On the Structure of the Cone of Positive Operators

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ABSTRACT

If K_1 is a proper cone in R^{n_1} and K_2 is a proper cone in R^{n_2}, then, as is well known, the set $\pi(K_1, K_2)$, which consists of all $n_2 \times n_1$ real matrices which take K_1 into K_2, forms a proper cone in the space R^{n_2, n_1}. In this paper a study of this cone is made, with particular emphasis on its faces and duality operator. A face of $\pi(K_1, K_2)$ is called simple if it is composed of all matrices in $\pi(K_1, K_2)$ which take some fixed face of K_1 into some fixed face of K_2. Maximal faces of $\pi(K_1, K_2)$ are characterized as a particular kind of simple faces. Relations between the duality operator of $\pi(K_1, K_2)$ and those of K_1 and K_2 are obtained. Among many other results, it is proved that $d_{\pi(K_1, K_2)}$, the duality operator of $\pi(K_1, K_2)$, is injective if and only if d_{K_2} is injective and each face of $\pi(K_1, K_2)$ is an intersection of simple faces. Two open questions are posed.

1. INTRODUCTION

Let K_1, K_2 be proper cones in R^{n_1} and R^{n_2} respectively, and let $\pi(K_1, K_2) = \{A \in R^{n_2, n_1} : AK_1 \subseteq K_2\}$, where R^{n_2, n_1} is the space of all $n_2 \times n_1$ real matrices. As is well known (see Schneider and Vidyasagar [30]), $\pi(K_1, K_2)$ forms a proper cone in R^{n_2, n_1}. Since 1975 the geometric properties of this

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cone have attracted the attention of many research workers. In particular, the determination of its extreme operators has been a focus of interest, and about ten papers have been written on the subject, though the problem is still far from being resolved. (See the References.) In this paper we return to a study of this cone, with particular emphasis on its faces and duality operator. In our investigation, we shall try to answer the following fundamental questions: How are the cones $\pi(K_1, K_2)$, K_1, and K_2 related? To what extent is the position of a matrix inside $\pi(K_1, K_2)$ determined by its action between the cones K_1 and K_2? Is there a simple way to describe the faces of $\pi(K_1, K_2)$? To answer the above questions, the duality operator of a cone comes in as a natural tool, since the injectivity or surjectivity of the duality operator of a cone is intimately related to its geometric properties.

We now describe the contents of this paper in more detail. The necessary definitions and notation are given in Section 2.

In Section 3, we give some fundamental results or observations on the cone $\pi(K_1, K_2)$ and its dual cone. In particular, for any $A \in \pi(K_1, K_2)$, we are able to characterize matrices belonging to the smallest exposed face of $\pi(K_1, K_2)$ containing A in terms of their actions between the cones K_1 and K_2.

In Section 4, simple faces of $\pi(K_1, K_2)$ are introduced. We determine the intersection of simple (as well as exposed simple) faces of $\pi(K_1, K_2)$ containing a fixed matrix A. Maximal faces of $\pi(K_1, K_2)$ are characterized as a particular kind of simple faces. Also we prove that each face of $\pi(K_1, K_2)$ is an intersection of simple faces if and only if the following nice property is satisfied: for any $A, B \in \pi(K_1, K_2)$, if $Bx \in \Phi(Ax)$ for all $x \in K_1$ then $B \in \Phi(A)$.

In Section 5, we first observe that the cone K_2 (as well as K_1^*) is linearly isomorphic with the intersection of the cone $\pi(K_1, K_2)$ with a linear subspace. Then we obtain some relations between the duality operators $d_{\pi(K_1, K_2)}$, d_{K_1}, and d_{K_2}. Among many other results, it is proved that $d_{\pi(K_1, K_2)}$ is injective if and only if d_{K_2} is injective and each face of $\pi(K_1, K_2)$ is an intersection of simple faces. A special property of the cone $\pi(K)$ [$= \pi(K, K)$] among general proper cones is also singled out (see Theorem 5.14).

Finally, in Section 6 we pose two open questions.

A minor portion of this work comes from Chapter 3 of the author's Ph.D. thesis [32]. Other results in this paper (at least in their primitive forms) were mostly obtained in the academic year 1978–79 when the author visited the University of Waterloo, and were contained in an earlier unpublished summary referred to by Barker [7, paragraph after Definition 2.A.8]. But some new insights and better proofs were also found in the process of writing up this paper.

2. PRELIMINARIES

We assume familiarity with the elementary properties of cones. For convenience and to fix notation, below we collect some of the definitions and cite relevant references.

Let K be a nonempty subset of a finite-dimensional real vector space V. K is called a (*convex*) *cone* if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \geqslant 0$. K is *pointed* if $K \cap (-K) = \{0\}$. K is *full* if its interior (in the usual topology of V) is nonempty; equivalently, $K - K = V$. If K is closed and satisfies all the above properties, K is called a *proper cone*. From now on, we always use K to denote a proper cone in the n-dimensional euclidean space R^n.

K induces a partial ordering on R^n by $x \overset{K}{\geqslant} y$ (we also write as $y \overset{K}{\leqslant} x$) if and only if $x - y \in K$.

A subcone F of K is called a *face* of K if $0 \overset{K}{\leqslant} y \overset{K}{\leqslant} x$ and $x \in F$ imply $y \in F$. $\{0\}$ and K itself are always faces of K, known as *trivial faces*. Other faces of K are said to be *nontrivial*. An arbitrary intersection of faces of K is always a face of K. If $S \subseteq K$, then the smallest face of K including S is called the *face generated by* S and is denoted by $\Phi(S)$. Sometimes we might consider several cones at the same time. Then it will be clear from the context of which cone $\Phi(S)$ is a face. If $S = \{x\}$, we write $\Phi(x)$ for simplicity. Then we have $\Phi(x) = \{ y \in K : \alpha y \overset{K}{\leqslant} x$ for some $\alpha > 0\}$. We denote the *relative interior* and *relative boundary* of $\Phi(x)$ respectively by relint $\Phi(x)$ and rbd $\Phi(x)$. It can be shown that for any face F of K and any vector $x \in K$, $\Phi(x) = F$ if and only if $x \in \operatorname{relint} F$. If x is a nonzero vector in K such that $\Phi(x) = \{\alpha x : \alpha \geqslant 0\}$, then $\Phi(x)$ is called an *extreme ray* and x an *extreme vector* of K. The zero vector is also treated as an extreme vector of K. K is said to be *polyhedral* if it has finitely many extreme rays. We denote by $\mathfrak{F}(K)$ the collection of all faces of K. Then $\mathfrak{F}(K)$ forms a complete lattice of finite length under the operations of meet and join given by $F \wedge G = F \cap G$ and $F \vee G = \Phi(F \cup G)$ (see Barker [3] and Loewy and Tam [26]).

By the *dual cone* of K, denoted by K^*, we mean the set $\{ z \in R^n : \langle z, x \rangle \geqslant 0$ for all $x \in K\}$, where $\langle z, x \rangle = x^T z$ denotes the usual inner product between the vectors x and z. (Vectors of R^n are represented as column vectors.) If K is a proper cone, then so is K^*; furthermore, $K^{**} = K$ and we have the following useful characterization:

$$\operatorname{int} K = \{ x \in K : \langle z, x \rangle > 0 \text{ for all nonzero } z \in K^* \}$$

(see Schneider and Vidyasagar [30]).

The concept of duality can be defined in a more general setting in terms of sets in a vector space and its dual space (see, for instance, Barker [5, 7]). But, since we are working with finite-dimensional real vector spaces, there is no loss of generality in restricting ourselves to euclidean spaces.

By the *duality operator* of K, we mean the mapping $d_K : \mathfrak{F}(K) \to \mathfrak{F}(K^*)$ given by $d_K(F) = (\text{span } F)^\perp \cap K^*$, where $(\text{span } F)^\perp$ is the orthogonal complement of the linear span of F. We call $d_K(F)$ the *dual face* of F. The duality operator of a cone was introduced and studied independently by Barker [5, 6] and Tam [32, 34, 36]. Ever since, it has been a useful tool in the study of cones. A face F of K is said to be *exposed* if it is the dual face of some face of K^*. Geometrically, a face of K, other than K itself, is exposed if and only if it is the intersection of K with a supporting hyperplane. It is known that the mapping $d_{K^*} \circ d_K$ is a closure operation on $\mathfrak{F}(K)$. We shall denote it by cl_K. It is not difficult to show that for any face F of K, $\text{cl}_K(F)$ is the smallest exposed face of K that includes F. Also, each face of K is exposed if and only if the duality operator d_K is injective.

