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Maximal exponents of polyhedral cones (II)

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Let K be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$. An $n \times n$ matrix A is said to be K-primitive if there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. For a polyhedral cone K, the maximum value of $\gamma(A)$, taken over all K-primitive matrices A, is called the exponent of K and is denoted by $\gamma(K)$. It is proved that the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones (i.e., cones having $n+1$ extreme rays) is $n^2 - n + 1$ if n is odd, and is $n^2 - n$ if n is even, the maximum value of the exponent being attained by a minimal cone with a balanced relation for its extreme vectors. The K-primitive matrices A such that $\gamma(A)$ attain the maximum value are identified up to cone-equivalence modulo positive scalar multiplication.

1. Introduction

This is the second of a sequence of papers studying the maximal exponents of K-primitive matrices over polyhedral cones. Here for a polyhedral (proper) cone K in $\mathbb{R}^n$ by a K-*primitive matrix* we mean a real square matrix A for which there exists a positive integer k such that A^k maps every nonzero vector of K into the interior of K; the least such k, denoted by $\gamma(A)$, is referred to as the *exponent* of A. In

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[12], the first paper in the sequence, it is proved that if K is an n-dimensional polyhedral cone with m extreme rays then its exponent $\gamma(K)$, which is defined to be $\max\{\gamma(A) : A \text{ is } K\text{–primitive}\}$, does not exceed $(n-1)(m-1)+1$, thus answering in the affirmative a conjecture posed by Steve Kirkland. [When $m = n$, the latter bound reduces to Wielandt's classical sharp bound [20] for exponents of (nonnegative) primitive matrices of a given order]. The general question of what the maximum value of $\gamma(K)$ is, when K is taken over all n-dimensional polyhedral cones with m extreme rays, for a given pair of positive integers m, n, remains unresolved. In this paper we take up the question for the minimal cone case, i.e., when $m = n+1$.

The upper bound $(n-1)(m-1)+1$ for $\gamma(K)$ obtained in [12] may suggest that for n-dimensional minimal cones K, $n^2 - n + 1$ is a sharp upper bound for $\gamma(K)$. It turns out that this is true when n is odd, but for even n the sharp upper bound is one less. In [12], in connection with the equality case of the upper bound $(n-1)(m-1)+1$ (or $(n-1)(m-1)$) for $\gamma(A)$, two special digraphs, represented by Figs. 1 and 2, respectively are singled out. They are precisely the two known primitive digraphs on n vertices (for some n) with the length of the shortest circuit equal to $n-1$. They will play an important role in this work.

We now describe the contents of this paper in some detail.

Section 2 contains most of the definitions, together with the relevant known results, which we need for the paper. For the sake of convenience, we collect together properties/results on minimal cones in Section 3. In particular, we show that for minimal cones, the concepts of "linearly isomorphic" and "combinatorially equivalent" are equivalent.

In Section 4 we prove that the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones is $n^2 - n + 1$ if n is odd, and is $n^2 - n$ if n is even. We also determine (up to linear isomorphism) the minimal cones K (and also the corresponding K-primitive matrices A) such that $\gamma(K)$ (and $\gamma(A)$) attains the maximum value. In particular, it is found that every minimal cone K whose exponent attains the maximum value has a balanced relation for its extreme vectors and also if A is a K-primitive matrix such that $\gamma(A) = \gamma(K)$ then necessarily the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, up to graph isomorphism, given by Figs. 1 or 2.

In Section 5 we consider the question of uniqueness of the minimal cone K and the corresponding K-primitive matrix A such that $\gamma(K)$ and $\gamma(A)$ attain the maximum value. It is proved that for every integer $n \geqslant 3$, there are (up to linear isomorphism) one or two n-dimensional optimal minimal cones, depending on whether n is odd or even. However, for each such minimal cone K, there are uncountably infinitely many pairwise non-cone-equivalent linearly independent optimal K-primitive matrices.

In Section 6, the final section, we give some open questions.

2. Preliminaries

We take for granted standard properties of nonnegative matrices, complex matrices and graphs that can be found in textbooks (see, for instance, [3,4,8,9,11]). A familiarity with elementary properties of finite-dimensional convex sets, convex cones and cone-preserving maps is also assumed (see, for instance,[2,14,17,21]). To fix notation and terminology, we give some definitions.

Let K be a nonempty subset of a finite-dimensional real vector space V. The set K is called a *convex cone* if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \geqslant 0$; K is *pointed* if $K \cap (-K) = \{0\}$; K is *full* if its interior $\operatorname{int} K$ (in the usual topology of V) is nonempty, equivalently, $K - K = V$. If K is closed and satisfies all of the above properties, K is called a *proper cone*.

In this paper, unless specified otherwise, we always use K to denote a proper cone in the n-dimensional Euclidean space $\mathbb{R}^n$.

We denote by $\geqslant^K$ the partial ordering of $\mathbb{R}^n$ induced by K, i.e., $x \geqslant^K y$ if and only if $x - y \in K$.

A subcone F of K is called a *face* of K if $x \geqslant^K y \geqslant^K 0$ and $x \in F$ imply $y \in F$. If $S \subseteq K$, we denote by $\Phi(S)$ the *face of K generated by S*, that is, the intersection of all faces of K including S. If $x \in K$, we write $\Phi(\{x\})$ simply as $\Phi(x)$. It is known that for any vector $x \in K$ and any face F of K, $x \in \operatorname{ri} F$ if and only if $\Phi(x) = F$; also, $\Phi(x) = \{y \in K : x \geqslant^K \alpha y \text{ for some } \alpha > 0\}$. (Here we denote by $\operatorname{ri} F$ the *relative interior of* F.) A vector $x \in K$ is called an *extreme vector* if either x is the zero vector or x is nonzero and

$\Phi(x) = \{\lambda x : \lambda \geqslant 0\}$; in the latter case, the face $\Phi(x)$ is called an *extreme ray*. We use $\operatorname{Ext} K$ to denote the set of all nonzero extreme vectors of K. Two nonzero extreme vectors are said to be *distinct* if they are not multiples of each other. The cone K itself and the set $\{0\}$ are always faces of K, known as *trivial faces*. Other faces of K are said to be *nontrivial*.

If S is a nonempty subset of a vector space, we denote by $\operatorname{pos} S$ the *positive hull* of S, i.e., the set of all possible nonnegative linear combinations of vectors taken from S.

A closed pointed cone K is said to be the *direct sum* of its subcones $K_1, \ldots, K_p$, and we write $K = K_1 \oplus \cdots \oplus K_p$ if every vector of K can be expressed uniquely as $x_1 + x_2 + \cdots + x_p$, where $x_i \in K_i$, $1 \leqslant i \leqslant p$. K is called *decomposable* if it is the direct sum of two nonzero subcones; otherwise, it is said to be *indecomposable*. It is well-known that every closed pointed cone K can be written as

$$K = K_1 \oplus \cdots \oplus K_p,$$

where each K_j is an indecomposable cone $(1 \leqslant j \leqslant p)$. Except for the ordering of the summands, the above decomposition is unique. We will refer to the K_j's as *indecomposable summands* of K.

By a *polyhedral cone* we mean a proper cone which has finitely many extreme rays. By the *dimension of a proper cone* we mean the dimension of its linear span. A polyhedral cone is said to be *simplicial* if the number of extreme rays is equal to its dimension. The nonnegative orthant $\mathbb{R}^n_+ := \{(\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n : \xi_i \geqslant 0\ \forall i\}$ is a typical example of a simplicial cone.

We denote by $\pi(K)$ the set of all $n \times n$ real matrices A (identified with linear mappings on $\mathbb{R}^n$) such that $AK \subseteq K$. Members of $\pi(K)$ are said to be *K-nonnegative* and are often referred to as *cone-preserving maps*. It is clear that $\pi\left(\mathbb{R}^n_+\right)$ consists of all $n \times n$ (entrywise) nonnegative matrices.

A matrix $A \in \pi(K)$ is said to be *K-irreducible* if A leaves invariant no nontrivial face of K; A is *K-positive* if $A(K \setminus \{0\}) \subseteq \operatorname{int} K$ and is *K-primitive* if there is a positive integer p such that A^p is K-positive. If A is K-primitive, then the smallest positive integer p for which A^p is K-positive is called the *exponent* of A and is denoted by $\gamma(A)$ (or by $\gamma_K(A)$ if the dependence on K needs to be emphasized). Note that when A is K-primitive, A^q is K-positive for every positive integer $q \geqslant \gamma(A)$.

It is known that the set $\pi(K)$ forms a proper cone in the space of $n \times n$ real matrices, the interior of $\pi(K)$ being the subset consisting of K-positive matrices. Also, $\pi(K)$ is polyhedral if and only if K is polyhedral. (See [17,15] or [1].)

A matrix A is said to be *an automorphism* of K if A is invertible and $A,\ A^{-1}$ are both K-nonnegative or, equivalently, $AK = K$.

Two proper cones $K_1,\ K_2$ are said to be *linearly isomorphic* if there exists a linear isomorphism $P : \operatorname{span} K_2 \longrightarrow \operatorname{span} K_1$ such that $PK_2 = K_1$.

It is clear that if K is a simplicial cone with n extreme rays then K is linearly isomorphic to $\mathbb{R}^n_+$.

Let $A \in \pi(K)$. In this work we need the digraph $(\mathcal{E}(K), \mathcal{P}(A, K))$, which is one of the four digraphs associated with A introduced by Barker and Tam [5,19]. It is defined in the following way: its vertex set is $\mathcal{E}(K)$, the set of all extreme rays of K; $(\Phi(x), \Phi(y))$ is an arc whenever $\Phi(y) \subseteq \Phi(Ax)$. If there is no danger of confusion, (in particular, within proofs) we write $(\mathcal{E}(K), \mathcal{P}(A, K))$ as $(\mathcal{E}, \mathcal{P}(A, K))$ or $(\mathcal{E}, \mathcal{P})$. It is readily checked that if K is the nonnegative orthant $\mathbb{R}^n_+$ then $(\mathcal{E}, \mathcal{P}(A, K))$ equals the usual digraph associated with A^T, the transpose of A. (If $B = (b_{ij})$ is an $n \times n$ matrix then by the usual digraph of B we mean the digraph with vertex set $\{1, \ldots, n\}$ such that (i, j) is an arc whenever $b_{ij} \neq 0$.)

