ECE 556— Microwave Engineering I
UNIVERSITY OF VIRGINIA

Lecture 2— Plane Waves

In the previous lecture, we saw that Kirchhoff's Laws do not strictly apply at sufficiently high frequencies. Faraday's Law and the continuity equation (upon which KVL and KCL are based) have time-dependent terms that are typically neglected in Kirchhoff's Laws. These terms, which represent the induced electromotive force (emf) due to current flowing around the circuit loop and the stray capacitance between the nodes of the circuit and ground, are difficult to account for and are usually ignored in circuit calculations. Furthermore, the fields that are responsible for a circuit's electrical behavior are functions of space as well as time. Classical circuit theory doesn't account for the spatial dependence of the electromagnetic fields or their corresponding electrical parameters, voltage and current. However, at at microwave frequencies, we need to account for both the spatial and temporal variations of electrical signals in a circuit. To do this, we will re-examine Maxwell's equations.

MAXWELL'S EQUATIONS AND PHASORS

In most applications of electrical engineering, we are concerned with the transfer of energy or information in the form of electrical signals. Like all electromagnetic phenomena, the propagation of electrical energy is described by Maxwell's equations and travelling waves. To describe electromagnetic propagation it is most convenient to work with Maxwell's equations in *differential* form. If we assume a uniform (ϵ, μ not functions of position), isotropic (ϵ, μ not functions of direction), time-invariant (ϵ, μ not functions of time), and source-free (current and charge are zero) medium then Maxwell's equations can be written in the form:

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \text{Faraday's Law}$$

$$\vec{\nabla} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \qquad \text{Ampere's Law}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon} = 0 \qquad \text{Gauss's Law}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

As a reminder, $\vec{\nabla}$ is an operator called "del" and is written as,

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

The *divergence* of a vector function $\vec{F}$ is a scalar given by,

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

and the *curl* of the vector function $\vec{F}$ is a vector given by,

$$\vec{\nabla} \times \vec{F} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix}$$

where "det" specifies the determinant.

Most typically, we will be interested in sinusoidal or *time-harmonic* signals, so it is convenient to use a phasor representation for the fields. Recall that for linear systems, the frequency of input and output signals are the same (i.e., a linear system *cannot* create new frequencies). Consequently, we only need to keep track of the magnitude and phase. A phasor is just a complex number representation that gives the magnitude and phase of a sinusoidal signal. As a review, recall that we may write an arbitrary sinusoidal (or "AC") quantity, $f(t)$, as

$$f(t) = \sqrt{2} F \cos(\omega t + \theta) = \sqrt{2} \mathrm{Re}\Big\{ \underline{F} \exp{(j\omega t)} \Big\}$$

In the above representation, the operator $\mathrm{Re}\{\cdot\}$ selects the real part of the quantity inside the brackets and

- $f(t)$ is called the "instantaneous" quantity,
- F is called the "rms" (root-mean-square) amplitude, and
- $\underline{F} = F \exp(j\theta)$ is called the "phasor" representation of $f(t)$.

When expressing vector fields as phasors, you should keep in mind that there will be *three* components — one corresponding to each component of the vector. We will indicate phasors with an underscore and vectors with an arrow. Thus, time-harmonic electric and magnetic fields will be written as,

$$\vec{E} = \sqrt{2} \mathrm{Re}\Big\{ \underline{\vec{E}} e^{j\omega t} \Big\} \qquad\qquad \vec{H} = \sqrt{2} \mathrm{Re}\Big\{ \underline{\vec{H}} e^{j\omega t} \Big\}$$

$\underline{\vec{F}}$ is just shorthand notation for,

$$\underline{\vec{F}} = \hat{x} \underline{F}_x + \hat{y} \underline{F}_y + \hat{z} \underline{F}_z,$$

where each component of the vector $\underline{\vec{F}}$ is a phasor quantity. You should keep in mind that each phasor component of a vector has, in general, a different magnitude and phase.

