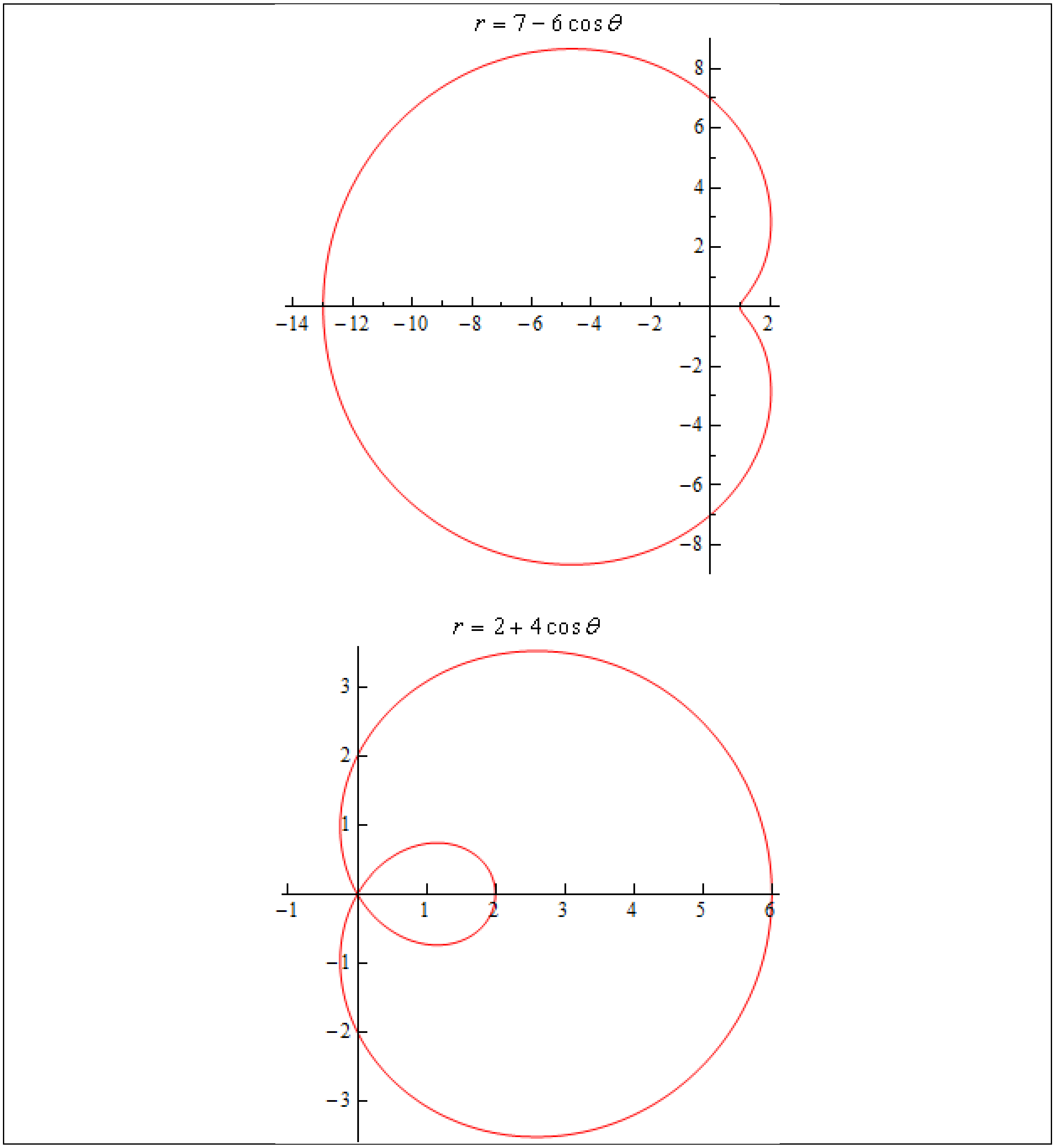
Example 5 Graph $r = 5 - 5\sin\theta$, $r = 7 - 6\cos\theta$, and $r = 2 + 4\cos\theta$.

