January 26–31, 2006

ECE 556— Microwave Engineering I
University of Virginia

Lecture 3— Plane Wave Propagation and Reflection

First, let's recap the important things from the last lecture that you need to remember:

- The Helmholtz equation is a phasor form of the wave equation:

$$\nabla^2 \begin{cases} \underline{\vec{E}} \\ \underline{\vec{H}} \end{cases} = -k^2 \begin{cases} \underline{\vec{E}} \\ \underline{\vec{H}} \end{cases}$$

- *Uniform plane waves* comprise one set of possible (and useful) solutions to the Helmholtz equation:

$$\underline{\vec{E}}, \underline{\vec{H}} \;\propto\; \exp(\pm jkz)$$

- The exponential terms above are called "propagation factors." They describe the change in the *phase* of the wave as it propagates over some distance z. The negative exponent describes a wave propagating in the $+z$ direction while the positive exponent describes a wave propagating the $-z$ direction.

- k is the wavenumber or "phase constant" of the wave. It describes the wave function's spatial periodicity and is related to the wavelength (λ) by,

$$\lambda = \frac{2\pi}{k}$$

  Note: in a *lossless* or *nondissipative* medium, the wavenumber is purely real.

- The phase velocity describes how fast the phase fronts travel in the direction of propagation. For plane waves, the phase velocity is,

$$v_p^{+,-} = \pm\frac{\omega}{k} = \pm\frac{1}{\sqrt{\mu\epsilon}}$$

  The "+" sign corresponds to waves travelling in the $+z$ direction while the "$-$" sign corresponds to $-z$ travelling waves.

- $\vec{E}$, $\vec{H}$, and the direction of propagation are mutually perpendicular. The cross produce $\vec{E} \times \vec{H}$ points in the direction of propagation.

- The ratio of the E-field and H-field phasors is the "wave impedance."

Planes Waves in Dissipative Media

So far in our discussion of plane waves, we have assumed that the medium in which the waves travel is uniform, isotropic, and time-invariant. When this is the case, the permittivity and permeability (also known as the medium's "electrical parameters") are scalars. For a lossless medium, these quantities are purely real, and so the wavenumber is also real. When this is the case, the only thing that happens as a wave propagates is that it's phase progresses according to its propagation factor,

$$\exp(\pm jkz)$$

Things get a little more interesting when the medium has losses. There are a number of different physical mechanisms that can result in loss, but the most common are usually classified as conductive (or "ohmic") losses and dielectric losses.

Conductive Losses

Conductive losses occur in media that have free charges that may move in response to an applied electric field. As a consequence, electric current flows in these materials, resulting in loss. If the material obeys Ohm's Law, then we can write the current density $\vec{J}$ as,

$$\vec{J} = \sigma \vec{E}$$

where σ is a material property called the "conductance." Taking this into account in Maxwell's equations leads to (assuming a charge-free medium),

$$\vec{\nabla} \times \underline{\vec{E}} = -j\omega\mu\underline{\vec{H}} \qquad \vec{\nabla} \cdot \underline{\vec{E}} = 0$$
$$\vec{\nabla} \times \underline{\vec{H}} = (\sigma + j\omega\epsilon)\underline{\vec{E}} \qquad \vec{\nabla} \cdot \underline{\vec{H}} = 0$$

Combining these to form the wave (Helmholtz) equation gives us,

$$\boxed{\begin{aligned} \nabla^2\underline{\vec{E}} &= j\omega\mu(\sigma + j\omega\epsilon)\underline{\vec{E}} = -k^2\underline{\vec{E}} \\ \nabla^2\underline{\vec{H}} &= j\omega\mu(\sigma + j\omega\epsilon)\underline{\vec{H}} = -k^2\underline{\vec{H}} \end{aligned}}$$

Essentially, the form of the Helmholtz equations are unchanged *except* that the wavenumber is now a complex quantity. That is,

$$k^2 = \omega^2 \mu\epsilon \Big(1 - j\frac{\sigma}{\omega\epsilon}\Big), \quad \text{or}$$

$$k = \omega\sqrt{\mu\epsilon}\Big(1 - j\frac{\sigma}{\omega\epsilon}\Big)^{\frac{1}{2}} = \beta - j\alpha$$