A significant advantage of phasors is that time-derivatives are simply replaced by factors of $j\omega$. Using this property, we can rewrite Maxwell's equations for instantaneous quantities (with explicit time-dependence) in phasor form:

$$\vec{\nabla} \times \underline{\vec{E}} = -j\omega\mu\underline{\vec{H}} \tag{1}$$

$$\vec{\nabla} \times \underline{\vec{H}} = j\omega\epsilon\underline{\vec{E}} \tag{2}$$

$$\vec{\nabla} \cdot \underline{\vec{E}} = 0 \tag{3}$$

$$\vec{\nabla} \cdot \underline{\vec{H}} = 0 \tag{4}$$

This is the form of Maxwell's equations that we will find most useful in this course.

The Wave Equation and Plane Waves

Maxwell's equations, as written above, are a set of coupled differential equations. To uncouple them, we take the curl of Faraday's Law (equation 1),

$$\vec{\nabla} \times (\vec{\nabla} \times \underline{\vec{E}}) = -j\omega\mu\vec{\nabla} \times \underline{\vec{H}}$$

Using the vector identity,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2\vec{F}$$

and substituting from Ampére's Law (equation 2) and Gauss's Law (equation 3) we have

$$-\nabla^2\underline{\vec{E}} = -j\omega\mu(j\omega\epsilon\underline{\vec{E}}),$$

or

$$\boxed{\nabla^2 \underline{\vec{E}} = -\omega^2 \mu\epsilon \underline{\vec{E}} \equiv -k^2 \underline{\vec{E}} \qquad (5)}$$

Repeating this process, but starting with Ampére's Law results in

$$\boxed{\nabla^2 \underline{\vec{H}} = -\omega^2 \mu\epsilon \underline{\vec{H}} = -k^2 \underline{\vec{H}} \qquad (6)}$$

Equations (5) and (6) are called the "homogeneous Helmholtz equations" and they are the phasor form of the "homogeneous wave equation:"

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

The parameter k is central to the theory of wave propagation and is called the *wavenumber* or *phase constant*. It is given by,

$$\boxed{k^2 = \omega^2 \mu\epsilon \qquad (7)}$$

Equation (7) is also very important is known as the "dispersion relation."

In general, to obtain a solution to the Helmholtz equations, we require boundary conditions. Some simple and particularly useful solutions can be found, however, by making a couple of simplifying assumptions. We will assume:

- Assume the vector field $\underline{\vec{E}}$ has only one component (say, $\underline{E}_x$). This assumption results in a *linearly polarized* field.
- Assume that $\underline{\vec{E}}$ varies with only one coordinate (e.g., z). This results in a "uniform" plane wave

You might think the above assumptions are rather unrealistic and, consequently, cannot represent a practical solution to the Helmholtz equation. You are right! However, there is a very good reason for studying these "idealized" solutions ... they are relatively simple. Furthermore, the objection raised above can, for the most part, be addressed:

- Plane waves are a good approximation to real waves in many practical situations,
- More complex wave solutions can be written in terms of a superposition of plane waves. In other words, plane waves can serve as a "basis" for constructing more complicated wave solutions that have realistic boundary conditions, and
- most of the basic phenomena and important properties of waves are easiest to study and understand in terms of plane waves.

Applying the assumptions outlined above reduces the Helmholtz equation to one-dimensional, second-order linear differential equation, i.e.,

$$\frac{d^2 \underline{E}_x}{dz^2} + k^2 \underline{E}_x = 0$$

There are two linearly-independent solutions to this type of equation and they can be expressed as complex exponentials. The general solution is a linear combination of these,

$$\underline{E}_x = A \exp(-jkz) + B \exp(jkz)$$

The complex exponents, $\exp(\pm jkz)$ are called "propagation factors." We can write the instantaneous form of the solution by multiplying by $\exp(j\omega t)$ and taking the real part:

$$\vec{E}(z,t) = \sqrt{2}\mathrm{Re}\Big\{ \Big(A \exp(-jkz) + B \exp(jkz) \Big) \exp(j\omega t) \Big\} \hat{x}, \quad \text{or}$$

$$\vec{E}(z,t) = \sqrt{2} A \cos(\omega t - kz)\hat{x} + \sqrt{2} B \cos(\omega t + kz)\hat{x}$$

Comment: At any given instant of time, the phase of $\vec{E}$ is constant over a set of planes perpendicular to the z-axis, hence the term "plane wave." A surface of constant phase is termed a "phase front" or "wavefront." The solutions represented above are "uniform plane waves" because the field amplitude is constant over any given phase front. A representation of the wavefronts of a plane wave is given in figure 1.