Following Loewy and Schneider [24], we say that a cone K is a *direct sum* of K_1 and K_2, and we write $K = K_1 \oplus K_2$, if (a) $\text{span } K_1 \cap \text{span } K_2 = \{0\}$ and (b) $K = K_1 + K_2$. (Then K_1 and K_2 are faces of K.) The cone K is said to be *decomposable* if there exist nonzero subsets K_1 and K_2 such that $K = K_1 \oplus K_2$. Otherwise, K is said to be *indecomposable*. The following result is fundamental (Fiedler and Ptak [19, (2, 6)]):

For any cone K, there exist indecomposable cones $K_1, \dots, K_r$ such that $K = K_1 \oplus \cdots \oplus K_r$ $(r \geqslant 1)$. This decomposition of K is unique (except for the order of its summands).

3. THE CONE $\pi(K_1, K_2)$ AND ITS DUAL CONE

Hereafter in this paper, we shall use K_1 to denote a proper cone in R^{n_1}, and K_2 a proper cone in R^{n_2}, where $n_1, n_2 \geqslant 1$. We denote by $\pi(K_1, K_2)$ the set of all $n_2 \times n_1$ real matrices A such that $AK_1 \subseteq K_2$. When $K_1 = K_2 = K$, we write $\pi(K_1, K_2)$ simply as $\pi(K)$. It is known that $\pi(K_1, K_2)$ forms a proper cone of R^{n_2, n_1} with $\text{int } \pi(K_1, K_2) = \{A \in R^{n_2, n_1} : A[K_1 \setminus \{0\}] \subseteq \text{int } K_2\}$ (see Barker [2, Proposition 1]).

Note that for any $A \in R^{n_2, n_1}$, $A \in \pi(K_1, K_2)$ if and only if $z^T A y \geqslant 0$ for all $y \in K_1$ and $z \in K_2^*$. Hence, matrices in $\pi(K_1, K_2)$ can be identified with bilinear functionals of $R^{n_1} \times R^{n_2}$ which are nonnegative on $K_1 \times K_2^*$. Indeed, it is possible to give our theory in the language of tensor products of cones. (See Barker [4] or [7] for further details.) Since we feel that this viewpoint is not helpful here, we keep our original approach.

If K_1 and K_2 are indecomposable, then (as can be shown by modifying a proof of Barker and Loewy [9, Lemma 2.2]), so is the cone $\pi(K_1, K_2)$. More generally, if $K_1 = C_{11} \oplus \cdots \oplus C_{1r}$ and $K_2 = C_{21} \oplus \cdots \oplus C_{2s}$ denote respectively the unique representations of K_1 and K_2 as direct sums of indecomposable subcones, then $\pi(K_1, K_2) = \bigoplus_{1 \leqslant i \leqslant r,\, 1 \leqslant j \leqslant s} S_{ij}$, where each subcone S_{ij} can be identified with the indecomposable cone $\pi(C_{1i}, C_{2j})$. (For an explanation, refer to the paragraph preceding Proposition 2.4 in Tam [34].) Thus, the study of a cone of the form $\pi(K_1, K_2)$ can be reduced to the special case where both cones K_1 and K_2 are indecomposable.

On R^{n_2, n_1} we introduce the usual inner product: $\langle A, B\rangle = \operatorname{tr}(B^T A)$. It is easily verified that, for any $A \in R^{n_2, n_1}$, $y \in R^{n_1}$, and $z \in R^{n_2}$, we have $\langle z, Ay\rangle = \langle zy^T, A\rangle$. Consequently, we obtain $\pi(K_1, K_2) = [\operatorname{pos}\{zy^T : y \in K_1 \text{ and } z \in K_2^*\}]^*$, where we use pos S to denote the *positive hull of* S (that is, the set of all possible nonnegative linear combinations of vectors taken from S); hence, $\pi(K_1, K_2)^* = \operatorname{cl}\operatorname{pos}\{zy^T : y \in K_1 \text{ and } z \in K_2^*\}$. This fact was first observed by Berman and Gaiha [13, Theorem 3.1(ii)]. The author [33, Theorem 1] completed the result by showing that the cone $\operatorname{pos}\{zy^T : y \in K_1 \text{ and } z \in K_2^*\}$ is closed. This result renders the study of the cone $\pi(K_1, K_2)$ and its dual cone more tractable.

For convenience, if $S_1 \subseteq R^{n_1}$ and $S_2 \subseteq R^{n_2}$, we shall use the notation $S_2 \otimes_p S_1$ to denote the set $\operatorname{pos}\{zy^T \in R^{n_2, n_1} : z \in S_2 \text{ and } y \in S_1\}$. In this notation, we have $\pi(K_1, K_2)^* = K_2^* \otimes_p K_1$.

It is not difficult to show that $K_2 \otimes_p K_1^*$ is precisely the cone generated by the rank-one matrices in $\pi(K_1, K_2)$. When $K_1 = K_2 = K$, it is clear that the identity operator $I \in \pi(K)$. In fact, the identity operator always lies outside the subcone $\pi(K^*)^*$ of $\pi(K)$, unless K is simplicial (see Tam [33, Theorems 3 and 4]). Besides this, we know very little about matrices of $\pi(K)$ which lie outside $\pi(K^*)^*$. Some such results have appeared either explicitly or implicitly in Sung and Tam [31] and Tam [33–35].

It is easy to check that the transposition map is an isometry between R^{n_2, n_1} and R^{n_1, n_2} which takes the cone $\pi(K_1, K_2)$ $[\pi(K_1, K_2)^*]$ onto the cone $\pi(K_2^*, K_1^*)$ $[\pi(K_2^*, K_1^*)^*]$. As a consequence, the duality operator $d_{\pi(K_1, K_2)}$ is injective (surjective) iff $d_{\pi(K_2^*, K_1^*)}$ is injective (surjective). As another consequence, we also have

THEOREM 3.1. *Let $A, B \in \pi(K_1, K_2)$. Then we have:*

(a) $B \in \Phi(A)$ *iff* $B^T \in \Phi(A^T)$.
(b) $B \in \operatorname{cl}_{\pi(K_1, K_2)}(\Phi(A))$ *iff* $B^T \in \operatorname{cl}_{\pi(K_2^*, K_1^*)}(\Phi(A^T))$.

THEOREM 3.2. *Let $A, B \in \pi(K_1, K_2)$. The following conditions are equivalent:*

(i) $B \in \operatorname{cl}_{\pi(K_1, K_2)}(\Phi(A))$.
(ii) *For every vector $y \in K_1$,* $By \in \operatorname{cl}_{K_2}(\Phi(Ay))$.

Proof. Condition (i) is equivalent to: for any $C \in \pi(K_1, K_2)^*$, if $\langle C, A\rangle = 0$ then $\langle C, B\rangle = 0$. But, as noted before, $\pi(K_1, K_2)^*$ is the positive hull of matrices of the form zy^T where $y \in K_1$ and $z \in K_2^*$; hence the condition is also equivalent to: for any vectors $y \in K_1$ and $z \in K_2^*$, we have that $\langle zy^T, A\rangle = 0$ implies $\langle zy^T, B\rangle = 0$. But $\langle zy^T, A\rangle = \langle z, Ay\rangle$, and a similar statement holds with A replaced by B, so the latter condition is equivalent to (ii). ∎

It is easy to verify that if $B \in \Phi(A)$, where $A, B \in \pi(K_1, K_2)$, then $By \in \Phi(Ay)$ for every vector $y \in K_1$. However, in contrast with Theorem 3.2(ii) $\Rightarrow$ (i), the converse of this result does not hold: see our Corollary 5.7.

The following result will be of frequent use.