It is not difficult to show that for any $A, B \in \pi(K)$, if $\Phi(A) = \Phi(B)$ then A, B are either both K-primitive or both not K-primitive, and if they are, then $\gamma(A) = \gamma(B)$. In Niu [13] it is proved that if K is a polyhedral cone then for any $A, B \in \pi(K)$, we have $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$ (as labelled digraphs) if and only if $\Phi(A) = \Phi(B)$. So it is not surprising to find that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ plays a role in determining a bound for $\gamma(A)$. (When K is nonpolyhedral, the situation is more subtle. We refer the interested readers to Tam [18] for the details.)

For a proper cone K, we say *K has finite exponent* if the set of exponents of K-primitive matrices is bounded; then we denote the maximum exponent by $\gamma(K)$ and refer to it as the *exponent of K*. If K has finite exponent, then a K-primitive matrix A is said to be *exp-maximal* if $\gamma(A) = \gamma(K)$. It is known that every polyhedral cone has finite exponent — a proof of this fact can be found in [12, Section 2].

For every pair of positive integers m, n with $3 \leqslant n \leqslant m$, we denote by $\mathcal{P}(m, n)$ the set of all n-dimensional polyhedral cones with m extreme rays. Note that we start with $n = 3$ as the cases $n = 1$ or

2 are trivial. Also, when $m = 3$, in order that $\mathcal{P}(m, n)$ is nonvacuous, n must be 3. We call a polyhedral cone $K_0 \in \mathcal{P}(m, n)$ an *exp-maximal cone* if $\gamma(K_0) = \max\{\gamma(K) : K \in \mathcal{P}(m, n)\}$.

To study the exponents of K-primitive matrices, we make use of the concept of local exponent defined in the following way. (For definition of local exponent of a primitive matrix, see [4, Section 3.5].) For any K-nonnegative matrix A, not necessarily K-primitive or K-irreducible, and any $0 \neq x \in K$, by the *local exponent of A at x*, denoted by $\gamma(A, x)$, we mean the smallest nonnegative integer k such that $A^k x \in \operatorname{int} K$. If no such k exists, we set $\gamma(A, x)$ equal ∞.

Two cone-preserving maps $A_1 \in \pi(K_1)$ and $A_2 \in \pi(K_2)$ are said to be *cone-equivalent* if there exists a linear isomorphism P such that $PK_2 = K_1$ and $P^{-1}A_1P = A_2$.

It is not difficult to establish the following:

Fact 2.1. Let K_1, K_2 be proper cones in $\mathbb{R}^n$. Suppose that $A_1 \in \pi(K_1)$ and $A_2 \in \pi(K_2)$ are cone-equivalent. Then:

(i) A_1 and A_2 are similar.
(ii) The cones $K_1,\ K_2$ are linearly isomorphic.
(iii) The digraphs $(\mathcal{E}(K_1), \mathcal{P}(A_1, K_1))$, $(\mathcal{E}(K_2), \mathcal{P}(A_2, K_2))$ are isomorphic.
(iv) Either A_1 is K_1-primitive and A_2 is K_2-primitive or they are not, and if they are, then $\gamma_{K_1}(A_1) = \gamma_{K_2}(A_2)$.
(v) For any $x \in K_2$, $\gamma(A_2, x) = \gamma(A_1, Px)$.

Also, it is clear that if K_1 and K_2 are linearly isomorphic cones, then either $K_1,\ K_2$ both have finite exponent or they both do not have, and if they both have, then $\gamma(K_1) = \gamma(K_2)$.

Under inclusion as the partial order, the set of all faces of K, denoted by $\mathcal{F}(K)$, forms a lattice with meet and join given respectively by: $F \wedge G = F \cap G$ and $F \vee G = \Phi(F \cup G)$. Two proper cones K_1, K_2 are said to be *combinatorially equivalent* if their face lattices $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$ are isomorphic (as lattices).

We will also need the following results which are established in [12]. (The definition and properties of a minimal cone will be given in the next section.)

Lemma 2.2 [12, Lemma 4.1]**.** *Let $K \in \mathcal{P}(m, n)$ $(3 \leqslant n \leqslant m)$ and let A be a K-primitive matrix. Then the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ equals $m - 1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, apart from the labelling of its vertices, given by Fig.* 1 *or Fig.* 2, *or* (*in case $m = n = 3$*) *by the digraph of order* 3 *whose arcs are precisely all possible arcs between every pair of distinct vertices.*
(*For simplicity, in Figs. 1 and 2 we label the vertex $\Phi(x_i)$ simply by x_i.*).

Remark 2.3. In what follows when we say the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 (or by other figures), we mean the digraph is given either by the figure up to graph isomorphism or by the figure as a labelled digraph. In most instances, we mean it in the former sense but in a few instances we mean it in the

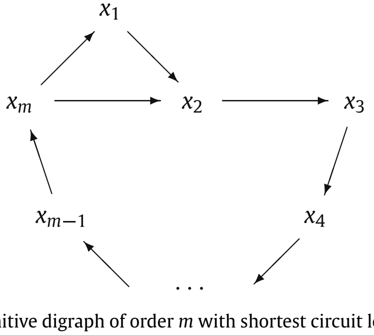

Fig. 1. A primitive digraph of order m with shortest circuit length $m - 1$.

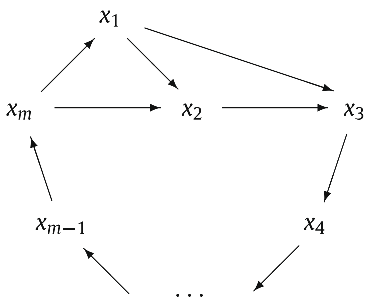

Fig. 2. Another primitive digraph of order m with shortest circuit length $m-1$.

latter sense. It should be clear from the context in what sense we mean. (For instance, in parts (i) and (iii) of the following Lemma 2.4 we mean the former sense, but in part (ii) we mean the latter sense.)

Lemma 2.4 [12, Lemma 4.2]. *Let $K \in \mathcal{P}(m,n)$ $(3 \leqslant n \leqslant m)$. Let A be a K-nonnegative matrix. Suppose that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig.* 1 *or Fig.* 2. *Then*:

(i) *A is K-primitive, nonsingular, non-derogatory, and has a unique annihilating polynomial of the form $t^m - ct - d$, where $c, d > 0$.*
(ii) *$\gamma(A)$ equals $\gamma(A, x_1)$ or $\gamma(A, x_2)$ depending on whether the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig.* 1 *or Fig.* 2. *In either case,* $\max_{1 \leqslant i \leqslant m} \gamma(A, x_i)$ *is attained at precisely one i.*
(iii) *Assume, in addition, that K is non-simplicial. If $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig.* 1 *then K must be indecomposable. If the digraph is given by Fig.* 2 *then either K is indecomposable or m is odd and K is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors (more specifically, with the vectors $x_1, x_4, x_6, \ldots, x_{m-1}$ lying on one side and the vectors $x_3, x_5, \ldots, x_m$ lying on the opposite side of the relation).*

For a square matrix C, we denote by m_C the *degree of the minimal polynomial of C*.

Theorem 2.5 [12, Theorem 5.2(i), (ii)]. *Let $K \in \mathcal{P}(m,n)$, where $m \geqslant 4$, and let A be a K-primitive matrix. Then*:

(i) *$\gamma(A) \leqslant (m_A - 1)(m-1) + 1$, where the equality holds only if the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig.* 1, *in which case $\gamma(A) = (n-1)(m-1) + 1$.*
(ii) *$\gamma(A) = (m_A - 1)(m-1)$ only if either $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig.* 1 *or Fig.* 2, *in which case $\gamma(A) = (n-1)(m-1)$, or $m_A = 3$.*

Corollary 2.6 [12, Corollary 5.4]. *For any $K \in \mathcal{P}(m,n)$ and any K-primitive matrix A, if the digraph of $(\mathcal{E}, \mathcal{P}(A,K))$ is not given by Fig.* 1 *or Fig.* 2 *or by a digraph of order* 3 *whose arc set consists of all possible arcs between every pair of distinct vertices, then $\gamma(A) \leqslant (n-1)(m-2) + 2$.*

Corollary 2.7 [12, Corollary 5.7]. *For any $K \in \mathcal{P}(m,n)$ with $m = n + k$, we have $\gamma(K) \leqslant (n-1)(m-1) + 1 = m^2 - (k+2)m + k + 2$. The equality holds only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig.* 1.

3. Minimal cones

The simplicial cones may be considered as the simplest kind of cones. The next simplest kind of cones, and also the one with which we will deal considerably in this work, are the minimal cones. Minimal cones were first introduced and studied by Fiedler and Pták [7]. We call an n-dimensional polyhedral cone *minimal* if it has precisely $n+1$ extreme rays. Clearly, if K is a minimal cone with (pairwise distinct) extreme vectors $x_1, \ldots, x_{n+1}$, then (up to multiples) these vectors satisfy a unique (linear) relation. Also, a minimal cone is indecomposable if and only if the relation for its extreme vectors is full, i.e., in the relation the coefficient of each extreme vector is nonzero (see [7, Theorem 2.25]). It is readily shown that every decomposable minimal cone is the direct sum of an indecomposable minimal cone and a simplicial cone.

