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Maximal exponents of polyhedral cones (I)

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ABSTRACT

Let K be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$. An $n \times n$ matrix A is said to be K-primitive if there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. For a polyhedral cone K, the maximum value of $\gamma(A)$, taken over all K-primitive matrices A, is called the exponent of K and is denoted by $\gamma(K)$. It is proved that if K is an n-dimensional polyhedral cone with m extreme rays then for any K-primitive matrix A, $\gamma(A) \leqslant (m_A - 1)(m - 1) + 1$, where m_A denotes the degree of the minimal polynomial of A, and the equality holds only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ associated with A (as a cone-preserving map) is equal to the unique (up to isomorphism) usual digraph associated with an $m \times m$ primitive matrix whose exponent attains Wielandt's classical sharp bound. As a consequence, for any n-dimensional polyhedral cone K with m extreme rays, $\gamma(K) \leqslant (n-1)(m-1)+1$. Our work answers in the affirmative a conjecture posed by Steve Kirkland about an upper bound of $\gamma(K)$ for a polyhedral cone K with a given number of extreme rays.

1. Introduction

If K is a polyhedral (proper) cone in $\mathbb{R}^n$ with m extreme rays, what is the maximum value of the exponents of K-primitive matrices? This question was posed by Steve Kirkland in an open problem session at the 8th ILAS conference held in Barcelona in July, 1999. Here by a *K-primitive matrix* we mean a real square matrix A for which there exists a positive integer k such that A^k maps every nonzero vector of K into the interior of K; the least such k is referred to as the *exponent of* A and is denoted by $\gamma(A)$. In view of Wielandt's classical sharp bound for exponents of (nonnegative) primitive matrices of a given order, Kirkland conjectured that $m^2 - 2m + 2$ is an upper bound for the maximum value considered in his question. This work is an outcome of our attempt to answer Kirkland's question.

In the classical nonnegative matrix case, the determination of upper bounds for the exponents of primitive matrices under various assumptions has been treated mainly by a graph-theoretic approach. Here for a K-primitive matrix A, we work with the digraph $(\mathcal{E}, \mathcal{P}(A, K))$, which is one of the four digraphs associated with A, introduced by Barker and Tam [6,23]. (Formal definitions will be given later.) Based on the same digraph, Niu [17] has started an initial study of the exponents of K-primitive matrices over a polyhedral cone K. His work has motivated partly the work of Tam [22] and our present work.

The study of K-primitive matrices in the general polyhedral cone case differs from the nonnegative matrix case (or, equivalently, the simplicial cone case) in at least two (not unrelated) respects. First, in the nonnegative matrix case the (distinct) extreme vectors of the underlying cone are linearly independent, whereas in the general polyhedral cone case the

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extreme vectors of the underlying cone satisfy certain nonzero (linear) relations. Second, in the nonnegative matrix case it is always possible to find a nonnegative matrix with a prescribed digraph as its associated digraph, whereas in the general polyhedral cone case we often need to treat first the realization problem, that is, to determine whether there is a polyhedral cone K for which there is a K-nonnegative matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by a prescribed digraph. As expected, and also illustrated by this work, the study of the polyhedral cone case is more difficult than the classical nonnegative matrix case.

We now describe the contents of this paper in some detail.

Section 2 contains most of the definitions, together with the relevant known results, which we need in the paper.

In Section 3 using cone-theoretic arguments we obtain a Sedláček–Dulmage–Mendelsohn type upper bound for the local exponents (see Section 2 for the definition), and hence also an upper bound for the exponent, of a K-primitive matrix A in terms of m_A, the degree of the minimal polynomial of A, and the lengths of circuits in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$. As an application we show that if A is K-irreducible then $(I+A)^{m_A-1}$ is K-positive, strengthening an early result of Schneider and Vidyasagar [20] and extending an improvement in the nonnegative matrix case due to Hartwig and Neumann [11]. Besides, using an argument involving the cone $\pi(K)$ of K-nonnegative matrices, we prove that if A is K-primitive and $m_A = 2$ then $\gamma(A)$ equals 1 or 2.