THEOREM 3.3. *Let $A \in \pi(K_1, K_2)$, and let $x, y \in K_1$.*

(a) *If $y \in \Phi(x)$ then $Ay \in \Phi(Ax)$.*
(b) *If $y \in \mathrm{cl}_{K_1}(\Phi(x))$ then $Ay \in \mathrm{cl}_{K_2}(\Phi(Ax))$.*

Proof. The proof of (a) is easy. To prove (b), let $z \in d_{K_2}(\Phi(Ax))$. Then $\langle A^Tz, x\rangle = \langle z, Ax\rangle = 0$, and hence $A^Tz \in d_{K_1}(\Phi(x))$. But $y \in \mathrm{cl}_{K_1}(\Phi(x))$, so $\langle Ay, z\rangle = \langle y, A^Tz\rangle = 0$. Since this is true for every vector $z \in d_{K_2}(\Phi(Ax))$, we have $Ay \in \mathrm{cl}_{K_2}(\Phi(Ax))$. ∎

Note that condition (ii) of Theorem 3.2 can be reformulated as: for any face F of K_1, $BF \subseteq \mathrm{cl}_{K_2}(\Phi(AF))$. Similarly, condition (b) of Theorem 3.3 can also be put in the form: for any face F of K_1, $A\,\mathrm{cl}_{K_1}(F) \subseteq \mathrm{cl}_{K_2}(\Phi(AF))$.

In passing, we give one interesting consequence of the above result. Recall that a matrix $A \in \pi(K)$ is said to be *K-reducible* if A leaves invariant a nontrivial face of K; otherwise, A is said to be *K-irreducible.* For other equivalent definitions, see Schneider and Vidyasagar [30].

COROLLARY 3.4. *Let $A \in \pi(K)$. If A is K-reducible, then so is B for every $B \in \mathrm{cl}_{\pi(K)}(\Phi(A))$.*

Proof. Suppose that A is K-reducible. Then there exists a nontrivial face F of K such that $AF \subseteq F$, and hence $\Phi(AF) \subseteq F$. Let $B \in \mathrm{cl}_{\pi(K)}(\Phi(A))$. Then by Theorem 3.2, we have $BF \subseteq \mathrm{cl}_K(\Phi(AF)) \subseteq \mathrm{cl}_K(F)$, and hence $\mathrm{cl}_K(\Phi(BF)) \subseteq \mathrm{cl}_K(F)$. On the other hand, by Theorem 3.3(b), we have $B\,\mathrm{cl}_K(F) \subseteq \mathrm{cl}_K(\Phi(BF))$. Thus B leaves invariant the nontrivial face $\mathrm{cl}_K(F)$, and hence is K-reducible. ∎

4. SIMPLE FACES

For any faces F of K_1 and G of K_2, we shall denote by $\pi_{F,G}$ the set $\{A \in \pi(K_1, K_2) : AF \subseteq G\}$. It is easily verified that $\pi_{F,G}$ is a face of $\pi(K_1, K_2)$. Any face of $\pi(K_1, K_2)$ of this form will be called *simple*. Simple faces were introduced independently by Barker [6] and by Tam [34]. For convenience, we collect below some elementary properties of simple faces.

THEOREM 4.1. *Let F, F_i be faces of K_1, and G, G_i be faces of K_2, $i = 1, 2$. Then each of the following holds*:

(a) *If $G_1 \subseteq G_2$ then $\pi_{F,G_1} \subseteq \pi_{F,G_2}$.*
(b) *If $F_1 \subseteq F_2$ then $\pi_{F_2,G} \subseteq \pi_{F_1,G}$.*
(c) *$\pi_{F,G} = \pi(K_1, K_2)$ iff $F = 0$ or $G = K_2$.*
(d) *$\pi_{F,G} = 0$ iff $F = K_1$ and $G = 0$.*
(e) *$\pi_{F,G_1 \wedge G_2} = \pi_{F,G_1} \wedge \pi_{F,G_2}$.*
(f) *$\pi_{F_1 \vee F_2,G} = \pi_{F_1,G} \wedge \pi_{F_2,G}$.*

Proof. The verifications of properties (a)–(f) are fairly easy. Just as an illustration, we give the proofs of (c) and (f).

(c): The "if" part is obvious. To prove the "only if" part, suppose that $F \neq 0$ and $G \neq K_2$. Choose some vector $y \in K_2 \setminus G$ and some vector $z \in \operatorname{int} K_1^*$. Then $yz^T \in \pi(K_1, K_2) \setminus \pi_{F,G}$.

(f): By (b), $\pi_{F_1 \vee F_2,G} \subseteq \pi_{F_1,G} \wedge \pi_{F_2,G}$. Let $A \in \pi_{F_1,G} \wedge \pi_{F_2,G}$. Then $AF_i \subseteq G$ for $i = 1, 2$, and hence $A(F_1 + F_2) \subseteq G$. In view of the fact that $F_1 \vee F_2 = \Phi(F_1 + F_2)$ (Barker [3, Proposition 3.2]) and that $\Phi(S) = \{y : 0 \overset{K}{\leqslant} y \overset{K}{\leqslant} x$ for some $x \in \operatorname{pos} S\}$ (Barker and Schneider [10, Lemma 2.8]), we obtain $A(F_1 \vee F_2) \subseteq G$, and hence $A \in \pi_{F_1 \vee F_2,G}$. Therefore, $\pi_{F_1 \vee F_2,G} = \pi_{F_1,G} \wedge \pi_{F_2,G}$. ■

If E is a face of K_1 and F a face of K_2^*, it is not difficult to verify that $F \otimes_p E$ is a face of $\pi(K_1, K_2)^*$. Also, extreme matrices of $F \otimes_p E$ are of the form zy^T where z is an extreme vector of F and y an extreme vector of E. Before we give results which relate the simple faces of $\pi(K_1, K_2)$ to faces of $\pi(K_1, K_2)^*$ of the above type under the action of the duality operators, we first prove a lemma.

LEMMA 4.2. *Let F, G be faces of K_1 and K_2 respectively. Let $A \in \pi(K_1, K_2)$ be such that $\Phi(A) = \pi_{F,G}$. Then we have*:

(a) *$Ay \in \operatorname{cl}_{K_2}(G) \setminus \operatorname{rbd} G$ for all nonzero $y \in \operatorname{cl}_{K_1}(F)$.*
(b) *$Ay \in \operatorname{int} K_2$ for all $y \in K_1 \setminus \operatorname{cl}_{K_1}(F)$.*

Proof. Choose vectors $x_0 \in \operatorname{int} K_2$, $x_1 \in \operatorname{relint} G$, $z_0 \in \operatorname{int} K_1^*$, and $z_1 \in \operatorname{relint} d_{K_1}(F)$. Denote the matrix $x_0 z_1^T + x_1 z_0^T$ by B. It is straightforward to verify that $B \in \pi_{F,G}$, that $By \in \operatorname{relint} G$ for all nonzero $y \in \operatorname{cl}_{K_1}(F)$, and that $By \in \operatorname{int} K_2$ whenever $y \in K_1 \setminus \operatorname{cl}_{K_1}(F)$. As $A \in \operatorname{relint} \pi_{F,G}$, there exists some $\alpha > 0$ such that $\alpha A - B \in \pi(K_1, K_2)$. Consequently, we have $Ay \notin \operatorname{rbd} G$ for any $0 \neq y \in \operatorname{cl}_{K_1}(F)$, and $Ay \in \operatorname{int} K_2$ for any $y \in K_1 \setminus \operatorname{cl}_{K_1}(F)$. Furthermore, by Theorem 3.3, for any $y \in \operatorname{cl}_{K_1}(F)$ we have $Ay \in \operatorname{cl}_{K_2}(\Phi(AF))$. But $AF \subseteq G$, so $Ay \in \operatorname{cl}_{K_2}(G)$. The proof is complete. ∎

One may ask whether the matrix $x_0 z_1^T + x_1 z_0^T$ constructed in the proof of Lemma 4.2 actually belongs to relint $\pi_{F,G}$. *If* the answer is in the affirmative, then Lemma 4.2(a) can be strengthened to $A[\operatorname{cl}_{K_1}(F) \setminus \{0\}] \subseteq \operatorname{relint} G$. Then it will follow that $\pi_{\operatorname{cl}_K(F), G} = \pi_{F,G}$. Unfortunately, the answer is no, as the following example will show.