This does not alter the form of the plane wave solutions to the Helmholtz equation, but does result in an exponential decrease in the wave amplitude as it propagates. k, as expressed above, is often called the "complex wavenumber." Its real part, β is the phase constant or propagation constant. The imaginary part α is the attenuation constant. The plane wave solutions in a lossy medium are thus written as (for a wave travelling in the $+z$ direction),

$$\underline{E}, \underline{H} \;\sim\; \exp(-jkz) = e^{-\alpha z} e^{-j\beta z}$$

In the special case that the medium is a "good" conductor, where "good" is defined by the condition

$$\sigma \gg \omega\epsilon$$

we can write the complex wavenumber as,

$$k = \omega\sqrt{\mu\epsilon}\Big(1 - j\frac{\sigma}{\omega\epsilon}\Big)^{\frac{1}{2}} \simeq \omega\sqrt{\mu\epsilon}\Big(-j\frac{\sigma}{\omega\epsilon}\Big)^{\frac{1}{2}} = \sqrt{\omega\mu\sigma}\,\underbrace{\sqrt{(-j)}}_{\exp(-j\frac{\pi}{4})}$$

$$k = \beta - j\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}\Big(1 - j\Big)$$

The quantity,

$$\delta_s \equiv \sqrt{\frac{2}{\omega\mu\sigma}}$$

is the distance over which the fields in the good conductor decay by an amount $1/e$. δ_s is known as the "skin depth."

Dielectric Losses

When an electric field is applied to a dielectric, the atoms making up the material become "polarized" in response to the field ... that is, the positive and negative charges are slightly displaced from one another, resulting in an induced "dipole moment." When the applied field varies with time, this polarization of the dielectric will vary with the same frequency. However, as the frequency increases, inertia and frictional forces will cause the induced polarization to lag behind the applied field. This is the origin of loss in dielectric materials, which are characterized as having a complex permittivity,

$$\epsilon = \epsilon' - j\epsilon''$$

The ratio of the real and imaginary components of the permittivity is called the *loss tangent*, denoted by the symbol $\tan\,\delta$:

$$\tan\,\delta \equiv \frac{\epsilon''}{\epsilon'}$$

As a result, the wavenumber inside a lossy dielectric is complex,

$$k = \beta - j\alpha = \omega\sqrt{\mu(\epsilon' - j\epsilon'')} = \omega\sqrt{\mu\epsilon'}\Big(1 - j\tan\,\delta\Big)^{\frac{1}{2}} \tag{1}$$

For low-loss dielectrics the loss tangent is quite small and this allows us to approximate the complex wavenumber using the Taylor expansion,

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x$$

Using this, we can write equation (1) as,

$$k = \omega\sqrt{\mu\epsilon'}\Big(1 - j\frac{\tan\,\delta}{2}\Big) \quad \Rightarrow \quad \beta = \omega\sqrt{\mu\epsilon'}, \quad \alpha = \frac{\omega\sqrt{\mu\epsilon'}}{2}\,\tan\,\delta$$

Boundary Conditions

In most of the interesting and important microwave circuits we will see, electromagnetic waves will not simply propagate in an unbounded medium but encounter obstacles or discontinuities. These discontinuities will generate "scattered" waves and it is important to understand how these scattered waves are generated and how they affect the behavior of the circuit.

Whenever electromagnetic energy is incident upon a boundary between two dissimilar regions of space, part of that energy will, in general, be reflected back and part will be transmitted into the new medium. The physical laws that govern this process are Maxwell's boundary conditions. These are usually derived in your electromagnetics courses and I won't repeat that derivation here. Mathematically, Maxwell's boundary conditions can be written as,

$$\hat{n} \cdot \left(\vec{D}_+ - \vec{D}_-\right) = \rho_s \tag{2}$$

$$\hat{n} \cdot \left(\vec{B}_+ - \vec{B}_-\right) = 0 \tag{3}$$

$$\hat{n} \times \left(\vec{E}_+ - \vec{E}_-\right) = 0 \tag{4}$$

$$\hat{n} \times \left(\vec{H}_+ - \vec{H}_-\right) = \vec{J}_s \tag{5}$$

The geometry relevant to these expression is shown in figure 1.

