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MAXIMAL EXPONENTS OF POLYHEDRAL CONES (III)

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ABSTRACT. Let K be a proper (i.e., closed, pointed, full, convex) cone in $\mathbb{R}^n$. An $n \times n$ matrix A is said to be K-primitive if $AK \subseteq K$ and there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. For a polyhedral cone K, the maximum value of $\gamma(A)$, taken over all K-primitive matrices A, is denoted by $\gamma(K)$. It is proved that for any positive integers $m, n, 3 \leq n \leq m$, the maximum value of $\gamma(K)$, as K runs through all n-dimensional polyhedral cones with m extreme rays, equals $(n-1)(m-1) + \frac{1}{2}\left(1 + (-1)^{(n-1)m}\right)$. For the 3-dimensional case, the cones K and the corresponding K-primitive matrices A such that $\gamma(K)$ and $\gamma(A)$ attain the maximum value are identified up to respectively linear isomorphism and cone-equivalence modulo positive scalar multiplication.

1. Introduction

Let K be a proper (i.e., closed, pointed, full, convex) cone in $\mathbb{R}^n$. An $n \times n$ matrix A is said to be *K-primitive* if $AK \subseteq K$ and there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; the least such k is referred to as the *exponent* of A and is denoted by $\gamma(A)$. When $K = \mathbb{R}^n_+$, the nonnegative orthant in $\mathbb{R}^n$, Wielandt's classical result (see [46]) states that the maximum of $\gamma(A)$, as A ranges over all (nonnegative entrywise) $n \times n$ primitive matrices, is $n^2 - 2n + 2$.

Here we consider polyhedral (i.e., finitely generated proper) cones in $\mathbb{R}^n$ having $m(\geq n)$ extreme rays. Given such a cone K, denote by $\gamma(K)$ the maximum of $\gamma(A)$ as A ranges over all K-primitive matrices. Generally, it is very difficult to compute $\gamma(K)$, so it is natural to ask what the maximum value of $\gamma(K)$ is as K ranges over all polyhedral cones in $\mathbb{R}^n$ having m extreme rays. Our main result, Theorem 4.1, determines this value.

This paper is a culmination of work done separately by the first and third authors on one hand, and the second author on the other hand. In earlier work of Loewy and Tam, [21] and [22], it has been shown that if K is a polyhedral cone in $\mathbb{R}^n$ with m extreme rays and A is K-primitive, then $\gamma(A) \leq (m_A - 1)(m-1) + 1$, where m_A is the degree of the minimal polynomial of A. It follows immediately that $\gamma(A) \leq (n-1)(m-1) + 1$. The case $m = n+1$, namely the so-called minimal cones, has been dealt with in detail. It turns out that in this case the maximum value of $\gamma(K)$ as K ranges over all n-dimensional cones with $n+1$ extreme rays is $n^2 - n + 1$ if n is odd and $n^2 - n$ if n is even. The cones K in this class attaining

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the maximum value are characterized, and for those cones K the corresponding K-primitive matrices A such that $\gamma(A)$ attain the value are also determined.

Further work led Loewy and Tam to conjecture that the maximum value of $\gamma(K)$ as K ranges over all n-dimensional polyhedral cones with m extreme rays is given by $(n-1)(m-1)+\frac{1}{2}\left(1+(-1)^{(n-1)m}\right)$, and they proved it in all cases except when n is even and m is odd. Later they were drawn to Grinberg's paper [11], which in turn led them to [28], which gives an English summary of [27], the Ph.D. thesis of Perles (written in Hebrew). It turns out that this conjecture is answered in the affirmative. However, the proof of this result, or any of the other many results in [27], has never been published in a journal paper.

The approach in [21] and [22] is based on a digraph depending on the given cone K and the K-primitive matrix A. This digraph is defined in the next section. The approach in [27] is basically lattice theoretic. Given a lattice L of finite length, certain monotone maps on L are considered. The thesis also deals with projective transformations acting on polytopes and contains many other results, but they are beyond the scope of this paper.

Note that the problem of determining the maximum value of $\gamma(K)$ as K ranges over all polyhedral cones in $\mathbb{R}^n$ with m extreme rays is equivalent to the problem of determining the maximum value of $\gamma(C)$ over all $(n-1)$-polytopes C with m extreme points, having the origin as an interior point, provided that we define $\gamma(C)$ to be the maximum value of the exponents of C-primitive matrices, where a C-primitive matrix and its exponent are defined in the obvious way. This follows from the Perron-Frobenius theory (see the proof of Theorem 5.3(i) of this paper). When C is a symmetric convex body, not necessarily a polytope, in $\mathbb{R}^n$, C can be used to define a norm $\|\cdot\|$ of $\mathbb{R}^n$. In that case $\gamma(C)$ is, in fact, equal to the critical exponent of the induced norm, which is defined as the smallest positive integer κ with the property that $\|A^\kappa\| = \|A\| = 1$ imply $\|A^l\| = 1$ for all positive integers l (or, equivalently, A has spectral radius 1). The critical exponents have been extensively studied by Pták and his collaborators (see [29] or [4, Chapter 2, Section 6 and Section 8]).

Our work on exponents of polyhedral cones can also be considered as a ramification of the geometric spectral theory of nonnegative linear operators, which is a study of the classical Perron-Frobenius theory of a nonnegative matrix and its generalizations to cone-preserving maps in the finite-dimensional setting from a cone-theoretic (geometric) viewpoint (see [45], [35], [41], [42], [43], [37], [38], [44]). For an interesting work that explores the connection between the results of Wielandt and of Perron and Frobenius on primitive matrices, see [19].

The paper is organized as follows. Section 2 contains most of the definitions, together with the relevant known results.

Section 3 is devoted to the 3-dimensional cone case of the problem. It is proved that the maximum value of $\gamma(K)$ as K runs through all 3-dimensional polyhedral cones with m extreme rays is $2m-1$, and for any 3-dimensional polyhedral cone K with m extreme rays and any K-primitive matrix $A, \gamma(A) = 2m-1$ if and only if $(\mathcal{E}, \mathcal{P}(A,K))$, the digraph associated with A, is (up to graph isomorphism) given by Figure 1. (The definition of $(\mathcal{E}, \mathcal{P}(A,K))$ and Figure 1 will be given in Section 2.) Furthermore, for every positive integer $m \geq 3$, we can construct, for every real number $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, a 3-dimensional polyhedral cone K_θ with m extreme rays and a K_θ-primitive matrix A_θ such that the digraph $(\mathcal{E}, \mathcal{P}(A_\theta, K_\theta))$

is given by Figure 1. The construction of the pair (K_θ, A_θ) makes use of roots of a polynomial of the form $t^m - ct - (1-c)$, where $0 < c < 1$. Evidently, there are some connections between our work in Section 3 and the work of Kirkland ([17], [18], [19]), who has considered polynomials of a more general form, namely, those of the form $t^m - \sum_{k=1}^m a_k t^{m-k}$, where $a_1, \dots, a_m$ are nonnegative real numbers with a sum equal to 1.

In the literature there are two main upper bounds for the exponent of a primitive matrix A, which are expressed in terms of the degree of the minimal polynomial and the diameter D of the usual digraph associated with A, namely, $\gamma(A) \leq (m_A - 1)^2 + 1$ and $\gamma(A) \leq D^2 + 1$ (see [26] and [34]). It is commonly agreed that these upper bounds belong to some of the best work in the area of exponents for nonnegative matrices. Unfortunately, it is not possible to extend these upper bounds to the setting of a cone-preserving map over a polyhedral cone. Our result for the 3-dimensional cone case clearly shows that the above upper bound given in terms of the degree of the minimal polynomial is invalid in the setting of a cone-preserving map. Since for a K-primitive matrix A the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ need not be strongly connected, the diameter of such a digraph can be infinite, so the other upper bound is out of the question.

In Section 4 we prove that for every pair of positive integers $m, n, 3 \leq n \leq m$, the maximum value of $\gamma(K)$, as K runs through all n-dimensional polyhedral cones with m extreme rays, equals $(n-1)(m-1)+1$ when m is even or m and n are both odd, and equals $(n-1)(m-1)$ when m is odd and n is even. Our argument relies on a certain geometric fact (Lemma 4.4), for which we offer the proof as given in [27]. In the Appendix we provide an alternative argument, which has independent interest.

In Section 5 we treat the question of uniqueness (up to linear isomorphism) of the cones K that maximize $\gamma(K)$, in the class of n-dimensional polyhedral cones with m extreme rays, and the uniqueness (in the sense of cone-equivalence modulo positive scalar multiplication, to be defined later) of the corresponding K-primitive matrices A whose exponents attain the maximum value for the case $n = 3$. Even for this case there are complications. It is proved that for every positive integer $m \geq 5$, up to linear isomorphism, the 3-dimensional polyhedral cones with m extreme rays that attain the maximum exponent are precisely the cones K_θ's introduced in Section 3, uncountably infinitely many of them. Also, when $m \geq 6$, we have: (1) for each $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, there is (up to multiples) only one K_θ-primitive matrix whose exponent attains the maximum value; (2) the automorphism group of K_θ consists of scalar matrices only; and (3) for any $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1}), \theta_1 \neq \theta_2$, the cones $K_{\theta_1}, K_{\theta_2}$ are not linearly isomorphic. The situation for the case $m = 5$ is more delicate: (3) is still true, but not (1) and (2). In this case the automorphism group of K_θ consists of the identity matrix and an involution P, different from the identity matrix, together with their positive multiples, and for each $\theta \in (\frac{2\pi}{5}, \frac{\pi}{2})$ there are (up to multiples) precisely two K_θ-primitive matrices whose exponent attains the maximum value, namely, A_θ and $P^{-1}A_\theta P$.

If one is interested in only the maximum value of $\gamma(K)$ but not in the uniqueness issue, then one can bypass Section 3. At the end of Section 4 we indicate how this can be done.

In Section 6, the final section, we give some further remarks and a few open questions.

2. Preliminaries

We take for granted standard properties of nonnegative matrices, complex matrices and graphs that can be found in textbooks (see, for instance, [5], [6], [14], [15], [20]). A familiarity with elementary properties of finite-dimensional convex sets, convex cones and cone-preserving maps is also assumed (see, for instance, [1], [30], [36], [47]). To fix notation and terminology, we give some definitions.

A nonempty subset K of a finite-dimensional real vector space V is called a *convex cone* if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \geq 0$; K is *pointed* if $K \cap (-K) = \{0\}$; K is *full* if its interior $\operatorname{int} K$ (in the usual topology of V) is nonempty, equivalently, $K - K = V$. If K is closed and satisfies all of the above properties, K is called a *proper cone.*

In this paper, unless specified otherwise, we always use K to denote a proper cone in the n-dimensional Euclidean space $\mathbb{R}^n$.

We denote by $\geq^K$ the partial ordering of $\mathbb{R}^n$ induced by K, i.e., $x \geq^K y$ if and only if $x - y \in K$.

A subcone F of K is called a *face* of K if $x \geq^K y \geq^K 0$ and $x \in F$ imply $y \in F$. If $S \subseteq K$, we denote by $\Phi(S)$ the *face of K generated by S*, that is, the intersection of all faces of K including S. (Occasionally, we write $\Phi_K(S)$ to indicate the dependence on K.) If $x \in K$, we write $\Phi(\{x\})$ simply as $\Phi(x)$. A vector $x \in K$ is called an *extreme vector* if either x is the zero vector or x is nonzero and $\Phi(x) = \{\lambda x \colon \lambda \geq 0\}$; in the latter case, the face $\Phi(x)$ is called an *extreme ray.* We use $\operatorname{Ext} K$ to denote the set of all nonzero extreme vectors of K. Two nonzero extreme vectors are said to be *distinct* if they are not multiples of each other. The cone K itself and the set $\{0\}$ are always faces of K, known as *trivial faces.* Other faces of K are said to be *nontrivial*.

If S is a nonempty subset of a vector space, we denote by $\operatorname{pos} S$ the *positive hull* of S, i.e., the set of all possible nonnegative linear combinations of vectors taken from S.

By a *polyhedral cone* we mean a proper cone which has finitely many extreme rays. By the *dimension of a proper cone* we mean the dimension of its linear span. A polyhedral cone is said to be *simplicial* if the number of extreme rays is equal to its dimension. The nonnegative orthant $\mathbb{R}^n_+ := \{(\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n \colon \xi_i \geq 0 \ \forall i\}$ is a typical example of a simplicial cone.

We denote by $\pi(K)$ the set of all $n \times n$ real matrices A (identified with linear mappings on $\mathbb{R}^n$) such that $AK \subseteq K$. Members of $\pi(K)$ are said to be *K-nonnegative* and are often referred to as *cone-preserving maps.* It is clear that $\pi(\mathbb{R}^n_+)$ consists of all $n \times n$ (entrywise) nonnegative matrices.

A matrix $A \in \pi(K)$ is said to be *K-irreducible* if A leaves invariant no nontrivial face of K; A is *K-positive* if $A(K \setminus \{0\}) \subseteq \operatorname{int} K$ and is *K-primitive* if there is a positive integer p such that A^p is K-positive. If A is K-primitive, then the smallest positive integer p for which A^p is K-positive is called the *exponent* of A and is denoted by $\gamma(A)$ (or by $\gamma_K(A)$ if the dependence on K needs to be emphasized).

Remark 2.1. The definition of a K-primitive matrix as given in the Preliminaries section of [21], [22] is correct, but in the abstract and introduction the needed assumption that A should be in $\pi(K)$ is missing (erroneously).

A matrix A is said to be *an automorphism* of K if A is invertible and A, A^{-1} both belong to $\pi(K)$ or, equivalently, $AK = K$.

Let $A \in \pi(K)$. In this work we need the digraph $(\mathcal{E}(K), \mathcal{P}(A,K))$, which is one of the four digraphs associated with A introduced by Barker and Tam ([7], [40]). It is defined in the following way: its vertex set is $\mathcal{E}(K)$, the set of all extreme rays of K; $(\Phi(x), \Phi(y))$ is an arc whenever $\Phi(y) \subseteq \Phi(Ax)$. If there is no danger of confusion (in particular, within proofs) we write $(\mathcal{E}(K), \mathcal{P}(A,K))$ simply as $(\mathcal{E}, \mathcal{P}(A,K))$ or $(\mathcal{E}, \mathcal{P})$. This graph was mentioned very briefly in [27], page 18, but was not used there.

For a proper cone K, we say K *has finite exponent* if the set of exponents of K-primitive matrices is bounded. Then we denote the maximum exponent by $\gamma(K)$ and refer to it as the *exponent of* K. If K has finite exponent, then a K-primitive matrix A is said to be *exp-maximal* if $\gamma(A) = \gamma(K)$.

For every pair of positive integers m, n with $3 \le n \le m$, we denote by $\mathcal{P}(m,n)$ the set of all n-dimensional polyhedral cones with m extreme rays. We call a polyhedral cone $K_0 \in \mathcal{P}(m,n)$ an *exp-maximal cone* if $\gamma(K_0) = \max\{\gamma(K) : K \in \mathcal{P}(m,n)\}$.

For any K-nonnegative matrix A, not necessarily K-primitive or K-irreducible, and any $0 \ne x \in K$, by the *local exponent of* A *at* x, denoted by $\gamma(A,x)$, we mean the smallest nonnegative integer k such that $A^k x \in \operatorname{int} K$. If no such k exists, we set $\gamma(A,x)$ equal ∞.

Proper cones K_1, K_2 are said to be *linearly isomorphic* if there exists a linear isomorphism $P\colon \operatorname{span} K_2 \to \operatorname{span} K_1$ such that $PK_2 = K_1$. If $A_1 \in \pi(K_1), A_2 \in \pi(K_2)$ are such that there exists a linear isomorphism P satisfying $PK_2 = K_1$ and $P^{-1}A_1P = A_2$, then we say A_1 and A_2 are *cone-equivalent*.

Fact 2.2. Let K_1, K_2 be proper cones in $\mathbb{R}^n$. Suppose that $A_1 \in \pi(K_1)$ and $A_2 \in \pi(K_2)$ are cone-equivalent. Then:

(i) The digraphs $(\mathcal{E}, \mathcal{P}(A_1,K_1)), (\mathcal{E}, \mathcal{P}(A_2,K_2))$ are isomorphic.
(ii) Either A_1 is K_1-primitive and A_2 is K_2-primitive or they are not, and if they are, then $\gamma_{K_1}(A_1) = \gamma_{K_2}(A_2)$.
(iii) For any $x \in K_2$, $\gamma(A_2,x) = \gamma(A_1,Px)$.

It is clear that if K_1 and K_2 are linearly isomorphic cones, then either K_1, K_2 both have finite exponent or they both do not have, and if they both have, then $\gamma(K_1) = \gamma(K_2)$.

Under inclusion as the partial order, the set of all faces of K, denoted by $\mathcal{F}(K)$, forms a lattice with meet and join given respectively by $F \wedge G = F \cap G$ and $F \vee G = \Phi(F \cup G)$. Proper cones K_1, K_2 are said to be *combinatorially equivalent* if their face lattices $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$ are isomorphic (as lattices).

In what follows when we say the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 (or by other figures), we mean the digraph is given either by the figure up to graph isomorphism or by the figure as a labelled digraph. In most instances, we mean it in the former sense, but in a few instances we mean it in the latter sense. It should be clear from the context in what sense we mean. (For instance, in part (i) of Lemma 2.4 we mean the former sense, but in part (ii) we mean the latter sense.)

We will need the following results which were established in [21, Lemma 4.1, Lemma 4.2, Theorem 5.2(i),(ii), Corollary 5.7]:

Lemma 2.3. *Let $K \in \mathcal{P}(m,n)$ $(3 \le n \le m)$ and let A be a K-primitive matrix. Then the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ equals $m-1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure* 1 *or Figure* 2*, or* (*in the* *case $m = n = 3$) by the digraph of order 3 whose arcs are precisely all possible arcs between every pair of distinct vertices*:

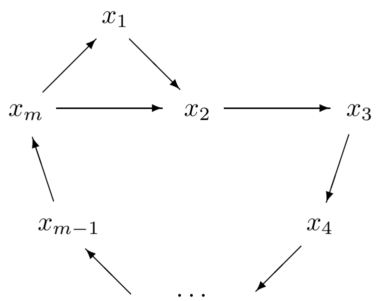

FIGURE 1

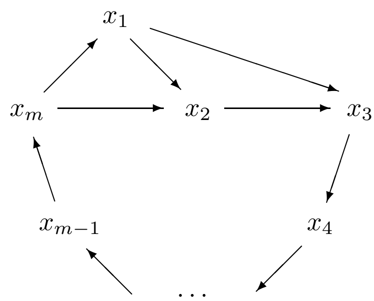

FIGURE 2

(*For simplicity, we label the vertex $\Phi(x_i)$ simply by x_i.*).