EXAMPLE 4.3. Let C be the convex set in R^2 given by

$$C = \left\{ (\xi_1, \xi_2)^T : \xi_1^2 + (\xi_2 - 1)^2 \leqslant 1,\ \xi_1 \leqslant 0 \right\}$$

$$\cup \left\{ (\xi_1, \xi_2)^T : 2\xi_1 + \xi_2 \leqslant 2,\ \xi_1, \xi_2 \geqslant 0 \right\}.$$

Let K be the proper cone in R^3 given by

$$K = \left\{ \alpha \begin{pmatrix} x \\ 1 \end{pmatrix} \in R^3 : x \in C,\ \alpha \geqslant 0 \right\}.$$

Denote by F the extreme ray $\Phi((0,0,1)^T)$ of K. Then $\operatorname{cl}_K(F)$ is the 2-dimensional face $\Phi((\frac{1}{2}, 0, 1)^T)$. Let $A = \operatorname{diag}(\frac{1}{2}, 1, 1)$. It is straightforward to show that $A \in \pi(K)$. In fact, $A \in \pi_{F,F}$ and $A[\operatorname{cl}_K(F) \setminus F] \subseteq \operatorname{relint} \operatorname{cl}_K(F)$. It follows that in this case, for any matrix $C \in \operatorname{relint} \pi_{F,F}$, we have $C[\operatorname{cl}_K(F) \setminus F] \subseteq \operatorname{relint} \operatorname{cl}_K(F)$. So the matrix $x_0 z_1^T + x_1 z_0^T$ constructed in the proof of Lemma 4.2 does not belong to relint $\pi_{F,F}$.

THEOREM 4.4. *Let F be a face of K_1, G a face of K_2, and E a face of K_2^*. Then each of the following holds:*

(a) $d_{\pi(K_1, K_2)}(\pi_{F,G}) = d_{K_2}(G) \otimes_p \operatorname{cl}_{K_1}(F)$.
(b) $d_{\pi(K_1, K_2)^*}(E \otimes_p F) = \pi_{F, d_{K_2^*}(E)}$.
(c) $\operatorname{cl}_{\pi(K_1, K_2)}(\pi_{F,G}) = \pi_{\operatorname{cl}_{K_1}(F), \operatorname{cl}_{K_2}(G)} = \pi_{F, \operatorname{cl}_{K_2}(G)}$.
(d) $\operatorname{cl}_{\pi(K_1, K_2)^*}(E \otimes_p F) = \operatorname{cl}_{K_2^*}(E) \otimes_p \operatorname{cl}_{K_1}(F)$.
(e) *If $F \neq 0$, then $\pi_{F,G}$ is an exposed face of $\pi(K_1, K_2)$ if and only if G is an exposed face of K_2.*

(f) $E \otimes_p F$ *is an exposed face of* $\pi(K_1, K_2)^*$ *if and only if E is an exposed face* K_2^* *and F is an exposed face of* K_1.

Proof. (a): Choose some $A \in \text{relint}\ \pi_{F,G}$. Consider any vectors $w \in d_{K_2}(G)$ and $x \in \text{cl}_{K_1}(F)$. Then $Ax \in \text{cl}_{K_2}(G)$, and hence we have $\langle wx^T, A\rangle = \langle w, Ax\rangle = 0$. This establishes the inclusion $d_{K_2}(G) \otimes_p \text{cl}_{K_1}(F) \subseteq d_{\pi(K_1, K_2)}(\pi_{F,G})$. To prove the reverse inclusion, it suffices to show that if $0 \neq w \in K_2^*$ and $0 \neq x \in K_1$ such that $wx^T \in d_{\pi(K_1,K_2)}(\pi_{F,G})$, then $w \in d_{K_2}(G)$ and $x \in \text{cl}_{K_1}(F)$. For any such vectors w and x, we have $\langle w, Ax\rangle = \langle wx^T, A\rangle = 0$. By Lemma 4.2(b), necessarily $x \in \text{cl}_{K_1}(F)$. Choose some vectors $z \in \text{int}\ K_1^*$ and $u \in \text{relint}\ G$. Then $uz^T \in \pi_{F,G}$ and satisfies $(uz^T)(x) \in \text{relint}\ G$. But $0 = \langle wx^T, uz^T\rangle = \langle w, (uz^T)(x)\rangle$, hence $w \in d_{K_2}(G)$.

(b): Straightforward verification.

(c): The first equality follows from (a) and (b). As for the second equality, the inclusion $\pi_{\text{cl}_{K_1}(F), \text{cl}_{K_2}(G)} \subseteq \pi_{F, \text{cl}_{K_2}(G)}$ is obvious; the reverse inclusion follows from Theorem 3.3(b).

(d): follows from (a) and (b).

Finally, (e) and (f) follow from (c) and (d) respectively. ∎

It is natural to ask whether there are faces of $\pi(K_1, K_2)$ that are not simple. Let us consider the simplest cases first. If K_1 is 1-dimensional (that is, equals R_+ or $-R_+$) the answer is no. It is clear that $\pi(R_+, K)$ can be identified with K, and under this identification the simple face $\pi_{R_+, F}$ of $\pi(R_+, K)$ corresponds to the face F of K (and the face $\pi_{0,F}$ corresponds to K). So each face of $\pi(R_+, K)$ ($\equiv K$) is simple. On the contrary, $\pi(K, R_+)$ has some faces that are not simple, so long as K^* has nonexposed faces. The reason is, $\pi(K, R_+)$ is just the cone of linear functionals nonnegative on K and can be identified with K^*. Under this identification, the face π_{F,R_+} corresponds to K^* [$= d_K(0)$], and the face $\pi_{F,0}$ corresponds to $d_K(F)$. So faces of $\pi(K, R_+)$ that correspond to nonexposed faces of K^* are not simple.

More generally, if K_1 and K_2 are both of dimension $\geqslant 2$, we have the following result:

THEOREM 4.5. *Let* K_1 *and* K_2 *be proper cones, both of dimension* $\geqslant 2$. *Then*:

(a) $\pi(K_1, K_2)$ *has a face which is not simple.*

(b) $\pi(K_1, K_2)^*$ *has a face which is not of the form* $E \otimes_p F$ *where E is a face of* K_2^* *and F a face of* K_1.

Proof. (a): Since $\dim K_1 \geqslant 2$, we can choose nonzero vectors $x_1, x_2 \in \partial K_1$ such that $x_1 + x_2 \in \text{int}\ K_1$. Choose a nonzero vector $z_1 \in d_{K_1}(\Phi(x_2))$. Then since $x_1 + x_2 \in \operatorname{int} K_1$, we have $\langle z_1, x_1\rangle = \langle z_1, x_1 + x_2\rangle > 0$. Replacing z_1 by a suitable positive multiple, we may assume that $\langle z_1, x_1\rangle = 1$. Similarly, we can choose a vector $z_2 \in d_{K_1}(\Phi(x_1))$ such that $\langle z_2, x_2\rangle = 1$. Now choose nonzero vectors $y_1, y_2 \in \partial K_2$ such that $y_1 + y_2 \in \operatorname{int} K_2$, and let $A = y_1 z_1^T + y_2 z_2^T$. Then clearly $A \in \pi(K_1, K_2)$. We are going to show that $\Phi(A)$ is not a simple face of $\pi(K_1, K_2)$. Assume to the contrary that $\Phi(A) = \pi_{F,G}$ for some faces F of K_1 and G of K_2. By our choice, the vectors x_i, z_i, $i = 1, 2$, satisfy $\langle z_i, x_j\rangle = \delta_{ij}$ (the Kronecker delta); so we have $Ax_i = y_i$, $i = 1, 2$. If $F = K_1$, then $y_1, y_2 \in G$; but $y_1 + y_2 \in \operatorname{int} K_2$, so $G = K_2$. Thus we have $A \in \operatorname{int} \pi(K_1, K_2)$ and hence $A[K_1 \setminus \{0\}] \subseteq \operatorname{int} K_2$, which however is not the case. So we have $F \neq K_1$, and hence $\operatorname{cl}_{K_1}(F) \neq K_1$. Since $x_1 + x_2 \in \operatorname{int} K_1$, we have either $x_1 \notin \operatorname{cl}_{K_1}(F)$ or $x_2 \notin \operatorname{cl}_{K_1}(F)$. But then, by Lemma 4.2(b), either Ax_1 or Ax_2 belongs to $\operatorname{int} K_2$. However, as shown above, $Ax_i = y_i$ for $i = 1, 2$, and by our choice, y_1 and y_2 both belong to ∂K_2; so again we arrive at a contradiction.