Solution

These will all graph out once in the range $0 \le \theta \le 2\pi$. Here is a table of values for each followed by graphs of each.

θ	$r = 5 - 5\sin\theta$	$r = 7 - 6\cos\theta$	$r = 2 + 4\cos\theta$
0	5	1	6
$\frac{\pi}{2}$	0	7	2
π	5	13	-2
$\frac{3\pi}{2}$	10	7	2
2π	5	1	6





There is one final thing that we need to do in this section. In the third graph in the previous example we had an inner loop. We will, on occasion, need to know the value of θ for which the graph will pass through the origin. To find these all we need to do is set the equation equal to zero and solve as follows,

$$0 = 2 + 4\cos\theta \qquad \Rightarrow \qquad \cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Section 3-5 : Surface Area with Parametric Equations

In this final section of looking at calculus applications with parametric equations we will take a look at determining the surface area of a region obtained by rotating a parametric curve about the *x* or *y*-axis.

We will rotate the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

about the *x* or *y*-axis. We are going to assume that the curve is traced out exactly once as *t* increases from α to β. At this point there actually isn't all that much to do. We know that the surface area can be found by using one of the following two formulas depending on the axis of rotation (recall the Surface Area section of the Applications of Integrals chapter).

$$S = \int 2\pi y \, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x \, ds \qquad \text{rotation about } y-\text{axis}$$

All that we need is a formula for *ds* to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t),\, y = g(t),\; \alpha \le t \le \beta$$

which is exactly what we need.

We will need to be careful with the *x* or *y* that is in the original surface area formula. Back when we first looked at surface area we saw that sometimes we had to substitute for the variable in the integral and at other times we didn't. This was dependent upon the *ds* that we used. In this case however, we will always have to substitute for the variable. The *ds* that we use for parametric equations introduces a *dt* into the integral and that means that everything needs to be in terms of *t*. Therefore, we will need to substitute the appropriate parametric equation for *x* or *y* depending on the axis of rotation.

Let's take a quick look at an example.

Example 1 Determine the surface area of the solid obtained by rotating the following parametric curve about the *x*-axis.

$$x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{d\theta} = -3\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

Before plugging into the surface area formula let's get the *ds* out of the way.

$$
\begin{aligned}
ds &= \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\, d\theta \\
&= 3\left|\cos\theta\sin\theta\right|\sqrt{\cos^2\theta + \sin^2\theta}\, d\theta \\
&= 3\cos\theta\sin\theta\, d\theta
\end{aligned}
$$

Notice that we could drop the absolute value bars since both sine and cosine are positive in this range of θ given.

Now let's get the surface area and don't forget to also plug in for the *y*.

$$
\begin{aligned}
S &= \int 2\pi y\, ds \\
&= 2\pi\int_0^{\frac{\pi}{2}} \sin^3\theta\left(3\cos\theta\sin\theta\right)\, d\theta \\
&= 6\pi\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\, d\theta \qquad\qquad u = \sin\theta \\
&= 6\pi\int_0^1 u^4\, du \\
&= \frac{6\pi}{5}
\end{aligned}
$$

Section 3-4 : Arc Length with Parametric Equations

In the previous two sections we've looked at a couple of Calculus I topics in terms of parametric equations. We now need to look at a couple of Calculus II topics in terms of parametric equations.

In this section we will look at the arc length of the parametric curve given by,

$$x = f(t) \qquad\qquad y = g(t) \qquad\qquad \alpha \le t \le \beta$$

We will also be assuming that the curve is traced out exactly once as *t* increases from α to β. We will also need to assume that the curve is traced out from left to right as *t* increases. This is equivalent to saying,

$$\frac{dx}{dt} \ge 0 \qquad\qquad \text{for } \alpha \le t \le \beta$$

So, let's start out the derivation by recalling the arc length formula as we first derived it in the arc length section of the Applications of Integrals chapter.