In dealing with (nonzero) relations on (nonzero) extreme vectors of a polyhedral cone, we find it convenient to write such relations in the form

$$\alpha_1 x_1 + \cdots + \alpha_p x_p = \beta_1 y_1 + \cdots + \beta_q y_q,$$

where the extreme vectors $x_1, \ldots x_p,\ y_1, \ldots, y_q$ are pairwise distinct and the coefficients $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$ are all positive. Clearly we have $p, q \geqslant 2$.

We call a relation on extreme vectors of a polyhedral cone *balanced* if the number of nonzero terms on its two sides differ by at most 1.

An indecomposable minimal cone is said to be *of type* (p, q), where $p,\ q$ are positive integers such that $2 \leqslant p \leqslant q$, if the number of (nonzero) terms on the two sides of the relation for its extreme vectors are respectively p and q. (We do not distinguish a relation with the one obtained from it by interchanging the left side with the right side.)

Given positive integers with $2 \leqslant p \leqslant q$ and $p+q = n+1$, one can construct as follows an n-dimensional indecomposable minimal cone of type (p, q). Choose any basis for $\mathbb{R}^n$, say $\{x_1, \ldots, x_n\}$, and let K be the polyhedral cone $\operatorname{pos}\{x_1, \ldots, x_n, x_{n+1}\}$, where $x_{n+1} = (x_1 + \cdots + x_p) - (x_{p+1} + \cdots + x_n)$. Then

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}$$

is the (essentially) unique relation for the vectors $x_1, \ldots, x_{n+1}$. As none of the vectors $x_1, \ldots, x_{n+1}$ can be written as a nonnegative linear combination of the remaining vectors, $x_1, \ldots, x_{n+1}$ are precisely (up to nonnegative scalar multiples) all the extreme vectors of K. Therefore, K is the desired indecomposable minimal cone.

Using the following easy result in linear algebra, one can show that indecomposable minimal cones of the same type are linearly isomorphic.

Lemma 3.1. *Let $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ be two families of vectors in finite-dimensional vector spaces V_1 and V_2 respectively. In order that there exists a linear mapping $T : V_1 \to V_2$ that satisfies $T(x_i) = y_i$ for $i = 1, \ldots, k$, it is necessary and sufficient that $\alpha_1 y_1 + \cdots + \alpha_k y_k = 0$ is a relation for $y_1, \ldots, y_k$ whenever the corresponding relation holds for $x_1, \ldots x_k$.*

We also need the following known characterization of maximal faces of an indecomposable minimal cone [16, Theorem 4.1]:

Theorem A. *Let K be an indecomposable minimal cone generated by extreme vectors $x_1, \ldots, x_{n+1}$ that satisfy*

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}.$$

Then for each pair (i, j), $1 \leqslant i \leqslant p$ and $p+1 \leqslant j \leqslant n+1$, $\operatorname{pos} M_{ij}$ is a maximal face of K, where $M_{ij} = \{x_1, \ldots, x_{n+1}\} \setminus \{x_i, x_j\}$. Moreover, each maximal face of K is of this form.

Note that by the preceding theorem every maximal face, and hence every nontrivial face, of an indecomposable minimal cone is a simplicial cone in its own right.

We take this opportunity to show that for minimal cones, the concepts of “linearly isomorphic" and “combinatorially equivalent" are equivalent.

Theorem 3.2. *Let $K_1,\ K_2$ be minimal cones of dimension n_1, n_2 respectively. Suppose that for $j = 1, 2$, $K_j = K_j' \oplus K_j''$, where K_j' is a simplicial cone and K_j'' is an indecomposable minimal cone of type (p_j, q_j). The following conditions are equivalent:*

(i) $n_1 = n_2$ *and* $(p_1, q_1) = (p_2, q_2)$.
(ii) $K_1,\ K_2$ *are linearly isomorphic*.
(iii) $K_1,\ K_2$ *are combinatorially equivalent*.

Proof. For $j = 1, 2$, let d_j be the dimension of K'_j.

(i) $\Rightarrow$ (ii): We have

$$d_1 = n_1 - \dim K''_1 = n_1 - (p_1 + q_1 - 1) = n_2 - (p_2 + q_2 - 1) = n_2 - \dim K''_2 = d_2;$$

so K'_1 and K'_2 are linearly isomorphic, being simplicial cones of the same dimension. On the other hand, K''_1 and K''_2 are also linearly isomorphic, as they are indecomposable minimal cones of the same type. Therefore, K_1 and K_2 are linearly isomorphic.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): As is well-known, if K is a polyhedral cone then K has faces of all possible dimensions from 0 to $\dim K$. So $\dim K$ is equal to the length of a maximal chain in the face lattice $\mathcal{F}(K)$ of K. Since the face lattices $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$ are isomorphic, they have maximal chains of the same length. Hence, we have $n_1 = n_2$.

To proceed further, we need to establish two assertions first:

Assertion 1. Let K be an n-dimensional minimal cone, $n \geqslant 3$. Suppose that $K = K' \oplus K''$, where K' is a d-dimensional simplicial cone $(0 \leqslant d \leqslant n - 3)$ and K'' is an $(n - d)$-dimensional indecomposable minimal cone of type (p, q). Then K has $d + pq$ maximal faces, d of which are minimal cones and the remaining are simplicial cones.

Proof. Since K is the direct sum of K' and K'', the maximal faces of K are precisely those of the form $M' \oplus K''$ or $K' \oplus M''$, where $M',\ M''$ denote respectively a maximal face of K' and a maximal face of K''. There are d maximal faces of the first kind, each of which, being the direct sum of a simplicial cone and a minimal cone, is a minimal cone in its own right. In view of Theorem A, maximal faces of the indecomposable minimal cone K'' are themselves simplicial cones and there are altogether pq of them; hence, K has pq maximal faces of the second kind, each of which is a simplicial cone. $\square$

Assertion 2. A minimal cone cannot be combinatorially equivalent to a simplicial cone.

Proof. As we have already noted, any two combinatorially equivalent polyhedral cones have the same dimension. Now an n-dimensional simplicial cone has n extreme rays, whereas an n-dimensional minimal cone has $n + 1$ extreme rays. So a minimal cone and a simplicial cone cannot be combinatorially equivalent. $\square$

Now back to the proof of the theorem. Let Ψ be a lattice isomorphism between $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$. Clearly Ψ provides a one-to-one correspondence between the maximal faces of K_1 and those of K_2. Note that if M_1 is a maximal face of K_1 that corresponds to the maximal face M_2 of K_2, then M_1 and M_2 are combinatorially equivalent, as Ψ induces a lattice isomorphism between $\mathcal{F}(M_1)$ and $\mathcal{F}(M_2)$. In view of Assertion 2, under Ψ, maximal faces of K_1 which are themselves minimal (respectively, simplicial) cones correspond to maximal faces of K_2 which are themselves minimal (respectively, simplicial) cones. By Assertion 1, we have $d_1 = d_2$ and $p_1q_1 = p_2q_2$. But we have already shown that $n_1 = n_2$ and also we have $p_j + q_j = n_j - d_j + 1$ for $j = 1, 2$, it follows that we have $(p_1, q_1) = (p_2, q_2)$. $\square$

4. Maximal exponents of minimal cones

In this section we are going to establish the following main result of our paper:

Theorem 4.1. *Let $n \geqslant 3$ be a given positive integer*.

(I) *The quantity* $\max\{\gamma(K) : K \in \mathcal{P}(n + 1, n)\}$ *equals $n^2 - n + 1$ if n is odd and equals $n^2 - n$ if n is even*.

(II) *Suppose n is odd.*

(i) *An n-dimensional minimal cone is exp-maximal if and only if the cone is indecomposable and the relation for its extreme vectors is balanced.*

(ii) *Let K be an n-dimensional exp-maximal minimal cone. A K-primitive matrix A is exp-maximal if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1.*

(III) *Suppose n is even.*

(i) *An n-dimensional minimal cone is exp-maximal if and only if either the cone is indecomposable and has a balanced relation for its extreme vectors, or it is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors.*

(ii) *Let K be an indecomposable exp-maximal minimal cone. A K-primitive matrix A is exp-maximal if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, upon relabelling its vertices suitably, given by Fig. 1 or Fig. 2, and in the latter case x_1, x_2 are required to appear on opposite sides of the relation for the extreme vectors of K.*

(iii) *Let K be a decomposable exp-maximal minimal cone. A K-primitive matrix A is exp-maximal if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 2.*

We need two lemmas for the proof of Theorem 4.1. The first lemma essentially says that if K is a minimal cone and A is a K-primitive matrix with the digraph $(\mathcal{E}, \mathcal{P})$ given by Fig. 1 or Fig. 2 then, depending on the parity of n, whether the digraph is given by Fig. 1 or Fig. 2 (and, in case given by Fig. 2, whether K is indecomposable or decomposable), we can completely specify the (unique) relation for the extreme vectors of K — which is necessarily balanced — as well as the action of A on the extreme vectors. Once the relation for the extreme vectors of K and the action of A on the extreme vectors are specified, the value of $\gamma(A)$ can be found as in the second lemma.

Remark 4.2. From now on scaling a vector or a matrix means multiplying it by a positive scalar.

Lemma 4.3. *Let $K \in \mathcal{P}(n+1, n)$, $n \geqslant 3$. Let A be a K-nonnegative matrix.*

(i) *Suppose that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1 (with $m = n + 1$, and K is necessarily indecomposable). If n is odd then, after scaling (the extreme vectors of K and A), the relation on* Ext K *and the matrix A are given respectively by (4.1) and (4.2):*

$$x_1 + x_3 + \cdots + x_{m-3} + x_{m-1} = x_2 + x_4 + \cdots + x_{m-2} + x_m. \tag{4.1}$$

$$\begin{aligned} Ax_1 &= (1+\alpha)x_2, \\ Ax_i &= x_{i+1} \quad \text{for } i = 2, 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2, \end{aligned} \tag{4.2}$$

where α is some positive scalar. If n is even then, after scaling, the relation on Ext K *and the matrix A are given respectively by (4.3) and (4.4):*

$$x_1 + x_2 + x_4 + \cdots + x_{m-3} + x_{m-1} = x_3 + x_5 + \cdots + x_{m-2} + x_m. \tag{4.3}$$

$$\begin{aligned} Ax_1 &= \alpha x_2, \\ Ax_i &= x_{i+1} \quad \text{for } i = 2, 3, \ldots, m-1, \\ Ax_m &= x_1 + (1+\alpha)x_2, \end{aligned} \tag{4.4}$$

where $\alpha > 0$.