(b): Choose vectors $x_1, x_2 \in \partial K_1$ such that $x_1 + x_2 \in \operatorname{int} K_1$. As shown in the proof of part (a), there exist vectors $z_1, z_2 \in \partial K_1^*$ such that $\langle z_i, x_j\rangle = \delta_{ij}$. Also choose vectors $w_1, w_2 \in \partial K_2^*$ such that $w_1 + w_2 \in \operatorname{int} K_2^*$. Then again we can find vectors $y_1, y_2 \in \partial K_2$ such that $\langle w_i, y_j\rangle = \delta_{ij}$. Let $A = y_1 z_1^T + y_2 z_2^T$. Clearly $A \in \pi(K_1, K_2)$. We contend that if the face $d_{\pi(K_1, K_2)}(\Phi(A))$ is of the form $E \otimes_p F$ for some faces E of K_2^* and F of K_1, then we shall arrive at a contradiction. Since $\langle w_1 x_2^T, A\rangle = \langle w_1, Ax_2\rangle = \langle w_1, y_2\rangle = 0$, we have $w_1 x_2^T \in d_{\pi(K_1, K_2)}(\Phi(A))$. Similarly, we also have $w_2 x_1^T \in d_{\pi(K_1, K_2)}(\Phi(A))$, and hence $w_1 x_2^T + w_2 x_1^T \in d_{\pi(K_1, K_2)}(\Phi(A))$. As $d_{\pi(K_1, K_2)}(\Phi(A)) = E \otimes_p F$, there exist nonzero vectors $u_i \in E$ and $v_i \in F$, $1 \leqslant i \leqslant p$, such that $w_1 x_2^T + w_2 x_1^T = \sum_{i=1}^p u_i v_i^T$. Multiply both sides of this equality on the right by a vector taken from $\operatorname{int} K_1^*$. The resulting expression says that some positive linear combination of the vectors w_1 and w_2 belongs to the face E of K_2^*. Hence, we have $w_1, w_2 \in E$. Similarly, we also have $x_1, x_2 \in F$. But, by our choice, $w_1 + w_2 \in \operatorname{int} K_2^*$ and $x_1 + x_2 \in \operatorname{int} K_1$, so we obtain $E = K_2^*$ and $F = K_1$; in other words, $d_{\pi(K_1, K_2)}(\Phi(A)) = \pi(K_1, K_2)^*$, and hence $A = 0$. Thus we have arrived at the desired contradiction. ∎

Observe that for any $A \in \pi(K_1, K_2)$, the set $\{B \in \pi(K_1, K_2) : Bx \in \Phi(Ax) \text{ for all } x \in K_1\}$ is a face of $\pi(K_1, K_2)$. Moreover, we have the following inclusion relations:

$$\begin{aligned}\Phi(A) &\subseteq \{B \in \pi(K_1, K_2) : Bx \in \Phi(Ax) \text{ for all } x \in K_1\}\\ &\subseteq \operatorname{cl}_{\pi(K_1, K_2)}(\Phi(A)).\end{aligned}$$

In terms of the simple faces of $\pi(K_1, K_2)$, in fact, more can be said.

THEOREM 4.6. *For any $A \in \pi(K_1, K_2)$, we have*

(a) $\{B \in \pi(K_1, K_2) : Bx \in \Phi(Ax)$ *for all* $x \in K_1\}$ *is equal to the intersection of all simple faces of* $\pi(K_1, K_2)$ *that include* A; *and*

(b) $\mathrm{cl}_{\pi(K_1, K_2)}(\Phi(A))$ *is equal to the intersection of all exposed simple faces of* $\pi(K_1, K_2)$ *that include* A.

Proof. (a): Denote the set $\{B \in \pi(K_1, K_2) : Bx \in \Phi(Ax)$ for all $x \in K_1\}$ by $\mathfrak{A}$. Let $B \in \mathfrak{A}$, and $\pi_{F,G}$ be a simple face of $\pi(K_1, K_2)$ that includes $\Phi(A)$. Write F as $\Phi(x)$. Then since $A \in \pi_{F,G}$, we have $AF \subseteq G$ and hence $\Phi(Ax) \subseteq G$. But $B \in \mathfrak{A}$, so we also have $BF \subseteq \Phi(Bx) \subseteq \Phi(Ax) \subseteq G$; in other words, $B \in \pi_{F,G}$. Hence $\mathfrak{A}$ is a subset of the intersection of all simple faces that include A.

Conversely, suppose that $B \in \pi_{F,G}$ whenever $A \in \pi_{F,G}$. For any vector $x \in K_1$, clearly we have $A \in \pi_{\Phi(x), \Phi(Ax)}$; hence $B\Phi(x) \subseteq \Phi(Ax)$, or equivalently, $Bx \in \Phi(Ax)$. This shows that $B \in \mathfrak{A}$.

(b): Since $\mathrm{cl}_{\pi(K_1, K_2)}(\Phi(A))$ is the smallest exposed face of $\pi(K_1, K_2)$ that includes A, it is clear that $\mathrm{cl}_{\pi(K_1, K_2)}(\Phi(A))$ is included in the intersection of all exposed simple faces including A. Conversely, let B belong to every exposed simple face that includes A. To prove that $B \in \mathrm{cl}_{\pi(K_1, K_2)}(\Phi(A))$, it suffices to show that $\langle zy^T, B\rangle = 0$ whenever $\langle zy^T, A\rangle = 0$, where $y \in K_1$ and $z \in K_2^*$. Consider any such vectors y, z. Then $\langle z, Ay\rangle = 0$, or $Ay \in d_{K_2^*}(\Phi(z))$, and hence $A \in \pi_{\Phi(y), d_{K_2^*}(\Phi(z))}$. As $d_{K_2^*}(\Phi(z))$ is an exposed face of K_2, by Theorem 4.4(e) $\pi_{\Phi(y), d_{K_2^*}(\Phi(z))}$ is an exposed face of $\pi(K_1, K_2)$ including A. Hence, by our hypothesis on B, we have $B \in \pi_{\Phi(y), d_{K_2^*}(\Phi(z))}$. It follows that $\langle zy^T, B\rangle = \langle z, By\rangle = 0$. The proof is complete. ∎

We readily deduce the following

COROLLARY 4.7. *The following conditions are equivalent:*

(i) *Every face of* $\pi(K_1, K_2)$ *can be written as an intersection of simple faces.*

(ii) *For any* $A, B \in \pi(K_1, K_2)$, $Bx \in \Phi(Ax)$ *for all* $x \in K_1$ *implies that* $B \in \Phi(A)$.

If K_1 and K_2 are general proper cones, the determination of the extreme matrices in $\pi(K_1, K_2)$ appears to be a difficult problem. On the other hand, the determination of the maximal faces of $\pi(K_1, K_2)$ is relatively easy.

THEOREM 4.8.

(a) *Every maximal face of* $\pi(K_1, K_2)$ *is simple.*

(b) $\pi_{F,G}$, *where* F *is a face of* K_1 *and* G *a face of* K_2, *is a maximal face of*

$\pi(K_1, K_2)$ if and only if $d_{K_1}(F)$ is a maximal face of K_1^* and G is a maximal face of K_2.

Proof. (a): Let $\Phi(A)$ be a maximal face of $\pi(K_1, K_2)$. Then there exists some nonzero extreme matrix B of $\pi(K_1, K_2)^*$ such that $d_{\pi(K_1, K_2)^*}(\Phi(B)) = \Phi(A)$. As such, B can be expressed in the form zy^T, where y, z are nonzero extreme vectors of K_1 and K_2^* respectively. By Theorem 4.4(b) we have $d_{\pi(K_1, K_2)^*}(\Phi(zy^T)) = \pi_{\Phi(y), d_{K_2^*}}(\Phi(z))$; hence $\Phi(A)$ is a simple face of $\pi(K_1, K_2)$.

