ECE 556— Microwave Engineering I
University of Virginia

Lecture 4— Plane Waves at Oblique Incidence

Thus far, we have only considered plane waves propagating along the z direction with the electric field directed along the x-axis. We may generalize these plane wave solutions to represent a *linearly polarized* plane wave travelling in an arbitrary direction as,

$$\boxed{\underline{\vec{E}} = \vec{E}_0 \exp(\pm j\vec{k}\cdot\vec{r}) \qquad \text{and} \qquad \underline{\vec{H}} = \vec{H}_0 \exp(\pm j\vec{k}\cdot\vec{r})} \tag{1}$$

In the above expressions, $\vec{r}$ is the "position" vector and is written in Cartesian coordinates as,

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z},$$

while $\vec{k}$ is the "wave vector,"

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$$

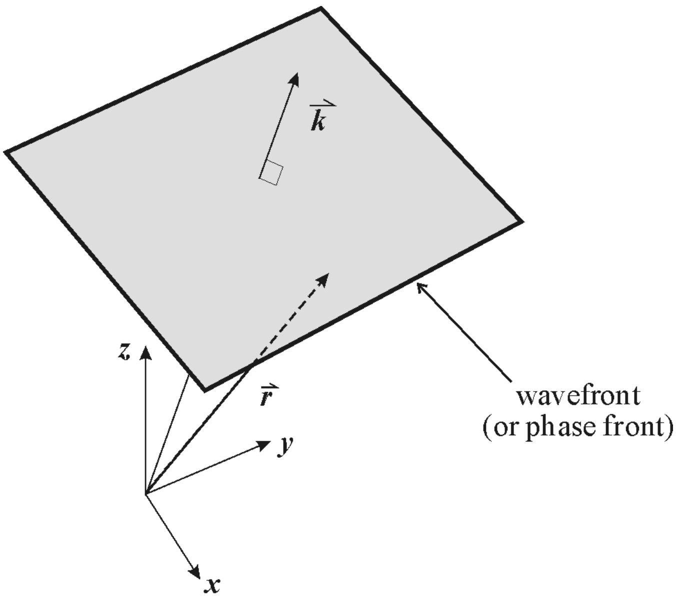

Fig. 1. Illustration of a "phase front" for a plane wave.

The fields given by equation (1) can be obtained from our previous wave solutions by a simple rotation of the Cartesian coordinate system and yields a plane wave in which the phase of the field, $\vec{k}\cdot\vec{r}$, is constant over a planar surface perpendicular to the vector $\vec{k}$. This is illustrated in figure 1. If the field amplitudes $\vec{E}_0$ and $\vec{H}_0$ are not functions of position, then the solutions are called "uniform plane waves." Recall that surfaces of constant phase are known as "phase fronts" or "wavefronts."

It is straightforward to show that equations (1) are solutions to the Helmholtz equations. Notice that operating on the plane wave solutions with $\vec{\nabla}$ is equivalent to multiplying by $\pm j\vec{k}$. That is,

$$\vec{\nabla}\times \longrightarrow \pm j\vec{k}\times$$
$$\vec{\nabla}\cdot \longrightarrow \pm j\vec{k}\cdot$$

Let us consider only the solution with the *negative* sign and apply it to Maxwell's equations in phasor form,

$$\vec{\nabla}\times\underline{\vec{E}} = -j\omega\mu\underline{\vec{H}} \longrightarrow \vec{k}\times\underline{\vec{E}}_0 = \omega\mu\underline{\vec{H}}_0 \tag{2}$$
$$\vec{\nabla}\times\underline{\vec{H}} = j\omega\epsilon\underline{\vec{E}}_0 \longrightarrow \underline{\vec{H}}_0\times\vec{k} = \omega\epsilon\underline{\vec{E}}_0 \tag{3}$$
$$\vec{\nabla}\cdot\underline{\vec{E}} = 0 \longrightarrow \vec{k}\cdot\underline{\vec{E}}_0 = 0 \tag{4}$$
$$\vec{\nabla}\cdot\underline{\vec{H}} = 0 \longrightarrow \vec{k}\cdot\underline{\vec{H}}_0 = 0 \tag{5}$$

Note that these expressions imply that $\underline{\vec{E}}_0$, $\underline{\vec{H}}_0$, and $\vec{k}$ are *mutually* orthogonal. In addition, we see upon forming the cross product

$$\vec{k}\times\left(\underline{\vec{H}}\times\vec{k}\right) = \omega\epsilon\left(\vec{k}\times\underline{\vec{E}}\right) = \omega^2\mu\epsilon\underline{\vec{H}}$$

Using the vector identity,

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \quad \text{we have}$$

$$\vec{k}\times\left(\underline{\vec{H}}\times\vec{k}\right)=k^2\underline{\vec{H}}=\omega^2\mu\epsilon\underline{\vec{H}} \quad \text{or,}$$

$$k^2=\omega^2\mu\epsilon$$

which is the dispersion relation.