Properties of Plane Waves

Phase Velocity

To understand how the wavefronts move, let us examine a fixed point on a particular wavefront and see how it moves with time. Suppose that at a time instant t, the *total phase* (i.e., the argument of the cosine functions above) for the two wave solutions are,

$$\omega t - kz = \text{constant} \quad \text{and} \quad \omega t + kz = \text{constant}$$

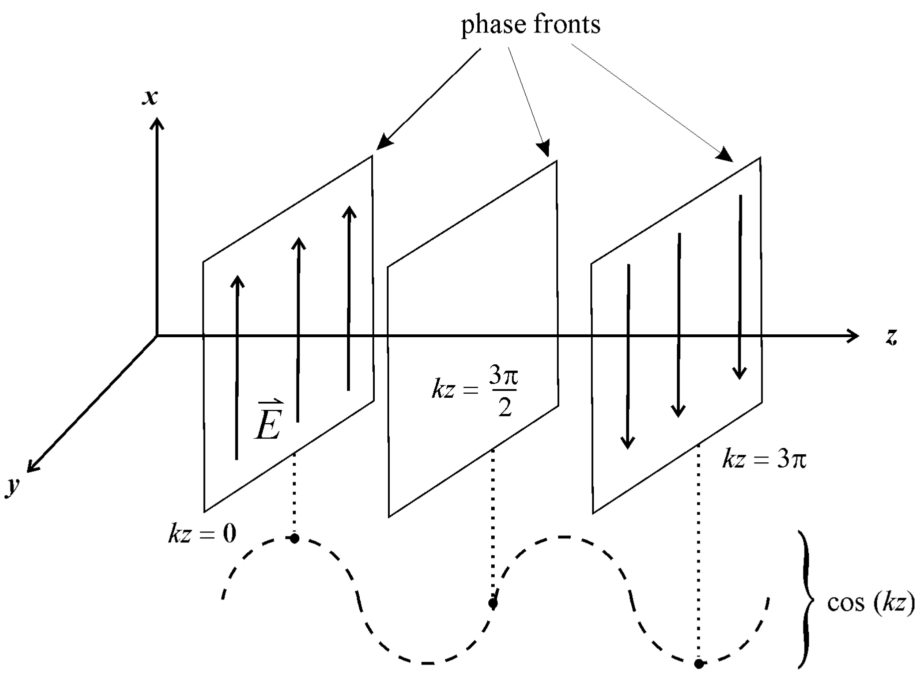

Fig. 1. Illustration of a "phase front" for a plane wave propagating in the z-direction.

Taking the derivative with respect to time, we find that

$$\omega - k\frac{dz}{dt} = 0 \quad \text{and} \quad \omega + k\frac{dz}{dt} = 0$$

.

Note that dz is just the change in position of the point in question over time interval dt. Thus, dz/dt is the velocity at which the phase fronts are moving. This is called the "phase velocity" and it is given by,

$$\frac{dz}{dt} = \frac{\omega}{k} \equiv v_p^+ = \frac{1}{\sqrt{\mu\epsilon}}$$

for the solution with argument $\omega t - kz$ and by

$$\frac{dz}{dt} = -\frac{\omega}{k} \equiv v_p^- = -\frac{1}{\sqrt{\mu\epsilon}}$$

for the solution with argument $\omega t + kz$. The negative sign indicates that the phase fronts move in the $-z$ direction. Thus, we see that the solution,

$$\underline{\vec{E}}_+ = A\exp(-jkz)$$

represents a plane wave propagating in the $+z$ direction with phase velocity v_p^+, and the solution

$$\underline{\vec{E}}_- = B\exp(+jkz)$$

represents a plane wave propagating in the $-z$ direction with phase velocity v_p^-.

Wavelength

Wavelength is another important parameter describing wave solutions. It is the spatial analogy of frequency (and it's reciprocal is sometimes called the "spatial frequency" or wavenumber). The wavelength (denoted by the symbol λ) is defined as the distance, at a given instant of time, over which the phase of the wave changes by 2π:

$$k\lambda = 2\pi \quad \Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

where ω is the "radial frequency" (units of radians per second) and f is the frequency.