Digraphs isomorphic to Figure 1 and Figure 2 are sometimes referred to respectively as the Wielandt digraph and the near-Wielandt digraph. They have appeared in the study of tournaments.

Recall that an $n \times n$ complex matrix A is said to be *nonderogatory* if every eigenvalue of A has geometric multiplicity 1 or, equivalently, if the minimal and characteristic polynomials of A are identical. (See, for instance, [15, Theorem 3.3.15].)

Lemma 2.4. *Let $K \in \mathcal{P}(m,n)$ $(3 \leq n \leq m)$. Let A be a K-nonnegative matrix. Suppose that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 or Figure 2. Then*:

(i) *A is K-primitive, nonsingular, nonderogatory, and has a unique annihilating polynomial of the form $t^m - ct - d$, where $c, d > 0$.*

(ii) *$\gamma(A)$ equals $\gamma(A, x_1)$ or $\gamma(A, x_2)$, depending on whether the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 or Figure 2. In either case, $\max_{1 \leq i \leq m} \gamma(A, x_i)$ is attained at precisely one i.*

For a square matrix A, we denote by m_A the *degree of the minimal polynomial of A*.

Theorem 2.5. *Let $K \in \mathcal{P}(m,n)$, where $m \geq 4$, and let A be a K-primitive matrix. Then*:

(i) *$\gamma(A) \leq (m_A - 1)(m-1) + 1$, where the equality holds only if the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1, in which case $\gamma(A) = (n-1)(m-1)+1$.*

(ii) *$\gamma(A) = (m_A - 1)(m-1)$ only if either $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 or Figure 2, in which case $\gamma(A) = (n-1)(m-1)$, or $m_A = 3$.*

Corollary 2.6. *For any $K \in \mathcal{P}(m,n)$, we have $\gamma(K) \leq (n-1)(m-1)+1$. The equality holds only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1.*

We would like to mention that the inequality (and its equality case in an equivalent form, without introducing a digraph) of Corollary 2.6 was obtained in [27, Theorems 2.28, 2.29] in the context of a join-endomorphism of a lattice. Later Grinberg [11, Theorems 4, 6] extended the result to the setting of a monotone mapping on a partially ordered set.

We will also need the following known result ([13, Lemma 5.3]):

Theorem 2.7. *Let $\{x_1, \dots, x_n\}$ be a basis for $\mathbb{R}^n$. Let $x_0 = \sum_{i=1}^n \alpha_i x_i$, where each α_i is different from 0. Let A and B be $n \times n$ real matrices. Suppose that A is nonsingular and also that Bx_j is a multiple of Ax_j for $j = 0, 1, \dots, n$. Then B is a multiple of A.*

3. The 3-dimensional case

The reader may skip this section if he/she is not interested in the uniqueness issue for the case $n = 3$, which is treated in Section 5. For an explanation, see the last paragraph of Section 4.

Two distinct extreme rays $\Phi(x), \Phi(y)$ (or, distinct extreme vectors x, y) of K are said to be *neighborly* if $x + y \in \partial K$.

Lemma 3.1. *Let K be a 3-dimensional polyhedral cone with extreme vectors $x_1, \dots, x_m$, where $m \geq 3$. Let $A \in \pi(K)$ and suppose that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2. Then:*

(i) *For $i = 1, \dots, m, \Phi(x_i)$ and $\Phi(x_{i+1})$ (where $\Phi(x_{m+1})$ is taken to be $\Phi(x_1)$) are neighborly extreme rays of K.*
(ii) *$\gamma(A)$ equals $2m - 1$ or $2m - 2$, depending on whether $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or by Figure 2.*

Note that for $i = 1, \dots, m, \Phi(x_i)$ and $\Phi(x_{i+1})$ are adjacent vertices of the digraph $(\mathcal{E}, \mathcal{P})$ (when it is given by Figure 1 or Figure 2). However, it is not clear that $\Phi(x_i)$ and $\Phi(x_{i+1})$ are neighborly extreme rays of the cone K.

Proof of Lemma 3.1. First, we consider the case when the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. It is not difficult to establish the following:

Assertion. *Let C be a convex polygon in $\mathbb{R}^2$ with extreme points $w_1, \dots, w_m$ and edges $\overline{w_i w_{i+1}}, i = 1, \dots, m$, where w_{m+1} is taken to be w_1. Let $\tilde{C}$ be the polygon with extreme points $w_1', w_2, w_3, \dots, w_m$, where $w_1' = (1 - \lambda)w_1 + \lambda w_2$ for some $\lambda, 0 < \lambda < 1$. Then the edges of $\tilde{C}$ are $\overline{w_1' w_2}, \overline{w_i w_{i+1}}, i = 2, \dots, m - 1$, and $\overline{w_m w_1'}$.*

The fact that $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 implies that Ax_i is a positive multiple of x_{i+1} for $i = 1, \dots, m - 1$ and, moreover, $\Phi(Ax_m)$ is a 2-dimensional face of K with extreme rays $\Phi(x_1)$ and $\Phi(x_2)$. Hence, $\Phi(x_1), \Phi(x_2)$ are neighborly extreme rays of K and we have $Ax_m = \alpha_1 x_1 + \alpha_2 x_2$ for some positive numbers α_1, α_2. So AK is generated by the (pairwise distinct) extreme vectors $x_1', x_2, x_3, \dots, x_m$, where $x_1' := Ax_m$. Using an equivalent formulation of the above assertion in terms of 3-dimensional polyhedral cones, we see that for all $i, j, 2 \leq i, j \leq m$, x_i, x_j are neighborly extreme vectors of AK if and only if they are neighborly extreme vectors of K. (Note that here we are not using (i), something that we have not yet established.)

By Lemma 2.4(i) A is nonsingular, so the cones K and AK are linearly isomorphic under A. Since x_1, x_2 are neighborly extreme vectors of K, Ax_1, Ax_2 are neighborly extreme vectors of AK. But Ax_1, Ax_2 are respectively positive multiples of x_2 and x_3, so x_2, x_3 are neighborly extreme vectors of AK and, in view of what we have done above, x_2, x_3 are also neighborly extreme vectors of K. By repeating the argument, we can show that for $i = 2, 3, \dots, m - 1$, x_i and x_{i+1} (also, x_i and

x_{i-1}) are neighborly extreme vectors of K. It is clear that the remaining extreme vector x_1 is neighborly to x_m (and x_2).

By direct calculation, $A^{2(m-1)}x_1$ is a positive linear combination of x_{m-1} and x_m, and as x_{m-1} and x_m are neighborly extreme vectors, $A^{2(m-1)}x_1 \in \partial K$. On the other hand, $A^{2m-1}x_1$ is a positive linear combination of x_m, x_1 and x_2, so it belongs to $\operatorname{int} K$. This shows that $\gamma(A, x_1) = 2m-1$. In view of Lemma 2.4(ii), we have $\gamma(A) = \gamma(A, x_1) = 2m-1$.

When the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, we employ a similar argument. For convenience, we denote the extreme vectors of AK by $y_1, \ldots, y_m$, where $y_1 = Ax_m, y_2 = Ax_1$ and $y_i = x_i$ for $i = 3, \ldots, m$. In this case, y_1 (respectively, y_2) is a positive linear combination of x_1 and x_2 (respectively, x_2 and x_3), and the extreme vectors x_2, x_1 of K are neighborly, and so are x_2 and x_3. Moreover, K and AK are still linearly isomorphic under A. Using an assertion similar to the one given above, we can show that for $i, j = 3, \ldots, m, y_i, y_j$ are neighborly extreme vectors of AK if and only if x_i, x_j are neighborly extreme vectors of K. Inductively we can show that x_i, x_{i+1} (also, x_i and x_{i-1}) are neighborly extreme vectors of K for $i = 3, \ldots, m-1$. Also, we can conclude that the remaining extreme vector x_1 of K is neighborly to x_m (and x_2).

By direct calculation, $A^{2m-3}x_2$ is a positive linear combination of x_{m-1} and x_m, whereas $A^{2m-2}x_2$ is a positive linear combination of x_m, x_1 and x_2; so $A^{2m-3}x_2 \in \partial K$ and $A^{2m-2}x_2 \in \operatorname{int} K$. By Lemma 2.4(ii) we have $\gamma(A) = \gamma(A, x_2) = 2m-2$. $\square$

According to Lemma 2.4(i), for any $K \in \mathcal{P}(m,n), A \in \pi(K)$, if $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 or Figure 2, then A has an annihilating polynomial of the form $t^m - ct - d$, where $c, d > 0$. Note that if, in addition, the spectral radius $\rho(A)$ of A equals 1, then necessarily $c + d = 1$. Conversely, if A is K-nonnegative and has an annihilating polynomial of the form $t^m - ct - (1-c)$, where $0 < c < 1$, then necessarily $\rho(A) = 1$. This follows from an application of the Perron-Frobenius theory to A, because then 1 is the only positive real root of the polynomial, in view of Descartes' rule of signs, which says that the number of positive roots of a polynomial either is equal to the number of its variations of sign or is less than that number by an even integer, a root of multiplicity k being counted as k roots. (The fact that 1 is the only positive real root of the polynomial can also be shown directly by proper factorization.)

We will see that in constructing examples of K and A such that $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1, polynomials of the form $t^m - ct - (1-c)$, where $0 < c < 1$, play a role. In our next result, we study the roots of a polynomial of said form.

Lemma 3.2. *Consider the polynomial*

$$h(t) = t^m - ct - (1-c),$$

where $m \geq 3$ and $c \in (0,1)$.

(i) *The roots of $h(t)$ are all simple unless m is odd and $c = c_m$, where c_m denotes the unique real root in $(0,1)$ of the equation*

$$\frac{(m-1)^{m-1}}{m^m} t^m = (t-1)^{m-1}.$$

(ii) *When m is even, $h(t)$ has precisely one real root other than 1. When m is odd, besides the root 1, $h(t)$ has precisely two real roots if $c > c_m$, one real* *root (which is a double root) if $c = c_m$ and no real roots if $c < c_m$. In all cases, each real root of $h(t)$ other than 1 lies in $(-1, 0)$.*

(iii) *When $m \geq 4$ or when $m = 3$ and $c < \frac{3}{4}(= c_3)$, the polynomial $h(t)$ has a unique complex root of the form $re^{i\theta}$, where $r > 0$ and $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Moreover, r and θ are related in that r is the unique positive real root, which is less than 1, of the polynomial $g_\theta(t)$ given by*

$$g_\theta(t) = \frac{\sin(m-1)\theta}{\sin\theta} t^m - \frac{\sin m\theta}{\sin\theta} t^{m-1} + 1.$$

Three remarks are in order. First, by Descartes' rule of signs, one can show that the polynomial $h(t)$, considered in the lemma, has exactly one positive real root, and also that it has exactly one negative real root if m is even and either two (counting multiplicities) or no negative real roots if m is odd. Moreover, since the coefficients of the polynomial $h(-t-1)$ are all positive if m is even and all negative if m is odd, the polynomial $h(t)$ always has no real root less than -1. Of course, this agrees with part (ii) of the lemma, but the lemma contains more information. Second, actually every root of $h(t)$ other than 1 has modulus strictly less than 1. Here is a one-line proof: If λ is a root of $h(t)$, then $|\lambda|^m = |c\lambda + (1-c)| \leq c|\lambda| + (1-c)$, which is possible only if $\lambda = 1$ or $|\lambda| < 1$. Third, part (iii) of the lemma follows from properly combining Theorem 2 in [18] and Lemma 3 in [17], at least for $m \geq 4$. (The assumptions made in [18], namely (2.2) there, seems to rule out the case $m = 3$ of our theorem. In the notation of [18] we have $d = n, k = 1, s = 1$ and $n = m$.) We give a proof for the sake of completeness.

Proof of Lemma 3.2. (i) Clearly, the roots of $h'(t)$ are precisely all the $(m-1)$th roots of $\frac{c}{m}$. A little calculation shows that if t_0 is a common root of $h(t)$ and $h'(t)$, then $\frac{ct_0}{m} = t_0^m = ct_0 + (1-c)$ and so $ct_0(\frac{1}{m} - 1) = 1 - c$, which implies that t_0 is the negative $(m-1)$th real root of $\frac{c}{m}$ and m is odd. So $h(t)$ and $h'(t)$ have a common root if and only if m is odd and the negative $(m-1)$th real root of $\frac{c}{m}$ is a root of $h(t)$. By calculation one finds that the latter condition, in turn, is equivalent to the condition that m is odd and c is a root of the polynomial $\alpha_m t^m - (t-1)^{m-1}$, where $\alpha_m = \frac{(m-1)^{m-1}}{m^m}$. When m is odd, by considering the said polynomial and its derivative we readily show that the polynomial has a unique real root in $(0, 1)$, which we denote by c_m. So we can conclude that the roots of $h(t)$ are all simple unless m is odd and $c = c_m$.

(ii) Rewriting $h(t)$, we have, $h(t) = (t^m - 1) - c(t-1) = (t-1)(p(t) - c)$, where $p(t) = t^{m-1} + t^{m-2} + \cdots + t + 1$. So 1 is always a root of $h(t)$, and for any complex number $w \neq 1$, w is a root of $h(t)$ if and only if w is a root of the equation $p(t) = c$.

When m is even, a consideration of the derivative of $p(t)$ shows that $p(t)$ is a strictly increasing function on the real line. But $p(-1) = 0, p(0) = 1$ and $c \in (0, 1)$, so the equation $p(t) = c$ has exactly one real root, and that real root belongs to $(-1, 0)$. Hence $h(t)$ has precisely one real root other than 1, and this root lies in $(-1, 0)$.

When m is odd, on the real line $p(t)$ is a strictly convex function, since its second derivative always takes positive values, as can be shown by some calculation. It is straightforward to show that a complex number z_0 is a common root of $h(t)$ and $h'(t)$ if and only if $p(z_0) = c$ and $p'(z_0) = 0$. On the other hand, by part (i) and its proof, $h(t)$ and $h'(t)$ have a common root if and only if $c = c_m$. In that case, the common root is unique and is equal to the negative $(m-1)$th real root of $\frac{c_m}{m}$. So

c_m is, in fact, the absolute minimum value of $p(t)$. Hence, the equation $p(t) = c$ has two distinct real roots if $c > c_m$, one real root (which is a double root) if $c = c_m$, and no real roots if $c < c_m$. As $p(-1) = p(0) = 1$ and $p(t)$ is strictly convex, each real root of the equation $p(t) = c$ or, in other words, each real root of $h(t)$ other than 1 must belong to $(-1, 0)$.

(iii) First, we establish the uniqueness of the root of $h(t)$ in the desired polar form. Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, where $r_1, r_2 > 0$ and $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, be two different roots of $h(t)$. Then from $z_1^m - cz_1 - (1-c) = 0$ and $z_2^m - cz_2 - (1-c) = 0$ we obtain $(z_1 - z_2)(\sum_{k=0}^{m-1} z_1^{m-1-k} z_2^k) = c(z_1 - z_2)$, which implies that $c = \sum_{k=0}^{m-1} z_1^{m-1-k} z_2^k$. For any nonzero complex number z, denote by $\arg(z)$ the argument of z that belongs to $(0, 2\pi]$. Since $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, we have $\arg(z_1^{m-1-k} z_2^k) \in (\frac{2\pi(m-1)}{m}, 2\pi)$ for $k = 0, \ldots, m-1$. Thus, for each k, the complex number $z_1^{m-1-k} z_2^k$ belongs to the relative interior of the convex cone in the complex plane generated by 1 and $e^{-\frac{2\pi}{m} i}$, and hence so does the sum $\sum_{k=0}^{m-1} z_1^{m-1-k} z_2^k$, which is a contradiction as c is a positive real number.

Next, we contend that for any real number $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, the polynomial $g_\theta(t)$ has a unique positive real root and this root is less than 1.

We have $g_\theta(0) = 1 > 0$ and

$$g_\theta(1) = \frac{\sin(m-1)\theta - \sin m\theta}{\sin\theta} + 1 = -\frac{\cos(m - \frac{1}{2})\theta}{\cos\frac{\theta}{2}} + 1 < 0,$$

where the second equality follows from the trigonometric identity $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ and the inequality holds as $2\pi - \frac{1}{2}\theta < (m - \frac{1}{2})\theta < 2\pi + \frac{\theta}{2}$ and $0 < \frac{\theta}{2} < \frac{\pi}{2}$. In addition, we also have

$$g'_\theta(t) = \frac{t^{m-2}}{\sin\theta}\,[mt\sin(m-1)\theta - (m-1)\sin m\theta] < 0$$

for all $t \in (0, \infty)$, as $\sin m\theta > 0$ and $\sin(m-1)\theta < 0$. So it is clear that the polynomial $g_\theta(t)$ has a unique positive real root and this root is less than 1. (The latter assertion can also be established by using Descartes' rule of signs.)

Now define a real-valued function ζ on $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$ by $\zeta(\theta) = r_\theta^{m-1}\frac{\sin m\theta}{\sin\theta}$, where r_θ denotes the unique positive real root of the polynomial $g_\theta(t)$. Note that ζ is a continuous function, as r_θ depends on θ continuously. Also, we have $0 < \zeta(\theta) < 1$, as $2\pi < m\theta < 2\pi + \theta$ and $0 < r_\theta < 1$.

It is readily checked that a complex number $re^{i\theta}$ (in polar form) is a root of the polynomial $h(t)$ if and only if we have

$$(1-c) + cr\cos\theta = r^m\cos m\theta, \tag{3.1}$$

$$cr\sin\theta = r^m\sin m\theta. \tag{3.2}$$

Rewriting (3.2) and adding $\cos\theta$ times (3.2) to $-\sin\theta$ times (3.1) (and noting that $\sin\theta, \cos\theta \neq 0$ for $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$), we find that (3.1) and (3.2) hold if and only if

$$c = r^{m-1}\frac{\sin m\theta}{\sin\theta} = \frac{r^m\sin(m-1)\theta}{\sin\theta} + 1.$$

So, for any $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, $r_\theta e^{i\theta}$ is a root of $h(t)$ if $c = \zeta(\theta)$. To complete our proof, it remains to show that the function ζ maps the open interval $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$ onto $(0, 1)$ when $m \geq 4$ and onto $(0, \frac{3}{4})$ when $m = 3$.