Classification of Plane Waves

It is convenient to classify plane waves according to the directions along which the field vectors are oriented. In general, we can classify three unique orientations or "polarizations" for a plane wave. These are denoted at TEM_z, TE_z and TM_z.

TEM_z Plane Waves

The classification $\mathrm{TEM_z}$ denotes a plane wave in which the electric field and magnetic field are perpendicular (or "transverse") to the z-direction. "TEM" stands for "transverse electromagnetic." $\mathrm{TEM_z}$ waves are simply the plane wave solutions that we have already been discussing. Figure 2(a) illustrates a $\mathrm{TEM_z}$ wave propagating in the $+z$ direction. Recall that the ratio of the electric field amplitude to the magnetic field amplitude for a propagating $\mathrm{TEM_z}$ wave is a constant of the medium and is given by,

$$Z_{\mathrm{TEM}}=\eta=\sqrt{\frac{\mu}{\epsilon}}$$

TM_z Plane Waves

A TM_z plane wave is one in which the magnetic field is perpendicular to the z direction. TM_z stands for "transverse magnetic" (to the z-axis). A TM_z wave is shown in figure 2(b).

For convenience, it is useful to define an intrinsic "TM_z wave impedance" that is the ratio of the electric field and magnetic field amplitudes that are perpendicular to the reference direction, z. In other words, we define Z_{TM} as,

$$Z_{TM} = \frac{\underline{E}_x}{\underline{H}_y}$$

We can find this geometrically, knowing that the ratio of the field amplitudes is just η. We can also use Maxwell's equations in algebraic form (equations (2–5)) to find an alternate and quite useful expression for the TM_z wave impedance. Using Faraday's Law,

$$\vec{k} \times \underline{\vec{E}}_0 = \omega\mu\underline{\vec{H}}_0$$

But noting that $\underline{E}_x = \underline{E}\cos\theta$, we have

$$\boxed{\frac{k\underline{E}_x}{\cos\theta} = \omega\mu\underline{H}_y \quad \Rightarrow \quad \frac{\underline{E}_x}{\underline{H}_y} = \frac{\omega\mu}{k}\cos\theta = \frac{k}{\omega\epsilon}\cos\theta = \frac{k_z}{\omega\epsilon} = \eta\cos\theta \qquad (6)}$$

Note that the TM_z wave impedance is less than the intrinsic wave impedance of the medium (η). This is expected since the electric field is not completely transverse to the z-axis, so $\underline{E}_x < \underline{E}$. The TM_z wave impedance is a characteristic of a given travelling TM-wave that is specified by its $\vec{k}$-vector.

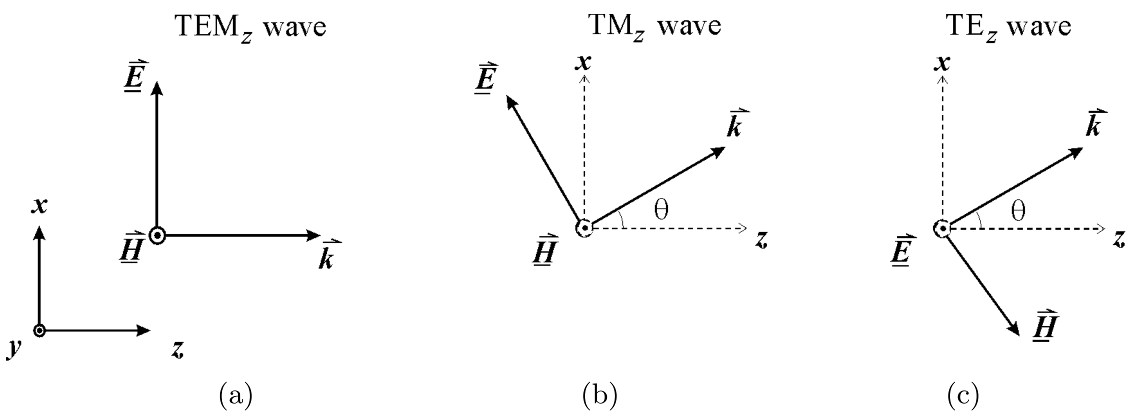

Fig. 2. Illustration of TEM, TM, and TE plane waves.