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

We will use the first *ds* above because we have a nice formula for the derivative in terms of the parametric equations (see the Tangents with Parametric Equations section). To use this we'll also need to know that,

$$dx = f'(t)\,dt = \frac{dx}{dt}\,dt$$

The arc length formula then becomes,

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\ \frac{dx}{dt}\,dt = \int_{\alpha}^{\beta} \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\ \frac{dx}{dt}\,dt$$

This is a particularly unpleasant formula. However, if we factor out the denominator from the square root we arrive at,

$$L = \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; \frac{dx}{dt} \, dt$$

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will allow us to cancel the two derivatives that are outside the square root and this gives,

Arc Length for Parametric Equations

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

Notice that we could have used the second formula for *ds* above if we had assumed instead that

$$\frac{dy}{dt} \ge 0 \qquad\qquad \text{for } \; \alpha \le t \le \beta$$

If we had gone this route in the derivation we would have gotten the same formula.

Let's take a look at an example.

Example 1 Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(t) \qquad\qquad y = 3\cos(t) \qquad\qquad 0 \le t \le 2\pi$$

Solution

We know that this is a circle of radius 3 centered at the origin from our prior discussion about graphing parametric curves. We also know from this discussion that it will be traced out exactly once in this range.

So, we can use the formula we derived above. We'll first need the following,

$$\frac{dx}{dt} = 3\cos(t) \qquad\qquad \frac{dy}{dt} = -3\sin(t)$$

The length is then,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)} \; dt \\ &= \int_0^{2\pi} 3\sqrt{\sin^2(t) + \cos^2(t)} \; dt \\ &= 3\int_0^{2\pi} dt \\ &= 6\pi \end{aligned}$$

Since this is a circle we could have just used the fact that the length of the circle is just the circumference of the circle. This is a nice way, in this case, to verify our result.

Let's take a look at one possible consequence if a curve is traced out more than once and we try to find the length of the curve without taking this into account.

Example 2 Use the arc length formula for the following parametric equations.

$$x = 3\sin\left(3t\right) \qquad y = 3\cos\left(3t\right) \qquad 0 \le t \le 2\pi$$

Solution

Notice that this is the identical circle that we had in the previous example and so the length is still 6π. However, for the range given we know it will trace out the curve three times instead once as required for the formula. Despite that restriction let's use the formula anyway and see what happens.

In this case the derivatives are,

$$\frac{dx}{dt} = 9\cos\left(3t\right) \qquad \frac{dy}{dt} = -9\sin\left(3t\right)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2\left(3t\right) + 81\cos^2\left(3t\right)}\, dt \\ &= \int_0^{2\pi} 9\, dt \\ &= 18\pi \end{aligned}$$

The answer we got form the arc length formula in this example was 3 times the actual length. Recalling that we also determined that this circle would trace out three times in the range given, the answer should make some sense.

If we had wanted to determine the length of the circle for this set of parametric equations we would need to determine a range of *t* for which this circle is traced out exactly once. This is, $0 \le t \le \frac{2\pi}{3}$. Using this range of *t* we get the following for the length.

$$\begin{aligned} L &= \int_0^{\frac{2\pi}{3}} \sqrt{81\sin^2\left(3t\right) + 81\cos^2\left(3t\right)}\, dt \\ &= \int_0^{\frac{2\pi}{3}} 9\, dt \\ &= 6\pi \end{aligned}$$

which is the correct answer.

Be careful to not make the assumption that this is always what will happen if the curve is traced out more than once. Just because the curve traces out *n* times does not mean that the arc length formula will give us *n* times the actual length of the curve!

Before moving on to the next section let's notice that we can put the arc length formula derived in this section into the same form that we had when we first looked at arc length. The only difference is that we will add in a definition for *ds* when we have parametric equations.