(ii) *Suppose K is indecomposable and $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 2. If n is even then, after scaling, the relation on* Ext K *and the matrix A are given respectively by (4.3) and (4.5), or by (4.6) and (4.7):*

$$\begin{aligned}
Ax_1 &= \alpha x_2 + (1-\beta)x_3,\\
Ax_2 &= \beta x_3,\\
Ax_i &= x_{i+1}, \quad \textit{for } i = 3, \ldots, m-1,\\
Ax_m &= x_1 + (1+\alpha)x_2,
\end{aligned} \tag{4.5}$$

where $\alpha > 0,\ 0 < \beta < 1$.

$$x_2 + x_3 + x_5 + \cdots + x_{m-2} + x_m = x_1 + x_4 + x_6 + \cdots + x_{m-3} + x_{m-1}. \tag{4.6}$$

$$\begin{aligned}
Ax_1 &= (1+\alpha)x_2 + (1+\beta)x_3,\\
Ax_2 &= \beta x_3,\\
Ax_i &= x_{i+1} \quad \textit{for } i = 3, \ldots, m-1,\\
Ax_m &= x_1 + \alpha x_2,
\end{aligned} \tag{4.7}$$

where $\alpha, \beta > 0$. *If* n *is odd then, after scaling, the relation on* Ext K *and the matrix* A *are given respectively by* (4.1) *and* (4.8):

$$\begin{aligned}
Ax_1 &= (1+\alpha)x_2 + \beta x_3,\\
Ax_2 &= (1+\beta)x_3,\\
Ax_i &= x_{i+1}, \quad i = 3, \ldots, m-1,\\
Ax_m &= x_1 + \alpha x_2,
\end{aligned} \tag{4.8}$$

where $\alpha, \beta > 0$.

(iii) *Suppose* K *is decomposable and* $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Fig.* 2. *Then, after scaling, the relation on* Ext K *and the matrix* A *are given respectively by* (4.9) *and* (4.10):

$$x_1 + x_4 + x_6 + \cdots + x_{m-3} + x_{m-1} = x_3 + x_5 + \cdots + x_{m-2} + x_m, \tag{4.9}$$

$$\begin{aligned}
Ax_1 &= \alpha x_2 + x_3,\\
Ax_2 &= \beta x_3,\\
Ax_i &= x_{i+1} \quad \textit{for } i = 3, \ldots, m-1,\\
Ax_m &= x_1 + \alpha x_2,
\end{aligned} \tag{4.10}$$

where $\alpha, \beta > 0$.

Proof

(i) Since K is non-simplicial and $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1, by Lemma 2.4(iii) K is indecomposable. So the (essentially unique) relation on Ext K, which we denote by R, is full. We contend that $x_m,\ x_1$ lie on different sides of R. Suppose not. Since Ax_m is a positive linear combination of $x_1,\ x_2$ and Ax_i is a positive multiple of x_{i+1} for $i = 1, \ldots, m-1$, we readily see that $x_1,\ x_2$ must lie on the same side of the relation obtained from R by applying A (and cancelling out common terms, if any, from the two sides). But the latter relation, which is nonzero (as the coefficients of $x_1,\ x_2$ are both nonzero), is just a multiple of R, so $x_1,\ x_2$ also lie on the same side of relation R. By applying A to the latter relation, we deduce that $x_2,\ x_3$ also lie on the same side of relation R. Continuing the argument, we can then show that $x_1, x_2, \ldots, x_m$ all lie on the same side of R, which is impossible, as K is a pointed cone. This proves our contention. The same argument, in fact, also shows that for $j = 2, 3, \ldots, m-1$, x_j, x_{j+1} lie on different sides of R. Now a simple parity count shows that $x_2,\ x_m$ lie on the same side or opposite sides of R, depending on whether n is odd or even. So when n is odd (i.e., m is even), after scaling the extreme vectors of K we may assume that relation R is given by (4.1).

As the digraph $(\mathcal{E},\mathcal{P})$ is given by Fig. 1, we have

$$Ax_1 = \beta x_2, Ax_i = \lambda_{i+1}x_{i+1} \quad \text{for } i = 2,\ldots,m-1 \quad \text{and} \quad Ax_m = \lambda_1 x_1 + \alpha x_2,$$

where α, β and $\lambda_1, \lambda_3, \lambda_4, \ldots, \lambda_m$ are some positive numbers. Substituting the values of the Ax_i's into the relation obtained from (4.1) by applying A, we obtain the relation:

$$\beta x_2 + \lambda_4 x_4 + \lambda_6 x_6 + \cdots + \lambda_m x_m = \lambda_3 x_3 + \lambda_5 x_5 + \cdots + \lambda_{m-1}x_{m-1} + (\lambda_1 x_1 + \alpha x_2).$$

But relation (4.1) and the above relation are positive multiples of each other, so it follows that we have $\lambda_1 = \lambda_3 = \lambda_4 = \cdots = \lambda_m$ and $\beta = \lambda + \alpha$, where we use λ to denote the common value of the λ_j's. Replacing A by a positive multiple, we may assume that $\lambda = 1$. Then A is given by (4.2).

When n is even, we can show in a similar way that the relation on Ext K and the matrix A are given by (4.3) and (4.4), respectively.

(ii) We consider the case when n is even first. By the same kind of argument that we have used for part (i) we can show that for $j = 3,\ldots,m$, the vectors $x_j,\ x_{j+1}$ lie on different sides of the relation on Ext K (where x_{m+1} is taken to be x_1). Hence, the vectors $x_3, x_5, \ldots, x_m$ lie on one side of the relation and the vectors $x_1, x_4, x_6, \ldots, x_{m-1}$ lie on the other side. As for the vector x_2 it can be on either side. If x_2 is on the same side as x_1 then, after scaling the extreme vectors of K, we may assume that the relation on Ext K is given by (4.3); if x_2 lies on the side opposite to x_1, we may assume that the relation is given by (4.6).

We treat the subcase when the relation is given by (4.3), the argument for the remaining subcase being similar. Since the digraph $(\mathcal{E},\mathcal{P})$ is given by Fig. 2, we have

$$\begin{aligned}&Ax_1 = \alpha x_2 + \gamma x_3, Ax_2 = \beta x_3, Ax_i = \lambda_{i+1}x_{i+1} \quad \text{for } i = 3,\ldots,m-1\\ &\text{and} \quad Ax_m = \lambda_1 x_1 + \delta x_2,\end{aligned}$$

where $\alpha,\ \beta,\ \delta,\ \gamma$ and $\lambda_1, \lambda_4, \ldots, \lambda_m$ are some positive numbers. Applying A to relation (4.3), we obtain the relation:

$$\begin{aligned}&\lambda_4 x_4 + \lambda_6 x_6 + \cdots + \lambda_{m-1}x_{m-1} + (\lambda_1 x_1 + \delta x_2)\\ &\quad = (\alpha x_2 + \gamma x_3) + \beta x_3 + \lambda_5 x_5 + \cdots + \lambda_{m-2}x_{m-2} + \lambda_m x_m.\end{aligned}$$

But relation (4.3) and the above relation are positive multiples of each other, it follows that we have $\lambda_4 = \lambda_5 = \cdots = \lambda_m = \lambda$, say, and $\lambda_1 = \lambda$, $\delta = \lambda + \alpha$ and $\gamma + \beta = \lambda$. Replacing A by a positive multiple, we may assume that $\lambda = 1$. Then A is given by Eq. (4.5).

Now we consider the case when n is odd. Again, we can show that for $j = 3,\ldots,m$, the vectors $x_j,\ x_{j+1}$ lie on different sides of the relation on Ext K (where x_{m+1} is taken to be x_1). Hence, $x_1, x_3, x_5, \ldots, x_{m-3}, x_{m-1}$ lie on one side of the unique relation and $x_4, x_6, \ldots, x_{m-2}, x_m$ lie on the other side. If $x_1,\ x_2$ lie on the same side of the relation then, since $x_1,\ x_3$ also lie on the same side, the same is true for the pair $x_2,\ x_3$. Then by applying A we find that $x_3,\ x_4$ also lie on the same side of the relation, which contradicts what we have observed above. So x_2 lies on the same side as $x_4, x_6, \ldots, x_m$, and after scaling we may assume that the unique relation is given by (4.1). In a similar way as before we can also show that after scaling A is given by (4.8).

(iii) Suppose K is decomposable and $(\mathcal{E},\mathcal{P}(A,K))$ is given by Fig. 2. By Lemma 2.4(iii), m is odd and K is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors. The last part of the proof for Lemma 2.4(iii) shows that in the relation on Ext (K) the vectors $x_1, x_4, x_6, \ldots, x_{m-1}$ lie on one side and the vectors $x_3, x_5, \ldots, x_m$ lie on the other side. After scaling the extreme vectors of K, we may assume that the relation on Ext K is given by relation (4.9). By the same kind of argument as before, we can also show that after scaling A is given by Eq. (4.10). $\square$

Lemma 4.4. *Let $K \in \mathcal{P}(m,n)$ be a minimal cone with extreme vectors $x_1,\ldots,x_m$ (where $m = n+1$), and let A be a K-nonnegative matrix. Let the relations (4.1), (4.3), (4.6), (4.9) on the extreme vectors of K and the equations (4.2), (4.4), (4.5), (4.7), (4.8), (4.10) on the action of A be as given in Lemma 4.3. Then:*

(i) $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Fig.* 1 (*and K is indecomposable*) *if and only if after scaling the relation on* Ext K *and the matrix A are given respectively by relation* (4.1) *and Eq.* (4.2), *in which case n is odd and* $\gamma(A) = n^2 - n + 1$, *or by relation* (4.3) *and Eq.* (4.4), *in which case n is even and* $\gamma(A) = n^2 - n$.