TE_z Plane Waves

A TE_z plane wave is one in which the electric field is perpendicular to the z direction. As you might now suspect, TE_z stands for "transverse electric" (to the z-axis). A typical TE_z wave is shown in figure 2(c).

As with the TM case, it is useful to define an intrinsic "TE_z wave impedance" that is the ratio of the electric field and magnetic field amplitudes that are perpendicular to the reference direction, z:

$$Z_{TE} = -\frac{\underline{E}_y}{\underline{H}_x}$$

Again, using Faraday's Law, and noting that $\underline{H}_x = -\underline{H}\cos\theta$, we have

$$\boxed{k\underline{E}_y = -\omega\mu\frac{\vec{\underline{H}}_x}{\cos\theta} \quad \Rightarrow \quad -\frac{\underline{E}_y}{\underline{H}_x} = \frac{\omega\mu}{k\cos\theta} = \frac{\omega\mu}{k_z} = \eta\sec\theta} \tag{7}$$

The TE_z wave impedance is *greater* than the intrinsic wave impedance of the medium (η) because the magnetic field is not completely transverse to the z-axis. Consequently $|\underline{H}_x| < |\underline{H}|$. As with the other types of plane waves, the TE_z wave impedance is a characteristic of a given travelling TE-wave and is determined exclusively by the electrical parameters of the medium (ϵ and μ) and the direction of propagation.

Comment:

We have seen the the intrinsic wave impedance, η, is complex if the medium of propagation is lossy. This is also true of Z_{TE} and Z_{TM}. However, it is also possible for Z_{TE} and Z_{TM} to be purely imaginary *even if the medium is lossless*! This can happen when a plane wave is incident obliquely upon a planar surface separating two media.

Oblique Incidence upon a Planar Interface

We have already considered the case where a plane wave is normally incident upon a planar interface between two different media. This case is often called "TEM" incidence. Now,

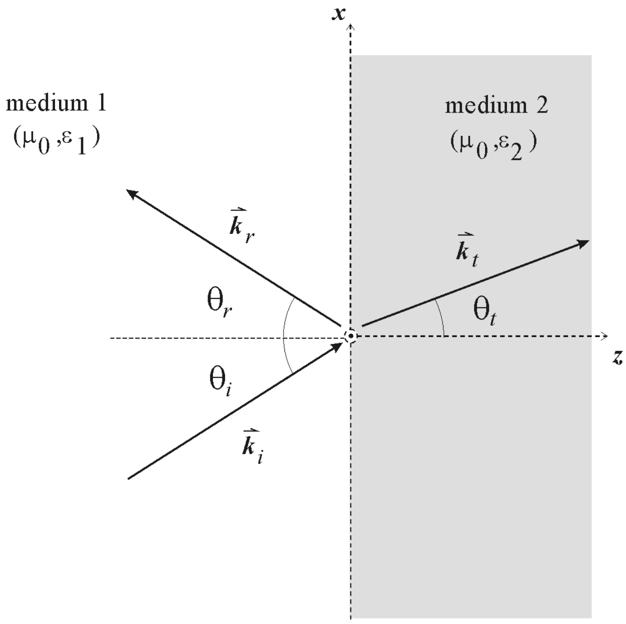

Fig. 3. Plane wave at oblique incidence.

let us turn our attention to a plane wave that is incident *obliquely* (at an angle) upon the interface. This is illustrated in figure 3. There are two separate cases we need to consider, TM_z incidence and TE_z incidence.

<u>TE_z **incidence** (or "perpendicular polarization")</u>

For a TE_z incident wave, the electric field is completely parallel to the interface. In physics courses, this is usually called "perpendicular polarization." As with the case of TEM incidence, we need to enforce Maxwell's boundary conditions — that is, the most make sure the field components tangent to the boundary are continuous.