The arc length formula can be summarized as,

$$L = \int ds$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y = f\left(x\right),\ a \le x \le b$$

$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x = h\left(y\right),\ c \le y \le d$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad \text{if } x = f\left(t\right), y = g\left(t\right),\ \alpha \le t \le \beta$$

Section 3-2 : Tangents with Parametric Equations

In this section we want to find the tangent lines to the parametric equations given by,

$$x = f(t) \qquad\qquad y = g(t)$$

To do this let's first recall how to find the tangent line to $y = F(x)$ at $x = a$. Here the tangent line is given by,

$$y = F(a) + m(x-a), \quad \text{where} \quad m = \left.\frac{dy}{dx}\right|_{x=a} = F'(a)$$

Now, notice that if we could figure out how to get the derivative $\frac{dy}{dx}$ from the parametric equations we could simply reuse this formula since we will be able to use the parametric equations to find the *x* and *y* coordinates of the point.

So, just for a second let's suppose that we were able to eliminate the parameter from the parametric form and write the parametric equations in the form $y = F(x)$. Now, plug the parametric equations in for *x* and *y*. Yes, it seems silly to eliminate the parameter, then immediately put it back in, but it's what we need to do in order to get our hands on the derivative. Doing this gives,

$$g(t) = F\left(f(t)\right)$$

Now, differentiate with respect to *t* and notice that we'll need to use the Chain Rule on the right-hand side.

$$g'(t) = F'\left(f(t)\right) f'(t)$$

Let's do another change in notation. We need to be careful with our derivatives here. Derivatives of the lower case function are with respect to *t* while derivatives of upper case functions are with respect to *x*. So, to make sure that we keep this straight let's rewrite things as follows.

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt}$$

At this point we should remind ourselves just what we are after. We needed a formula for $\frac{dy}{dx}$ or $F'(x)$ that is in terms of the parametric formulas. Notice however that we can get that from the above equation.

Derivative for Parametric Equations

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided} \quad \frac{dx}{dt} \neq 0$$

Notice as well that this will be a function of *t* and not *x*.

As an aside, notice that we could also get the following formula with a similar derivation if we needed to,

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}, \qquad \text{provided} \qquad \frac{dy}{dt} \neq 0$$

Why would we want to do this? Well, recall that in the arc length section of the Applications of Integral section we actually needed this derivative on occasion.

So, let's find a tangent line.

Example 1 Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad\qquad y = t^2$$

at $(0,4)$.

Solution

Note that there is apparently the potential for more than one tangent line here! We will look into this more after we're done with the example.

The first thing that we should do is find the derivative so we can get the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

At this point we've got a small problem. The derivative is in terms of *t* and all we've got is an *x*-*y* coordinate pair. The next step then is to determine the value(s) of *t* which will give this point. We find these by plugging the *x* and *y* values into the parametric equations and solving for *t*.

$$\begin{aligned} 0 &= t^5 - 4t^3 = t^3\left(t^2 - 4\right) && \Rightarrow && t = 0, \pm 2 \\ 4 &= t^2 && \Rightarrow && t = \pm 2 \end{aligned}$$

Any value of *t* which appears in both lists will give the point. So, since there are two values of *t* that give the point we will in fact get two tangent lines. That's definitely not something that happened back in Calculus I and we're going to need to look into this a little more. However, before we do that let's actually get the tangent lines.

$t = -2$:

Since we already know the *x* and *y*-coordinates of the point all that we need to do is find the slope of the tangent line.

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

The tangent line (at $t = -2$) is then,

$$y = 4 - \frac{1}{8}x$$

$t = 2$:

Again, all we need is the slope.

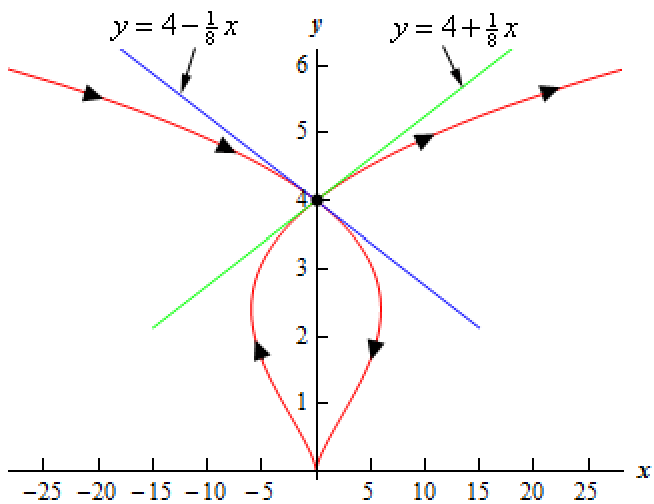
$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

The tangent line (at $t = 2$) is then,

$$y = 4 + \frac{1}{8}x$$

Before we leave this example let's take a look at just how we could possibly get two tangents lines at a point. This was definitely not possible back in Calculus I where we first ran across tangent lines.