The results of Section 3 may suggest that for a K-primitive matrix A the longer the shortest circuit in $(\mathcal{E}, \mathcal{P}(A, K))$ is, the more likely it is that $\gamma(A)$ has a larger value. In Section 4 we single out digraphs on m ($\geqslant 4$) vertices, with the length of the shortest circuit equal to $m-1$, the largest possible value, that may be realized as $(\mathcal{E}, \mathcal{P}(A, K))$ for some K-primitive matrix A, where K is a polyhedral cone with m extreme rays (see Lemma 4.1). It is found that, up to graph isomorphism, there are two of them, represented by Fig. 1 and Fig. 2 respectively. (These figures will be given later in the paper.) It turns out that they are precisely the two known so-called primitive digraphs on m vertices with the length of the shortest circuit equal to $m-1$. When A is a K-nonnegative matrix such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1 or Fig. 2, we make some interesting observations on A, and by delicate manipulation with the relations on the extreme vectors, we also obtain certain geometric properties of K (see Lemma 4.2).

In Section 5 we apply the results in the preceding two sections to obtain further upper bounds for exponents of K-primitive matrices. In particular, it is proved that if K is an n-dimensional polyhedral cone with m extreme rays then its exponent $\gamma(K)$, which is defined to be $\max\{\gamma(A)\colon A \text{ is } K\text{-primitive}\}$, does not exceed $(n-1)(m-1)+1$, and the equality holds only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1. As a consequence, we answer in the affirmative the above-mentioned conjecture posed by Kirkland.

An example of a proper cone that does not have finite exponent is given in Section 6. An open problem is also posed in Section 7, the final section.

This paper lays the groundwork for further study of maximal exponents of polyhedral cones. The general question of what the maximum value of $\gamma(K)$ is when K is taken over all n-dimensional polyhedral cones with m extreme rays for a given pair of positive integers m, n will be treated in [15] and [16].

2. Preliminaries

We take for granted standard properties of nonnegative matrices, complex matrices and graphs that can be found in textbooks (see, for instance, [4,5,9,10,14]). A familiarity with elementary properties of finite-dimensional convex sets, convex cones and cone-preserving maps is also assumed (see, for instance, [2,18,21,25]). To fix notation and terminology, we give some definitions.

Let K be a nonempty subset of a finite-dimensional real vector space V. The set K is called a *convex cone* if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \geqslant 0$; K is *pointed* if $K \cap (-K) = \{0\}$; K is *full* if its interior $\operatorname{int} K$ (in the usual topology of V) is nonempty, equivalently, $K - K = V$. If K is closed and satisfies all of the above properties, K is called a *proper cone*.

In this paper, unless specified otherwise, we always use K to denote a proper cone in the n-dimensional Euclidean space $\mathbb{R}^n$.

We denote by $\geqslant^K$ the partial ordering of $\mathbb{R}^n$ induced by K, i.e., $x \geqslant^K y$ if and only if $x - y \in K$.

A subcone F of K is called a *face* of K if $x \geqslant^K y \geqslant^K 0$ and $x \in F$ imply $y \in F$. If $S \subseteq K$, we denote by $\Phi(S)$ the *face of K generated by S*, that is, the intersection of all faces of K including S. If $x \in K$, we write $\Phi(\{x\})$ simply as $\Phi(x)$. It is known that for any vector $x \in K$ and any face F of K, $x \in \operatorname{ri} F$ if and only if $\Phi(x) = F$; also, $\Phi(x) = \{y \in K\colon x \geqslant^K \alpha y \text{ for some } \alpha > 0\}$. (Here we denote by $\operatorname{ri} F$ the *relative interior of F*.) A vector $x \in K$ is called an *extreme vector* if either x is the zero vector or x is nonzero and $\Phi(x) = \{\lambda x\colon \lambda \geqslant 0\}$; in the latter case, the face $\Phi(x)$ is called an *extreme ray*. We use $\operatorname{Ext} K$ to denote the set of all nonzero extreme vectors of K. Two nonzero extreme vectors are said to be *distinct* if they are not multiples of each other. The cone K itself and the set $\{0\}$ are always faces of K, known as *trivial faces*. Other faces of K are said to be *nontrivial*.

If S is a nonempty subset of a vector space, we denote by $\operatorname{pos} S$ the *positive hull* of S, i.e., the set of all possible nonnegative linear combinations of vectors taken from S.

A closed pointed cone K is said to be the *direct sum* of its subcones $K_1, \ldots, K_p$, and we write $K = K_1 \oplus \cdots \oplus K_p$ if every vector of K can be expressed uniquely as $x_1 + x_2 + \cdots + x_p$, where $x_i \in K_i$, $1 \leqslant i \leqslant p$. K is called *decomposable* if it is the direct sum of two nonzero subcones; otherwise, it is said to be *indecomposable*. It is well known that every closed pointed cone K can be written as

$$K = K_1 \oplus \cdots \oplus K_p,$$

where each K_j is an indecomposable cone ($1 \leqslant j \leqslant p$). Except for the ordering of the summands, the above decomposition is unique. We will refer to the K_j's as *indecomposable summands* of K.