(ii) $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Fig.* 2 *and K is indecomposable if and only if after scaling the relation on* Ext K *and the matrix A are given respectively by relation* (4.6) *and Eq.* (4.7), *in which case n is even and* $\gamma(A) = n^2 - n$, *or by relation* (4.3) *and Eq.* (4.5), *in which case n is even and* $\gamma(A) = n^2 - n - 1$, *or by relation* (4.1) *and Eq.* (4.8), *in which case n is odd and* $\gamma(A) = n^2 - n$.

(iii) $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Fig.* 2 *and K is decomposable if and only if after scaling the relation on* Ext K *and the matrix A are given respectively by relation* (4.9) *and Eq.* (4.10), *in which case n is even and* $\gamma(A) = n^2 - n$.

Proof. The "only if" parts of (i), (ii) and (iii) are done in Lemma 4.3. It remains to treat the "if" parts and the parts concerning the value of $\gamma(A)$.

(i) First, suppose the relation on Ext K and the matrix A are given respectively by relation (4.1) and Eq. (4.2). It is clear that m is even, and so n is odd. Note that A is well-defined, as it preserves the relation on Ext K. We contend that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1. It is clear that for $i = 1, \ldots, m-1$, $(\Phi(x_i), \Phi(x_{i+1}))$ is the only outgoing arc from vertex $\Phi(x_i)$. By definition $Ax_m = x_1 + \alpha x_2$, so $\Phi(Ax_m)$ equals $\Phi(x_1 + x_2)$, which is the smallest face of K containing x_1, x_2. Since relation (4.1), the (unique) relation on the extreme vectors of K, is full, K is indecomposable. As x_1, x_2 lie on opposite sides of (4.1), by Theorem A, x_1, x_2 lie on a common maximal face of K, i.e., $\Phi(x_1 + x_2)$ is a nontrivial face. But every nontrivial face of an indecomposable minimal cone is simplicial, so x_1, x_2 are the only extreme vectors of $\Phi(x_1 + x_2)$. It follows that $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$ are the only outgoing arcs from vertex $\Phi(x_m)$. This proves our contention. Now a straightforward calculation yields the following: $A^{m-1}x_1 = (1+\alpha)x_m$; $A^m x_1 = (1+\alpha)(x_1 + \alpha x_2)$, i.e., $\Phi(A^m x_1) = \Phi(x_1 + x_2)$; and $\Phi(A^{j(m-1)}x_1) = \Phi(x_{m-j+1} + x_{m-j+2} + \cdots + x_{m-1} + x_m)$ for $j = 1, \ldots, m-2$. So $A^{(n-1)(m-1)}x_1$ is a positive linear combination of $x_3, x_4, \ldots, x_m$ and by Theorem A it belongs to the relative interior of a maximal face of K. On the other hand, $A^{(n-1)(m-1)+1}x_1$ belongs to int K as it can be written as a positive linear combination of all x_i except x_3. Thus $\gamma(A, x_1) = (n-1)(m-1) + 1 = n^2 - n + 1$. But $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1, so by Lemma 2.4(ii), $\gamma(A) = \gamma(A, x_1) = n^2 - n + 1$.

Next, suppose that the relation on Ext K and the matrix A are given by relation (4.3) and Eq. (4.4) respectively. Then K is indecomposable. Note that the left side of relation (4.3) has one term more than its right side and it contains both x_1, x_2. Since $m (\geqslant 4)$ is odd, the left side has at least three terms; so $Ax_m = x_1 + (1+\alpha)x_2 \in \partial K$. As before one can verify that $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1. Also, a straightforward calculation shows that $A^{(n-1)(m-1)-1}x_1$, being a positive linear combination of $x_2, x_3, \ldots, x_{m-1}$, belongs to ∂K, whereas $A^{(n-1)(m-1)}x_1$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to int K. Since $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1, we have $\gamma(A) = \gamma(A, x_1) = (n-1)(m-1) = n^2 - n$.

(ii) Suppose the relation on Ext K and the matrix A are given by relation (4.6) and Eq. (4.7) respectively. Since relation (4.6) is full, K is indecomposable. Note that the left side of (4.6) has at least three terms and it contains both x_2 and x_3, so Ax_1, which is a positive linear combination of x_2, x_3, belongs to ∂K. Similarly, $Ax_m (= x_1 + \alpha x_2)$ also belongs to ∂K, as x_1, x_2 lie on opposite sides of (4.6). By the same kind of argument as given in the proof for part (i), one readily verifies that $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2. Also, $A^{(n-1)(m-1)-1}x_2$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to ∂K, whereas $A^{(n-1)(m-1)}x_2$ belongs to int K, as it can be written as a positive linear combination of all the x_i's except x_3. Since $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2, we have $\gamma(A) = \gamma(A, x_2) = (n-1)(m-1) = n^2 - n$.

When the relation on Ext K and the matrix A are given by relation (4.3) and Eq. (4.5) respectively, we can show that K is indecomposable and also that $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2. In this case, $A^{(n-1)(m-1)-2}x_2$, being a positive linear combination of $x_2, x_3, \ldots, x_{m-1}$, belongs to ∂K, whereas $A^{(n-1)(m-1)-1}x_2$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to int K. It follows that we have $\gamma(A) = \gamma(A, x_2) = (n-1)(m-1) - 1 = n^2 - n - 1$.

Similarly, we can show that when the relation and the matrix A are given by relation (4.1) and Eq. (4.8) respectively, K is indecomposable, $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2 and $\gamma(A) = n^2 - n$.

(iii) Suppose the relation on $\operatorname{Ext} K$ and the matrix A are given by relation (4.9) and Eq. (4.10) respectively. In this case $K = \operatorname{pos}\{x_2\} \oplus \operatorname{pos}\ \{x_1, x_j, 3 \leqslant j \leqslant m\}$. We readily check that $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2. A straightforward calculation shows that $A^{(m-1)(m-2)-1}x_2$ is a positive linear combination of $x_3, x_4, \ldots, x_m$. So $A^{(n-1)(m-1)-1}x_2$ belongs to the indecomposable summand $\operatorname{pos}\{x_1, x_j, 3 \leqslant j \leqslant m\}$ of K and hence lies in ∂K. On the other hand, $A^{(n-1)(m-1)}x_2$ belongs to $\operatorname{int} K$, as it can be written as a positive linear combination of all x_i's except x_3. Hence we have $\gamma(A) = \gamma(A, x_2) = (n-1)(m-1) = n(n-1)$. $\square$

Proof of Theorem 4.1. We first observe that when n is even, there is no minimal cone K such that $\gamma(K) = n^2 - n + 1$. Assume to the contrary that there is one such K. Choose a K-primitive matrix A that satisfies $\gamma(A) = n^2 - n + 1$. By Corollary 2.7 $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1. Since n is even, by the second half of Lemma 4.3(i), after scaling, the relation on $\operatorname{Ext} K$ and the matrix A are given by relation (4.3) and Eq. (4.4) respectively. So by Lemma 4.4(i), we have $\gamma(A) = n^2 - n$, which is a contradiction.

For any positive integer n, by Corollary 2.7, $\gamma(K) \leqslant n^2 - n + 1$ for every n-dimensional minimal cone K.

Let n be odd. Take any n-dimensional indecomposable cone K for which the relation on its extreme vectors has the same number of terms on its two sides. After re-indexing and scaling the extreme vectors $x_1, \ldots, x_m$ of K, we may assume that the relation on $\operatorname{Ext} K$ is given by (4.1). Let A be the $n \times n$ real matrix given by (4.2). By Lemma 4.4(i) A is K-primitive and $\gamma(A) = n^2 - n + 1$. So we have $\gamma(K) = n^2 - n + 1$. This establishes (I) for odd n as well as the "if" part of (II)(i).

Now let n be even. In view of the above observations, the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones is at most $n^2 - n$. We are going to show that the value $n^2 - n$ can be attained.

Take any indecomposable minimal cone K such that in the relation on $\operatorname{Ext} K$ the number of vectors on its two sides differ by 1. Scaling the extreme vectors of K, we may assume that the relation is given by (4.3). Let A be the matrix given by (4.4). By Lemma 4.4(i) we have $\gamma(A) = n^2 - n$. For this K, certainly we have $\gamma(K) = n^2 - n$. This establishes (I) for even n and completes the proof for (I).

If K is the direct sum of a ray and an indecomposable minimal cone for which the relation on its extreme vectors has same number of terms on its two sides, then necessarily n is even and after scaling we may assume that the relation is given by (4.9). By Lemma 4.4(iii), the matrix A defined by (4.10) satisfies $\gamma(A) = n^2 - n$.

So we have also established the “if" part of (III)(i).

To prove the “only if" part of (II)(i), assume that n is odd and let K be an n-dimensional minimal cone that satisfies $\gamma(K) = n^2 - n + 1$. By Corollary 2.7 there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1. By Lemma 2.4(iii), K is indecomposable. By Lemma 4.3(i), after scaling the extreme vectors of K, we may assume that the relation on $\operatorname{Ext} K$ is given by relation (4.1). So the relation has the same number of terms on its two sides.

The “only if" part of (II)(ii) follows from part(i) of Theorem 2.5 (by taking $m = n + 1$ and $m_A = n$), whereas its “if" part is a consequence of Lemma 4.4(i).

The proof of part(II) is complete.