Note that the function ζ is one-to-one. Otherwise, there exist $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, $\theta_1 \neq \theta_2$, such that $\zeta(\theta_1) = \zeta(\theta_2)$. However, then $r_{\theta_1}e^{i\theta_1}$ and $r_{\theta_2}e^{i\theta_2}$ are distinct roots of the polynomial $t^m - \zeta(\theta_1)t - (1-\zeta(\theta_1))$, both with argument belonging to $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$, which contradicts what we have obtained at the beginning of the proof of part (iii). As ζ is one-to-one and continuous, it is either strictly increasing or strictly decreasing on $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$. In view of the intermediate value property of a real-valued continuous function, it suffices to show that $\lim_{\theta \to \frac{2\pi}{m}^+} \zeta(\theta) = 0$ and $\lim_{\theta \to \frac{2\pi}{m-1}^-} \zeta(\theta)$ equals 1 when $m \geq 4$ and equals $\frac{3}{4}$ when $m = 3$.

By the definition of ζ and the fact that $0 < r_\theta < 1$, it is readily seen that we have $\lim_{\theta \to \frac{2\pi}{m}^+} \zeta(\theta) = 0$. On the other hand, from the condition $g_\theta(r_\theta) = 0$ we also have $\zeta(\theta) = \frac{r_\theta^m \sin\,(m-1)\theta}{\sin\theta} + 1$, which implies that $\lim_{\theta \to \frac{2\pi}{m-1}^-} \zeta(\theta) = 1$, provided that $m \geq 4$. When $m = 3$, the polynomial equation given in (i) becomes $\frac{2^2}{3^3}t^3 = (t-1)^2$. Since $\frac{3}{4}$ is a root of the latter equation, we have $c_3 = \frac{3}{4}$. Note that the polynomial $g_\theta(t)$ tends to $-2t^3 - 3t^2 + 1$ as θ tends to π from the left. Also, $\frac{1}{2}$ is a root of the latter polynomial. So $\lim_{\theta\to\pi^-} r_\theta = \frac{1}{2}$, and we have

$$\lim_{\theta\to\pi^-} \zeta(\theta) = \lim_{\theta\to\pi^-} r_\theta^2 \frac{\sin 3\theta}{\sin\theta} = \frac{3}{4},$$

as desired. □

In the course of the proof of Lemma 3.2 we have also established the following (see also [17, Lemma 3]):

Corollary 3.3. (i) *For any $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1}), r_\theta e^{i\theta}$, where r_θ is the unique positive real root of the polynomial $g_\theta(t)$ given in Lemma* 3.2*, is a root of the polynomial $t^m - ct - (1-c)$ where $c = r_\theta^{m-1}\frac{\sin m\theta}{\sin\theta}$. Moreover, the pair (r_θ, θ) satisfies the relations* (3.1) *and* (3.2)*, which appear in the proof of Lemma* 3.2*.*

(ii) *For every positive integer $m \geq 4$* (*respectively, $m = 3$*)*, every real number c in $(0,1)$* (*respectively, in $(0, \frac{3}{4})$*) *can be expressed uniquely as $r_\theta^{m-1}\frac{\sin m\theta}{\sin\theta}$, where $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$.*

Theorem 3.4. (i) *For every positive integer $m \geq 3$,*

$$\max\{\gamma(K) : K \in \mathcal{P}(m,3)\} = 2m-1.$$

(ii) *For every $K \in \mathcal{P}(m,3)$, K is exp-maximal if and only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure* 1*.*

(iii) *Let $m \geq 3$ be a positive integer. For any $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, let r_θ denote the unique positive real root of the polynomial $g_\theta(t)$ as defined in Lemma* 3.2(iii)*. Let K_θ be the polyhedral cone in $\mathbb{R}^3$ generated by the vectors*

$$x_j(\theta) := \begin{bmatrix} r_\theta^{j-1}\cos(j-1)\theta \\ r_\theta^{j-1}\sin(j-1)\theta \\ 1 \end{bmatrix}, \ j = 1, \ldots, m.$$

Also, let $A_\theta = r_\theta \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \oplus [1]$. Then $x_1(\theta), \ldots, x_m(\theta)$ are the extreme vectors of K_θ, K_θ is an exp-maximal polyhedral cone and A_θ is an exp-maximal K_θ-primitive matrix.

Note that, for simplicity, we suppress the dependence of K_θ on m.

Proof. For every $K \in \mathcal{P}(m,3)$, by Corollary 2.6 (with $n = 3$) we have $\gamma(K) \leq 2m-1$, where the equality holds only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. In view of Lemma 3.1(ii), parts (i) and (ii) will follow if we can construct a polyhedral cone $K \in \mathcal{P}(m,3)$ for which there exists a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1. To complete the proof, we are going to establish part (iii) and at the same time show that the digraph $(\mathcal{E}, \mathcal{P}(A_\theta, K_\theta))$ is given by Figure 1.

Consider any fixed $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Hereafter we denote r_θ simply by r. By Corollary 3.3(i), $re^{i\theta}$ is a root of the polynomial $t^m - ct - (1-c)$, where $c = r^{m-1}\frac{\sin m\theta}{\sin\theta}$. Consider the m points $y_1, \ldots, y_m$ in $\mathbb{R}^2$ given by

$$y_j = (r^{j-1}\cos(j-1)\,\theta, r^{j-1}\sin(j-1)\,\theta)^T \text{ for } j = 1, \ldots, m.$$

Take B to be the 2×2 matrix $r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Note that $y_j = B^{j-1}y_1$ for $j = 2, \ldots, m$. By Corollary 3.3(i) the equations (3.1) and (3.2), which appear in the proof of Lemma 3.2, are satisfied. So we have

$$By_m = \begin{pmatrix} r^m\cos m\theta \\ r^m \sin m\theta \end{pmatrix} = c\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} + (1-c)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = cy_2 + (1-c)y_1.$$

Let C denote the convex polygon $\operatorname{conv}\{y_1, \ldots, y_m\}$. It is clear that we have $K_\theta = \operatorname{pos}(C \times \{1\})$ and $A_\theta = B \oplus [1]$. In the above we have established that $BC \subseteq C$, so A_θ is K_θ-nonnegative. We contend that $x_1(\theta), \ldots, x_m(\theta)$ are all the extreme vectors of K_θ or, equivalently, $y_1, \ldots, y_m$ are all the extreme points of C.

Clearly the extreme points of C are among $y_1, \ldots, y_m$. Since the Euclidean norm of y_1 is 1, whereas that of y_j, for $j = 2, \ldots, m$, is less than 1, y_1 is certainly an extreme point of C. If y_2 is not an extreme point, then the same holds for $y_3 = By_2, \ldots, y_m = B^{m-2}y_2$, so y_1 is the only extreme point of C, which is impossible. In view of the definition of the y_i's, it is clear that none of the points $y_3, \ldots, y_m$ can be written as a positive linear combination of y_1 and y_2. (This can be seen, for instance, by considering the arguments of the complex numbers corresponding to these points.) So the line segment $\overline{y_1y_2}$ forms a side of the polygon C. If $y_3, y_4, \ldots, y_m$ are not all extreme points of C, let $3 \leq k \leq m$ be the smallest integer such that y_k is not extreme. Then, applying powers of B on y_k, we conclude that $y_{k+1}, \ldots, y_m$ are also not extreme. Hence $y_1, \ldots, y_{k-1}$ are all the extreme points of C. Since $B^{m-k+1}y_k = cy_2 + (1-c)y_1$ lies on the side $\overline{y_1y_2}$, necessarily y_k belongs to a side of C, so there exist integers $i, j, 1 \leq i < j < k$ such that $y_k = \alpha y_i + (1-\alpha)y_j$, where $0 < \alpha < 1$. Applying B^{m-k+1} to the preceding relation, we obtain

$$(1-c)y_1 + cy_2 = \alpha y_{m-k+i+1} + (1-\alpha)y_{m-k+j+1}.$$

Since $\overline{y_1y_2}$ is a side of C, we have, $y_{m-k+j+1} \in \overline{y_1y_2}$, which is a contradiction, as $3 \leq m-k+j+1 \leq m$.

In the above we have proved that $\Phi(x_1(\theta)), \ldots, \Phi(x_m(\theta))$ are all the extreme rays of K_θ. At the same time, we have also shown that $\Phi(x_1(\theta))$ and $\Phi(x_2(\theta))$ are neighborly extreme rays. By noting that the digraph $(\mathcal{E}, \mathcal{P}(A_\theta, K_\theta))$ is given by Figure 1, we complete the proof. □

In contrast to Theorem 3.4, we have the following:

Theorem 3.5. *For every positive integer $m \geq 5$ there exists a 3-dimensional polyhedral cone K with m extreme rays for which there is no K-primitive matrix A with the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ given by Figure 1.*

Proof. Let K be the polyhedral cone in $\mathbb{R}^3$ with extreme vectors

$$y_j = (\cos \frac{2j\pi}{m}, \sin \frac{2j\pi}{m}, 1)^T, j = 1, \dots, m.$$

We contend that there is no K-primitive matrix A for which $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, where $x_1, x_2, \dots, x_m$ is a rearrangement of $y_1, \dots, y_m$.

We assume to the contrary that there is one such A. By Lemma 3.1, for $i = 1, \dots, m$, x_i and x_{i+1} are neighborly extreme vectors of K (where x_{m+1} is taken to be x_1). On the other hand, for each j, the extreme vectors neighborly to y_j are y_{j+1} and y_{j-1}. [We adopt the convention that for each integer j, y_j equals y_k where k is the unique integer that satisfies $1 \leq k \leq m, k \equiv j \pmod{m}$.] Suppose $x_1 = y_{j_1}$, where $1 \leq j_1 \leq m$. Since x_2 is neighborly to x_1, x_2 must be either y_{j_1+1} or y_{j_1-1}. First consider the case when $x_2 = y_{j_1+1}$. Since x_3 is neighborly to x_2, it is equal to either y_{j_1+2} or y_{j_1}. However, we have already had $x_1 = y_{j_1}$, so x_3 must be y_{j_1+2}. Continuing the argument, we can show that $x_j = y_{j_1+j-1}$ for $j = 1, \dots, m$. Then we take $\hat{A}$ to be $\begin{bmatrix} \cos \frac{2\pi}{m} & -\sin \frac{2\pi}{m} \\ \sin \frac{2\pi}{m} & \cos \frac{2\pi}{m} \end{bmatrix} \oplus [1]$. If $x_2 = y_{j_1-1}$, we can show in a similar manner that $x_j = y_{j_1-j+1}$ for $j = 1, \dots, m$. In this case, we take $\hat{A}$ to be $\begin{bmatrix} \cos \frac{2\pi}{m} & \sin \frac{2\pi}{m} \\ -\sin \frac{2\pi}{m} & \cos \frac{2\pi}{m} \end{bmatrix} \oplus [1]$. In any case, $\hat{A}$ satisfies $\hat{A}x_i = x_{i+1}$ for $i = 1, \dots, 4$. On the other hand, since the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 and $m \geq 5$, Ax_i is a positive multiple of x_{i+1} for $i = 1, 2, 3, 4$. So $\hat{A}x_i$ is a positive multiple of Ax_i for $i = 1, \dots, 4$. By Theorem 2.7 it follows that A and $\hat{A}$ are positive multiples of each other. So we arrive at a contradiction, as $\hat{A}$ is clearly not K-primitive. $\square$

As we will see, Theorem 3.5 is superseded by Theorem 5.3(i), which implies that for every positive integer $m \geq 5$, for almost every $K \in \mathcal{P}(m, 3)$, there do not exist K-primitive matrices A for which $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1.

Let K_1, K_2 be linearly isomorphic proper cones. If D is a digraph that can be realized as $(\mathcal{E}, \mathcal{P}(A_1, K_1))$ for some K_1-nonnegative matrix A_1, then clearly (up to graph isomorpism) D can also be realized as $(\mathcal{E}, \mathcal{P}(A_2, K_2))$ for some K_2-nonnegative matrix A_2. On the other hand, if K_1, K_2 are assumed to be combinatorially equivalent only, then the same cannot be said.

Remark 3.6. Let K_1, K_2 be combinatorially equivalent proper cones. Then:

(i) If G is a digraph such that $G = (\mathcal{E}(K_1), \mathcal{P}(A_1, K_1))$ for some K_1-primitive matrix A_1, then there need not exist a K_2-primitive matrix A_2 such that $(\mathcal{E}(K_2), \mathcal{P}(A_2, K_2))$ is isomorphic with G.
(ii) The values of $\gamma(K_1), \gamma(K_2)$ need not be the same.

Since any two 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent, the preceding remark follows from Theorem 3.4 and Theorem 3.5.

4. The higher-dimensional case

In this section we are going to establish the following main result of this paper.

Theorem 4.1. *For every pair of positive integers $m, n, 3 \leq n \leq m$, we have*

$$\max\{\gamma(K) : K \in \mathcal{P}(m,n)\} = (n-1)(m-1) + \frac{1}{2}\left(1 + (-1)^{(n-1)m}\right).$$

According to Corollary 2.6, the inequality $\max\{\gamma(K) : K \in \mathcal{P}(m,n)\} \leq (n-1)(m-1) + \frac{1}{2}\left(1+(-1)^{(n-1)m}\right)$ is valid when m is even or m and n are both odd. For this inequality, it remains to prove the following:

Lemma 4.2. *Let m, n be positive integers with n even, m odd, $4 \leq n \leq m$. For every $K \in \mathcal{P}(m,n)$, we have $\gamma(K) \leq (n-1)(m-1)$.*

Proof. Assume to the contrary that there exists a pair of positive integers m, n with n even, m odd, $4 \leq n \leq m$, such that $\gamma(K) = (n-1)(m-1)+1$. Let A be a K-primitive matrix that satisfies $\gamma(A) = (n-1)(m-1)+1$. According to Theorem 2.5(i), we have $m_A = n$, and the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1. In view of Lemma 2.4(ii), $\gamma(A, x_1) = \gamma(A)$, so $A^{(n-1)(m-1)}x_1 \in \partial K$. We need a formula for $\Phi(A^k x_1)$ for $k = 1, \ldots, (n-1)(m-1)+1$. Clearly, every such integer k can be expressed uniquely as $i(m-1)+j$, where i, j are integers with $0 \leq i \leq n-1, 1 \leq j \leq m-1$. We want to show that for every pair of integers $i, j, 0 \leq i \leq n-1, 1 \leq j \leq m-1$, we have

$$\Phi(A^{i(m-1)+j}x_1) = \begin{cases} \Phi(x_{m-(i-j-1)} + \cdots + x_m + x_1 + \cdots + x_{j+1}) & \text{for } j < i, \\ \Phi(x_{j-i+1} + \cdots + x_{j+1}) & \text{for } j \geq i. \end{cases} \tag{4.1}$$

We proceed by induction on k. The formula clearly holds for $k = 1$. Consider any positive integer $k \geq 2$. Assume that the formula holds for every positive integer less than k. We make use of the fact that $\Phi(A^{l+1}w) = \Phi(A\Phi(A^l w))$ for any positive integer l and any vector $w \in K$, and calculate $\Phi(A^{i(m-1)+(j-1)}x_1)$ by the induction assumption, dividing our argument into four cases: $1 = j < i$; $2 \leq j < i$; $j > i$ and $j = i$. For instance, for the first case, by the induction assumption we have $\Phi(A^{i(m-1)}x_1) = \Phi(A^{(i-1)(m-1)+m-1}x_1) = \Phi(x_{m-i+1} + \cdots + x_m)$, so $\Phi(A^{i(m-1)+1}x_1) = \Phi(A\Phi(x_{m-i+1} + \cdots + x_m)) = \Phi(x_{m-i+2} + \cdots + x_m + x_1 + x_2)$, as desired. Similarly, we can handle the other cases.

Note that the above derivation of formula (4.1) is always valid, irrespective of the parities of m, n.

One consequence of (4.1) is that we have

$$\Phi(A^{(n-2)m}x_1) = \Phi(A^{(n-2)(m-1)+(n-2)}x_1) = \Phi(x_1 + \cdots + x_{n-1}).$$

Since $A^{(n-1)(m-1)}x_1 \in \partial K$ and $A^{(n-1)(m-1)}x_1 = A^{m-n+1}(A^{(n-2)m}x_1)$, necessarily $A^{(n-2)m}x_1 \in \partial K$. The fact that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 implies that Ax_i is a positive multiple of x_{i+1} for $i = 1, \ldots, n-1$, that $Ax_m = \alpha x_1 + \beta x_2$ for some positive scalars α, β, that A is nonsingular, and also that $\{x_1, Ax_1, \ldots, A^{n-1}x_1\}$ forms a basis for $\mathbb{R}^n$ (cf. the proof of [21, Lemma 4.2]). As $\Phi(A^{(n-2)m}x_1)$ contains the vectors $x_1, \ldots, x_{n-1}$, it follows that it is an $(n-1)$-dimensional face of K. Hence span$\{x_1, \ldots, x_{n-1}\}$ is a supporting hypersubspace for K. Now we have the following calculations on determinants:

$$\begin{aligned} & \operatorname{sgn}(|A|)\operatorname{sgn}(\left| x_m \quad x_1 \quad \cdots \quad x_{n-1} \right|) = \operatorname{sgn}(\left| Ax_m \quad Ax_1 \quad \cdots \quad Ax_{n-1} \right|) \\ = \; & \operatorname{sgn}(\left| \alpha x_1 + \beta x_2 \quad x_2 \quad \cdots \quad x_n \right|) = \operatorname{sgn}(\left| x_1 \quad x_2 \quad \cdots \quad x_n \right|) \\ = \; & (-1)^{n-1}\operatorname{sgn}(\left| x_n \quad x_1 \quad \cdots \quad x_{n-1} \right|) \neq 0. \end{aligned}$$

As the determinants $\left| x_m \quad x_1 \quad \cdots \quad x_{n-1} \right|$ and $\left| x_n \quad x_1 \quad \cdots \quad x_{n-1} \right|$ are both nonzero by the preceding calculations, x_m and x_n must lie on the same open half-space determined by $\operatorname{span}\{x_1, \ldots, x_{n-1}\}$, so the two said determinants must have the same sign. Hence we obtain $\operatorname{sgn}(|A|) = (-1)^{n-1}$.