In equations (2)–(5), $\hat{n}$ is a unit vector that is normal to the boundary and points from the lower (labelled with a "$-$") region to the upper (labelled with a "$+$") region. The dot products and cross products select the components of the fields that are either normal (dot product) or tangent (cross product) to the boundary. ρ_s represents the *free* surface charge density on the boundary while $\vec{J}_s$ is the free surface current density flowing on the boundary.

In the absence of free surface current or charge, equations (4) and (5) tell us that the E and H-field components tangent to the boundary *must be continuous.* This is the boundary condition that we will find most useful in what follows.

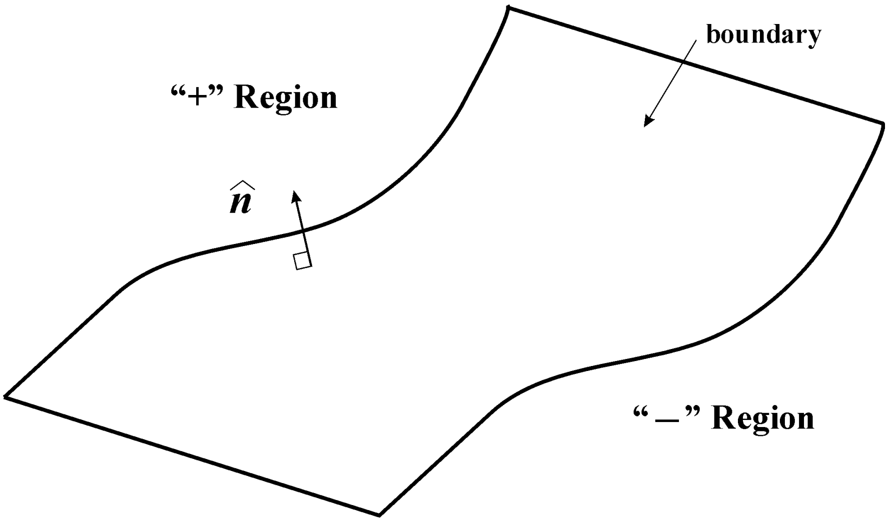

Fig. 1. A boundary separating two regions of space (labelled "+" and "−"). $\hat{n}$ is a unit normal vector pointing from the "−"-region to the "+"-region.

Normal Incidence onto a Planar Boundary

Let us consider a plane wave propagating in medium 1 that is characterized by permittivity ϵ_1 and permeability μ_0. Given these electrical parameters, the wavenumber (k_1) and intrinsic wave impedance (η_1) are easily determined:

$$k_1 = \omega\sqrt{\mu_0 \epsilon_1} \qquad\qquad \eta_1 = \sqrt{\frac{\mu_0}{\epsilon_1}}$$

Suppose this wave encounters a planar boundary separating medium 1 from medium 2 (with electrical parameters μ_0 and ϵ_2). This situation is illustrated in figure 2. Clearly a single $+z$-travelling wave solution will not satisfy the boundary conditions. However, we can satisfy the boundary conditions if we have a $+z$ directed wave in medium 2 (denoted $\vec{E}_t$ for "transmitted"),

$$\underline{\vec{E}}_t = E_t \exp(-jk_2 z)\hat{x}, \qquad k_2 = \omega\sqrt{\mu_0 \epsilon_2}$$

and a *second* wave in medium 1 that travels in the $-z$ direction (denoted $\vec{E}_r$ for "reflected"):