To prove the “only if" part of (III)(i), assume that n is even and let K be an n-dimensional minimal cone such that $\gamma(K) = n^2 - n$. Choose a K-primitive matrix A such that $\gamma(A) = \gamma(K)$. By part (ii) of Theorem 2.5, in this case we have $m_A = n$ and either $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2, or $m_A = 3$. The case $m_A = 3$ cannot happen; otherwise, $n = 3$, contradicting the assumption that n is even. Then, by Lemma 2.4(iii), K is either indecomposable or is the direct sum of a ray and an indecomposable minimal cone for which the relation on its extreme vectors has the same number of terms on its two sides. In the latter case, we are done. In the former case, $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2. If the digraph is given by Fig. 1 then, since K is indecomposable minimal, by Lemma 4.3(i), after scaling, the relation on $\operatorname{Ext} K$ is given by (4.3). If the digraph is given by Fig. 2, then by part (ii) of the same lemma, after scaling, the relation on $\operatorname{Ext} K$ and the matrix A are given respectively by (4.3) and (4.5) or (4.6) and (4.7). (We have to rule out the former possibility, because by Lemma 4.4(ii) we have $\gamma(A) = n^2 - n - 1$, which is a contradiction.) In any case, in the relation on Ext K the number of terms on its two sides differ by 1.

To prove the “only if" part of (III)(ii), let K be an indecomposable minimal cone such that in the relation on its extreme vectors the number of terms on its two sides differ by 1, and suppose that A is a K-primitive matrix such that $\gamma(A) = n^2 - n$. By the above proof for the “only if" part of (III)(i), $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2. If it is given by Fig. 2 then after scaling the relation on Ext K is given by relation (4.6), hence $x_1,\ x_2$ lie on opposite sides of the relation.

To prove the “if" part of (III)(ii), first suppose that $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1. Since n is even, by Lemma 4.4(i), after scaling, the relation on Ext K and the matrix A are given respectively by (4.3) and (4.4), and we have $\gamma(A) = n^2 - n$. If the digraph is given by Fig. 2 and $x_1,\ x_2$ appear on opposite sides of the relation on Ext K, then since K is indecomposable and n is even, by Lemma 4.4(ii) after scaling the relation on Ext K and the matrix A are given respectively by (4.6) and (4.7), and we have $\gamma(A) = n^2 - n$.

The “if" part of (III)(iii) follows from Lemma 4.4(iii). To prove its “only if" part, let A be a K-primitive matrix such that $\gamma(A) = n^2 - n$. As explained in the above proof for the “only if" part of (III)(i), the digraph $(\mathcal{E}, \mathcal{P})$ is given by either Fig. 1 or Fig. 2. By Lemma 2.4(iii), if it is given by Fig. 1 then K is necessarily indecomposable. But now K is decomposable, so the digraph is given by Fig. 2.

The proof is complete. $\square$

In an open problem session at the Barcelona ILAS Conference held in July, 1999 Steve Kirkland conjectured that for a polyhedral cone K with m extreme rays $m^2 - 2m + 2$ is an upper bound for the exponents of K-primitive matrices.

By Corollary 2.7 and Theorem 4.1(I) we readily deduce the following result, which is an improvement of the already-proved Kirkland’s conjecture.

Corollary 4.5. *For any positive integer $m \geqslant 4$, the maximum value of $\gamma(K)$ as K runs through non-simplicial polyhedral cones with m extreme rays and of all possible dimensions is $m^2 - 3m + 3$ if m is even, and is $m^2 - 3m + 2$ if m is odd.*

5. Uniqueness of exp-maximal minimal cones and their exp-maximal primitive matrices

Given positive integers m, n with $3 \leqslant n \leqslant m$, up to linear isomorphism, how many exp-maximal cones are there in $\mathcal{P}(m, n)$? For a given exp-maximal cone K in $\mathcal{P}(m, n)$, up to cone-equivalence modulo positive scalar multiplication, how many exp-maximal K-primitive matrices are there? In this section we are going to address these questions for the class of minimal cones.

The problem of identifying exp-maximal minimal cones has already begun in Section 4. According to Theorem 4.1, a cone $K \in \mathcal{P}(n+1, n)$ is exp-maximal if and only if K is an indecomposable minimal cone with a balanced relation for its extreme vectors or n is even and K is the direct sum of a ray and an $(n-1)$-dimensional indecomposable minimal cone with a balanced relation for its extreme vectors. On the other hand, by Theorem 3.2 for every positive integer $n \geqslant 3$, there is, up to linear isomorphism, only one n-dimensional indecomposable minimal cone with a balanced relation for its extreme vectors. Summarizing, we have the following:

Remark 5.1. For every positive integer $n \geqslant 3$, up to linear isomorphism, there are precisely one or two n-dimensional exp-maximal minimal cones, depending on whether n is odd or even.

Next, we are going to identify, up to cone-equivalence and scalar multiples, the exp-maximal primitive matrices for exp-maximal minimal cones. As a matter of fact, in identifying exp-maximal minimal cones in Section 3, we have already provided (implicitly in Lemma 4.3), up to cone-equivalence and scalar multiples, all the exp-maximal primitive matrices for minimal cones. It remains to show that there are no more.

We begin with a general result.

Lemma 5.2. *Let $K \in \mathcal{P}(m, n)$ be indecomposable. If $A, \widetilde{A}$ are different K-nonnegative matrices such that the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\widetilde{A}, K))$ are given both by Fig. 1 or both by Fig. 2 (as labelled digraphs), then $\widetilde{A}$ and A are not cone-equivalent.*

Proof. We prove the equivalent statement: If A and $\widetilde{A}$ are cone-equivalent K-nonnegative matrices such that the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\widetilde{A}, K))$ are given both by Fig. 1 or both by Fig. 2 then $A = \widetilde{A}$. Since A and $\widetilde{A}$ are cone-equivalent K-nonnegative matrices, there exists an automorphism P of K such that $P\widetilde{A} = AP$. We contend that for $j = 1, \ldots, m$, P maps x_j to a positive multiple of itself. Once this is proved, it will follow that $P \in \Phi(I)$. But K is indecomposable, so by [10, Theorem 3.3] $\Phi(I)$ is an extreme ray of $\pi(K)$; hence P is a positive multiple of I and we have $\widetilde{A} = A$, as desired.

To prove our contention, we first deal with the case when the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\widetilde{A}, K))$ are both given by Fig. 1. As P is an automorphism of K, P permutes the extreme rays of K among themselves. According to Lemma 2.4(ii), the maximum value of $\gamma(A, x)$, for $x = x_1, \ldots, x_m$, is attained at x_1 only. When A is replaced by $\widetilde{A}$, the same can be said. Since A and $\widetilde{A}$ are cone-equivalent, by Fact 2.1(v) P must map x_1 to a positive multiple of itself. Making use of the relation $P\widetilde{A} = AP$ and the fact that Ax_i (also $\widetilde{A}x_i$) is a positive multiple of x_{i+1} for $i = 1, \ldots, m-1$ and proceeding inductively, we readily show that P maps each x_i to a positive multiple of itself, which is our contention.

If $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\widetilde{A}, K))$ are both given by Fig. 2, again by Fact 2.1(v) we readily see that Px_2 must be a positive multiple of itself. Then we proceed in a similar manner to establish the desired contention. $\square$

Next, a remark on the automorphisms of an indecomposable minimal cone is in order.

Let K be an indecomposable minimal cone in $\mathbb{R}^n$ with extreme vectors $x_1, \ldots, x_{n+1}$ that satisfy the relation

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}.$$

Let σ be a permutation on $\{1, \ldots, n+1\}$ that maps $\{1, \ldots, p\}$ and $\{p+1, \ldots, n\}$ each onto itself, or interchanges the first set with the second set (in which case n is odd and $p = \frac{n+1}{2}$). Then σ determines a (unique) automorphism P_σ of K with spectral radius 1 ($x_1 + \cdots + x_p$ being the corresponding eigenvector) which is given by: $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \ldots, n+1$. Conversely, every automorphism of K whose spectral radius is 1 arises in this way.

Now we consider the exp-maximal primitive matrices for indecomposable exp-maximal minimal cones first. We give two results, one for the odd dimensional case and the other for the even dimensional case. The relations (4.1)–(4.7) that are mentioned in these results have already appeared in Lemma 4.3.

Theorem 5.3. *Let K be an n-dimensional indecomposable exp-maximal minimal cone, where n is odd. Suppose that the extreme vectors $x_1, \ldots, x_{n+1}$ of K satisfy relation (4.1) (with $m = n+1$). For every $\alpha > 0$, let A_α be the exp-maximal K-primitive matrix given by (4.2) (but with A replaced by A_α). Then:*

(i) *$\Phi(A_\alpha)$ is a 2-dimensional face, independent of the choice of the positive scalar α; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha}}$ with $\tilde{\alpha} > 0$.*
(ii) *Every exp-maximal K-primitive matrix is cone-equivalent to a positive multiple of some A_α and thus is a positive multiple of a matrix of the form $P_\sigma^{-1} A_\alpha P_\sigma$, where P_σ is the automorphism of K given by $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \ldots, n+1$, σ being a permutation on $\{1, \ldots, n+1\}$ that maps $\{1, 3, \ldots, n-2, n\}$ onto itself or onto $\{2, 4, \ldots, n-1, n+1\}$.*
(iii) *For distinct positive scalars α_1, α_2, A_{α_1} and A_{α_2} or their positive multiples are not cone-equivalent.*

Proof

(i) It is clear that $\Phi(A_\alpha) = \Phi(A_{\tilde{\alpha}})$ for any $\alpha, \tilde{\alpha} > 0$ as $\Phi(A_\alpha x_j) = \Phi(A_{\tilde{\alpha}} x_j)$ for each j and K is polyhedral. We assert that $\Phi(A_\alpha)$ is equal to the 2-dimensional face generated by the matrices B, C of $\pi(K)$ determined respectively by:

$$Bx_i = x_{i+1} \quad \text{for } i = 1, \ldots, m, \text{ where } x_{m+1} \text{ is taken to be } x_1,$$

and

$$Cx_1 = x_2 = Cx_m, \quad Cx_i = 0 \quad \text{for } i = 2, \ldots, m-1.$$

It is readily checked that B and C each preserve relation (4.1); so they are well-defined and K-nonnegative. Since $A_\alpha = B + \alpha C$, we have $B, C \in \Phi(A_\alpha)$ and hence $\text{pos}\{B, C\} \subseteq \Phi(A_\alpha)$. To complete the argument, we contend that every matrix in ri $\Phi(A_\alpha)$ is a positive multiple of $A_{\tilde{\alpha}}$ for some $\tilde{\alpha} > 0$ (and hence belongs to pos$\{B, C\}$). Since $\Phi(A_\alpha) = \text{cl}[\text{ri}\,\Phi(A_\alpha)]$, once this is proved, it will follow that $\Phi(A_\alpha) = \text{pos}\{B, C\}$ and so our assertion follows.