Noting that $\Phi(A^{m-1}x_1) = \Phi(x_m)$ and $\Phi(A^{m-1}x_i) = \Phi(A^{i-1}x_m) = \Phi(\alpha x_{i-1} + \beta x_i)$ for $2 \leq i \leq m$, we also have the following calculations:

$$\begin{aligned} & \operatorname{sgn}(|A|)^{m-1}\operatorname{sgn}(\left| x_{m-n+2} \quad x_{m-n+3} \quad \cdots \quad x_m \quad x_1 \right|) \\ & = \operatorname{sgn}(\left| \alpha x_{m-n+1} + \beta x_{m-n+2} \quad \alpha x_{m-n+2} + \beta x_{m-n+3} \quad \cdots \quad \alpha x_{m-1} + \beta x_m \quad x_m \right|) \\ & = \operatorname{sgn}(\left| x_{m-n+1} \quad x_{m-n+2} \quad \cdots \quad x_{m-1} \quad x_m \right|) \\ & \neq 0, \end{aligned}$$

the last determinant being nonzero, as the set $\{x_{m-n+1}, \ldots, x_m\}$ is, apart from positive scalar multiples, obtained from the linearly independent set $\{x_1, \ldots, x_n\}$ by applying the nonsingular transformation A^{m-n}. Applying A once more, we find that

$$(\operatorname{sgn}(|A|))^m \operatorname{sgn}(\left| x_{m-n+2} \quad \cdots \quad x_m \quad x_1 \right|),\ \operatorname{sgn}(\left| x_{m-n+2} \quad \cdots \quad x_m \quad \alpha x_1 + \beta x_2 \right|)$$

are equal and are both nonzero. Now note that $\operatorname{span}\Phi(x_{m-n+2} + \cdots + x_m)$ is a supporting hypersubspace for K as we have $\Phi(A^{(n-1)(m-1)}x_1) = \Phi(x_{m-n+2} + \cdots + x_m)$ by (4.1), $A^{(n-1)(m-1)}x_1 \in \partial K$ and $\{x_{m-n+2}, \ldots, x_m\}$ is a linearly independent set. So x_1 and $\alpha x_1 + \beta x_2$ lie on the same open half-space determined by the said hypersubspace, and we have $(\operatorname{sgn}(|A|))^m = 1$. But we have already shown that $\operatorname{sgn}(|A|) = (-1)^{n-1}$, hence we obtain $(-1)^{(n-1)m} = 1$ or $(n-1)m \equiv 0 \pmod 2$, which is a contradiction. $\square$

According to formula (4.1) in the proof of Lemma 4.2, when $K \in \mathcal{P}(m,n)$ and A is a K-primitive matrix such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1, we have

$$\Phi(A^{(n-1)(m-1)-1}x_1) = \Phi(x_{m-n+1} + x_{m-n+2} + \cdots + x_{m-1})$$

and

$$\Phi(A^{(n-1)(m-1)}x_1) = \Phi(x_{m-n+2} + x_{m-n+3} + \cdots + x_m).$$

To complete the proof of Theorem 4.1, *it remains to establish the following*:

> *When m is even or when m and n are both odd* (*respectively, when m is odd and n is even*), *it is possible to construct a polyhedral cone $K \in \mathcal{P}(m,n)$ and a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure* 1 *and $x_{m-n+2} + x_{m-n+3} + \cdots + x_m \in \partial K$* (*respectively, $x_{m-n+1} + x_{m-n+2} + \cdots + x_{m-1} \in \partial K$*).

To proceed, we first show that for every pair of positive integers $m, n, 3 \leq n \leq m$, the digraph given by Figure 1 can always be realized as $(\mathcal{E}, \mathcal{P}(A,K))$, where $K \in \mathcal{P}(m,n)$ and A is a K-primitive matrix. In fact, we will obtain more than what is required for the proof of Theorem 4.1.

Lemma 4.3. *For every pair of positive integers $m, n, 3 \leq n \leq m$, and any real number $c \in (0,1)$ (with $c \in [c_m, 1)$ in the case m is odd and n is even), there is a polyhedral cone $K \in \mathcal{P}(m,n)$ for which there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 and $t^m - ct - (1-c)$ is an annihilating polynomial for A.*

Proof. For the given c, let $h(t)$ denote the polynomial $t^m - ct - (1-c)$.

First, we treat the case when m, n are both odd. Write $m = 2k+1$ and $n = 2p+1$. Then $1 \leq p \leq k$. By part (ii) of Lemma 3.2 $h(t)$ has no or two real roots (counting multiplicities) other than 1, depending on whether $c < c_m$ or $c \geq c_m$, where $c_m \in (0,1)$ is the positive real number defined in the lemma. So for $c < c_m$ (respectively, $c \geq c_m$) $h(t)$ has k (respectively, $k-1$) pairs of nonreal conjugate complex roots, say, $r_j e^{\pm i\theta_j}$, with $r_j > 0$ and $0 < \theta_j < \pi$, for $j = 1, \ldots, k$ (respectively, for $j = 1, \ldots, k-1$), arranged in any fixed order. By part (iii) of the above-mentioned lemma, when $m \geq 4$ or $m = 3(= n)$ and $c < c_3(= \frac{3}{4})$, we may assume that $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. When $m = n = 3$ and $c \geq c_3$, the roots of $h(t)$ are all real. This subcase can be verified separately. (See Remark 5.2(ii), (iii).) Hereafter, we assume that $m \geq 4$ or $m = n = 3$ and $c < c_3$. Denoting the lth standard unit vector of $\mathbb{R}^n$ by e_l, let K be the polyhedral cone in $\mathbb{R}^n$ given by $K = \operatorname{pos}\{x_1, \ldots, x_m\}$, where for $j = 1, \ldots, m$: x_j equals

$$\sum_{l=1}^{p} \left(r_l^{j-1} \cos(j-1)\theta_l \, e_{2l-1} + r_l^{j-1} \sin(j-1)\theta_l \, e_{2l} \right) + e_n$$

if $c < c_m$; x_j equals

$$\sum_{l=1}^{p-1} \left(r_l^{j-1} \cos(j-1)\theta_l \, e_{2l-1} + r_l^{j-1} \sin(j-1)\theta_l \, e_{2l} \right) + a_1^{j-1} e_{n-2} + a_2^{j-1} e_{n-1} + e_n$$

if $c > c_m$, where a_1, a_2 are the real roots of $h(t)$ other than 1; and x_j equals

$$\sum_{l=1}^{p-1} \left(r_l^{j-1} \cos(j-1)\theta_l \, e_{2l-1} + r_l^{j-1} \sin(j-1)\theta_l \, e_{2l} \right) + t_0^{j-1} e_{n-2} + (j-1) t_0^{j-2} e_{n-1} + e_n$$

if $c = c_m$, where t_0 is the double root of $h(t)$ (when $c = c_m$).

It is clear that K is a pointed cone, as the nth component of every nonzero vector of K is positive. We are going to prove that K is a full cone by showing that the $n \times n$ matrix whose jth column is x_j, for $j = 1, \ldots, n$, is nonsingular. Depending on $c < c_m$ or $c \geq c_m$, we pre-multiply the latter matrix by the $n \times n$ nonsingular matrix

$$\underbrace{\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{p \text{ times}} \oplus [1]$$

or

$$\underbrace{\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{(p-1) \text{ times}} \oplus I_3.$$

When $c < c_m$, we obtain the Vandermonde matrix associated with $z_1, \bar{z_1}, z_2, \bar{z}_2, \ldots,$ $z_p, \bar{z}_p, 1$, where $z_j = r_j e^{i\theta_j}$, i.e., the matrix $[v_{ab}]$ given by

$$v_{ab} = \begin{cases} z_j^{b-1} & \text{for } a = 2j-1,\ a < n, \\ \bar{z_j}^{b-1} & \text{for } a = 2j,\ a < n, \\ 1 & \text{for } a = n. \end{cases}$$

When $c > c_m$, we obtain the Vandermonde matrix associated with $z_1, \bar{z_1}, z_2, \bar{z}_2, \ldots,$ $z_{p-1},\ \bar{z}_{p-1}, a_1, a_2, 1$. As the roots of $h(t)$ are simple, provided that $c \neq c_m$ (see Lemma 3.2), these Vandermonde matrices are nonsingular. When $c = c_m$, the matrix we obtain is almost a Vandermonde matrix except for the fact that the jth entry of its $(n-1)$th row is equal to $(j-1)t_0^{j-2}$ for $j = 1, \ldots, n$. In this case the matrix obtained is still nonsingular, as it is the coefficient matrix of the $n \times n$ linear system associated with the problem of finding a Hermite interpolation polynomial (see, for instance, [24, p. 607]).

Now take A to be the $n \times n$ matrix given by:

$$A = r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [1]$$

for $c < c_m$;

$$A = r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_{p-1} \begin{bmatrix} \cos\theta_{p-1} & -\sin\theta_{p-1} \\ \sin\theta_{p-1} & \cos\theta_{p-1} \end{bmatrix} \oplus [a_1] \oplus [a_2] \oplus [1]$$

for $c > c_m$; and

$$A = r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_{p-1} \begin{bmatrix} \cos\theta_{p-1} & -\sin\theta_{p-1} \\ \sin\theta_{p-1} & \cos\theta_{p-1} \end{bmatrix} \oplus \begin{bmatrix} t_0 & 0 \\ 1 & t_0 \end{bmatrix} \oplus [1]$$

for $c = c_m$.

As can be readily checked, $Ax_j = x_{j+1}$ for $j = 1, \ldots, m-1$. Also, Ax_m is equal to

$$\sum_{l=1}^{p} (r_l^m \cos m\theta_l\, e_{2l-1} + r_l^m \sin m\theta_l\, e_{2l}) + e_n$$

or

$$\sum_{l=1}^{p-1} (r_l^m \cos m\theta_l\, e_{2l-1} + r_l^m \sin m\theta_l\, e_{2l}) + a_1^m e_{n-2} + a_2^m e_{n-1} + e_n$$

or

$$\sum_{l=1}^{p-1} (r_l^m \cos m\theta_l\, e_{2l-1} + r_l^m \sin m\theta_l\, e_{2l}) + t_0^m e_{n-2} + m t_0^{m-1} e_{n-1} + e_n,$$

depending on whether $c < c_m, c > c_m$ or $c = c_m$. As $r_j e^{\pm i\theta_j}$ are conjugate complex roots of $h(t)$ for $j = 1, \ldots, p$ (or $p-1$) and a_1, a_2 are real roots of $h(t)$ (in the case $c > c_m$) and t_0 is a double root of $h(t)$ (in the case $c = c_m$), one can check that in all cases we have $Ax_m = (1-c)x_1 + cx_2$. Hence A is K-nonnegative. As $A^m x_1 =$ $(1-c)x_1 + cAx_1$ we have $(A^m - cA - (1-c)I)x_i = A^{i-1}(A^m - cA - (1-c)I)x_1 = 0$ for $i = 2, \ldots, m$. But $\operatorname{span}\{x_1, \ldots, x_m\} = \mathbb{R}^n$, so $h(t)$ is an annihilating polynomial for A.

It remains to show that $x_1, \ldots, x_m$ are precisely the pairwise distinct extreme vectors of K (the polyhedral cone generated by them), and the face $\Phi(x_1 + x_2)$ contains (up to multiples) only x_1, x_2 as its extreme vectors. Once this is done, it will follow that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, as desired.

For $j = 1, \ldots, m$, let u_j denote the subvector of x_j formed by its 1st, 2nd and last components. Since $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and $r_1 = r_{\theta_1}$, by Theorem 3.4(iii) $u_1, \ldots, u_m$ are precisely the pairwise distinct extreme vectors of the polyhedral cone $\mathrm{pos}\{u_1, \ldots, u_m\}$. So it is clear that each x_j cannot be written as a nonnegative linear combination of the remaining x_l's or, in other words, $x_1, \ldots, x_m$ are precisely the extreme vectors of K. Also, the proof of Theorem 3.4(iii) guarantees that u_1, u_2 are neighborly extreme vectors of the 3-dimensional polyhedral cone $\mathrm{pos}\{u_1, \ldots, u_m\}$, which means that there is no representation of $u_1 + u_2$ as a positive linear combination of $u_1, \ldots, u_m$, in which at least one of the vectors $u_3, \ldots, u_m$ is involved. As a consequence, there is also no representation of $x_1 + x_2$ as a positive linear combination of $x_1, \ldots, x_m$, in which at least one of the vectors $x_3, \ldots, x_m$ is involved. Hence, the face of K generated by $x_1 + x_2$ is 2-dimensional, as desired.

Now we consider the problem of constructing the desired pair (K, A) for even m. By Lemma 3.2 the polynomial $h(t)$ has precisely two distinct real roots, namely, 1 and, say, a. We write m as $2k + 2$ and let the nonreal complex roots of $h(t)$ be $r_j e^{\pm i\theta_j}$, where $r_j > 0$ and $0 < \theta_j < \pi$, for $j = 1, \ldots, k$ (and $r_1 e^{i\theta_1}$ is the unique root with $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$). Now write n as $2p+2$ or $2p+1$ (with $1 \le p \le k$), depending on whether n is even or odd. Let K be the polyhedral cone in $\mathbb{R}^n$ given by

$$K = \mathrm{pos}\{x_1, \ldots, x_m\},$$

where for $j = 1, \ldots, m$,

$$x_j = \sum_{l=1}^{p} \left(r_l^{j-1} \cos(j-1)\theta_l \, e_{2l-1} + r_l^{j-1} \sin(j-1)\theta_l \, e_{2l} \right) + a^{j-1} \, e_{n-1} + e_n$$

or

$$\sum_{l=1}^{p} \left(r_l^{j-1} \cos(j-1)\theta_l \, e_{2l-1} + r_l^{j-1} \sin(j-1)\theta_l \, e_{2l} \right) + e_n,$$

depending on whether n is even or odd.

Now take A to be the $n \times n$ matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [a] \oplus [1]$$

or

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [1],$$

again depending on whether n is even or odd. Using the same argument as before, we can show that K is a proper cone, $x_1, \ldots, x_m$ are its extreme vectors, A is K-nonnegative, $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, and $h(t)$ is an annihilating polynomial for A.

It remains to consider the case when m is odd and n is even. Note that we rule out the possibility that $c \in (0, c_m)$, because then by Lemma 3.2(ii) $h(t)$ has precisely one real root and so $h(t)$ cannot be an annihilating polynomial for a K-nonnegative matrix with $K \in \mathcal{P}(m, n)$.

For $c \in [c_m, 1)$, according to Lemma 3.2(ii), the polynomial $h(t)$ has three real roots, namely, 1, and say a_1, a_2, where $a_1, a_2 \in (-1, 0)$, and $a_1 = a_2$ if and only if $c = c_m$. Let the remaining $2k$ (where $m - 3 = 2k$) nonreal complex roots be $r_j e^{\pm i\theta_j}, j = 1, \ldots, k$. By part (iii) of the same lemma, we may assume that

$\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Write n as $2p+2$. Then $1 \le p \le k$. Let K be the polyhedral cone in $\mathbb{R}^n$ given by

$$K = \mathrm{pos}\{x_1, \ldots, x_m\},$$

where for $j = 1, \ldots, m$,

$$x_j = \sum_{l=1}^{p} \left(r_l^{j-1} \cos(j-1)\theta_l \, e_{2l-1} + r_l^{j-1} \sin(j-1)\theta_l \, e_{2l} \right) + a^{j-1} e_{n-1} + e_n,$$

where a is equal to a_1 or a_2. Now let A be the matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [a] \oplus [1].$$

The subsequent arguments are similar to those for the previous cases. We omit the details. $\square$

Our next step is to show the following crucial lemma.

Lemma 4.4. *Let m, n be positive integers such that $3 \le n \le m$, where n is odd or n, m are both even. Let K_0 be the cone in $\mathbb{R}^n$ with extreme vectors $y_1, \ldots, y_m$ given by*

$$y_j = \sum_{l=1}^{\frac{n-1}{2}} \left(\cos \frac{2l(j-1)\pi}{m} \, e_{2l-1} + \sin \frac{2l(j-1)\pi}{m} \, e_{2l} \right) + e_n, 1 \le j \le m,$$

when n is odd and by

$$y_j = \sum_{l=1}^{\frac{n-2}{2}} \left(\cos \frac{2l(j-1)\pi}{m} \, e_{2l-1} + \sin \frac{2l(j-1)\pi}{m} \, e_{2l} \right) + (-1)^{j-1} e_{n-1} + e_n, 1 \le j \le m,$$

when n, m are both even. Then $\sum_{j=m-n+2}^{m} y_j$ lies in ∂K_0 and generates a simplicial face of K_0.

Before giving the proof of Lemma 4.4 we show how it can be used to complete the proof of Theorem 4.1.

First, consider the case when m is even or m, n are both odd. We slightly modify the construction of K for $c < c_m$ as given in the proof of Lemma 4.3, and use continuity argument. For $c > 0$, sufficiently small, we can write the roots of $h(t)$ as $r_j e^{\pm i\theta_j}, j = 1, \ldots, \lceil \frac{m}{2} \rceil - 1$, where $0 < \theta_1 < \cdots < \theta_{\lceil \frac{m}{2} \rceil - 1} < \pi$, together with 1 and a negative real number a greater than -1, or just 1, depending on whether m is even or odd. By choosing c sufficiently close to 0, we can make $r_j e^{i\theta_j}$, for $j = 1, \ldots, \lceil \frac{m}{2} \rceil - 1$ (respectively, a), very close to $e^{i\frac{2j\pi}{m}}$ (respectively, -1), so that the extreme vectors $x_1, \ldots, x_m$ of K are as close as we please to, respectively, the extreme vectors $y_1, \ldots, y_m$ of K_0. By Lemma 4.4 the vector $y_{m-n+2} + y_{m-n+3} + \cdots + y_m$ generates a simplicial face of K_0. Hence $\mathrm{span}\{y_{m-n+2}, y_{m-n+3}, \ldots, y_m\}$ is a hypersubspace and the remaining extreme vectors $y_1, \ldots, y_{m-n+1}$ all lie in the same open half-space determined by this hypersubspace. By continuity, it follows that when c is sufficiently close to 0, $\mathrm{span}\{x_{m-n+2}, x_{m-n+3}, \ldots, x_m\}$ is a hypersubspace and the remaining extreme vectors of K all lie on the same open half-space determined by it. The latter implies, in particular, that $x_{m-n+2} + x_{m-n+3} + \cdots + x_m \in \partial K$, which is what we want.