Writing out the incident, reflected, and transmitted waves explicitly,

$$\underline{\vec{E}}_{\text{inc}} = \underline{E}_0 e^{-j\vec{k}_i\cdot\vec{r}}\hat{y},$$

$$\underline{\vec{E}}_{\text{ref}} = \underline{E}_r e^{-j\vec{k}_r\cdot\vec{r}}\hat{y},$$

$$\underline{\vec{E}}_{\text{tran}} = \underline{E}_0 e^{-j\vec{k}_t\cdot\vec{r}}\hat{y},$$

we require that at $z = 0$,

$$\underline{E}_0 e^{-j\vec{k}_i\cdot\vec{r}} + \underline{E}_r e^{-j\vec{k}_r\cdot\vec{r}} = \underline{E}_t e^{-j\vec{k}_t\cdot\vec{r}}\Big|_{z=0}$$

This relation must hold for any arbitrary position on the boundary (i.e., for all x, y for which $z = 0$). This can only happen if the phase of each of the phasors has the same *transverse* (x and y) dependence, i.e.,

$$\vec{k}_i \cdot \vec{r}\Big|_{z=0} = \vec{k}_r \cdot \vec{r}\Big|_{z=0} = \vec{k}_t \cdot \vec{r}\Big|_{z=0}$$

The first equality can be written as,

$$\vec{k}_{i,\|} = \vec{k}_{r,\|} \qquad \longrightarrow \qquad k_i \sin\,\theta_i = k_i \sin\,\theta_r, \quad \text{or}$$

$$\boxed{\theta_i = \theta_r}$$

where θ_i and θ_r are the angles of incidence and reflection, respectively. $\vec{k}_\|$ denotes the component of the $\vec{k}$-vector parallel to the interface (i.e., $\vec{k}_\| = \hat{x}k_x + \hat{y}k_y$). This relation is known as the "law of reflection." The other equality can be written as,

$$\vec{k}_{i,\|} = \vec{k}_{t,\|} \qquad \longrightarrow \qquad k_i \sin\,\theta_i = k_t \sin\,\theta_t, \quad \text{or}$$

$$\boxed{k_i \sin\,\theta_i = k_2 \sin\,\theta_t, \quad \text{or} \quad n_i \sin\,\theta_i = n_t \sin\,\theta_t}$$

where n_i and n_t are the indicies of refraction in the medium of incidence and transmission, respectively. This is known as "Snell's law of refraction."

Returning to Maxwell's boundary conditions, we need to enforce continuity of the tangential magnetic fields. However, we can use the definition of the TE_z wave impedance to help us do this. Recall that Z_{TE} is defined in terms of the field components *perpendicular* to z — that is, *parallel* to the boundary! Thus we can enforce continuity of the tangential magnetic fields by writing the $\vec{H}$-fields in terms of the $\vec{E}$-fields *and* the appropriate TE wave impedances:

$$\frac{\underline{E}_0}{Z_{\mathrm{TE}}^{(1)}} e^{-j\vec{k}_i\cdot\vec{r}} - \frac{\underline{E}_r}{Z_{\mathrm{TE}}^{(1)}} e^{-j\vec{k}_r\cdot\vec{r}} = \frac{\underline{E}_t}{Z_{\mathrm{TE}}^{(2)}} e^{-j\vec{k}_t\cdot\vec{r}}\Big|_{z=0}$$

As we have seen, the arguments of all the exponents must be identical in the $z = 0$ plane. this leads to the following two relations:

$$\underline{E}_0 + \underline{E}_r = \underline{E}_t \quad \text{and}$$

$$\frac{\underline{E}_0}{Z_{\mathrm{TE}}^{(1)}} - \frac{\underline{E}_r}{Z_{\mathrm{TE}}^{(1)}} = \frac{\underline{E}_t}{Z_{\mathrm{TE}}^{(2)}}$$

Taking the ratio of these relations, dividing out E_0 and defining the "TE reflection coefficient," Γ_{TE} as,

$$\Gamma_{TE} = \frac{\underline{E}_r}{\underline{E}_0}$$

we have

$$\boxed{Z_{TE}^{(1)} \frac{1+\Gamma_{TE}}{1-\Gamma_{TE}} = Z_{TE}^{(2)} \qquad \Longrightarrow \qquad \Gamma_{TE} = \frac{Z_{TE}^{(2)} - Z_{TE}^{(1)}}{Z_{TE}^{(2)} + Z_{TE}^{(1)}}}$$

which is <u>exactly</u> the same form as the reflection coefficient for the TEM case! Writing this explicitly in terms of the angles of incidence and transmission, we have

$$\Gamma_{TE} = \frac{\eta_2 \sec\theta_t - \eta_1 \sec\theta_i}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i}$$