Associated Magnetic Field

Maxwell's equations couple electric fields and magnetic fields ... that is, for time-varying fields, an electric field has an associated magnetic field. We can easily find the magnetic field associated with the plane wave solutions given above by invoking Faraday's Law:

$$\vec{\nabla} \times \underline{\vec{E}} = -j\omega\mu\underline{\vec{H}}$$

$$\vec{\nabla} \times \Big(A\exp(-jkz) + B\exp(jkz)\Big)\hat{x} = \frac{d}{dz}\Big\{A\exp(-jkz) + B\exp(jkz)\Big\}\hat{y}$$

$$= -jk\Big\{A\exp(-jkz) - B\exp(jkz)\Big\}\hat{y}$$

Thus, we have

$$\boxed{\underline{\vec{H}} = \frac{k}{\omega\mu}Ae^{-jkz}\hat{y} - \frac{k}{\omega\mu}Be^{jkz}\hat{y}}$$

As usual, we obtain the instantaneous form of the wave by multiplying through by $\exp(j\omega t)$ and taking the real part:

$$\vec{H}(z,t) = \sqrt{2}\frac{k}{\omega\mu}A\cos(\omega t - kz)\hat{y} - \sqrt{2}\frac{k}{\omega\mu}B\cos(\omega t + kz)\hat{y}$$

Now for a few comments about the wave solutions we found above:

- The $\vec{E}$ and $\vec{H}$ fields for a plane wave are orthogonal to one another. In addition, they are both perpendicular to the direction of propagation (z). Note that the direction of propagation is given by the vector $\vec{E} \times \vec{H}$.

- Each linearly independent solution to the wave equation, i.e.,

$$A \exp(-jkz) \quad \text{and} \quad B \exp(jkz)$$

  is called a "travelling wave." A superposition of these two solutions (for example, the general solution given above) is called a "standing wave."

- $\vec{E}$ and $\vec{H}$ for a *travelling wave* are in phase. That is, they have the *same* phase argument.

- Notice that there is a minus sign associated with the wave travelling in the $-z$ direction. This arises because the direction of propagation is given by $\vec{E} \times \vec{H}$. Thus, to express a wave travelling in the opposite direction, it is necessary to reverse the sign of one of the field components. This is illustrated in figure 2.

- The ratio of E to H has units of ohms and is called the "wave impedance." For a travelling wave, e.g.,

$$\underline{E}_x = Ae^{-jkz}, \quad \underline{H}_y = \frac{k}{\omega\mu} Ae^{-jkz}$$

  we have

$$\frac{\underline{E}_x}{\underline{H}_y} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} \equiv \eta$$

This quantity is called the "intrinsic wave impedance" and it is strictly a property of the medium in which the wave propagates.

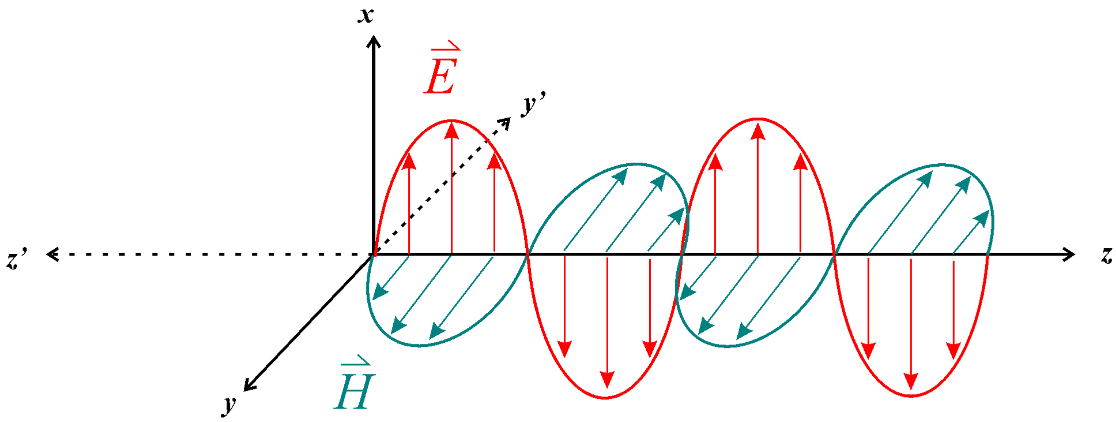

Fig. 2. Diagram of a plane wave propagating in the $+z$ direction or the $-z'$ direction.