When m is odd and n is even, we still use the construction for K and A as given in the proof of Lemma 4.3, but we take a to be the smaller of a_1 and a_2 (so that

a tends to -1 as c tends to 1) and assume that $\theta_1, \ldots, \theta_k$ are in increasing order (again with $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$). However, instead of K_0 we make use of the polyhedral cone K_1 of $\mathbb{R}^n$ with extreme vectors $y_1, \ldots, y_{m-1}$ given by

$$y_j = \sum_{l=1}^{\frac{n-2}{2}} \left(\cos \frac{2l(j-1)\pi}{m-1}\, e_{2l-1} + \sin \frac{2l(j-1)\pi}{m-1}\, e_{2l} \right) + (-1)^{j-1} e_{n-1} + e_n.$$

Since $m-1$ is even, by what we have just done, $\operatorname{span}\{y_{m-n+1}, y_{m-n+2}, \cdots, y_{m-1}\}$ is a hypersubspace and the remaining extreme vectors of K_1 all lie in the same open half-space determined by it. We choose c less than 1 but sufficiently close to 1 so that the extreme vectors $x_1, \ldots, x_{m-1}$ of K are very close to, respectively, the extreme vectors $y_1, \ldots, y_{m-1}$ of K_1 and such that $\operatorname{span}\{x_{m-n+1}, x_{m-n+2}, \cdots, x_{m-1}\}$ is a hypersubspace and the remaining extreme vectors of K all lie in the same open half-space determined by it (noting, in particular, that x_m and x_1 both tend to y_1 as c tends to 1). Then we have $x_{m-n+1} + x_{m-n+2} + \cdots + x_{m-1} \in \partial K$, as desired.

Now we come to the proof of Lemma 4.4. The lemma clearly holds for the case $m = n$. Hereafter we assume that $m > n$. We want to show that $\sum_{j=m+2-n}^{m} y_j$ lies in ∂K_0 and generates a simplicial face. For this purpose, it suffices to show that for $t = 1, \ldots, m+1-n$, the determinants $|y_t \quad y_{m+2-n} \cdots y_m|$ are nonzero and have the same sign.

We will need the following known result due to Scott (see [32] or [12, p. 23, hint to Exercise 23]):

Lemma 4.5. *Let p be a given positive integer. For any $\theta \in \mathbb{R}$, denote by $x(\theta)$ the vector $(\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cdots, \cos p\theta, \sin p\theta)^T$ of $\mathbb{R}^{2p}$. For any $\theta_0, \theta_1, \ldots, \theta_{2p} \in \mathbb{R}$, we have*

$$\begin{vmatrix} x(\theta_0) & x(\theta_1) & \cdots & x(\theta_{2p}) \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 4^{p^2} \prod_{0 \le i < j \le 2p} \sin \frac{1}{2}(\theta_j - \theta_i).$$

To prove Lemma 4.4 we first treat the case when n is odd. Assume $n = 2p+1, p \ge 1$. For $1 \le t \le m+1-n$, applying Lemma 4.5 with $\theta_j = \frac{2j\pi}{m}$ for $j = 0, \ldots, m-1$ and noting that $y_j = (x(\theta_{j-1}), 1)^T$ for $j = 1, \ldots, m$, we have

$$\begin{aligned} &\begin{vmatrix} y_t & y_{m+2-n} & \cdots & y_m \end{vmatrix} \\ &\quad = 4^{p^2} \prod_{m+2-n \le i < j \le m} \sin \frac{1}{2}(\theta_{j-1} - \theta_{i-1}) \prod_{j=m+2-n}^{m} \sin \frac{1}{2}(\theta_{j-1} - \theta_{t-1}) > 0. \end{aligned}$$

Here all the factors are positive because $0 < \frac{1}{2}(\theta_{j-1} - \theta_{i-1}) = \frac{(j-i)\pi}{m} < \pi$ for $1 \le i < j \le m$.

Now consider the case when n and m are both even. Assume $n = 2p+2, p \ge 1$. We have

$$\begin{vmatrix} y_t & y_{m+2-n} & \cdots & y_m \end{vmatrix} = \begin{vmatrix} x(\theta_{t-1}) & x(\theta_{m+1-n}) & \cdots & x(\theta_{m-1}) \\ (-1)^{t-1} & (-1)^{m+1-n} & \cdots & (-1)^{m-1} \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

To compute the determinant on the right, add the nth row to the $(n-1)$th row and expand the resulting determinant by its $(n-1)$th row. The determinant becomes

$$(1 + (-1)^{t-1}) D_1 + 2 \sum_{k=2}^{p+1} D_k,$$

where D_k denotes the $(n-1, 2k-1)$-minor of the said resulting determinant and hence is of the form

$$\begin{vmatrix} x(\phi_0) & \cdots & x(\phi_{2p}) \\ 1 & \cdots & 1 \end{vmatrix},$$

with $0 \leq \phi_0 < \phi_1 < \cdots < \phi_{2p} < 2\pi$ (the ϕ_j's depending on the k). By the argument we have used for the case when n is odd, each D_k is positive. So the determinant $\begin{vmatrix} y_t & y_{m+2-n} & \cdots & y_m \end{vmatrix}$ is always positive, independent of the choice of t from $\{1, 2, \ldots, m+1-n\}$.

We have completed the proof of Theorem 4.1.

In the proof of Theorem 4.1 we have shown that in all cases we can find an exp-maximal cone $K \in \mathcal{P}(m,n)$ and an exp-maximal K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1. Next, we are going to show that when n is even and m is odd, it is also possible to choose the optimal pair K, A in such a way that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 2, as suggested by Theorem 2.5(ii).

We borrow a construction mentioned at the end of Section 7 of [21]. Let K, A be an optimal pair as constructed in the proof of Theorem 4.1 such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 and the extreme vectors $x_{m-n+1}, x_{m-n+2}, \ldots, x_{m-1}$ generate an $(n-1)$-dimensional simplicial face of K. Note that the remaining extreme vectors $x_1, \ldots, x_{m-n}, x_m$ all lie in the same open half-space determined by the hypersubspace $\operatorname{span}\{x_{m-n+1}, x_{m-n+2}, \ldots, x_{m-1}\}$. Choose $\alpha > 1$, sufficiently close to 1, so that the vector $(1-\alpha)x_1 + \alpha x_m$ also belongs to the said open half-space and the vectors $(1-\alpha)x_1 + \alpha x_m, x_1, x_2, \ldots, x_{m-1}$ form the extreme vectors of a polyhedral cone, which we denote by $\tilde{K}$. It is readily checked that for the same A, A is $\tilde{K}$-nonnegative and $(\mathcal{E}, \mathcal{P}(A, \tilde{K}))$ is isomorphic to Figure 2 (under the isomorphism given by $\Phi((1-\alpha)x_1 + \alpha x_m) \mapsto \Phi(x_1), \Phi(x_j) \mapsto \Phi(x_{j+1})$ for $j = 1, \ldots, m-1$). As shown in the proof of Theorem 4.1 (see the discussion preceding Lemma 4.3) we have $A^{(n-1)(m-1)-1}x_1 \in \Phi_K(x_{m-n+1} + \cdots + x_{m-1})$. By our choice of α, the extreme vectors $x_1, \ldots, x_{m-n}, (1-\alpha)x_1 + \alpha x_m$ of $\tilde{K}$ all lie in the same open half-space determined by the hypersubspace $\operatorname{span}\{x_{m-n+1}, x_{m-n+2}, \ldots, x_{m-1}\}$. Thus $\Phi_{\tilde{K}}(x_{m-n+1} + \cdots + x_{m-1})$ is a simplicial face of $\tilde{K}$ and equals $\Phi_K(x_{m-n+1} + \cdots + x_{m-1})$. Hence we have $A^{(n-1)(m-1)-1}x_1 \in \partial\tilde{K}$, and so $\gamma_{\tilde{K}}(A) \geq (n-1)(m-1)$. In view of Lemma 4.2 we obtain $\gamma_{\tilde{K}}(A) = (n-1)(m-1)$, as desired.

The proofs of Lemmas 4.2 and 4.4 are based on [27]. A different proof of Lemma 4.4, due to the first and third authors, involves certain generalized Vandermonde matrices, the complete symmetric polynomials, the Jacobi-Trudi determinant, and a nontrivial result about polynomials with nonnegative coefficients (as given in [2]). As the proof might be of independent interest, we include it in the Appendix.

If we were not interested in the uniqueness issue (for the case $n = 3$), which will be treated in our next section, we could have bypassed Section 3 and Lemma 4.3 and proved Theorem 4.1 directly as follows — this is essentially the way done in [27]: First, establish Lemma 4.2 and Lemma 4.4. Construct the pair (K, A) as in the proof for Lemma 4.3 for c sufficiently close to 0 (respectively, 1) when m, n are both odd or m is even (respectively, when m is odd and n is even), and finish the proof in the same way as before. The construction is feasible because in every case the polynomial $h(t)$ has the right number of real roots and with the right parity (namely, one positive root when m, n are both odd, one positive root and one negative root when m is even, and one positive root and two negative roots when

m is odd and n is even). This follows from the continuity argument by considering the polynomial $t^m - 1$ (or the polynomial $t^m - t$ in the case m is odd and n is even) — there is no need to apply Lemma 3.2(ii). When m is even or m, n are both odd, by continuity it is clear that $x_1, \ldots, x_m$ are the extreme vectors of K, as $y_1, \ldots, y_m$ are the extreme vectors of K_0. When m is odd and n is even, the same argument does not apply, since x_m, x_1 both tend to y_1. In that case we can argue as follows. Suppose that not all of the vectors $x_1, \ldots, x_m$ are extreme vectors of K. Then K has less than m extreme rays. By Corollary 2.6 $\gamma(K) \leq (n-1)(m-2)+1$. On the other hand, by what we have done before (see the discussion following the statement of Lemma 4.4 for the case m odd, n even, and the proof of Lemma 4.2), we have

$$\Phi(A^{(n-1)(m-1)-1}x_1) = \Phi(x_{m-n+1} + \cdots + x_{m-1}) \text{ and } x_{m-n+1} + \cdots + x_{m-1} \in \partial K.$$

Hence $\gamma(A) \geq \gamma(A, x_1) \geq (n-1)(m-1)$. But $(n-1)(m-1) > (n-1)(m-2)+1$, so we arrive at a contradiction.

5. Uniqueness of exp-maximal cones and their exp-maximal primitive matrices

Given positive integers m, n with $3 \leq n \leq m$, up to linear isomorphism, how many exp-maximal cones are there in $\mathcal{P}(m,n)$? For a given exp-maximal cone K in $\mathcal{P}(m,n)$, up to cone-equivalence modulo positive scalar multiplication, how many exp-maximal K-primitive matrices are there? In this section we are going to address these questions for the cases $m = n$ and $n = 3$. The corresponding questions for the case $m = n+1$ have already been settled in [22]; it is proved that for every integer $n \geq 3$, there are (up to linear isomorphism) one or two n-dimensional exp-maximal minimal cones, depending on whether n is odd or even, and for each such minimal cone K, there are uncountably infinitely many exp-maximal K-primitive matrices which are pairwise linearly independent and noncone-equivalent.

Since there is, up to linear isomorphism, only one simplicial cone of a given dimension, we need not treat the problem of identifying exp-maximal cones in $\mathcal{P}(m,n)$ when $m = n$. As expected, we have the following result, which describes all the exp-maximal K-primitive matrices for K in $\mathcal{P}(n,n), n \geq 3$:

Remark 5.1. Let $K \in \mathcal{P}(n,n), n \geq 3$, and let A be a K-primitive matrix with $\rho(A) = 1$. Then A is exp-maximal K-primitive if and only if there exists $c \in (0,1)$ such that A is cone-equivalent to $C_h (\in \pi(\mathbb{R}^n_+))$, the companion matrix of the polynomial $h(t) = t^n - ct - (1-c)$, i.e.,

$$C_h = \begin{bmatrix} 0 & & & & 1-c \\ 1 & 0 & & \mathbf{0} & c \\ & 1 & \ddots & & 0 \\ & \mathbf{0} & \ddots & \ddots & \vdots \\ & & & 1 & 0 \end{bmatrix}_{n \times n}.$$

Proof. The "if" part is obvious as C_h is exp-maximal $\mathbb{R}^n_+$-primitive (see, for instance, [6, Theorem 3.5.6]). To show the "only if" part, we may assume that $K = \mathbb{R}^n_+$. Since $(\mathcal{E}, \mathcal{P}(A, \mathbb{R}^n_+))$ is, up to isomorphism, given by Figure 1 (and equals the usual digraph associated with A^T), there exists a permutation matrix P such that $P^TAP = B$, where B is a nonnegative matrix with the same zero-nonzero pattern as the companion matrix C_h. It is not difficult to find a diagonal matrix D with positive diagonal entries such that $D^{-1}BD = C_h$ for some $c \in (0,1)$. However, PD is an automorphism of $\mathbb{R}^n_+$, so it follows that A is cone-equivalent to C_h. $\square$

The exp-maximal primitive matrices for $\mathcal{P}(n,n)$ in the special case $n = 3$ deserve special attention because the behavior of the roots of the polynomial $t^n - ct - (1-c)$ (where $0 < c < 1$) for $n = 3$ is somewhat different from that for $n \geq 4$. According to Lemma 3.2, when $n \geq 4$, the polynomial $t^n - ct - (1-c)$ always has a (unique) complex root of the form $re^{i\theta}$, where $r > 0$ and $\theta \in (\frac{2\pi}{n}, \frac{2\pi}{n-1})$. For $n = 3$, the said polynomial has a complex root of the required form only if $0 < c < \frac{3}{4}$; for $\frac{3}{4} \leq c < 1$, the roots of the polynomial are all real, and with a double root when $c = \frac{3}{4}$. It is not difficult to establish the following:

Remark 5.2. Let $K \in \mathcal{P}(3,3)$ and let A be a K-primitive matrix with $\rho(A) = 1$. Then A is exp-maximal K-primitive if and only if A is cone-equivalent to one of the following:

(i) $A_\theta \in \pi(K_\theta)$, with $\theta \in (\frac{2\pi}{3}, \pi)$, where $K_\theta \in \mathcal{P}(3,3)$ and $A_\theta \in \pi(K_\theta)$ are defined in the same way as before;

(ii) $\mathrm{diag}(\alpha_1, \alpha_2, 1) \in \pi(\tilde{K})$, where for some $c \in (\frac{3}{4}, 1), \alpha_1, \alpha_2$ are the (distinct) real roots, other than 1, of the polynomial $t^3 - ct - (1-c)$, and $\tilde{K}$ is the polyhedral cone in $\mathbb{R}^3$ generated by the extreme vectors $x_1 = (1,1,1)^T, x_2 = (\alpha_1, \alpha_2, 1)^T$ and $x_3 = (\alpha_1^2, \alpha_2^2, 1)^T$;

(iii) $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \pi(K_0)$, where $K_0 = \mathrm{pos}\{x_1, Ax_1, A^2x_1\}$ with $x_1 = (1,1,1)^T$.

Next, we consider the 3-dimensional cone case. Some work on identifying exp-maximal 3-dimensional cones has already been done in Section 3. By Theorem 3.4(iii), for every positive integer $m \geq 3$, there are uncountably infinitely many exp-maximal cones K_θ in $\mathcal{P}(m,3)$, one for each $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$.

Theorem 5.3. *Let $m \geq 4$ be a positive integer. For each $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, let K_θ and A_θ be respectively the exp-maximal cone and exp-maximal K_θ-primitive matrix as defined in Theorem* 3.4(iii).

(i) *If $K \in \mathcal{P}(m,3)$ is an exp-maximal polyhedral cone and A is an exp-maximal K-primitive matrix with $\rho(A) = 1$, then there exists a unique $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ such that A is cone-equivalent to A_θ (and K is linearly isomorphic to K_θ). Moreover, the eigenvalues of A are $1, r_\theta e^{\pm i\theta}$, where r_θ has the same meaning as given in Corollary* 3.3.

(ii) *When $m \geq 6, A_\theta$ is, up to positive scalar multiples, the only exp-maximal K_θ-primitive matrix.*

Proof. (i) Since A is exp-maximal, by Theorem 2.5(i) we may assume that as a labelled digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1. Let $v \in \mathrm{int}\, K^*$ be a Perron vector of A^T and denote by C the complete cross-section $\{x \in K : \langle x, v \rangle = 1\}$ of K, which is a polygon with m extreme points. By the proof of Lemma 2.4(i) (as given in [21, Lemma 4.2]) and in view of Lemma 3.1, we may assume that $x_1, \ldots, x_m$ are precisely all the extreme points of C and x_i, x_{i+1} are neighborly extreme points for $i = 1, \ldots, m$ (where x_{m+1} is taken to be x_1). Also, $Ax_j = x_{j+1}$ for $j = 1, \ldots, m-1$ and $Ax_m = (1-c)x_1 + cx_2$ for some $c \in (0,1)$, and moreover $t^m - ct - (1-c)$ is an annihilating polynomial for A. Let u be the Perron vector of A that belongs to C and denote by $\hat{C}$ the polygon $C - u$. Note that $\operatorname{span} \hat{C}$ equals $(\operatorname{span}\{v\})^\perp$, and so it is invariant under A (as v is an eigenvector of A^T). Let λ_1, λ_2 denote the eigenvalues of the restriction of A to $(\operatorname{span}\{v\})^\perp$. It is clear that the eigenvalues of A are 1 (the Perron root) and λ_1, λ_2. As A is K-primitive, by the Perron-Frobenius theory, $|\lambda_j| < 1$ for $j = 1, 2$. We contend that λ_1, λ_2 form a conjugate pair of nonreal complex numbers.

For $j = 1, \ldots, m$, denote by y_j the point $x_j - u$. Clearly, $y_1, \ldots, y_m$ are all the extreme points of $\hat{C}$ and y_i, y_{i+1} are neighborly extreme points for $i = 1, \ldots, m$ (where y_{m+1} is taken to be y_1), and $0 \in \operatorname{ri} \hat{C}$ as $u \in \operatorname{ri} C$. Since $Au = u$, the action of A on C induces a corresponding action on $\hat{C}$: we have $Ay_j = y_{j+1}$ for $j = 1, \ldots, m-1$, and $Ay_m = (1-c)y_1 + cy_2$. Note that since $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1, A maps no extreme points of C into $\operatorname{ri} C$ and maps the relative interior of precisely one side of C into $\operatorname{ri} C$. The preceding conclusion is still true if C is replaced by $\hat{C}$.