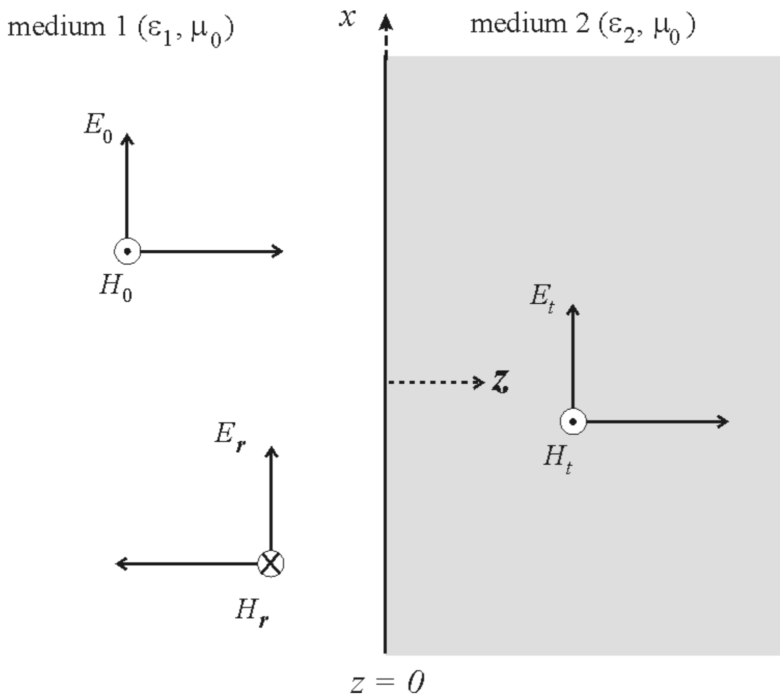

Fig. 2.

$$\underline{\vec{E}}_r = E_r \exp(jk_1 z)\hat{x}, \qquad k_1 = \omega\sqrt{\mu_0\epsilon_1}$$

Assuming the boundary lies at $z = 0$, continuity of the tangential fields at the boundary requires,

$$\underline{E}_i + \underline{E}_r = \underline{E}_t\Big|_{z=0}, \qquad \text{and}$$

$$\underline{H}_i + \underline{H}_r = \underline{H}_t\Big|_{z=0}$$

We will define the ratio of the electric field phasor amplitude to the magnetic field phasor amplitude as the "wave impedance," Z. Note that this is different than the *intrinsic wave impedance* (η) which relates the electric and magnetic fields for a travelling wave. Using the concept of wave impedance, we can write

$$Z_1 \equiv \frac{\underline{E}_i + \underline{E}_r}{\underline{H}_i + \underline{H}_r}\Big|_{z=0} = \frac{\underline{E}_t}{\underline{H}_t}\Big|_{z=0} \equiv Z_2 \tag{6}$$

where Z_1 and Z_2 are the wave impedances at $z = 0$ in medium 1 and medium 2, respectively. Substituting the explicit plane wave solutions into equation (6) gives us,

$$\left.\frac{E_0 \exp(-jk_1 z) + E_r \exp(jk_1 z)}{H_0 \exp(-jk_1 z) + H_r \exp(jk_1 z)}\right|_{z=0} = \left.\frac{E_t \exp(-jk_2 z)}{H_t \exp(-jk_2 z)}\right|_{z=0}$$

Noting that,

$$\frac{E_0}{H_0} = \eta_1, \quad -\frac{E_r}{H_r} = \eta_1, \text{ and } \quad \frac{E_t}{H_t} = \eta_2$$

we have

$$\eta_1 \frac{E_0 + E_r}{E_0 - E_r} = \eta_2 \tag{7}$$

The *reflection coefficient*, Γ, is defined as the ratio of the reflected E-field phasor to the incident E-field phasor, i.e.,

$$\Gamma \equiv \frac{E_r \exp(jkz)}{E_0 \exp(-jkz)} = \Gamma_0 \exp(j2kz)$$

where Γ_0 is the reflection coefficient at $z = 0$. Using this definition, we can write (7) as

$$\eta_1 \frac{1 + \Gamma_0}{1 - \Gamma_0} = \eta_2$$

Upon rearranging, we obtain the most important formula in this course,

$$\boxed{\Gamma_0 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$

The *transmission coefficient*, τ, is defined as the ratio of the transmitted electric field amplitude to the incident electric field amplitude at the interface,

$$\tau \equiv \frac{E_t}{E_0}$$

Using the boundary condition, $E_0 + E_r = E_t$, we have

$$\boxed{\tau = \frac{E_0 + E_r}{E_0} = 1 + \Gamma_0 = \frac{2\eta_2}{\eta_1 + \eta_2}}$$