A quick graph of the parametric curve will explain what is going on here.



So, the parametric curve crosses itself! That explains how there can be more than one tangent line. There is one tangent line for each instance that the curve goes through the point.

The next topic that we need to discuss in this section is that of horizontal and vertical tangents. We can easily identify where these will occur (or at least the t's that will give them) by looking at the derivative formula.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Horizontal tangents will occur where the derivative is zero and that means that we'll get horizontal tangent at values of *t* for which we have,

Horizontal Tangent for Parametric Equations

$$\frac{dy}{dt} = 0, \quad \text{provided} \quad \frac{dx}{dt} \neq 0$$

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values of *t* for which we have,

Vertical Tangent for Parametric Equations

$$\frac{dx}{dt} = 0, \quad \text{provided} \quad \frac{dy}{dt} \neq 0$$

Let's take a quick look at an example of this.

Example 2 Determine the *x-y* coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$$x = t^3 - 3t \qquad\qquad y = 3t^2 - 9$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = 3t^2 - 3 = 3\left(t^2 - 1\right) \qquad\qquad \frac{dy}{dt} = 6t$$

Horizontal Tangents

We'll have horizontal tangents where,

$$6t = 0 \qquad\qquad \Rightarrow \qquad\qquad t = 0$$

Now, this is the value of *t* which gives the horizontal tangents and we were asked to find the *x-y* coordinates of the point. To get these we just need to plug *t* into the parametric equations. Therefore, the only horizontal tangent will occur at the point $\left(0,-9\right)$.

