ECE 556— Microwave Engineering I
UNIVERSITY OF VIRGINIA

Lecture 18 — Microwave Networks

SCATTERING PARAMETERS

Scattering parameters are basically a generalization of the notion of reflection coefficient. In general, a microwave circuit (or "network") has multiple ports and the "scattering matrix" relates the incident and scattered waves at these ports. By the way, a "port" is a pair of terminals across which we can apply a voltage.

Scattering parameters are used at microwave frequencies for a number of reasons. Because signals propagate as waves in microwave circuits, it tends to be more natural to describe these networks in terms of wave parameters and scattering coefficients. In addition, at microwave frequencies, it is more convenient to measure these wave amplitudes and scattering parameters than impedance or admittance parameters. To understand what scattering parameters *actually* describe, consider the multiport network shown in fig. 1. An "incident" wave at port i is denoted by a_i while a "scattered" wave at port j is denoted b_j. You can think of "incident" as synonymous with "into the network" while "scattered" means "out of the network." a_i and b_j are called "wave amplitudes" and they can be related to incident and scattered voltage waves as follows:

$$a_i \equiv \frac{V_i^+}{\sqrt{Z_i}} \qquad \text{and} \qquad b_j \equiv \frac{V_j^-}{\sqrt{Z_j}}$$

In the above expressions, the "+" and "−" denote incident and scattered respectively. Z_i and Z_j are the normalizing impedances of ports i and j. In principle, these can be whatever you like, but usually it is most convenient to choose them as the terminating impedances of those ports. When the ports are connected to transmission lines (which is often the case), the normalizing impedance at port i is usually taken as the characteristic impedance of the line connected to that port.

Notice that a_i and b_j are phasors and carry both amplitude and phase information. In addition, they are defined with respect to normalizing impedances, just as reflection coefficients are. The reason that wave amplitudes are defined this way is it allows them to be related to power very simply. Consider port i. We can express the voltage and current at port i in terms of the wave amplitudes as,

$$V_i = \sqrt{Z_i}(a_i + b_i) \qquad \text{and} \qquad I_i = \frac{(a_i - b_i)}{\sqrt{Z_i}}$$

Consequently, the average power flowing into port i is,

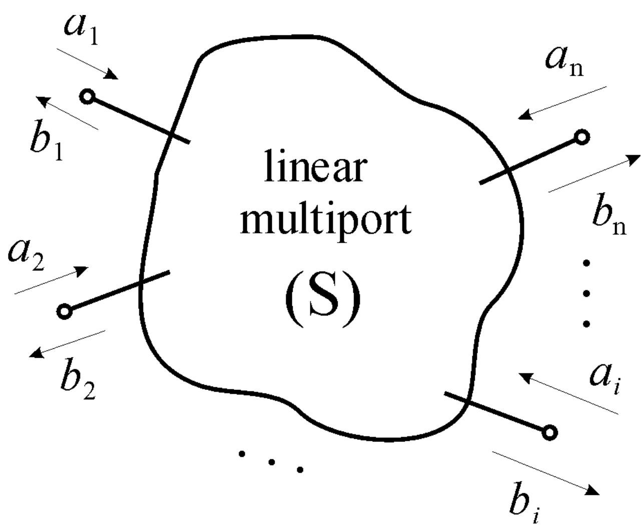


Fig. 1.

$$P_i = Re\Big\{V_i I_i^*\Big\} = Re\{(a_i + b_i)(a_i - b_i)\} = |a_i|^2 - |b_i|^2$$

The expression above lends itself to a very nice interpretation. The magnitude squared of the incident wave amplitude, $|a_i|^2$, is just the incident power, while the magnitude squared of the scattered wave amplitude, $|b_i|^2$, is the reflected or scattered power. This relation between power and wave amplitudes is convenient because we can use it to define wave amplitudes for non-TEM transmission media in which the concept of voltage is somewhat ambiguous or ill-defined.

The expression for power is easily extended to find the total power delivered to a multiport network,

$$P_{\text{total}} = \sum_i P_i = \sum_i |a_i|^2 - |b_i|^2 = \vec{a}^{\,\dagger}\vec{a} - \vec{b}^{\,\dagger}\vec{b}$$

where we have used vector notation to describe the wave amplitudes. $\vec{a}$ and $\vec{b}$ are column vectors whose elements are simply the wave amplitudes at the various ports:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The † symbol denotes the *Hermetian conjugate* or conjugate transpose of the vector, i.e.,

$$\vec{a}^{\,\dagger} = (\,a_1^* \quad a_2^* \quad \ldots \quad a_n^*\,)\,.$$

The *scattering matrix* or S-matrix of a multiport network relates the incident and scattered wave amplitudes at its ports:

$$\vec{b} = S\vec{a}$$

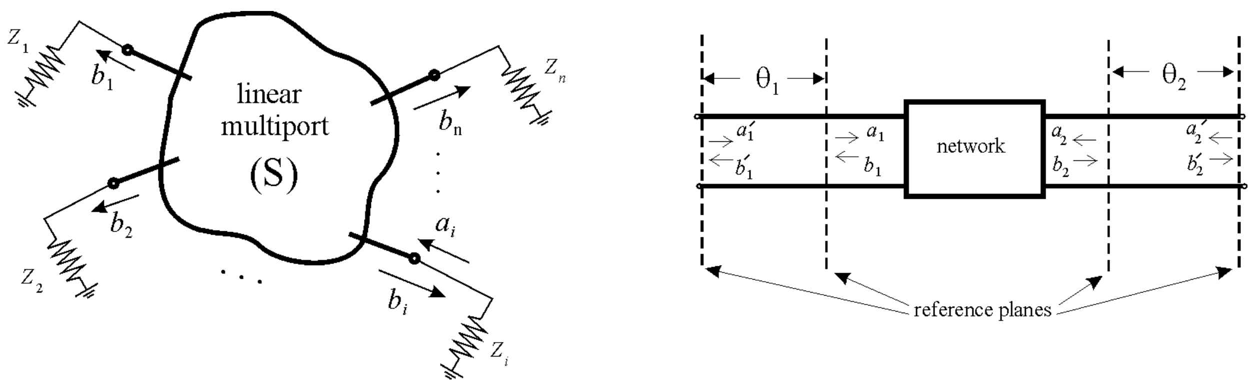
The components of S are known as scattering parameters and are written s_{ij} where i denotes the output port and j refers to the input port:

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{pmatrix}$$

Measuring s-Parameters

Scattering parameters are measured with instruments known as "network analyzers." A variety of measurement schemes have been developed over the years, but most of these instruments are based on sampling a portion of the incident and scattered waves using directional couplers. These sampled waves are then fed to a phase-tracking heterodyne receiver (called a "vector voltmeter") to measure the relative amplitudes and phases of the signals. Modern instruments use on-board microprocessors and elaborate calibration procedures to remove systematic errors in the measurement and achieve a high degree of accuracy.

To measure the scattering parameters of a given network, all of the ports must be terminated with their appropriate normalizing impedance. This eliminates reflections from the ports and allows the S-matrix to be determined from simple ratios, i.e. referring to fig. 2 we can write,



(a) (b)

Fig. 2.

$$s_{ij} = \left.\frac{b_i}{a_j}\right|_{\text{all ports matched}}.$$

You should notice that the s-parameters of a network depend on the reference plane at which the wave amplitudes are measured. Changing reference planes will introduce a phase shift in the measurements as demonstrated for the two-port network of fig. 2(b). We can relate the s-parameters at different reference planes as follows (refer to fig. 2(b)):

At plane 1

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

and at plane 2

$$\begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} = \begin{pmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{pmatrix} \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}.$$

The wave amplitudes on a uniform section of transmission line are related by the propagation (or phase) factors,

$$a_1 = a'_1 e^{-j\theta_1} \qquad a_2 = a'_2 e^{-j\theta_2}$$

$$b_1 = b'_1 e^{j\theta_1} \qquad b_2 = b'_2 e^{j\theta_2}.$$

Thus,

$$\begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} = \begin{pmatrix} e^{-j\theta} & 0 \\ 0 & e^{-j\theta_2} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} e^{-j\theta} & 0 \\ 0 & e^{-j\theta_2} \end{pmatrix} \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}, \text{ or}$$

$$S' = \begin{pmatrix} s_{11}e^{-j2\theta_1} & s_{12}e^{-j(\theta_1+\theta_2)} \\ s_{21}e^{-j(\theta_1+\theta_2)} & s_{22}e^{-j2\theta_2} \end{pmatrix}$$

Properties of s-Parameters

Often, we will know beforehand or *a priori* that a network satisfies certain prescribed conditions. One example is when the network is comprised entirely of lossless elements. Without knowing the exact details of how the circuit elements in the network are connected or configured, we can find a relationship between the network's scattering parameters imposed by the prescribed condition. This idea is quite powerful and will allow us to reach some general conclusions about scattering parameters based solely on the overall properties of the network. We will look at two specific network properties: *reciprocity* and *lossless networks.*