Consider any K-nonnegative matrix $\widetilde{A}$ that satisfies $\Phi(\widetilde{A}) = \Phi(A_\alpha)$. Since $\Phi(\widetilde{A}x_m) = \Phi(A_\alpha x_m)$ and $\Phi(A_\alpha x_m)$ is the 2-dimensional face of K generated by x_1, x_2, after scaling $\widetilde{A}$, we may assume that $\widetilde{A}x_m = x_1 + \tilde{\alpha}x_2$ for some $\tilde{\alpha} > 0$. Similarly, we may assume that $\widetilde{A}x_i = \tilde{\alpha}_{i+1}x_{i+1}$ for $i = 1, \ldots, m-1$. Substituting the values of the $\widetilde{A}x_i$'s into the relation obtained from (4.1) by applying $\widetilde{A}$, we obtain

$$\tilde{\alpha}_2x_2 + \tilde{\alpha}_4x_4 + \cdots + \tilde{\alpha}_{m-2}x_{m-2} + \tilde{\alpha}_mx_m = \tilde{\alpha}_3x_3 + \tilde{\alpha}_5x_5 + \cdots + \tilde{\alpha}_{m-1}x_{m-1} + x_1 + \tilde{\alpha}x_2.$$

Since (4.1) is, up to multiples, the only relation for the extreme vectors of K, we have

$$\tilde{\alpha}_i = 1 = \tilde{\alpha}_2 - \tilde{\alpha} \quad \text{for } i = 3, \ldots, m.$$

Hence $\widetilde{A}$ is given by

$$\widetilde{A}x_1 = (1 + \tilde{\alpha})x_2, \widetilde{A}x_i = x_{i+1} \quad \text{for } i = 2, \ldots, m-1, \quad \text{and} \quad \widetilde{A}x_m = x_1 + \tilde{\alpha}x_2,$$

for some $\tilde{\alpha} > 0$. This proves that, after scaling, $\widetilde{A}$ equals some $A_{\tilde{\alpha}}$, which is our contention.

(ii) Let A be an exp-maximal K-primitive matrix. In view of Theorem 4.1(II)(ii), the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1, except that $x_1, \ldots, x_m$ are to be replaced by $x_{\sigma(1)}, \ldots, x_{\sigma(m)}$ respectively, where σ is some permutation on $\{1, \ldots, m\}$. By Lemma 4.3(i) we can find positive scalars α and λ_j, $j = 1, \ldots, m$, such that, after scaling A, the relation on Ext K and the matrix A are given respectively by the relations obtained from (4.1) and (4.2) by replacing each x_j by $\lambda_jx_{\sigma(j)}$. Since the relation on $x_1, \ldots, x_m$ is, up to multiples, unique, it follows that $\lambda_1, \ldots, \lambda_m$ are the same and moreover σ maps the set $\{1, 3, \ldots, n\}$ onto itself or onto $\{2, 4, \ldots, n+1\}$. It is readily checked that we have $A = P_\sigma^{-1}A_\alpha P_\sigma$. So, after scaling, A is equivalent to some A_α.

(iii) As can be readily seen, if α_1, α_2 are distinct positive scalars, then A_{α_1} and A_{α_2} are linearly independent. Since the digraphs $(\mathcal{E}, \mathcal{P}(A_{\alpha_1}, K))$ and $(\mathcal{E}, \mathcal{P}(A_{\alpha_2}, K))$ are both given by Fig. 1, by Lemma 5.2 a positive multiple of A_{α_1} cannot be cone-equivalent to a positive multiple of A_{α_2}. $\square$

Theorem 5.4. *Let K be an n-dimensional indecomposable exp-maximal minimal cone, where n is even. Suppose that the extreme vectors $x_1, \ldots, x_m$ of K (with $m = n + 1$) satisfy relation (4.3). For every $\alpha > 0$, let A_α be the exp-maximal K-primitive matrix given by (4.4) (but with A replaced by A_α). For every $\alpha, \beta > 0$, let $A_{\alpha,\beta}$ be the K-nonnegative matrix whose action on the vectors x_i for $i = 1, \ldots, n+1$ are given by:*

$$\begin{aligned}
A_{\alpha,\beta}x_1 &= \beta x_2,\\
A_{\alpha,\beta}x_3 &= (1+\alpha)x_1 + (1+\beta)x_2,\\
A_{\alpha,\beta}x_n &= x_3 + \alpha x_1,\\
A_{\alpha,\beta}x_i &= \begin{cases} x_{i+3} & \text{when } i \text{ is even, } i \neq n,\\ x_{i-1} & \text{when } i \text{ is odd, } i \neq 1, 3.\end{cases}
\end{aligned}$$

Then:

(i) *$\Phi(A_\alpha)$ is a 2-dimensional face, independent of the choice of the positive scalar α; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha}}$ with $\tilde{\alpha} > 0$.*

(ii) *$\Phi(A_{\alpha,\beta})$ is a 3-dimensional face, independent of the choice of the positive scalars α, β; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha},\tilde{\beta}}$ with $\tilde{\alpha}, \tilde{\beta} > 0$.*

(iii) *Every exp-maximal K-primitive matrix is cone-equivalent to a positive multiple of some A_α or some $A_{\alpha,\beta}$ and thus is a positive multiple of a matrix of the form $P_\sigma^{-1}A_\alpha P_\sigma$ or $P_\sigma^{-1}A_{\alpha,\beta}P_\sigma$, where P_σ is the automorphism of K given by $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \ldots, n+1$, σ being a permutation on $\{1, \ldots, n+1\}$ that maps the set $\{1, 2, 4, \ldots, n-2, n\}$ onto itself.*

(iv) *The A_α's, $A_{\alpha,\beta}$'s or their positive multiples are pairwise not cone-equivalent.*

Proof

(i) Use the same argument as that for Theorem 5.3(i).

(ii) Use the same kind of argument as that for Theorem 5.3(i); in this case we can show that $\Phi(A_{\alpha,\beta})$ is the 3-dimensional face of $\pi(K)$ with distinct extreme matrices B, C, D, whose actions on the vectors x_i for $i = 1, \ldots, n+1$ are given respectively by:

$$Bx_1 = 0, \quad Bx_3 = x_1 + x_2, \quad Bx_n = x_3, \quad \text{and}$$
$$Bx_i = \begin{cases} x_{i+3} & \text{when } i \text{ is even, } i \neq n \\ x_{i-1} & \text{when } i \text{ is odd, } i \neq 1, 3 \end{cases},$$
$$Cx_3 = x_1 = Cx_n, \quad Cx_i = 0 \quad \text{for } i \neq 3, n,$$

and

$$Dx_1 = x_2 = Dx_3, \quad Dx_i = 0 \quad \text{for } i \neq 1, 3.$$

(iii) Let $\widetilde{K}$ denote the n-dimensional indecomposable minimal cone with extreme vectors $\tilde{x}_1, \ldots, \tilde{x}_{n+1}$ that satisfy relation (4.6) (with $x_1, \ldots, x_m$ replaced by $\tilde{x}_1, \ldots, \tilde{x}_m$ respectively), and let $\widetilde{A}$ be the $\widetilde{K}$-nonnegative matrix defined by (4.7) (but with A replaced by $\widetilde{A}$ and x_is replaced by $\tilde{x}_i$s). Let P be the matrix from span K to span $\widetilde{K}$ given by $Px_1 = \tilde{x}_2, Px_3 = \tilde{x}_1$, and

$$Px_j = \begin{cases} \tilde{x}_{j-1} & \text{when } j \text{ is odd, } j \neq 1, 3, \\ \tilde{x}_{j+1} & \text{when } j \text{ is even.} \end{cases}$$

Then, as can be readily checked, P is a linear isomorphism which maps K onto $\widetilde{K}$ and, moreover, we have $A_{\alpha,\beta} = P^{-1}\widetilde{A}P$. So $A_{\alpha,\beta}$ is cone-equivalent to $\widetilde{A}$.

Let A be an exp-maximal K-primitive matrix. In view of Theorem 4.1(III)(ii), the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1 or Fig. 2 (with $m = n+1$), except that $x_1, \ldots, x_{n+1}$ are to be replaced by $x_{\sigma(1)}, \ldots, x_{\sigma(n+1)}$ respectively, where σ is some permutation on $\{1, \ldots, n+1\}$. In the former case, following the argument given in the proof for Theorem 5.3(ii), we can show that A is a positive multiple of a matrix of the form $P_\sigma^{-1}A_\alpha P_\sigma$ where P_σ is the automorphism of K given by $P_\sigma x_j = x_{\sigma(j)}$ for $j = 1, \ldots, n+1$, σ being a permutation on $\{1, \ldots, n+1\}$ that maps the set $\{1, 2, 4, \ldots, n-2, n\}$ onto itself, noting that σ cannot interchange the sets $\{1, 2, 4, \ldots, m-3, m-1\}$ and $\{3, 5, \ldots, m-2, m\}$ as their cardinality differ by 1. So, in this case, A is cone-equivalent to a positive multiple of some A_α.