(b): "if" part: Write $\pi_{F,G}$ as $\Phi(A)$, where $A \in \operatorname{relint} \pi_{F,G}$. To establish the maximality of $\pi_{F,G}$, it suffices to show that for any $B \in \pi(K_1, K_2) \setminus \pi_{F,G}$ we have $A + B \in \operatorname{int} \pi(K_1, K_2)$, or equivalently, $(A + B)[K_1 \setminus \{0\}] \subseteq \operatorname{int} K_2$. Consider a nonzero vector $y \in K_1$. If $y \notin \operatorname{cl}_{K_1}(F)$, then by Lemma 4.2, $Ay \in \operatorname{int} K_2$ and hence $(A + B)y \in \operatorname{int} K_2$. If $y \in \operatorname{cl}_{K_1}(F)$, then again by Lemma 4.2, $Ay \in \operatorname{cl}_{K_2}(G) \setminus \operatorname{rbd} G$. Since G is maximal, $\operatorname{cl}_{K_2}(G) = G$, so $Ay \in \operatorname{relint} G$. If we can show that $By \notin G$, then by the maximality of G it will follow that $(A + B)y \in \operatorname{int} K_2$. Now we have $d_{K_1}(F) \subseteq d_{K_1}(\Phi(y))$, since $y \in \operatorname{cl}_{K_1}(F)$. Also, as y is nonzero, $d_{K_1}(\Phi(y)) \neq K_1^*$. Hence by the maximality of $d_{K_1}(F)$, we have $d_{K_1}(\Phi(y)) = d_{K_1}(F)$. Write F as$\Phi(x)$. Then $x \in \operatorname{cl}_{K_1}(\Phi(y))$, and by Theorem 3.3 $Bx \in \operatorname{cl}_{K_2}(\Phi(By))$. If $By \in G$, then $Bx \in \operatorname{cl}_{K_2}(G) = G$, contradicting the assumption that $B \notin \pi_{F,G}$. Thus $By \notin G$, which is our desired conclusion.

"Only if" part: Suppose that $d_{K_1}(F)$ is properly included in a maximal face of K_1^*, say H. Write $d_{K_1^*}(H)$ as $\Phi(y)$. Then $d_{K_1}(\Phi(y)) = \operatorname{cl}_{K_1^*}(H) = H$, since every maximal face is exposed. Note that we may assume that $\operatorname{cl}_{K_2}(G) \neq K_2$; otherwise, $G = K_2$ and we have $\pi_{F,G} = \pi(K_1, K_2)$. So by Theorems 4.1(c), $\pi_{\Phi(y), \operatorname{cl}_{K_2}(G)} \neq \pi(K_1, K_2)$. Since $d_{K_1}(F) \subseteq H = d_{K_1}(\Phi(y))$, we have $y \in \operatorname{cl}_{K_1}(F)$, and hence by Theorems 4.4(c) and 4.1(b), we have $\pi_{F,G} \subseteq \pi_{\operatorname{cl}_{K_1}(F), \operatorname{cl}_{K_2}(G)} \subseteq \pi_{\Phi(y), \operatorname{cl}_{K_2}(G)}$. Choose vectors $y' \in K_2 \setminus G$ and $z \in H \setminus d_{K_1}(F)$. Then $y'z^T(y) = 0$, as $d_{K_1}(\Phi(y)) = H$; and hence, $y'z^T \in \pi_{\Phi(y), \operatorname{cl}_{K_2}(G)}$. By our choice of the vector z, for any vector $w \in \operatorname{relint} F$, we have $z^T w > 0$, and hence $(y'z^T)(w) \notin G$; so $y'z^T \notin \pi_{F,G}$. This shows that $y'z^T \in \pi_{\Phi(y), \operatorname{cl}_{K_2}(G)} \setminus \pi_{F,G}$, and hence $\pi_{F,G}$ is not a maximal face of $\pi(K_1, K_2)$.

Now suppose that G is properly included in a maximal face of K_2, say H. It is not difficult to show that then $\pi_{F,H}$ is a nontrivial face of $\pi(K_1, K_2)$ properly including $\pi_{F,G}$. Hence $\pi_{F,G}$ is not a maximal face. ∎

In connection with part (b) of the above theorem, note that a sufficient condition for $d_{K_1}(F)$ to be a maximal face of K_1^* is that F is an exposed extreme ray of K_1 (see our Lemma 5.11). The condition, however, is not necessary, as one can readily find a 3-dimensional proper cone K_1 as a counterexample.

5. RELATIONS BETWEEN THE DUALITY OPERATORS

There is a natural way to "imbed" K_2, as well as K_1^*, into $\pi(K_1, K_2)$. For any $z \in R^{n_1}$, we denote by τ_z the linear transformation from R^{n_2} to R^{n_2, n_1} given by $\tau_z(x) = xz^T$. In case $0 \neq z \in K_1^*$, it is easily verified that the cones K_2 and $\mathfrak{R}(\tau_z) \cap \pi(K_1, K_2)$ are linearly isomorphic under τ_z, where $\mathfrak{R}(\tau_z)$ denotes the range space of τ_z. Similarly, K_1^* is linearly isomorphic with $\mathfrak{R}(\lambda_x) \cap \pi(K_1, K_2)$ under λ_x, provided that $0 \neq x \in K_2$, where λ_x is the linear transformation from R^{n_1} to R^{n_2, n_1} given by $\lambda_x(z) = xz^T$. With these in mind, we are ready to prove:

THEOREM 5.1.

(a) *If $d_{\pi(K_1, K_2)}$ is injective, then d_{K_1} is surjective and d_{K_2} is injective.*
(b) *If $d_{\pi(K_1, K_2)}$ is surjective, then d_{K_1} is injective and d_{K_2} is surjective.*

Proof. (a): Choose some nonzero vector $z \in K_1^*$, and let τ_z have the same meaning as above. Since the duality operator of $\pi(K_1, K_2)$ is given to be injective and the duality operator of a linear subspace is always injective, the duality operator of $\mathfrak{R}(\tau_z) \cap \pi(K_1, K_2)$ is also injective (see Tam [36, Corollary 4.14]). But the latter cone is linearly isomorphic with K_2, so the duality operator d_{K_2} is also injective. By a similar argument we deduce that $d_{K_1^*}$ is injective, and hence d_{K_1} is surjective (see Tam [36, Proposition 2.5(a)]).

[Actually, the injectivity of d_{K_2} also follows from Theorem 4.4(e). Furthermore, since the cone $\pi(K_2^*, K_1^*)$ is isometric with $\pi(K_1, K_2)$ (under the transposition map), the injectivity of $d_{K_1^*}$ also follows. But our above proof is more illuminating.]

(b): Since $d_{\pi(K_1, K_2)}$ is surjective, $d_{\pi(K_1, K_2)^*}$ is injective. Let τ_y and λ_z have the same meanings as before. As $\pi(K_1, K_2)^* = K_2^* \oplus_p K_1$, it is readily seen that K_2^* is linearly isomorphic with $\pi(K_1, K_2)^* \cap \mathfrak{R}(\tau_y)$ under τ_y, provided that $0 \neq y \in K_1$ and that K_1 is linearly isomorphic with $\pi(K_1, K_2)^* \cap \mathfrak{R}(\lambda_z)$ under λ_z, provided that $0 \neq z \in K_2^*$. So the argument for part (a) also works here. ∎

Note however that the injectivity of $d_{\pi(K_1, K_2)}$ does not guarantee the surjectivity of d_{K_2}, nor the injectivity of d_{K_1}. Indeed, if $K_1 = R_+$, then $\pi(K_1, K_2)$ can be identified with K_2, and we may have d_{K_2} being injective, but not surjective. If $K_2 = R_+$, then $\pi(K_1, K_2)$ can be identified with K_1^*, and we may have d_{K_1} being surjective but not injective. But if K_2 (K_1) is a proper cone such that $d_{\pi(K_1, K_2)}$ is injective for all proper cones K_1 (K_2), then necessarily d_{K_2} (d_{K_1}) is bijective, because then we may choose $K_1 = K_2$ and apply the following result.