Assume that λ_1, λ_2 are real. As $t^m - ct - (1-c)$ is an annihilating polynomial for A, it contains λ_1, λ_2 as its roots. By Lemma 3.2(ii) we have $\lambda_1, \lambda_2 \in (-1, 0)$. Let $w \in (\operatorname{span}\{v\})^\perp$ be an eigenvector of A corresponding to λ_1. Since $\operatorname{span}\{w\}$ contains the relative interior point 0 of $\hat{C}$, it meets the relative boundary of $\hat{C}$ at two points, say, z_1, z_2. We may assume that $z_1 = a_1 w$ and $z_2 = a_2 w$ with $a_1 a_2 < 0$ and $|a_2| \geq |a_1|$. A little calculation shows that we have $Az_1 = \frac{a_1\lambda_1}{a_2} z_2 \in \operatorname{ri} \hat{C}$, as $0 < \frac{a_1\lambda_1}{a_2} < 1$ and $0 \in \operatorname{ri} \hat{C}$. If z_1 is an extreme point of $\hat{C}$, we already obtain a contradiction, as A sends no extreme point of $\hat{C}$ to $\operatorname{ri} \hat{C}$. So suppose that z_1 lies in the relative interior of a side of the polygon $\hat{C}$. Now the point Az_2, which is a positive multiple of z_1, either lies in $\operatorname{int} \hat{C}$ or is equal to z_1. Suppose $Az_2 \in \operatorname{int} \hat{C}$. Since A sends no extreme point of $\hat{C}$ into $\operatorname{int} \hat{C}$, z_2 must lie in the relative interior of a side of the polygon $\hat{C}$. Then A maps the relative interior of two different sides of $\hat{C}$ into its relative interior, which is a contradiction. So we must have $Az_2 = z_1$. If z_2 is an extreme point, then necessarily $z_2 = y_m$, and the side of $\hat{C}$ that contains z_1 is the line segment $\overline{y_1 y_2}$. As we have $Ay_1 = y_2$ and $Ay_2 = y_3$, it follows that $Az_1 \in \operatorname{ri} \overline{y_2 y_3}$. On the other hand, we have already shown that $Az_1 \in \operatorname{ri} \hat{C}$. So we arrive at a contradiction. Hence z_2 must lie in the relative interior of a side of $\hat{C}$. Then necessarily the side of $\hat{C}$ that contains z_2 is $\overline{y_{m-2} y_{m-1}}$, whereas the side that contains z_1 is $\overline{y_{m-1} y_m}$. Since $z_2 \in \operatorname{ri} \overline{y_{m-2} y_{m-1}}$, we have $z_1 = Az_2 \in \operatorname{ri} \overline{y_{m-1} y_m}$, and hence $Az_1 \in \operatorname{ri} \operatorname{conv}\{y_m, y_1, y_2\}$. Note that Az_1 also belongs to $\operatorname{ri} \operatorname{conv}\{y_{m-2}, y_{m-1}, y_m\}$ as it lies in $\operatorname{ri} \overline{z_1 z_2}$, and $z_1 \in \operatorname{ri} \overline{y_{m-1} y_m}, z_2 \in \operatorname{ri} \overline{y_{m-2} y_{m-1}}$. However, $\operatorname{ri} \operatorname{conv}\{y_m, y_1, y_2\} \cap \operatorname{ri} \operatorname{conv}\{y_{m-2}, y_{m-1}, y_m\} = \emptyset$ as $m \geq 4$, so we arrive at a contradiction.

In the above we have shown that the eigenvalues λ_1, λ_2 of A form a conjugate pair of nonreal complex numbers, say $\pm re^{i\theta}$. Then clearly $r < 1$. Now choose a basis $\{u_1, u_2\}$ for $(\operatorname{span}\{v\})^\perp$ with $u_1 = y_1$ such that

$$\begin{aligned} Au_1 &= r\cos\theta \, u_1 + r\sin\theta \, u_2, \\ Au_2 &= -r\sin\theta \, u_1 + r\cos\theta \, u_2. \end{aligned}$$

Let $\alpha_1 = 1, \beta_1 = 0$, and for $j = 2, \ldots, m$, let $y_j = \alpha_j u_1 + \beta_j u_2$. Then we have

$$\begin{pmatrix}\alpha_{j+1}\\ \beta_{j+1}\end{pmatrix} = r\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}$$

for $j = 1, \ldots, m-1$, as $Ay_j = y_{j+1}$ for every such j. Since $y_1, \ldots, y_m$ form the consecutive vertices of a polygon in $(\mathrm{span}\{v\})^{\perp}$, the points $(\alpha_j, \beta_j)^T, j = 1, \ldots, m$, also form the consecutive vertices of a polygon in $\mathbb{R}^2$. Now it should be clear that we have $(m-1)\theta < 2\pi$ and $2\pi < m\theta$, which implies that $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$.

As $(\alpha_1, \beta_1)^T = (1, 0)^T$, a little calculation gives $Ay_m = r^m \cos m\theta\, u_1 + r^m \sin m\theta\, u_2$. Since u_1, u_2 are linearly independent, the relation $Ay_m = (1-c)y_1 + cy_2$ leads to relations (3.1) and (3.2) that appear in the proof of Lemma 3.2. Eliminating c from these two relations, we obtain

$$\frac{\sin(m-1)\theta}{\sin\theta} r^m - \frac{\sin m\theta}{\sin\theta} r^{m-1} + 1 = 0;$$

hence r equals r_θ, the unique positive real root of the polynomial $g_\theta(t)$. Now let P be the 3×3 matrix given by: $Pu_j = e_j$ for $j = 1, 2, 3$, where $u_3 = u$, the Perron vector of A that belongs to C, and e_j is the jth standard unit vector of $\mathbb{R}^3$. It is readily checked that P is a nonsingular matrix that maps K onto K_θ. Moreover, we have $PA = A_\theta P$. So the cone-preserving maps A and A_θ are cone-equivalent (and the cones K and K_θ are linearly isomorphic).

(ii) Let B be an exp-maximal K_θ-primitive matrix. Then $\gamma(B) = 2m-1$, and by Theorem 2.5(i) the digraph $(\mathcal{E}, \mathcal{P}(B, K_\theta))$ is, apart from the labelling of its vertices, given by Figure 1. Hereafter, for simplicity, we denote $x_j(\theta)$ by x_j for $j = 1, \ldots, m$. Also, we adopt the convention that for any integer $j \notin \{1, \ldots, m\}, x_j$ is taken to be x_k, where k is the unique integer that satisfies $1 \le k \le m, k \equiv j (\mathrm{mod}\, m)$. According to Lemma 3.1, adjacent vertices of the digraph $(\mathcal{E}, \mathcal{P}(B, K_\theta))$ correspond to neighboring extreme rays of K_θ. Using an argument similar to the one given in the proof of Theorem 3.5, we can show that there exists $p, 1 \le p \le m$, such that one of the following holds:

(I) Bx_j is a positive multiple of x_{j+1} for $j = p, p+1, \ldots, p+m-2$ and Bx_{p+m-1} is a positive linear combination of x_{p+m} and x_{p+m+1} or

(II) Bx_j is a positive multiple of x_{j-1} for $j = p, p-1, \ldots, p-m+2$ and Bx_{p-m+1} is a positive linear combination of x_{p-m} and x_{p-m-1}.

We first consider the case when (I) holds. Then Bx_j is a positive multiple of $A_\theta x_j$ for all $j = 1, \ldots, m$ except for $j = m, p-1$ (except for $j = m$ in the case $p = 1$). Since there are $m-2$ ($m-1$ in the case $p = 1$) such x_j's and $m - 2 \ge 4$ (as $m \ge 6$), by Theorem 2.7, B must be a positive multiple of A_θ, which is what we want. (In fact, then necessarily $p = 1$.)

Now consider the case when (II) holds. Then Bx_j is a positive multiple of $A_\theta^{-1} x_j$ for all $j = 1, \ldots, m$, except for $j = 1, p+1$ (except for $j = 1$ in the case $p = m$). By Theorem 2.7 again we infer that B is a positive multiple of A_θ^{-1}, which is impossible, as A_θ^{-1} is not K_θ-nonnegative. □

Corollary 5.4. *Let $m \ge 6$ be a positive integer, and let $K_\theta \in \mathcal{P}(m, 3)$ be the exp-maximal cone as defined before. Then*:

(i) *The automorphism group of K_θ consists only of scalar matrices.*
(ii) *For any $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1}), \theta_1 \neq \theta_2$, the cones $K_{\theta_1}, K_{\theta_2}$ are not linearly isomorphic.*

Proof. (i) We first establish the following (which is true for $m \geq 4$):

Assertion. *Every automorphism of K_θ that commutes with A_θ is a scalar matrix.*

Proof. Let P be an automorphism of K_θ that commutes with A_θ. For simplicity, denote $x_j(\theta)$ by x_j for $j = 1, \ldots, m$. Since P is an automorphism of K_θ, P permutes the extreme rays of K_θ among themselves. Consider the relation $A_\theta P x_m = P A_\theta x_m$. Note that the right side is not an extreme vector of K_θ as $A_\theta x_m$ is a positive linear combination of x_1 and x_2, whereas the left side is an extreme vector if Px_m is a positive multiple of x_j for $j = 1, \ldots, m-1$. Hence Px_m must be a positive multiple of x_m. By considering the relations $A_\theta P x_j = P A_\theta x_j$ for $j = m-1, m-2, \ldots, 1$ (and in this order), in a similar way we infer that Px_j is a positive multiple of x_j for $j = m-1, m-2, \ldots, 1$. By Theorem 2.7 it follows that P is a scalar matrix.

Now back to the proof of (i). Let P be an automorphism of K_θ. Since A_θ is an exp-maximal K_θ-primitive matrix, so is $P^{-1} A_\theta P$. By Theorem 5.3(ii) we have $P^{-1} A_\theta P = \alpha A_\theta$ for some $\alpha > 0$. As $P^{-1} A_\theta P$ and A_θ are similar and A_θ is nonsingular, necessarily $\alpha = 1$. So we have $A_\theta P = P A_\theta$, and by the above Assertion it follows that P is a scalar matrix.

(ii) Let $\theta_1, \theta_2 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ be such that the cones $K_{\theta_1}, K_{\theta_2}$ are linearly isomorphic, say P is a linear isomorphism that maps K_{θ_2} onto K_{θ_1}. Since A_{θ_1} is exp-maximal K_{θ_1}-primitive, clearly $P^{-1} A_{\theta_1} P$, which is cone-equivalent to A_{θ_1}, is exp-maximal K_{θ_2}-primitive. In view of Theorem 5.3(ii) and the fact $\rho(A_{\theta_j}) = 1$ for $j = 1, 2$, we have $P^{-1} A_{\theta_1} P = A_{\theta_2}$. Now for $j = 1, 2$, the eigenvalues of A_{θ_j} are 1 and $r_{\theta_j} e^{\pm i\theta_j}$. So we must have $\theta_1 = \theta_2$. $\square$

Note that part (i) of Theorem 5.3 is not true for $m = 3$. This is because every A_θ has a pair of conjugate non-real complex eigenvalues, whereas an exp-maximal $\mathbb{R}^3_+$-primitive matrix need not have non-real eigenvalues (see Remark 5.2).

Parts (i) and (ii) of Corollary 5.4 are both invalid when $m = 3$ or 4: the automorphism group of $\mathbb{R}^3_+$ or of a 3-dimensional minimal cone clearly contains members which are not scalar matrices; and all K_θ's are linearly isomorphic to $\mathbb{R}^3_+$ or to a 3-dimensional minimal cone, depending on whether $m = 3$ or 4.

Part (ii) of Theorem 5.3 cannot be extended to cover the cases $m = 3, 4$. For $m = 3, 4$, by what is known for the simplical cone case (see Remark 5.1) and the minimal cone case (see, [22, Theorem 5.3, Theorem 5.4, Theorem 5.6]) there are uncountably infinitely many pairwise linearly independent noncone-equivalent exp-maximal K-primitive matrices.

In contrast, for the case $m = 5$, we have the following result:

Theorem 5.5. *Let $K \in \mathcal{P}(5, 3)$ be an exp-maximal cone.*

(i) *There are precisely two exp-maximal K-primitive matrices with spectral radius 1, and they are cone-equivalent.*
(ii) *The automorphism group of K consists of the identity matrix and an involution, different from the identity matrix, together with their positive multiples.*

Proof. First, we can find an exp-maximal K-primitive matrix A with $\rho(A) = 1$ such that as a labelled digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1, and we have

$$Ax_1 = x_2, \quad Ax_2 = x_3, \quad Ax_3 = x_4, \quad Ax_4 = x_5 \text{ and } \quad Ax_5 = a_1 x_1 + a_2 x_2,$$

where $a_1, a_2 > 0, a_1 + a_2 = 1$. (See the first paragraph in the proof of Theorem 5.3(i).)

(i) We proceed by deriving a necessary condition for a K-primitive matrix which is not a multiple of A to be exp-maximal, and then we show that (up to multiples) there is only one K-primitive matrix that satisfies the condition.

Let B be an exp-maximal K-primitive matrix which is not a multiple of A. Then, apart from the labelling of its vertices, the digraph $(\mathcal{E}, \mathcal{P}(B,K))$ is also given by Figure 1. According to the proof of Theorem 5.3(ii), there exists an integer $p, 1 \le p \le 5$, such that one of the following holds:

(I) Bx_j is a positive multiple of x_{j+1} for $j = p, p+1, p+2, p+3$ and Bx_{p+4} is a positive linear combination of x_{p+5} and x_{p+6}, where for any integer $j \notin \{1, \dots, 5\}$, x_j is taken to be x_k, k being the unique integer that satisfies $1 \le k \le 5, k \equiv j (\mathrm{mod}\, 5)$ or

(II) Bx_j is a positive multiple of x_{j-1} for $j = p, p-1, p-2, p-3$ and Bx_{p-4} is a positive linear combination of x_{p-5} and x_{p-6}.

We first show that case (I) cannot happen at all. If $p = 1$, then Ax_j and Bx_j are linearly dependent for $j = 1, 2, 3, 4$, and by Theorem 2.7 it follows that B is a multiple of A, which is a contradiction. Hereafter we consider $p = 2, 3, 4$ or 5.

Since $(\mathcal{E}, \mathcal{P}(B,K))$ is given by Figure 1 (up to isomorphism), by Lemma 2.4(i) B is nonsingular. Define $C = B^{-1}A$.

For $p = 2$, we have

$$Bx_2 = \beta_3 x_3, \quad Bx_3 = \beta_4 x_4, \quad Bx_4 = \beta_5 x_5, \quad Bx_5 = \beta_1 x_1, \quad Bx_1 = \beta_2 x_2 + \gamma_3 x_3,$$

where γ_3 and all β_i's are positive. Then $Cx_2 = B^{-1}Ax_2 = \beta_3^{-1}x_2$. Similarly, $Cx_3 = \beta_4^{-1}x_3$ and $Cx_4 = \beta_5^{-1}x_4$. As $B(\gamma_3 x_2 - \beta_3 x_1) = -\beta_2\beta_3 x_2$, we have $B^{-1}x_2 = \beta_2^{-1}x_1 - \beta_2^{-1}\beta_3^{-1}\gamma_3 x_2$. Hence

$$Cx_1 = B^{-1}Ax_1 = B^{-1}x_2 = \beta_2^{-1}x_1 - \beta_2^{-1}\beta_3^{-1}\gamma_3 x_2$$

and

$$Cx_5 = B^{-1}Ax_5 = a_1\beta_1^{-1}x_5 + a_2\beta_2^{-1}x_1 - a_2\beta_2^{-1}\beta_3^{-1}\gamma_3 x_2.$$

Suppose $x_1 = f_1 x_2 + f_2 x_3 + f_3 x_4$. Then

$$Cx_1 = f_1 Cx_2 + f_2 Cx_3 + f_3 Cx_4 = f_1\beta_3^{-1}x_2 + f_2\beta_4^{-1}x_3 + f_3\beta_5^{-1}x_4,$$

and from the previous relation for Cx_1 we also obtain

$$Cx_1 = (\beta_2^{-1}f_1 - \beta_2^{-1}\beta_3^{-1}\gamma_3)x_2 + \beta_2^{-1}f_2 x_3 + \beta_2^{-1}f_3 x_4.$$

By comparing the coefficients of x_3 and x_4 and using the fact that f_1, f_2, f_3 are all nonzero, we find that $\beta_4 = \beta_5$. Now define $T = C - \beta_4^{-1}I$. Then x_3 and x_4 are in the null space of T, and as T is nonzero, $\mathrm{rank}T = 1$. Hence there exists a linear functional φ of $\mathbb{R}^3$ such that $Tu = \varphi(u)x_2$ for all $u \in \mathbb{R}^3$ (noting that Tx_2 is a nonzero multiple of x_2). In particular, we have

$$\varphi(x_5)x_2 = Tx_5 = (a_1\beta_1^{-1} - \beta_4^{-1})x_5 + a_2\beta_2^{-1}x_1 - a_2\beta_2^{-1}\beta_3^{-1}\gamma_3 x_2,$$

which is impossible, as any three of the x_i's are linearly independent.

For $p = 3$, we have

$$Bx_3 = \beta_4 x_4, \quad Bx_4 = \beta_5 x_5, \quad Bx_5 = \beta_1 x_1, \quad Bx_1 = \beta_2 x_2, \quad Bx_2 = \beta_3 x_3 + \gamma_4 x_4,$$

where γ_4 and all β_i's are positive. A little calculation yields

$$Cx_3 = \beta_4^{-1}x_3, \quad Cx_4 = \beta_5^{-1}x_4, \quad Cx_1 = \beta_2^{-1}x_1,$$
$$Cx_5 = a_1\beta_1^{-1}x_5 + a_2\beta_2^{-1}x_1, \quad \text{ and } Cx_2 = \beta_3^{-1}x_2 - \beta_3^{-1}\beta_4^{-1}\gamma_4 x_3.$$

Suppose $x_2 = f_1x_1 + f_2x_3 + f_3x_4$. As for the previous case, by considering the coefficients of x_1 and x_4 in two different relations for Cx_2, we infer that $\beta_2 = \beta_5$. Then it is readily seen that $T := C - \beta_5^{-1}I$ is of rank one and with range space spanned by x_3. So there exists a linear functional φ of $\mathbb{R}^3$ such that $Ty = \varphi(y)x_3$ for all $y \in \mathbb{R}^3$. In particular, we have $\varphi(x_5)x_3 = Tx_5 = (a_1\beta_1^{-1} - \beta_5^{-1})x_5 + a_2\beta_2^{-1}x_1$, which is a contradiction.

By a similar argument we also dispose of the remaining possibilities $p = 4$ and $p = 5$.

Now we consider case (II). Here we define $C = AB$ (instead of $C = B^{-1}A$). We are going to show that only the subcase $p = 2$ can happen.