Reciprocity

The idea of reciprocity is somewhat subtle but has far-reaching consequences. Reciprocity in circuits is based on the *Lorentz Reciprocity Theorem* from electromagnetic field theory and we will not go into the details here. Basically reciprocity means that you can exchange the excitation and measurement points in a circuit or network and the relationship between the input and output signals will not change. In terms of s-parameters, a reciprocal network has a symmetric S-matrix:

$$s_{ij} = s_{ji} \tag{1}$$

Obviously, not all networks are reciprocal and it is not difficult to think of examples where equation (1) is not satisfied. Amplifiers are a good example $\rightarrow$ they usually don't work as well if you put them in the circuit backwards! For a network to be reciprocal, it has to satisfy four basic criteria. These come from the assumptions implicit in the Lorentz Reciprocity Theorem and are as follows:

(1) The network must be *linear*,

(2) the network must be *time-invariant*,

(3) the network must be made of "reciprocal materials" — that is, the materials from which the circuit is fabricated must have symmetric ϵ and μ tensors, and

(4) the network cannot contain current or voltage sources.

Transistors, circulators, and isolators are examples of non-reciprocal circuits that violate one or more of the conditions outlined above. In general, circuits made of lumped or distributed passive elements (e.g., filters, couplers, antennas, and matching networks) will be reciprocal.

Lossless Networks

A lossless network is one that dissipates no energy. In other words, the total incident power entering the network must equal the total scattered power exiting the network. In terms of wave amplitudes, we can express this as,

$$\vec{b}^{\,\dagger}\vec{b} = \vec{a}^{\,\dagger}\vec{a},$$

where, again, the dagger denotes the Hermitian conjugate. In terms of the scattering matrix, we can write

$$\vec{b}^{\,\dagger}\vec{b} = (\vec{a}^{\,\dagger} S^{\dagger})\ (S\vec{a}) = \vec{a}^{\,\dagger}\vec{a}, \quad \text{or}$$

$$S^{\dagger} S = I \tag{2}$$

where I is the identity matrix. In mathematics, a matrix satisfying equation (2) is said to be *unitary* and its inverse is the same as its Hermitian conjugate.

example

As an example of how we can apply the above concepts, consider a lossless reciprocal two-port network as shown in fig. 3. Because the network is reciprocal, $s_{21} = s_{12}$ and we can write the S-matrix as,

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}.$$

From the lossless condition, $S^{\dagger}\ S = I$, or,

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} s_{11}^* & s_{12}^* \\ s_{12}^* & s_{22}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which gives us the following set of relations:

$$\begin{aligned} |s_{11}|^2 + |s_{12}|^2 &= 1 \qquad & s_{11}s_{12}^* + s_{12}s_{22}^* &= 0 \\ |s_{12}|^2 + |s_{22}|^2 &= 1 \qquad & s_{11}^*s_{12} + s_{12}^*s_{22} &= 0 \end{aligned}$$

From the first set of equations (on the left), we clearly see that $|s_{11}| = |s_{22}|$. The second set of equations (on the right) are simply complex conjugates of each other.

In the special case that the network is *symmetric*, then $s_{11} = s_{22}$ and the relations above reduce to,

$$s_{11}s_{12}^* + s_{12}s_{22}^* = s_{11}s_{12}^* + s_{12}s_{11}^* = 2Re\Big\{s_{11}s_{12}^*\Big\} = 0 \tag{3}$$

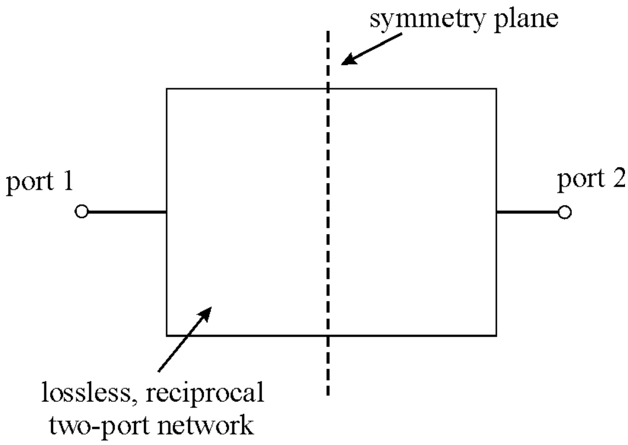


Fig. 3.

This implies that the parameters s_{11} and s_{12} are $90°$ out of phase. To see this, you can write the s-parameters in polar form:

$$s_{11} = \rho_{11} e^{j\phi_{11}} \quad \text{and} \quad s_{12} = \rho_{12} e^{j\phi_{12}}$$

Inserting this into equation (3) gives us,

$$2\rho_{11}\rho_{12} Re\left\{e^{j(\phi_{11}-\phi_{12})}\right\} = 0 \quad \Rightarrow \quad \phi_{11} - \phi_{12} = \pm\frac{\pi}{2}$$