By a *polyhedral cone* we mean a proper cone which has finitely many extreme rays. By the *dimension of a proper cone* we mean the dimension of its linear span. A polyhedral cone is said to be *simplicial* if the number of extreme rays is equal to its dimension. The nonnegative orthant $\mathbb{R}^n_+ := \{(\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n\colon\ \xi_i \geqslant 0\ \forall i\}$ is a typical example of a simplicial cone.

We denote by $\pi(K)$ the set of all $n \times n$ real matrices A (identified with linear mappings on $\mathbb{R}^n$) such that $AK \subseteq K$. Members of $\pi(K)$ are said to be *K-nonnegative* and are often referred to as *cone-preserving maps*. It is clear that $\pi(\mathbb{R}^n_+)$ consists of all $n \times n$ (entrywise) nonnegative matrices.

A matrix $A \in \pi(K)$ is said to be *K-irreducible* if A leaves invariant no nontrivial face of K; A is *K-positive* if $A(K \setminus \{0\}) \subseteq \operatorname{int} K$ and is *K-primitive* if there is a positive integer p such that A^p is K-positive. If A is K-primitive, then the smallest positive integer p for which A^p is K-positive is called the *exponent* of A and is denoted by $\gamma(A)$ (or by $\gamma_K(A)$ if the dependence on K needs to be emphasized).

It is known that the set $\pi(K)$ forms a proper cone in the space of $n \times n$ real matrices, the interior of $\pi(K)$ being the subset consisting of K-positive matrices. Also, $\pi(K)$ is polyhedral if and only if K is polyhedral. (See [21,20] or [1].)

It is clear that if K is a simplicial cone with n extreme rays then K is linearly isomorphic to $\mathbb{R}^n_+$. The simplicial cones may be considered as the simplest kind of cones. The next simplest kind of cones, and also the one with which we will deal considerably in this work, are the minimal cones. Minimal cones were first introduced and studied by Fiedler and Pták [8]. We call an n-dimensional polyhedral cone *minimal* if it has precisely $n+1$ extreme rays. Clearly, if K is a minimal cone with (pairwise distinct) extreme vectors $x_1, \ldots, x_{n+1}$, then (up to multiples) these vectors satisfy a unique (linear) relation. For instance, the vectors $(1,0,0)^T, (0,1,0)^T, (1,0,1)^T$ and $(0,1,1)^T$ form the distinct nonzero extreme vectors of a minimal cone in $\mathbb{R}^3$. They satisfy a unique relation, namely, $(1,0,0)^T + (0,1,1)^T = (0,1,0)^T + (1,0,1)^T$. It is known that a minimal cone is indecomposable if and only if the relation for its extreme vectors is full, i.e., in the relation the coefficient of each extreme vector is nonzero (see [8, Theorem 2.25]). Also, every decomposable minimal cone is the direct sum of an indecomposable minimal cone and a simplicial cone.

In dealing with (nonzero) relations on (nonzero) extreme vectors of a polyhedral cone, we find it convenient to write such relations in the form

$$\alpha_1 x_1 + \cdots + \alpha_p x_p = \beta_1 y_1 + \cdots + \beta_q y_q,$$

where the extreme vectors $x_1, \ldots x_p, y_1, \ldots, y_q$ are pairwise distinct and the coefficients $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ are all positive. Clearly we have $p, q \geqslant 2$.

We call a relation on extreme vectors of a polyhedral cone *balanced* if the number of nonzero terms on its two sides differ by at most 1.

Let $A \in \pi(K)$. In this work we need the digraph $(\mathcal{E}, \mathcal{P}(A,K))$, which is one of the four digraphs associated with A introduced by Barker and Tam [6,23]. It is defined in the following way: its vertex set is $\mathcal{E}$, the set of all extreme rays of K; $(\Phi(x), \Phi(y))$ is an arc whenever $\Phi(y) \subseteq \Phi(Ax)$. If there is no danger of confusion, (in particular, within proofs) we write $(\mathcal{E}, \mathcal{P}(A,K))$ simply as $(\mathcal{E}, \mathcal{P})$. It is readily checked that if K is the nonnegative orthant $\mathbb{R}^n_+$ then $(\mathcal{E}, \mathcal{P}(A,K))$ equals the usual digraph associated with A^T, the transpose of A. (If $B = (b_{ij})$ is an $n \times n$ matrix then by the usual digraph of B we mean the digraph with vertex set $\{1, \ldots, n\}$ such that (i,j) is an arc whenever $b_{ij} \neq 0$.)