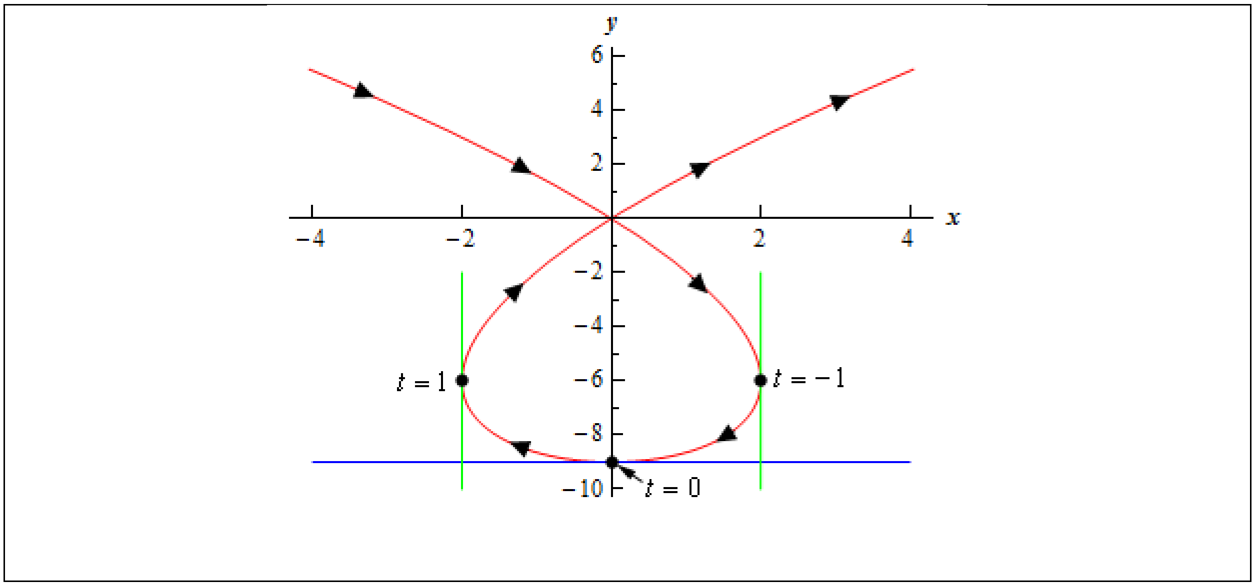
Vertical Tangents

In this case we need to solve,

$$3\left(t^2 - 1\right) = 0 \qquad\qquad \Rightarrow \qquad\qquad t = \pm 1$$

The two vertical tangents will occur at the points $\left(2,-6\right)$ and $\left(-2,-6\right)$.

For the sake of completeness and at least partial verification here is the sketch of the parametric curve.



The final topic that we need to discuss in this section really isn't related to tangent lines but does fit in nicely with the derivation of the derivative that we needed to get the slope of the tangent line.

Before moving into the new topic let's first remind ourselves of the formula for the first derivative and in the process rewrite it slightly.

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}}$$

Written in this way we can see that the formula actually tells us how to differentiate a function *y* (as a function of *t*) with respect to *x* (when *x* is also a function of *t*) when we are using parametric equations.

Now let's move onto the final topic of this section. We would also like to know how to get the second derivative of *y* with respect to *x*.

$$\frac{d^2y}{dx^2}$$

Getting a formula for this is fairly simple if we remember the rewritten formula for the first derivative above.

Second Derivative for Parametric Equations

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

It is important to note that,

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

Let's work a quick example.

Example 3 Find the second derivative for the following set of parametric equations.

$$x = t^5 - 4t^3 \qquad\qquad y = t^2$$

Solution

This is the set of parametric equations that we used in the first example and so we already have the following computations completed.

$$\frac{dy}{dt} = 2t \qquad\qquad \frac{dx}{dt} = 5t^4 - 12t^2 \qquad\qquad \frac{dy}{dx} = \frac{2}{5t^3 - 12t}$$

We will first need the following,

$$\frac{d}{dt}\left(\frac{2}{5t^3 - 12t}\right) = \frac{-2\left(15t^2 - 12\right)}{\left(5t^3 - 12t\right)^2} = \frac{24 - 30t^2}{\left(5t^3 - 12t\right)^2}$$

The second derivative is then,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{24 - 30t^2}{\left(5t^3 - 12t\right)^2}}{5t^4 - 12t^2} \\ &= \frac{24 - 30t^2}{\left(5t^4 - 12t^2\right)\left(5t^3 - 12t\right)^2} \\ &= \frac{24 - 30t^2}{t\left(5t^3 - 12t\right)^3} \end{aligned}$$

So, why would we want the second derivative? Well, recall from your Calculus I class that with the second derivative we can determine where a curve is concave up and concave down. We could do the same thing with parametric equations if we wanted to.

Example 4 Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.

$$x = 1 - t^2 \qquad\qquad y = t^7 + t^5$$

Solution

To compute the second derivative we'll first need the following.

$$\frac{dy}{dt} = 7t^6 + 5t^4 \qquad\qquad \frac{dx}{dt} = -2t \qquad\qquad \frac{dy}{dx} = \frac{7t^6 + 5t^4}{-2t} = -\frac{1}{2}\left(7t^5 + 5t^3\right)$$

Note that we can also use the first derivative above to get some information about the increasing/decreasing nature of the curve as well. In this case it looks like the parametric curve will be increasing if $t < 0$ and decreasing if $t > 0$.

Now let's move on to the second derivative.

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\left(35t^4 + 15t^2\right)}{-2t} = \frac{1}{4}\left(35t^3 + 15t\right)$$

It's clear, hopefully, that the second derivative will only be zero at $t = 0$. Using this we can see that the second derivative will be negative if $t < 0$ and positive if $t > 0$. So the parametric curve will be concave down for $t < 0$ and concave up for $t > 0$.

Here is a sketch of the curve for completeness sake.