For $p = 3$, we have

$$Bx_3 = \beta_2x_2, \quad Bx_2 = \beta_1x_1, \quad Bx_1 = \beta_5x_5, \quad Bx_5 = \beta_4x_4, \quad Bx_4 = \beta_3x_3 + \gamma_2x_2,$$

where γ_2 and all β_i's are positive. Hence

$$Cx_2 = ABx_2 = \beta_1x_2, \quad Cx_3 = ABx_3 = \beta_2x_3, \quad Cx_5 = \beta_4x_5,$$
$$Cx_1 = ABx_1 = a_1\beta_5x_1 + a_2\beta_5x_2, \text{ and } \quad Cx_4 = ABx_4 = \beta_3x_4 + \gamma_2x_3.$$

Suppose $x_1 = f_1x_2 + f_2x_3 + f_5x_5$. Then we have

$$Cx_1 = \beta_1f_1x_2 + \beta_2f_2x_3 + \beta_4f_3x_5$$

and

$$Cx_1 = (a_1\beta_5f_1 + a_2\beta_5)x_2 + (a_1\beta_5f_2)x_3 + (a_1\beta_5f_3)x_5,$$

from which we obtain $\beta_2 = \beta_4$. It is readily seen that $T := C - \beta_2I$ is of rank one and with range space spanned by x_2. Hence there exists a linear functional φ of $\mathbb{R}^3$ such that $Ty = \varphi(y)x_2$ for all $y \in \mathbb{R}^3$. In particular, we have $\varphi(x_4)x_2 = T(x_4) = (\beta_3 - \beta_2)x_4 + \gamma_2x_3$, which is a contradiction.

For $p = 4$, we let

$$Bx_4 = \beta_3x_3, \quad Bx_3 = \beta_2x_2, \quad Bx_2 = \beta_1x_1, \quad Bx_1 = \beta_5x_5, \quad Bx_5 = \beta_4x_4 + \gamma_3x_3,$$

where γ_3 and all β_i's are positive. By the same argument as before, we obtain $\beta_2 = \beta_3$. Then by considering Tx_5, where $T := C - \beta_2I$, we readily arrive at a contradiction.

For $p = 5$, we have

$$Bx_5 = \beta_4x_4, \quad Bx_4 = \beta_3x_3, \quad Bx_3 = \beta_2x_2, \quad Bx_2 = \beta_1x_1, \quad Bx_1 = \beta_5x_5 + \gamma_4x_4,$$

where γ_4 and all β_i's are positive. Then

$$Cx_2 = \beta_1x_2, \quad Cx_3 = \beta_2x_3, \quad Cx_4 = \beta_3x_4, \quad Cx_5 = \beta_4x_5,$$
$$\text{and } Cx_1 = A(\beta_5x_5 + \gamma_4x_4) = a_1\beta_5x_1 + a_2\beta_5x_2 + \gamma_4x_5.$$

By Theorem 2.7, from the first four equality relations we infer that C is a scalar matrix. This contradicts the last equality relation, as x_1, x_2, x_5 are linearly independent.

For $p = 1$, we have

$$Bx_1 = \beta_5x_5, \quad Bx_5 = \beta_4x_4, \quad Bx_4 = \beta_3x_3, \quad Bx_3 = \beta_2x_2, \quad Bx_2 = \beta_1x_1 + \gamma_5x_5,$$

where (as in previous cases) all the coefficients that appear above are positive. Then,

$$Cx_3 = \beta_2 x_3, \quad Cx_4 = \beta_3 x_4, \quad Cx_5 = \beta_4 x_5,$$
$$Cx_1 = \beta_5 Ax_5 = a_1\beta_5 x_1 + a_2\beta_5 x_2,$$
$$\text{and } Cx_2 = A(\beta_1 x_1 + \gamma_5 x_5) = a_1\gamma_5 x_1 + (a_2\gamma_5 + \beta_1)x_2.$$

Note that x_3, x_4, x_5 are eigenvectors of C. Moreover, Cx_1, Cx_2 are each a linear combination of x_1 and x_2 with positive coefficients. By the Perron-Frobenius theory, there exist $\mu_1, \mu_2 > 0$ such that the vector $u = \mu_1 x_1 + \mu_2 x_2$ is also an eigenvector of C. As $\Phi(x_1), \Phi(x_2)$ are neighborly extreme rays of K, it is readily seen that u is a linear combination of x_3, x_4, x_5 with nonzero coefficients. So by Theorem 2.7 C is a scalar matrix, which is a contradiction.

Now consider the remaining subcase $p = 2$. We contend that there exists (up to positive multiples) a unique 3×3 real matrix B that satisfies

$$Bx_2 = \beta_1 x_1, \quad Bx_1 = \beta_5 x_5, \quad Bx_5 = \beta_4 x_4, \quad Bx_4 = \beta_3 x_3, \quad Bx_3 = \beta_2 x_2 + \gamma_1 x_1$$

or, equivalently, a unique 3×3 real matrix C that satisfies

$$Cx_2 = \beta_1 x_2, \quad Cx_4 = \beta_3 x_4, \quad Cx_5 = \beta_4 x_5,$$
$$Cx_1 = a_1\beta_5 x_1 + a_2\beta_5 x_2, \quad Cx_3 = \beta_2 x_3 + \gamma_1 x_2,$$

and, moreover, γ_1 and the β_i's can be chosen all positive. Suppose

$$x_1 = f_1 x_2 + f_2 x_4 + f_3 x_5 \quad \text{and} \quad x_3 = g_1 x_2 + g_2 x_4 + g_3 x_5.$$

By computing Cx_1 and Cx_3 in two different ways, we obtain respectively

$$\beta_1 f_1 x_2 + \beta_3 f_2 x_4 + \beta_4 f_3 x_5 = (a_1\beta_5 f_1 + a_2\beta_5)x_2 + a_1\beta_5 f_2 x_4 + a_1\beta_5 f_3 x_5$$

and

$$\beta_1 g_1 x_2 + \beta_3 g_2 x_4 + \beta_4 g_3 x_5 = (\beta_2 g_1 + \gamma_1)x_2 + \beta_2 g_2 x_4 + \beta_2 g_3 x_5.$$

Then by comparing the coefficients of x_2, x_4 and x_5 we obtain the following square homogeneous system of linear equations in the unknowns $\beta_1, \dots, \beta_5, \gamma_1$:

$$\begin{bmatrix} f_1 & 0 & 0 & 0 & -a_1 f_1 - a_2 & 0 \\ 0 & 0 & f_2 & 0 & -a_1 f_2 & 0 \\ 0 & 0 & 0 & f_3 & -a_1 f_3 & 0 \\ g_1 & -g_1 & 0 & 0 & 0 & -1 \\ 0 & -g_2 & g_2 & 0 & 0 & 0 \\ 0 & -g_3 & 0 & g_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \gamma_1 \end{bmatrix} = \mathbf{0}.$$

It is not difficult to see that the first four rows together with the last row of the coefficient matrix of the said system form a linearly independent set. So the coefficient matrix has rank at least 5. On the other hand, one can check that the vector $w := (1 + f_1^{-1}a_2 a_1^{-1}, 1, 1, 1, a_1^{-1}, g_1 f_1^{-1} a_2 a_1^{-1})^T$ is a solution vector for the system. Hence, the solution of the system is spanned by w.

Now by Lemma 3.1, for $i = 1, \dots, 5$, $\Phi(x_i)$ and $\Phi(x_{i+1})$ (where $\Phi(x_6)$ is taken to be $\Phi(x_1)$) are neighborly extreme rays of K. From the geometry it is clear that a certain positive linear combination of x_1, x_4 can be rewritten as a positive linear combination of x_2 and x_5. As a consequence, when x_1 is expressed as a linear combination of x_2, x_4, x_5, the coefficients of x_2 are positive. So $f_1 > 0$. By a similar argument we can also show that $g_1 > 0$. As a_1, a_2 are positive, w has positive entries. So our contention follows. This establishes part (ii), except that we have not yet shown that A is cone-equivalent to a positive multiple of B.

(ii) Let P be a nonscalar automorphism of K. Then $B := P^{-1}AP$ is an exp-maximal K-primitive matrix. If $A = B$, then P commutes with A, and by the Assertion established in the proof of Corollary 5.4(i) P is a scalar matrix, which is a contradiction. So B is different from A, and also not a multiple of A, as A and B are similar and are nonnilpotent. By the proof of part (i) (for Case (II) with $p = 2$) B must satisfy the following equality relations:

$$Bx_2 = \beta_1 x_1, Bx_1 = \beta_5 x_5, Bx_5 = \beta_4 x_4, Bx_4 = \beta_3 x_3, Bx_3 = \beta_2 x_2 + \gamma_1 x_1, \tag{5.1}$$

where the scalars γ_1 and β_i's are all positive. In view of the definition of B, the last condition in relation (5.1) can be rewritten as $APx_3 = \beta_2 Px_2 + \gamma_1 Px_1$. Since P is an automorphism, P permutes the extreme rays of K. As $\Phi(x_5)$ is the only extreme ray of K that is not mapped to an extreme ray under A, Px_3 must be a positive multiple of x_5. From the last but one condition in (5.1) we also have $APx_4 = \beta_3 Px_3$, and as $Ax_4 = x_5$ (and A is nonsingular), it follows that Px_4 is a positive multiple of x_4. Similarly, from the third and second condition in (5.1) we can also show that Px_5 is a positive multiple of x_3 and Px_1 is a positive multiple of x_2. Now it should be clear that Px_2 is a positive multiple of x_1, and also P^2 maps every extreme ray of K onto itself. By Theorem 2.7, replacing P by a suitable positive multiple, we may assume that $P^2 = I$. Hence P must satisfy the following set of conditions:

$$\begin{gathered} Px_1 = \mu_2 x_2, \quad Px_2 = \mu_1 x_1, \quad Px_3 = \mu_5 x_5, \quad Px_4 = x_4, \quad Px_5 = \mu_3 x_3, \\ \text{where } \mu_1, \mu_2, \mu_3, \mu_5 > 0, \quad \mu_1\mu_2 = \mu_3\mu_5 = 1. \end{gathered}$$

It is clear that if P is a matrix that satisfies the above set of conditions, then P is an automorphism of K, which is an involution, different from the identity matrix. To complete the proof, it remains to show that there exists a unique matrix P that satisfies the above set of conditions or, equivalently, to show that there is a unique way to choose a positive scalar μ_3 such that the 3×3 matrix P determined by

$$P[x_3 \quad x_4 \quad x_5] \;=\; [\mu_3^{-1}x_5 \quad x_4 \quad \mu_3 x_3]$$

satisfies $Px_1 = \mu_1^{-1}x_2$ for some $\mu_1 > 0$. (Then the condition $Px_2 = \mu_1 x_1$ is automatically satisfied because by the defining condition P is an involution.)

Now write

$$x_1 = \varphi_1 x_3 + \varphi_2 x_4 + \varphi_3 x_5 \text{ and } x_2 = \psi_1 x_3 + \psi_2 x_4 + \psi_3 x_5. \tag{5.2}$$

By comparing the coefficients of x_3, x_4, x_5 in the representations of Px_1 and x_2 in terms of x_3, x_4, x_5, the problem is reduced to proving that there exists a unique positive scalar μ_3 satisfying

$$\frac{\varphi_3\mu_3}{\psi_1} = \frac{\varphi_2}{\psi_2} = \frac{\varphi_1\mu_3^{-1}}{\psi_3} > 0.$$

After a little calculation one can see that the problem is equivalent to showing that the numbers $\frac{\varphi_2}{\psi_2}$ and $\frac{\varphi_1\psi_1}{\varphi_3\psi_3}$ are positive and we have $\varphi_1\varphi_3\psi_2^2 = \psi_1\psi_3\varphi_2^2$. (Then $\mu_3 = \sqrt{\frac{\varphi_1\psi_1}{\varphi_3\psi_3}}$.)

The scalars φ_i, ψ_i, $i = 1, 2, 3$, can be computed from (5.2) by using Cramer's rule. Noting that they all have a common denominator (namely, $|\ x_3 \quad x_4 \quad x_5\ |$), after some calculations we obtain

$$\frac{\varphi_2}{\psi_2} = \frac{|\ x_1 \quad x_3 \quad x_5\ |}{|\ x_2 \quad x_3 \quad x_5\ |}, \quad \frac{\varphi_1\psi_1}{\varphi_3\psi_3} = \frac{|\ x_1 \quad x_4 \quad x_5\ ||\ x_2 \quad x_4 \quad x_5\ |}{|\ x_1 \quad x_4 \quad x_3\ ||\ x_2 \quad x_4 \quad x_3\ |},$$
$$\text{and} \quad \frac{\varphi_1\varphi_3\psi_2^2}{\psi_1\psi_3\varphi_2^2} = \frac{|\ x_1 \quad x_4 \quad x_5\ ||\ x_3 \quad x_4 \quad x_1\ ||\ x_3 \quad x_2 \quad x_5\ |^2}{|\ x_2 \quad x_4 \quad x_5\ ||\ x_3 \quad x_4 \quad x_2\ ||\ x_3 \quad x_1 \quad x_5\ |^2}.$$

By Cramer's rule again, $\frac{|\ x_1 \quad x_3 \quad x_5\ |}{|\ x_2 \quad x_3 \quad x_5\ |}$ is equal to the coefficient of x_2 when x_1 is written as a linear combination of x_2, x_3 and x_5. From the geometry (using the kind of argument that has appeared near the end of the proof for part (i)) it is clear that this coefficient is positive. So $\frac{\varphi_2}{\psi_2} > 0$. Similarly, we can show that $\frac{|\ x_1 \quad x_4 \quad x_5\ |}{|\ x_1 \quad x_4 \quad x_3\ |}$ and $\frac{|\ x_2 \quad x_4 \quad x_5\ |}{|\ x_2 \quad x_4 \quad x_3\ |}$ are both negative. So we also have $\frac{\varphi_1\psi_1}{\varphi_3\psi_3} > 0$. It remains to show that

$$\begin{aligned} &|\ x_1 \quad x_4 \quad x_5\ ||\ x_3 \quad x_4 \quad x_1\ ||\ x_3 \quad x_2 \quad x_5\ |^2 \\ = &|\ x_2 \quad x_4 \quad x_5\ ||\ x_3 \quad x_4 \quad x_2\ ||\ x_3 \quad x_1 \quad x_5\ |^2. \end{aligned} \tag{5.3}$$

It will take some calculations to arrive at the latter equality relation. To proceed further, we choose K and A to be K_θ and A_θ, respectively — it is clear that there is no loss of generality in doing this. So we have $x_j = (r_\theta^{j-1}\cos\theta, r_\theta^{j-1}\cos\theta, 1)^T$ for $j = 1, \ldots, 5$, where r_θ has the same meaning as before. We find it more convenient to work in the complex domain. Let C denote the matrix $\begin{bmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is clear that relation (5.3) is equivalent to the one obtained from it by replacing each determinant that appears in the relation by the determinant multiplied by $|C|$. Now note that we have

$$|C|\Big|\ x_3 \quad x_1 \quad x_5\ \Big| = \begin{vmatrix} z^2 & 1 & z^4 \\ \bar{z}^2 & 1 & \bar{z}^4 \\ 1 & 1 & 1 \end{vmatrix}, \quad |C|\Big|\ x_3 \quad x_2 \quad x_5\ \Big| = \begin{vmatrix} z^2 & z & z^4 \\ \bar{z}^2 & \bar{z} & \bar{z}^4 \\ 1 & 1 & 1 \end{vmatrix}, \text{etc.},$$

where $z = r_\theta e^{i\theta}$, and by straightforward calculations we also obtain

$$\begin{aligned}
\begin{vmatrix} 1 & z^3 & z^4 \\ 1 & \bar{z}^3 & \bar{z}^4 \\ 1 & 1 & 1 \end{vmatrix} &= (\bar{z}-z)(1-z)(1-\bar{z})[(\bar{z}^2+z^2)+z\bar{z}(\bar{z}+z)+\bar{z}z+\bar{z}^2z^2], \\
\begin{vmatrix} z^2 & z^3 & 1 \\ \bar{z}^2 & \bar{z}^3 & 1 \\ 1 & 1 & 1 \end{vmatrix} &= (\bar{z}-z)(1-z)(1-\bar{z})(\bar{z}+z+z\bar{z}), \\
\begin{vmatrix} z^2 & z & z^4 \\ \bar{z}^2 & \bar{z} & \bar{z}^4 \\ 1 & 1 & 1 \end{vmatrix} &= -r_\theta^2(\bar{z}-z)(1-z)(1-\bar{z})(1+\bar{z}+z), \\
\begin{vmatrix} z & z^3 & z^4 \\ \bar{z} & \bar{z}^3 & \bar{z}^4 \\ 1 & 1 & 1 \end{vmatrix} &= r_\theta^2(\bar{z}-z)(1-z)(1-\bar{z})(\bar{z}+z+z\bar{z}),
\end{aligned}$$

$$\begin{vmatrix} z^2 & z^3 & z \\ \bar{z}^2 & \bar{z}^3 & \bar{z} \\ 1 & 1 & 1 \end{vmatrix} = r_\theta^2(\bar{z}-z)(1-z)(1-\bar{z}),$$

and

$$\begin{vmatrix} z^2 & 1 & z^4 \\ \bar{z}^2 & 1 & \bar{z}^4 \\ 1 & 1 & 1 \end{vmatrix} = -(\bar{z}-z)(\bar{z}+z)(1+z)(1-z)(1-\bar{z})(1+\bar{z}).$$

Substitute the values of $\begin{vmatrix} 1 & z^3 & z^4 \\ 1 & \bar{z}^3 & \bar{z}^4 \\ 1 & 1 & 1 \end{vmatrix}$, etc. into the said relation obtained from relation (5.3). Upon simplification we obtain

$$[(\bar{z}^2+z^2)+\bar{z}z(\bar{z}+z)+\bar{z}z+\bar{z}^2z^2](1+\bar{z}+z)^2 = [(1+\bar{z})(1+z)(\bar{z}+z)]^2.$$

The latter relation can be rewritten as

$$[2r^2(2c^2-1)+r^2(2cr)+r^2+r^4](1+2cr)^2 = (1+2cr+r^2)^2(2cr)^2,$$

where for simplicity we write $\cos\theta$ as c and r_θ as r. Upon further simplification, the relation becomes

$$(8c^3-4c)r^3+(4c^2-1)r^2+2cr+1 = 0. \tag{5.4}$$

We are going to show that relation (5.4) always holds, thus establishing relation (5.3). By Corollary 3.3 $z = re^{i\theta}$ is a root of the polynomial $t^5 - \hat{c}t - (1-\hat{c})$, where $\hat{c} = r^4\frac{\sin 5\theta}{\sin\theta}$. So we have $\frac{z^5-1}{z-1} = \frac{\hat{c}z+(1-\hat{c})-1}{z-1} = \hat{c}$, which implies $\mathrm{Im}(z^4+z^3+z^2+z) = 0$. Making use of the trigonometric identities $\frac{\sin 2\theta}{\sin\theta} = 2\cos\theta$, $\frac{\sin 3\theta}{\sin\theta} = 4\cos^2\theta - 1$, and $\frac{\sin 4\theta}{\sin\theta} = 4\cos\theta(2\cos^2\theta-1)$, after further calculations we find that the latter relation becomes relation (5.4).