We can also find the transmission coefficient (defined as $\underline{E}_t/\underline{E}_0$ by enforcing continuity of the tangential electric field at the interface:

$$\boxed{\tau_{TE} \equiv \frac{\underline{E}_t}{\underline{E}_0} = \frac{\underline{E}_0 + \underline{E}_r}{\underline{E}_0} = 1 + \Gamma_{\mathrm{TE}} = \frac{2Z_{\mathrm{TE}}^{(2)}}{Z_{\mathrm{TE}}^{(1)} + Z_{\mathrm{TE}}^{(2)}} = \frac{2\eta_2 \sec\theta_t}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i}}$$

TM_z **incidence** (or "parallel polarization")

For a TM_z incident wave, the magnetic field is parallel to the interface. This is also called "parallel polarization" because the electric field is in the "plane of incidence" (the plane that contains the $\vec{k}$-vectors). Proceeding as before, we require that at $z = 0$,

$$\underline{H}_0 + \underline{H}_r = \underline{H}_t$$

We can enforce continuity of the tangential electric fields by using the same approach as before and applying the definition of the TM_z wave impedance. We write a continuity equation for the tangential electric fields by expressing the $\vec{E}$-fields in terms of the $\vec{H}$-fields *and* the appropriate TM wave impedances:

$$Z_{\mathrm{TM}}^{(1)}\underline{H}_0 - Z_{\mathrm{TM}}^{(1)}\underline{H}_r = Z_{\mathrm{TM}}^{(2)}\underline{H}_t$$

Taking the ratio of these relations, dividing out H_0 and defining the "TM reflection coefficient," Γ_{TM} as,

$$\Gamma_{TM} = \frac{\underline{E}_r}{\underline{E}_0} = -\frac{\underline{H}_r}{\underline{H}_0}$$

we have

$$\boxed{Z_{TM}^{(1)} \frac{1 - \frac{\underline{H}_r}{\underline{H}_0}}{1 + \frac{\underline{H}_r}{\underline{H}_0}} = Z_{TM}^{(2)} \qquad \Longrightarrow \qquad \Gamma_{TM} = \frac{Z_{TM}^{(2)} - Z_{TM}^{(1)}}{Z_{TM}^{(2)} + Z_{TM}^{(1)}}}$$

Again, we get the familiar formula for reflection coefficient. You should note, however, that there is one "twist" to the TM case. The reflection and transmission coefficients are normally defined with respect to the *total* electric field amplitudes, not just the components parallel to the interface. For this reason, we need to multiply the reflection and transmission coefficients for the TM case by a ratio of cosines to get the total field components. This subtlety arises for the TM case because the electric field vectors are not completely parallel to the boundary between the two media. This doesn't affect the formula for the reflection coefficient because the ratio of cosines is unity (angle of incidence = angle of reflection). However, it does have an effect on the formula you might anticipate for the transmission coefficient. We will see this below.

Enforcing continuity of the tangential fields at the interface allows us to write,

$$\underline{E}_{x,t} = \underline{E}_{x,0} + \underline{E}_{x}, r$$

where the subscript x denotes the x-component of the field. Dividing through by $\underline{E}_{x,0}$ gives us,

$$\frac{\underline{E}_{x,t}}{\underline{E}_{x,0}} = 1 + \Gamma_{\mathrm{TM}}$$

<u>However</u>, the TM transmission coefficient (τ_{TM}) is defined as,

$$\tau_{TM} \equiv \frac{\underline{E}_t}{\underline{E}_0} = \frac{\underline{E}_{x,t}}{\underline{E}_{x,0}} \cdot \frac{\cos\,\theta_i}{\cos\,\theta_t}$$

Thus,

$$\tau_{TM} = 1 + \Gamma_{TM} \cdot \frac{\cos\,\theta_i}{\cos\,\theta_t} = \frac{2Z_{\mathrm{TM}}^{(2)}}{Z_{\mathrm{TM}}^{(1)} + Z_{\mathrm{TM}}^{(2)}} \cdot \frac{\cos\,\theta_i}{\cos\,\theta_t} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$