COROLLARY 5.2. *If $d_{\pi(K)}$ is injective or surjective, then d_K is bijective.*

In fact, by our above method more can be said of the connection between the cones K_1, K_2, and $\pi(K_1, K_2)$. But before we come to that, we give a general result first.

LEMMA 5.3. *Let K be a proper cone, and let W be a linear subspace in R^n. Let $x, y \in K \cap W$. Denote the face of $K \cap W$ generated by y by $\Phi_{K \cap W}(y)$. Then:*

(a) *$x \in \Phi(y)$ if and only if $x \in \Phi_{K \cap W}(y)$.*

(b) *If $P_W[K^*]$ is closed, where P_W denotes the orthogonal projection of R^n onto W, then we have*

$$x \in \mathrm{cl}_K\big(\Phi(y)\big) \quad \text{if and only if} \quad x \in \mathrm{cl}_{K \cap W}\big(\Phi(y)\big).$$

Proof. The verification of (a) is straightforward.

(b): It is known (and not difficult to show) that the dual of $K \cap W$ in W is given by $(K \cap W)^D = \mathrm{cl}\, P_W[K^*]$.

To prove the "if" part, let $z \in d_K(\Phi(y))$. Then $P_W(z) \in (K \cap W)^D$, and since $\langle P_W(z), y\rangle = \langle z, y\rangle = 0$, we have $P_W(z) \in d_{K \cap W}(\Phi(y))$. Thus $\langle z, x\rangle = \langle P_W(z), x\rangle = 0$, where the last equality holds because $x \in \mathrm{cl}_{K \cap W}(\Phi(y))$. This shows that $x \in \mathrm{cl}_K(\Phi(y))$.

The "only if" part can be established by reversing the argument for the "if" part, but we need the closedness assumption on $P_W[K^*]$ here. ■

Without the closedness assumption, the "only if" part of Lemma 5.3(b) no longer holds. For a counterexample, choose a 3-dimensional proper cone K with a nonexposed extreme ray $\Phi(y)$. Take W to be the 2-dimensional subspace $\mathrm{span}\,\mathrm{cl}_K(\Phi(y))$, and $\Phi(x)$ to be the extreme ray of $\mathrm{cl}_K(\Phi(y))$ besides $\Phi(y)$. Then by our choice, $x \in \mathrm{cl}_K(\Phi(y))$. However, $x \notin \mathrm{cl}_{K \cap W}(\Phi(y))$.

THEOREM 5.4. *Let $x, y \in K_2$, and let K_1 be any proper cone. The following conditions are equivalent:*

(i) *$x \in \Phi(y)$.*

(ii) *$xz^T \in \Phi(yz^T)$ for some nonzero (or for all) $z \in K_1^*$, where $\Phi(yz^T)$ denotes the face of $\pi(K_1, K_2)$ generated by yz^T.*

(iii) *$xz^T \in \tilde{\Phi}(yz^T)$ for some nonzero (or for all) $z \in K_1^*$, where $\tilde{\Phi}(yz^T)$ denotes the face of $\pi(K_1^*, K_2^*)^*$ generated by yz^T.*

Proof. Let $0 \neq z \in K_1^*$. As noted before, K_2 is linearly isomorphic with $\Re(\tau_z) \cap \pi(K_1, K_2)$ under τ_z, where τ_z has the same meaning as before. Hence

$$x \in \Phi(y) \quad \text{iff} \quad \tau_z(x) \in \Phi_{\Re(\tau_z) \cap \pi(K_1, K_2)}(\tau_z(y)) \quad \text{iff} \quad xz^T \in \Phi(yz^T),$$

where the last "iff" follows from Lemma 5.3(a). This established the equivalence of (i) and (ii). Noting that $\pi(K_1^*, K_2^*)^* = K_2 \otimes_p K_1^*$, the equivalence of (i) and (iii) can be established in a similar way. ■

THEOREM 5.5. *Let $x, y \in K_2$, and let K_1 be any proper cone. The following conditions are equivalent*:

(i) $x \in \mathrm{cl}_{K_2}(\Phi(y))$.

(ii) $xz^T \in \mathrm{cl}_{\pi(K_1, K_2)}(\Phi(yz^T))$ *for some nonzero* (*or for all*) $z \in K_1^*$, *where* $\Phi(yz^T)$ *denotes the face of* $\pi(K_1, K_2)$ *generated by* yz^T.

(iii) $xz^T \in \mathrm{cl}_{\pi(K_1^*, K_2^*)^*}(\tilde{\Phi}(yz^T))$ *for some nonzero* (*or for all*) $z \in K_1^*$, *where* $\tilde{\Phi}(yz^T)$ *denotes the face of* $\pi(K_1^*, K_2^*)^*$ *generated by* yz^T.

Proof. (i) $\Leftrightarrow$ (ii): Let $0 \neq z \in K_1^*$. In view of Lemma 5.3(b) and the proof of Theorem 5.4, it suffices to show that $P_W[\pi(K_1, K_2)^*]$ is closed, where $W = \Re(\tau_z)$. Now $W \cap \pi(K_1, K_2) = \Re(\tau_z) \cap \pi(K_1, K_2) = K_2 \otimes_p \{z\}$, and for any vectors $x \in K_2$ and $y \in R^{n_2}$, $\langle xz^T, yz^T \rangle = (x^T y)(z^T z)$. Hence, we have $[W \cap \pi(K_1, K_2)]^D = K_2^* \otimes_p \{z\}$. But by the known result mentioned in the proof of Lemma 5.3(b), cl $P_W[\pi(K_1, K_2)^*]$ is equal to $[W \cap \pi(K_1, K_2)]^D$, the dual of $W \cap \pi(K_1, K_2)$ in W. So the proof is complete if we can show that $K_2^* \otimes_p \{z\} \subseteq P_W[\pi(K_1, K_2)^*]$. Choose some vector $y_0 \in K_1$ such that $y_0^T z = z^T z$. Let $w \in K_2^*$. We claim that $P_W(wy_0^T) = wz^T$; since $wy_0^T \in \pi(K_1, K_2)^*$, once this is proved, we are done. But $wz^T \in W$ and $\langle wz^T, wy_0^T - wz^T \rangle = (w^T w)(y_0 - z)^T z = 0$, so our assertion follows.

(i) $\Leftrightarrow$ (iii): The proof is similar. ■

It should be noted that we could have established Theorems 5.4 and 5.5 by direct calculations and then used them to deduce Theorem 5.1. We chose our present approach because it reveals more clearly the connections between the cones K_1, K_2, and $\pi(K_1, K_2)$.

THEOREM 5.6. *Let K_1 be any proper cone. The following conditions are equivalent*:

(i) d_{K_1} *is surjective*.

(ii) *For any (or for some) proper cone* K_2, *if* $A, B \in \pi(K_1, K_2)$ *such that* $Bx \in \Phi(Ax)$ *for all* $x \in K_1$, *then* $B^T z \in \Phi(A^T z)$ *for all* $z \in K_2^*$.

Proof. (i) $\Rightarrow$ (ii): Let $A, B \in \pi(K_1, K_2)$ such that $Bx \in \Phi(Ax)$ for all $x \in K_1$. Then $Bx \in \mathrm{cl}_{K_2}(\Phi(Ax))$ for all $x \in K_1$. So by Theorems 3.1, 3.2 and the injectivity of $d_{K_1^*}$, we readily obtain, for all $z \in K_2^*$, $B^T z \in \mathrm{cl}_{K_1^*}(\Phi(A^T z)) = \Phi(A^T z)$.