We have proved the existence of a unique automorphism P of K, which is an involution, different from the identity matrix. Our argument also shows that $P^{-1}AP$ is a positive multiple of B. So the last part of part (i) is also established. □

Corollary 5.6. *Part* (ii) *of Corollary* 5.4 *can be extended to cover the case* $m = 5$.

Proof. Let $\theta, \phi \in (\frac{2\pi}{5}, \frac{\pi}{2})$ and suppose that there is an isomorphism P that maps K_ϕ onto K_θ. Then $P^{-1}A_\theta P$ is an ex-maximal K_ϕ-primitive matrix. By Theorem 5.5(i) $P^{-1}A_\theta P$ equals either A_ϕ or some K_ϕ-primitive matrix which is cone-equivalent to A_ϕ. In any case A_θ is cone-equivalent to A_ϕ. So by Theorem 5.3(i) we have $\theta = \phi$. □

To conclude this section, we would like to mention the following known fact: almost all proper cones in $\mathbb{R}^n_+$ are smooth and strictly convex and admit only trivial automorphisms (see [13, Theorem 5.5 and the paragraph following it]).

6. Remarks and open questions

Recall that a closed pointed cone K is said to be the *direct sum* of its subcones $K_1, \ldots, K_p$, and we write $K = K_1 \oplus \cdots \oplus K_p$ if every vector of K can be expressed uniquely as $x_1 + \cdots + x_p$, where $x_i \in K_i, 1 \le i \le p$. K is said to be *decomposable* if it is the direct sum of two nonzero subcones; otherwise, K is said to be *indecomposable*.

It is known that every n-dimensional indecomposable minimal cone with distinct extreme vectors $x_1, \ldots, x_{n+1}$ satisfies a unique relation of the form $\alpha_1 x_1 + \alpha_2 x_2 +$

$\cdots + \alpha_{n+1} x_{n+1} = 0$, where the scalars $\alpha_1, \ldots, \alpha_{n+1}$ are all nonzero (see [9, Theorem 2.25]). If the number of positive α_i's and the number of negative α_i's differ by at most one, then the relation is said to be *balanced*.

One consequence of Theorem 4.1 is the following:

Remark 6.1. A polyhedral cone $K \in \mathcal{P}(m,n)$ is decomposable, nonsimplicial and exp-maximal if and only if the integer n is even and K is a minimal cone which is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors.

The "if" part of the preceding remark is a consequence of [22, Theorem 4.1(III)(i)]. The fact that the "only if" part also holds can be shown as follows.

Let $K \in \mathcal{P}(m,n)$ be an exp-maximal decomposable nonsimplicial polyhedral cone and let A be an exp-maximal K-primitive matrix. According to Theorem 4.1, $\gamma(A)$ equals $(n-1)(m-1)+1$, unless m is odd and n is even, and in the latter case $\gamma(A)$ equals $(n-1)(m-1)$. Here we cannot have $\gamma(A) = (n-1)(m-1)+1$, because by Theorem 2.5(i) the latter condition implies that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1, and then by [21, Lemma 2.4(iii)] K is indecomposable. So we have $\gamma(A) = (n-1)(m-1)$. Then, by the same result, one of the following must hold: $n = m_A = 3$, or $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1, or $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 2 and K is an even-dimensional minimal cone which is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors. As the first two cases cannot happen, we obtain the desired condition on K.

Question 6.1. Identify indecomposable exp-maximal cones in $\mathcal{P}(m,n)$ for $m > n$, $m \neq n+1$ and $n \neq 3$.

In this work we are able to identify the exp-maximal cones and the corresponding exp-maximal primitive matrices only for the extreme cases $m = n, m = n+1$ and $n = 3$. To deal with the other cases, one may first consider the following question, which has been posed as an open question in [21]:

Question 6.2. Given positive integers m, n with $3 \leq n \leq m$, characterize the $n \times n$ real matrices A with the property that there exists $K \in \mathcal{P}(m,n)$ such that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1.

It is known that an $n \times n$ real matrix A is K-nonnegative for some proper cone K in $\mathbb{R}^n$ if and only if A satisfies the Perron-Schaefer condition, i.e., the spectral radius of A is an eigenvalue, with index not less than that of any other eigenvalue with the same modulus. A substantial amount of work has been done on real matrices that leave invariant a proper cone and have certain specific properties. We refer the interested reader to [41], [44] or [37].

Using the results (or proof techniques) of this paper, it is not difficult to obtain the following answers to the preceding question for two special cases, namely, $n = 3, m \geq 4$, and $m = n$. For convenience, we normalize the matrices under consideration.

Remark 6.2. Let A be a 3×3 real matrix with $\rho(A) = 1$, and let $m \geq 4$ be a given positive integer. Then there exists $K \in \mathcal{P}(m,3)$ such that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 (and hence A is exp-maximal K-primitive) if and only if the eigenvalues of A are $1, r_\theta e^{\pm i\theta}$, where $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and r_θ is the unique positive real root of the polynomial $g_\theta(t)$ given in Lemma 3.2.

Remark 6.3. Let A be an $n \times n$ real matrix with $\rho(A) = 1$. Then there exists $K \in \mathcal{P}(n,n)$ such that A is K-nonnegative and the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1 (and hence A is exp-maximal K-primitive) if and only if A is nonderogatory and the characteristic polynomial of A is of the form $t^n - ct - (1-c)$, where $c \in (0,1)$.

We would like to add that in the "if" part of Remark 6.3, except for the case $c = c_n, n$ being odd, we may omit the assumption that A be nonderogatory. The point is, for $c \neq c_n$, by Lemma 3.2(i) A has simple eigenvalues and hence is, necessarily, nonderogatory. As an illustration for the exceptional case, consider the matrix $A = \operatorname{diag}(-\frac{1}{2}, -\frac{1}{2}, 1)$. Its characteristic polynomial is $t^3 - \frac{3}{4}t - \frac{1}{4}$, which is of the said form. (Recall that $c_3 = \frac{3}{4}$.) Since A is not nonderogatory, by Lemma 2.4(i), there does not exist $K \in \mathcal{P}(n,n)$ such that A is K-nonnegative and the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Figure 1.

Appendix. Alternative proof of Lemma 4.4

It is clear that there is an automorphism of K_0 that takes y_j to $y_{m+j+1-n}$ for $j = 1, \ldots, m$ (where for $r > m$, y_r is taken to be y_s with $s \equiv r (\operatorname{mod} m), 1 \leq s \leq m$). To prove Lemma 4.4 it suffices to show that $\sum_{j=1}^{n-1} y_j$ lies in ∂K_0 and generates a simplicial face.

For $p = n, \ldots, m$, let Q_p denote the $n \times n$ matrix $\begin{bmatrix} y_1 & y_2 & \cdots & y_p \end{bmatrix}$. We need to show that the determinants $|Q_p|$ are all nonzero and have the same sign.

We find it more convenient to work in the complex domain. Write $e^{2\pi i/m}$ as ω and denote by $V(\lambda_1, \ldots, \lambda_n)$ the $n \times n$ Vandermonde matrix whose kth row is $\begin{bmatrix} 1 & \lambda_k & \cdots & \lambda_k^{n-1} \end{bmatrix}$. Pre-multiplying the matrix Q_p by

$$\underbrace{\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{\frac{n-2}{2} \text{ times}} \oplus I_2 \text{ when } n \text{ is even}$$

or by

$$\underbrace{\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{\frac{n-1}{2} \text{ times}} \oplus (1) \text{ when } n \text{ is odd}$$

yields the matrix obtained from the Vandermonde matrix $V(\omega, \bar{\omega}, \omega^2, \bar{\omega}^2, \ldots, \omega^{\frac{n-2}{2}}, \bar{\omega}^{\frac{n-2}{2}}, -1, 1)$ by replacing its last column by

$$\begin{bmatrix} \omega^{p-1} & \bar{\omega}^{p-1} & \cdots & \omega^{\frac{(n-2)(p-1)}{2}} & \bar{\omega}^{\frac{(n-2)(p-1)}{2}} & (-1)^{p-1} & 1 \end{bmatrix}^T$$

when m, n are both even or yields the matrix obtained from the Vandermonde matrix $V(\omega, \bar{\omega}, \omega^2, \bar{\omega}^2, \ldots, \omega^{\frac{n-1}{2}}, \bar{\omega}^{\frac{n-1}{2}}, 1)$ by replacing its last column by

$$\begin{bmatrix} \omega^{p-1} & \bar{\omega}^{p-1} & \cdots & \omega^{\frac{(n-1)(p-1)}{2}} & \bar{\omega}^{\frac{(n-1)(p-1)}{2}} & 1 \end{bmatrix}^T$$

when n is odd. Denote the determinant of the matrix by e_p.

Note that for each $p = n, n+1, \ldots, m, e_p$ is equal to $\det Q_p$ times a nonzero constant, which depends on n but not on p. More specifically, the said nonzero constant is $\pm 2^{\frac{n-2}{2}}$ if $n \equiv 2 \pmod 4$, $\pm 2^{\frac{n-2}{2}} i$ if $n \equiv 0 \pmod 4$, $\pm 2^{\frac{n-1}{2}}$ if $n \equiv 1 \pmod 4$, and $\pm 2^{\frac{n-1}{2}} i$ if $n \equiv 3 \pmod 4$. We want to show all e_p's are nonzero real numbers with the same sign if $n \equiv 2 \pmod 4$ or $n \equiv 1 \pmod 4$, and if $n \equiv 0$ (mod

4) or $n \equiv 3 \pmod 4$, then all ie_p's are also nonzero real numbers with the same sign.

We now consider a kind of generalized Vandermonde determinant on the indeterminates $t_1, \ldots, t_n$. Let $n \geq 3$. For every integer $p \geq n-1$, let $f_p(t_1, \ldots, t_n)$ denote the polynomial function $f_p(t_1, \ldots, t_n)$ given by the determinant of the matrix obtained from the Vandermonde matrix $V(t_1, t_2, \ldots, t_n)$ by replacing its last column by $\begin{bmatrix} t_1^p & t_2^p & \cdots & t_n^p \end{bmatrix}^T$.

By the kth *complete* (*homogeneous*) *symmetric polynomial* in $t_1, \ldots, t_n$, denoted by $h_k(t_1, t_2, \ldots, t_n)$, we mean the polynomial which is the sum of all monomials in $t_1, \ldots, t_n$ of degree k, where each monomial appears exactly once. For instance,

$$h_2(t_1, t_2, t_3) = t_1t_2 + t_1t_3 + t_2t_3 + t_1^2 + t_2^2 + t_3^2.$$

By definition, $h_0(t_1, \ldots, t_n) \equiv 1$. It is convenient to define $h_r(t_1, \ldots, t_n)$ to be 0 for $r < 0$.

Claim 1. For every integer $p \geq n-1$, we have

$$f_p(t_1, t_2, \ldots, t_n) = h_{p+1-n}(t_1, t_2, \ldots, t_n) \prod_{1 \leq i < j \leq n} (t_j - t_i). \tag{6.1}$$

The above claim can be deduced from a formula that expresses a Schur function $s_\lambda(t_1, \ldots, t_n)$ as a determinant involving complete symmetric polynomials $h_r(t_1, \ldots, t_n)$, known as the *Jacobi-Trudi determinant* (see Macdonald [23, p. 25] or Sagan [31, pp. 154-159]). Recall that if $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a *partition* of length $\leq n$ (i.e. a finite or infinite sequence of nonnegative integers in nonincreasing order such that the number of nonzero λ_i is at most n), then the quotient

$$\frac{|[t_i^{j-1+\lambda_{n-j+1}}]_{1 \leq i,j \leq n}|}{\prod_{1 \leq i < j \leq n}(t_j - t_i)}$$

is a symmetric polynomial in $t_1, \ldots, t_n$. It is called the *Schur function associated with* λ and is denoted by $s_\lambda(t_1, \ldots, t_n)$. For $p \geq n-1$, the polynomial $f_p(t_1, \ldots, t_n)$ introduced above satisfies

$$\frac{f_p(t_1, \ldots, t_n)}{\prod_{1 \leq i < j \leq n}(t_j - t_i)} = s_\lambda(t_1, \ldots, t_n),$$

where λ is the partition $(p - (n-1), 0, \ldots)$. According to the said determinantal formula, for any partition λ of length $\leq n$, we have

$$s_\lambda(t_1, \ldots, t_n) = |[h_{\lambda_i - i + j}(t_1, \ldots, t_n)]_{1 \leq i,j \leq n}|.$$

A little calculation shows that when $\lambda = (p - (n-1), 0, \ldots)$, $[h_{\lambda_i - i + j}]_{1 \leq i,j \leq n}$ is an upper triangular matrix whose $(1,1)$-entry is $h_{p-(n-1)}$ and all of whose other diagonal entries are equal to 1. It follows that we have $s_\lambda(t_1, \ldots, t_n) = h_{p+1-n}(t_1, \ldots, t_n)$, as desired. $\square$

For any set S of n complex numbers, we denote by $h_j(S)$ the value of $h_j(t_1, \ldots, t_n)$ evaluated at an n-tuple formed by all of the members of S, taken in any order. Similarly, we use $\sigma_j(S)$ to denote the *jth elementary symmetric function* on S.

Claim 2. Let S be a nonempty proper subset of $\mathbb{Z}_m$, the set of all mth roots of unity. Denote by S^c the complement of S in $\mathbb{Z}_m$. Then

$$h_j(S) = (-1)^j \sigma_j(S^c)$$

for $j = 1, \ldots, m-n$, where n is the cardinality of S.

Proof. For any set T of n complex numbers, as is known (see, for instance, [23]), the generating function for the elementary symmetric functions on T is given by

$$E(t;T) = \sum_{r=0}^{n} \sigma_r(T)t^r = \prod_{x\in T}(1+xt)$$

(where $\sigma_0(T)$ is taken to be 1). Also, the generating function for the complete symmetric polynomials on T is given by

$$H(t;T) = \sum_{r=0}^{\infty} h_r(T)t^r = \prod_{x\in T}\frac{1}{1-xt}.$$

In view of the relation $\prod_{x\in\mathbb{Z}_m}(1-xt) = 1-t^m$, for the given set S, we have

$$\begin{aligned}\sum_{r=0}^{m-n}(-1)^r\sigma_r(S^c)t^r = E(-t,S^c) = \prod_{x\in S^c}(1-xt) &= \frac{1-t^m}{\prod_{x\in S}(1-xt)}\\ &= (1-t^m)\Sigma_{r=0}^{\infty}h_r(S)t^r.\end{aligned}$$

By comparing the coefficients, our claim follows. $\square$

In view of Claim 1 and the discussion preceding the claim, it remains to show the following:

Claim 3. For $r = 1,\ldots,m-n$, $h_r(S) > 0$, where S is the subset of the set of mth roots of unity given by

$$S = \begin{cases}\{\omega,\bar\omega,\ldots,\omega^{\frac{n-2}{2}},\bar\omega^{\frac{n-2}{2}},-1,1\} \text{ when } m,n \text{ are both even},\\ \{\omega,\bar\omega,\ldots,\omega^{\frac{n-1}{2}},\bar\omega^{\frac{n-1}{2}},1\} \text{ when } n \text{ is odd}.\end{cases}$$

Now, by Claim 2 we have

$$\prod_{x\in S^c}(t-x) = t^{m-n} + \sum_{j=1}^{m-n}(-1)^j\sigma_j(S^c)t^{m-n-j} = t^{m-n} + \sum_{j=1}^{m-n}h_j(S)t^{m-n-j}.$$

To establish Claim 3, it suffices to show that the coefficients of the polynomial $\prod_{x\in S^c}(t-x)$ are all positive. We complete our argument by applying the following interesting nontrivial result due to Barnard et al. [2, Theorem 1]. (Or see [3, Theorem 2.4.5] or [8, Theorem 4].)

Theorem 6.4. *Let $p(t)$ be a polynomial of degree n, $p(0) = 1$, with nonnegative coefficients and zeros $a_1,\ldots,a_n$. For $\tau \geq 0$ write*

$$p_\tau(t) = \prod_{\substack{1\leq j\leq n\\ |\arg(a_j)|>\tau}}(1-t/a_j),$$

where $\arg(z)$ *is defined so that* $\arg(z)\in[-\pi,\pi)$. *If* $p_\tau(t)\neq p(t)$, *then the coefficients of $p_\tau(t)$ are all positive.*

Let us consider the case when m,n are both even first. In this case we have $S^c = \{\omega^{\frac{n}{2}},\bar\omega^{\frac{n}{2}},\ldots,\omega^{\frac{m}{2}-1},\bar\omega^{\frac{m}{2}-1}\}$. Take $p(t)$ to be the following polynomial, which has nonnegative coefficients and constant term 1:

$$\prod_{\substack{1\leq j\leq m-1\\ j\neq\frac{m}{2}}}(t-\omega^j) = \frac{t^m-1}{(t-1)(t+1)} = t^{m-2}+t^{m-4}+\cdots+t^2+1.$$

Choose any positive number τ from $(\frac{n-2}{m}\pi, \frac{n}{m}\pi)$. Then

$$p_\tau(t) = \prod_{j=\frac{n}{2}}^{\frac{m}{2}-1} (1 - \frac{t}{\omega^j})(1 - \frac{t}{\bar{\omega}^j}) = \prod_{j=\frac{n}{2}}^{\frac{m}{2}-1} (t - \omega^j)(t - \bar{\omega}^j).$$

By Theorem 6.4, $p_\tau(t)$ is a polynomial with positive coefficients. So $\prod_{x \in S^c}(t - x)$ equals $p_\tau(t)$ and is a polynomial with positive coefficients.

When m is even and n is odd, we take $p(t)$ to be the same polynomial, but choose τ from $(\frac{n-1}{m}\pi, \frac{n+1}{m}\pi)$. A little calculation shows that in this case $\prod_{x \in S^c}(t - x)$ is equal to $(t + 1)p_\tau(t)$, and so it also has positive coefficients.

When m, n are both odd, we take $p(t)$ to be the polynomial $t^{m-1} + t^{m-2} + \cdots + 1$ and apply a similar argument. $\square$

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