It is not difficult to show that for any $A, B \in \pi(K)$, if $\Phi(A) = \Phi(B)$—where $\Phi(A), \Phi(B)$ are faces of $\pi(K)$—then A, B are either both K-primitive or both not K-primitive, and if they are, then $\gamma(A) = \gamma(B)$. In Niu [17] it is proved that if K is a polyhedral cone then for any $A, B \in \pi(K)$, we have $(\mathcal{E}, \mathcal{P}(A,K)) = (\mathcal{E}, \mathcal{P}(B,K))$ (as labelled digraphs) if and only if $\Phi(A) = \Phi(B)$. So it is not surprising to find that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ plays a role in determining a bound for $\gamma(A)$. (When K is nonpolyhedral, the situation is more subtle. We refer the interested readers to Tam [22] for the details.)

Let K be a polyhedral cone. Then $\pi(K)$ is also polyhedral and hence has finitely many faces. Since K-primitive matrices that belong to the relative interior of the same face of $\pi(K)$ share a common exponent, it follows that there are only finitely many (integral) values that can be attained by the exponents of K-primitive matrices.

For a proper cone K, we say K *has finite exponent* if the set of exponents of K-primitive matrices is bounded; then we denote the maximum exponent by $\gamma(K)$ and refer to it as the *exponent of K*. By the above discussion, every polyhedral cone has finite exponent. An example of a proper cone in $\mathbb{R}^3$ which does not have finite exponent will be given in Section 6.

We will make use of the concept of a *primitive digraph*, which can be defined as a digraph for which there is a positive integer k such that for every pair of vertices i, j there is a directed walk of length k from i to j; the least such k is referred to as the *exponent of the digraph*. It is clear that a nonnegative matrix is primitive if and only if its usual digraph is primitive. It is also well known that primitive digraphs are precisely strongly connected digraphs with the greatest common divisor of the lengths of their circuits equal to 1.

It is known that for a K-nonnegative matrix A, if A^p is K-positive then so is A^q for every $q \geqslant p$. This follows from the fact that if B is a K-nonnegative matrix such that $Bu \in \partial K$ for some $u \in \operatorname{int} K$ then we have $BK \subseteq \Phi(Bu) \subseteq \partial K$. The same fact also implies that the action of a K-nonnegative matrix A on a vector x in K enjoys a similar property—if $A^i x$ belongs to $\operatorname{int} K$, then so does $A^j x$ for all positive integers $j > i$.

If A is a K-nonnegative matrix and if p is a positive integer such that $A^p F \subseteq F$ for some nontrivial face F, then $A^{kp} F \subseteq F$ for all positive integers k and hence A cannot be K-primitive. This shows that the positive powers of a K-primitive matrix are all K-irreducible.

To study the exponents of K-primitive matrices, we make use of the concept of local exponent defined in the following way. (For definition of local exponent of a primitive matrix, see [5, Section 3.5].) For any K-nonnegative matrix A, not necessarily K-primitive or K-irreducible, and any $0 \neq x \in K$, by the *local exponent of A at x*, denoted by $\gamma(A, x)$, we mean the smallest nonnegative integer k such that $A^k x \in \operatorname{int} K$. If no such k exists, we set $\gamma(A, x)$ equal ∞. (If A is a primitive matrix and e_j is the jth standard unit vector, then $\gamma(A, e_j)$ equals the smallest integer k such that all elements in column j of A^k are nonzero.) Clearly, A is K-primitive if and only if the set of local exponents of A is bounded; in this case, $\gamma(A)$ is equal to $\max\{\gamma(A, x)\colon 0 \neq x \in K\}$, which is also the same as the maximum taken over all nonzero extreme vectors of K. By a compactness argument Barker [1] has shown that the K-primitivity of A is equivalent to the apparently weaker condition—which is also the definition adopted by him for K-primitivity—that all local exponents of A are finite.

By the definition of $(\mathcal{E}, \mathcal{P}(A, K))$, we have

Fact 2.1. If there is a path in $(\mathcal{E}, \mathcal{P}(A, K))$ of length k from $\Phi(x)$ to $\Phi(y)$, then $\Phi(A^k x) \supseteq \Phi(y)$.

By Fact 2.1 we obtain the following:

Fact 2.2. Let K be a polyhedral cone. If A is a K-nonnegative matrix such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is primitive, then A is a K-primitive matrix with exponent less than or equal to that of the primitive digraph $(\mathcal{E}, \mathcal{P}(A, K))$.