When the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 2 (but with vertices relabelled as indicated above), by Lemma 4.3(ii) we can find positive scalars α, β and $\lambda_j, j = 1, \ldots, m$, such that, after scaling A, the relation on Ext K and the matrix A are given by the relations obtained from (4.6) and (4.7) respectively by replacing each x_j by $\lambda_j x_{\sigma(j)}$. Let Q be the matrix from span$K$ to span$\widetilde{K}$ given by: $Q(\lambda_j x_{\sigma(j)}) = \tilde{x}_j$. As can be readily checked, Q is a linear isomorphism which maps K onto $\widetilde{K}$ and, moreover, we have $A = Q^{-1}\widetilde{A}Q$, where $\widetilde{A}$ is the matrix introduced at the beginning of the proof for this part. So A is cone-equivalent to $\widetilde{A}$, and hence is also cone-equivalent to $A_{\alpha,\beta}$, as desired.

(iv) As done in the proof for Theorem 5.3(iii), if α_1, α_2 are distinct positive scalars, then a positive multiple of A_{α_1} and a positive multiple of A_{α_2} are linearly independent, and so by Lemma 5.2 they are not cone-equivalent. For a similar reason, a positive multiple of A_{α_1,β_1} is also not cone-equivalent to a positive multiple of A_{α_2,β_2}, provided that $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$. Moreover, a matrix of the form A_α and one of the form $A_{\beta,\gamma}$, or their positive multiples, are also not cone-equivalent, because $(\mathcal{E}, \mathcal{P}(A_\alpha))$ is given by Fig. 1 whereas $(\mathcal{E}, \mathcal{P}(A_{\beta,\gamma}))$ is given by Fig. 2, and the two digraphs are not isomorphic. $\square$

Now we consider the exp-maximal primitive matrices for a decomposable exp-maximal minimal cone. In this case, Lemma 5.2 no longer applies. What we have is the following:

Lemma 5.5. *Let $K \in \mathcal{P}(n+1, n)$ be an exp-maximal decomposable minimal cone with extreme vectors $x_1, \ldots, x_{n+1}$ (where n is even). Suppose that $K = \mathrm{pos}\{x_2\} \oplus \mathrm{pos}\{x_1, x_3, x_4, \ldots, x_{n+1}\}$, where*

$x_1, x_3, x_4, \ldots, x_{n+1}$ satisfy the relation given by (4.9) (with $m = n + 1$). Let A and $\widetilde{A}$ be the K-nonnegative matrices defined respectively by (4.10) and by the relation obtained from (4.10) by replacing A, α, β by $\widetilde{A}, \tilde{\alpha}, \tilde{\beta}$, respectively. Then for any $\omega > 0$, $\widetilde{A}$ and ωA are cone-equivalent if and only if $\omega = 1$ and $\alpha\beta = \tilde{\alpha}\tilde{\beta}$.

Proof. "Only if" part: First, note that the given assumptions guarantee that the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(\widetilde{A}, K))$ are both given by Fig. 2 (see Lemma 4.4(iii)).

Suppose that $\widetilde{A}$ and ωA are cone-equivalent. Let P be an automorphism of K such that $P\widetilde{A} = \omega AP$. By the argument given in the proof of Lemma 5.2, we can show that P takes x_2 to a positive multiple of itself. Then from the relation $P\widetilde{A}x_2 = \omega APx_2$ we infer that P also maps x_3 to a positive multiple of itself. Proceeding inductively, we can show that P maps each x_j to a positive multiple of itself. Say, we have $Px_i = \lambda_i x_i$ for $i = 1, \ldots, m$. Substituting the values of the Px_i's into the relation obtained from (4.9) by applying P and using the fact that, up to multiples, (4.9) is the only relation for the extreme vectors of K, we conclude that all the λ_j's, for $j = 1, \ldots, m, j \neq 2$, are equal. Denote their common value by λ. Then P is given by $Px_i = \lambda x_i$ for $i = 1, \ldots, m, i \neq 2$, and $Px_2 = \mu x_2$, where μ denotes λ_2. Now by the given assumptions on A and $\widetilde{A}$ we have

$$P\widetilde{A}x_1 = P(\tilde{\alpha}x_2 + x_3) = \tilde{\alpha}\mu x_2 + \lambda x_3 \text{ and } \omega APx_1 = \omega A(\lambda x_1) = \omega\lambda(\alpha x_2 + x_3).$$

But $P\widetilde{A}x_1 = \omega APx_1$, so we obtain $\omega = 1$ and $\tilde{\alpha}/\alpha = \lambda/\mu$. Then the relation $P\widetilde{A} = \omega AP$ reduces to $P\widetilde{A} = AP$. Similarly, from the relation $P\widetilde{A}x_2 = APx_2$ we obtain $\beta/\tilde{\beta} = \lambda/\mu$. Hence we have $\alpha\beta = \tilde{\alpha}\tilde{\beta}$.

Conversely, suppose that $\alpha\beta = \tilde{\alpha}\tilde{\beta}$. Choose positive scalars λ, μ such that $\lambda/\mu = \tilde{\alpha}/\alpha (= \beta/\tilde{\beta})$. Let P be the automorphism of K determined by $Px_2 = \mu x_2$ and $Px_i = \lambda x_i$ for all $i \neq 2$. Then, as can be readily checked, $P\widetilde{A}x_i = APx_i$ for every i. Hence we have $P\widetilde{A} = AP$, i.e., A and $\widetilde{A}$ are cone-equivalent. □

In view of Lemma 5.5 and using the kind of argument as given in the proofs for Theorem 5.3 and Theorem 5.4, we can establish the following, whose proof we omit:

Theorem 5.6. *Let $K \in \mathcal{P}(n+1, n)$ be an exp-maximal decomposable minimal cone with extreme vectors $x_1, \ldots, x_{n+1}$ (where n is even). Suppose that $K = \mathrm{pos}\{x_2\} \oplus \mathrm{pos}\{x_1, x_3, x_4, \ldots, x_{n+1}\}$, where $x_1, x_3, x_4, \ldots, x_{n+1}$ satisfy the relation given by (4.9) (with $m = n + 1$). For every $\alpha, \beta > 0$, let $A_{\alpha,\beta}$ be the exp-maximal K-primitive matrix defined by (4.10) (but with A replaced by $A_{\alpha,\beta}$). Then:*

- (i) *$\Phi(A_{\alpha,\beta})$ is a 3-dimensional simplicial face, independent of the choice of the positive scalars α, β; its relative interior consists of positive multiples of matrices of the form $A_{\tilde{\alpha},\tilde{\beta}}$.*
- (ii) *Every exp-maximal K-primitive matrix is cone-equivalent to a positive multiple of some $A_{1,\beta}$.*
- (iii) *For distinct positive scalars β_1, β_2, the matrices $A_{1,\beta_1}, A_{1,\beta_2}$, or their positive multiples, are pairwise not cone-equivalent.*

In view of the preceding theorems, we can conclude that for every exp-maximal minimal cone K, indecomposable or not, there are uncountably infinitely many exp-maximal K-primitive matrices which are pairwise linearly independent and non-cone-equivalent.

6. Open questions

Let E_n denote the set of values attained by the exponents of primitive matrices of order n. Dulmage and Mendelsohn [6] have found intervals in the set $\{1, 2, \ldots, (n-1)^2 + 1\}$ containing no integer which is the exponent of a primitive matrix of order n. These intervals have been called *gaps* in E_n. The problem of determining E_n or the gaps is an intricate problem, but it has been completely resolved. (See, for instance, [4].)

For a given polyhedral cone (or a proper cone) K, we can consider a similar problem — to determine the set of values attained by the exponents of K-primitive matrices. We expect that for every polyhedral cone K of dimension greater than 2 there are gaps in the set of values attained by the exponents of

K-primitive matrices (but at present we *do not* have a proof for this claim). As an illustration, consider an indecomposable minimal cone $K \in \mathcal{P}(m, n)$ with a balanced relation for its extreme vectors, where n is an odd integer $\geqslant 5$. For a K-primitive matrix A, if the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2 then $\gamma(A)$ equals $n^2 - n + 1$ or $n^2 - n$ (see Theorem 4.1(II) and Lemma 4.4(ii)). On the other hand, if the digraph is not given by Fig. 1 or Fig. 2, then by Lemma 2.2 the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is at most $n - 1 (= m - 2)$ and by Corollary 2.6 it follows that $\gamma(A) \leqslant (n-1)^2 + 2$. So in this case any integer lying in the closed interval $[n^2 - 2n + 4, n^2 - n - 1]$ cannot be attained as the exponent of some K-primitive matrix.

Perhaps, a less difficult problem is the following:

Question 6.1. Let $m \geqslant 4$ be a positive integer. Determine the set of integers that can be attained as the exponent of a K-primitive matrix for some n-dimensional polyhedral cone K with m extreme rays, where $3 \leqslant n \leqslant m$.

A natural question related to this work is the following:

Question 6.2. If K is an n-dimensional minimal cone such that the relation for its extreme vectors has p vectors on one side and q vectors on the other side, where $p, q \geqslant 2$, $p + q \leqslant n + 1$, what is $\gamma(K)$?

In this paper we have been able to answer the preceding question for the special case $|p - q| \leqslant 1$. Our success depends on the fact that if A is an exp-maximal primitive matrix for such a minimal cone, then the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ must be given by Fig. 1 or Fig. 2, and in that case we can apply Lemma 2.4. However, when $|p - q| > 1$, our method no longer applies. In that case we *do not* know (at least at present) how to describe the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ — but, definitely, it is different from Fig. 1 or Fig. 2, in view of Theorem 4.1.

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