(ii) $\Rightarrow$ (i): Assume that d_{K_1} is not surjective. Then there exists $z \in K_1^*$ such that $\mathrm{cl}_{K_1^*}(\Phi(z)) \neq \Phi(z)$. Write $\mathrm{cl}_{K_1^*}(\Phi(z))$ as $\Phi(w)$. Choose a nonzero vector $u \in K_2$. Note that $uz^T, uw^T \in \pi(K_1, K_2)$. Furthermore, as can be readily checked, for any vector $x \in K_1$, we have $(uw^T)x \in \Phi((uz^T)x)$. However, since $w \notin \Phi(z)$, if $v \in \mathrm{int}\, K_2^*$ then $(uw^T)^T v \notin \Phi((uz^T)^T v)$. Hence, condition (ii) is not satisfied. ■

COROLLARY 5.7. *d_{K_1} is surjective provided that there exists a proper cone K_2 such that for any $A, B \in \pi(K_1, K_2)$, $Bx \in \Phi(Ax)$ for all $x \in K_1$ implies that $B \in \Phi(A)$.*

Proof. Follows from Theorem 3.1(a) and the preceding theorem. ■

The nice condition "For any $A, B \in \pi(K_1, K_2)$, $Bx \in \Phi(Ax)$ for all $x \in K_1$ implies that $B \in \Phi(A)$," however, guarantees none of the following: the injectivity of d_{K_1}, that of d_{K_2}, and the surjectivity of d_{K_2}. It is not difficult to see that this nice condition is always satisfied whenever K_1 is polyhedral. This explains why it does not guarantee the injectivity or the surjectivity of d_{K_2}. When $K_2 = R_+$, the cone $\pi(K_1, R_+)$ can be identified with K_1^*. Then this nice condition is equivalent to the injectivity of $d_{K_1^*}$, which clearly does not imply the injectivity of d_{K_1}.

THEOREM 5.8. *Let K_1 and K_2 be given proper cones. The following conditions are equivalent:*

(i) *d_{K_1} is surjective and d_{K_2} is injective.*
(ii) *For any $A, B \in \pi(K_1, K_2)$, one has $Bx \in \Phi(Ax)$ for all $x \in K_1$ if and only if $B^T z \in \Phi(A^T z)$ for all $z \in K_2^*$.*

Proof. Follows readily from Theorem 5.6. ■

COROLLARY 5.9. *The following conditions are equivalent:*

(i) *d_K is bijective.*
(ii) *For any $A, B \in \pi(K)$, one has $Bx \in \Phi(Ax)$ for all $x \in K$ if and only if $B^T z \in \Phi(A^T z)$ for all $z \in K^*$.*

Proof. Follows readily from Theorem 5.8. ∎

THEOREM 5.10. *Let K_1 and K_2 be given proper cones. The following conditions are equivalent*:

(i) $d_{\pi(K_1, K_2)}$ *is injective.*
(ii) d_{K_2} *is injective, and $\pi(K_1, K_2)$ satisfies the equivalent conditions of Corollary* 4.7.

Proof. (i) ⇒ (ii): The injectivity of d_{K_2} follows from Theorem 5.1(a). Note that Theorem 4.6(b) also says that every exposed face of $\pi(K_1, K_2)$ is an intersection of simple faces. Since $d_{\pi(K_1, K_2)}$ is injective, every face of $\pi(K_1, K_2)$ is exposed, so condition (i) of Corollary 4.7 is satisfied.

(ii) ⇒ (i): By Theorem 4.4(e) the injectivity of K_2 implies that every simple face of $\pi(K_1, K_2)$ is exposed. So Theorem 4.6 and condition (i) of Corollary 4.8 together imply that, for any face $\mathfrak{C}$ of $\pi(K_1, K_2)$, we have

$$\mathfrak{C} = \mathrm{cl}_{\pi(K_1, K_2)}(\mathfrak{C});$$

hence $d_{\pi(K_1, K_2)}$ is injective. ∎

LEMMA 5.11. *The dual face of each exposed extreme ray of K is a maximal face of K^*.*

Proof. Let $\Phi(x)$ be an exposed extreme ray of K. Then $d_K(\Phi(x))$ is a nontrivial face of K^*, say, included in the maximal face M. Then $0 \neq d_{K^*}(M) \subseteq d_{K^*} \circ d_K(\Phi(x)) = \Phi(x)$, where the last equality follows from the exposedness of $\Phi(x)$. But $\Phi(x)$ is an extreme ray, so we have $d_{K^*}(M) = \Phi(x)$. Hence, we have $d_K(\Phi(x)) = d_K \circ d_{K^*}(M)$. As a maximal face of K^*, M is certainly exposed; so $d_K \circ d_{K^*}(M) = M$. Thus our assertion follows. ∎

LEMMA 5.12. *If d_K is surjective, then each exposed face of K is an (finite) intersection of maximal faces.*

Proof. Let F be an exposed face of K. Choose extreme rays $\Phi(z_1), \ldots, \Phi(z_p)$ of K^* such that $\bigvee_{i=1}^{p} \Phi(z_i) = d_K(F)$. Then $F = d_{K^*} \circ d_K(F) = d_{K^*}(\bigvee_{i=1}^{p} \Phi(z_i)) = \bigwedge_{i=1}^{p} d_{K^*}(\Phi(z_i))$. As d_K is surjective, each $\Phi(z_i)$, $1 \leqslant i \leqslant p$, is an exposed extreme ray of K^*. Hence by Lemma 5.11 each $d_{K^*}(\Phi(z_i))$ is a maximal face of K, and our assertion follows. ∎

THEOREM 5.13. *If d_{K_1} is injective and d_{K_2} is surjective, then every exposed face of $\pi(K_1, K_2)$, other than $\pi(K_1, K_2)$ itself, can be written as an intersection of maximal faces.*

Proof. (a): By Theorem 4.6(b) every exposed face of $\pi(K_1, K_2)$ is an intersection of exposed, simple faces. So it suffices to prove that each exposed, simple face of $\pi(K_1, K_2)$, other than $\pi(K_1, K_2)$ itself, can be written as an intersection of maximal faces. Now consider an exposed simple face $\pi_{F,G}$, different from $\pi(K_1, K_2)$. By Theorem 4.1(c) and Theorem 4.4(e), F is a nonzero face of K_1, G is an exposed face of K_2, and $G \neq K_2$. Express F as the join of finitely many (exposed) extreme rays, say $F_1, \dots, F_p$. Then by Lemma 5.11 each $d_{k_1}(F_i)$ is a maximal face of K_1^*. Since d_{K_2} is surjective, by Lemma 5.12 we can also express G as $\bigwedge_{j=1}^{q} M_j$, where each M_j is a maximal face of K_2. From Theorem 4.1(e) and (f), we readily obtain

$$\pi_{F,G} = \bigwedge_{\substack{1 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant q}} \pi_{F_i, M_j},$$

where each π_{F_i, M_j} is a maximal face of $\pi(K_1, K_2)$, in view of Theorem 4.8. The proof is complete. ■

THEOREM 5.14. *The following conditions are equivalent:*

(i) *$d_{\pi(K)}$ is injective.*

(ii) *d_K is injective (or bijective), and each face of $\pi(K)$ can be written as an intersection of simple faces.*

(iii) *Each face of $\pi(K)$, other than $\pi(K)$ itself, can be written as an intersection of maximal faces.*

Proof. The equivalence of (i) and (ii) follows from Theorem 5.10 (and Corollary 5.2). The implication (iii) ⇒ (i) is obvious. Finally, the implication (i) ⇒ (iii) follows from Theorem 5.13, because if $d_{\pi(K)}$ is injective, then each face of $\pi(K)$ is exposed and d_K is bijective. ■

It should be noted that for a general proper cone K, when d_K is injective, it is not necessarily true that every nontrivial face of K is an intersection of maximal faces.

6. OPEN QUESTIONS

The results of this paper (for instance, Theorem 5.4 and Corollary 4.7) show that if the duality operator $d_{\pi(K)}$ is injective then the cone π(K) possesses many nice properties. In [34, Corollary 5.5] the author also proved that the face semiring $\mathscr{F}(\pi(K))$ is simple if and only if $d_{\pi(K)}$ is injective. It seems worthwhile to find some conditions on K alone which are sufficient for $d_{\pi(K)}$ to be injective. One trivial sufficient condition is that K is polyhedral, because then $\pi(K)$ is also polyhedral and hence its duality operator is bijective. But for nonpolyhedral cones, at present, the situation is unclear. We pose two related open questions:

OPEN QUESTION 1. Let K be a proper cone. If d_K is bijective, does it follow that $d_{\pi(K)}$ is injective or surjective?

(We are asking the converse of Corollary 5.2.)

OPEN QUESTION 2. Does there exist a nonpolyhedral proper cone K such that every face of $\pi(K)$ can be written as an intersection of simple faces?

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