Some Special Cases

Let's now examine two special cases:

1. Incidence onto a perfect (lossless) dielectric:

   In this case, the intrinsic wave impedances are purely real and given by,

   $$\eta = \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}} = \frac{\eta_0}{n}$$

   where n is the *index of refraction*, defined by

   $$n \equiv \sqrt{\epsilon_r} = \sqrt{\frac{\epsilon}{\epsilon_0}}$$

   The reflection and transmission coefficients are thus,

   $$\Gamma_0 = \frac{n_1 - n_2}{n_1 + n_2}, \qquad \tau = \frac{2n_1}{n_1 + n_2}$$

2. Incidence onto a perfect conductor:

   A perfect conductor has infinite conductivity and thus a wave impedance of zero. In other words, the boundary condition for a perfect conductor is the tangential electric field must vanish on its surface. This is ensured if the reflected wave has the same magnitude as the incident wave but is phase-shifted by 180°. That is,

   $$\Gamma_0 = \frac{0 - \eta_1}{0 + \eta_1} = -1, \qquad \tau = 1 + \Gamma_0 = 0$$

   As expected, there is no wave transmitted into a perfect conductor.

Some Comments

Wave Impedance in Lossy Media

We noted earlier in this lecture that k for a wave propagating in a lossy medium is complex. As a result, the intrinsic wave impedance is also complex. Recall that we found the intrinsic wave impedance to be,

$$\eta = \frac{\omega\mu}{k}$$

If $k = \beta - j\alpha$, then the wave impedance will have both real and imaginary parts. Physically, this means that the electric and magnetic fields for a plane wave travelling in the medium will no longer be in phase. For good conductors and low-loss dielectrics, the wave impedance is *inductive*, meaning the electric field will lead the magnetic field in phase.

Consider, for example, a wave travelling in a conductive medium where $\sigma \gg \omega\epsilon$. The wavenumber is given by,

$$k = \sqrt{\frac{\omega\mu\sigma}{2}}(1 - j)$$

which gives a wave impedance, Z, of

$$Z = \sqrt{\frac{\omega\mu}{2\sigma}}(1 + j)$$

Here we have used Z for the wave impedance since the symbol "η" is customarily used for lossless media.

Power

The power associated with a plane wave is found from the *Poynting vector*, $\vec{S}$, defined to be

$$\vec{S} = \vec{E} \times \vec{H} \quad \text{or} \quad \underline{\vec{S}} = \underline{\vec{E}} \times \underline{\vec{H}}^{*}$$

where $\underline{\vec{S}}$ is the *complex* Poynting vector used for phasor quantities. The Poynting vector has units of power density and it points in the direction of energy flow. To obtain the total power associated with a wave, you need to integrate the Poynting vector over a surface perpendicular to the direction of propagation, i.e.,

$$P = \oint_S \vec{S} \cdot \hat{n} da$$

For a plane wave propagating in a lossless medium (with wavenumber k and intrinsic impedance η),

$$\underline{\vec{S}} = E_0 e^{-jkz} \hat{x} \times \left(\frac{E_0}{\eta} e^{-jkz} \right)^* \hat{y} = \frac{|E_0|^2}{\eta} \hat{z}$$

Since the power density is constant, a uniform plane wave carries infinite energy. If the medium is lossy, then the power associated with the wave decays as the wave propagates and energy is dissipated:

$$\underline{\vec{S}} = E_0 e^{-\alpha z} e^{-j\beta z} \hat{x} \times \left(\frac{E_0}{Z} e^{-\alpha z} e^{-j\beta z} \right)^* \hat{y} = \frac{|E_0|^2}{Z^*} e^{-2\alpha z} \hat{z}$$

where Z denotes the intrinsic wave impedance in the lossy medium. At a *lossless* interface (that is, an interface that dissipates no energy), power is conserved,

$$P_{\text{inc}} = P_{\text{ref}} + P_{\text{trans}}$$

In terms of the reflection and transmission coefficients, this can be written as

$$|\Gamma_0|^2 + \frac{\eta_1}{\eta_2} |\tau|^2 = 1$$

We will see this relation again in the course when we discuss lossless microwave networks.