To show the preceding fact, let k denote the exponent of the primitive digraph $(\mathcal{E}, \mathcal{P})$ and let $x \in \operatorname{Ext} K$. Then there is a path of length k in the said digraph from $\Phi(x)$ to $\Phi(y)$ for any $y \in \operatorname{Ext} K$. By Fact 2.1 we have $\Phi(A^k x) \supseteq \Phi(y)$. Since this is true for every $y \in \operatorname{Ext} K$, it follows that $A^k x \in \operatorname{int} K$. But x is an arbitrary nonzero extreme vector of K, so A^k is K-positive. Hence A is K-primitive and $\gamma(A) \leqslant k$.

Fact 2.1 implies also the following:

Fact 2.3. Let $A \in \pi(K)$ and let $x, y \in \operatorname{Ext} K$. Suppose that $\gamma(A, y)$ is finite. If there is a path in $(\mathcal{E}, \mathcal{P}(A, K))$ of length k from $\Phi(x)$ to $\Phi(y)$, then $\gamma(A, x)$ is also finite and we have $\gamma(A, x) \leqslant k + \gamma(A, y)$.

3. Upper bounds for exponents

Hereafter, for every pair of positive integers m, n with $3 \leqslant n \leqslant m$, we denote by $\mathcal{P}(m, n)$ the set of all n-dimensional polyhedral cones with m extreme rays. Note that we start with $n = 3$ as the cases $n = 1$ or 2 are trivial. Also, when $m = 3$, in order that $\mathcal{P}(m, n)$ is nonvacuous, n must be 3.

The following theorem of Sedláček [19] and Dulmage and Mendelsohn [7] (see, for instance, [5, Theorem 3.5.4]) gives an upper bound for the exponent of a primitive matrix A in terms of lengths of circuits in the digraph of A.

Theorem A. *Let A be an $n \times n$ primitive matrix. If s is the length of the shortest circuit in the digraph of A, then $\gamma(A) \leqslant n + s(n-2)$.*

By setting $s = n-1$ in Theorem A, one recovers the sharp general upper bound $(n-1)^2 + 1$, due to Wielandt [24], for exponents of $n \times n$ primitive matrices.

The next lemma gives an analogous result on the local exponents of a cone-preserving map, which is essential to our work.

If D is a digraph, v is a vertex of D and W is a nonempty subset of the vertex set of D, then we say *v has access to W* if there is a path from v to a vertex of W. In this case, the length (i.e., the number of edges) of the shortest path from v to a vertex of W is referred to as the *distance* from v to W. If v belongs to W, the distance is taken to be zero.

For a square matrix C, we denote by m_C the *degree of the minimal polynomial of C*.

Lemma 3.1. *Let K be a proper cone and let $A \in \pi(K)$. Let $\Phi(x)$ be a vertex of $(\mathcal{E}, \mathcal{P}(A, K))$ which is at a distance w ($\geqslant 0$) to a circuit $\mathcal{C}$ of length l. Suppose that A^l is K-irreducible, or that the circuit $\mathcal{C}$ contains a vertex $\Phi(u)$ for which $\gamma(A, u)$ is finite. Then $\gamma(A, x)$ is finite and*

$$\gamma(A, x) \leqslant w + (m_{A^l} - 1)l \leqslant w + (m_A - 1)l \leqslant w + (n-1)l.$$

Proof. Let $\mathcal{C}$: $\Phi(x_1) \to \Phi(x_2) \to \cdots \to \Phi(x_l) \to \Phi(x_1)$ be the circuit under consideration. (Here, for convenience, we represent the arc $(\Phi(x), \Phi(y))$ by $\Phi(x) \to \Phi(y)$.) Without loss of generality, we may assume that the distance from $\Phi(x)$ to $\Phi(x_1)$ is w. Since there is a path of length l from $\Phi(x_1)$ to itself, by Fact 2.1 we have $\Phi(A^l x_1) \supseteq \Phi(x_1)$, which implies the following chain of inclusions:

$$\Phi(x_1) \subseteq \Phi\bigl(A^l x_1\bigr) \subseteq \Phi\bigl(A^{2l} x_1\bigr) \subseteq \cdots \subseteq \Phi\bigl(A^{jl} x_1\bigr) \subseteq \Phi\bigl(A^{(j+1)l} x_1\bigr) \subseteq \cdots.$$

Let p denote the dimension of the subspace $\operatorname{span}\{(A^l)^j x_1\colon j = 0, 1, \ldots\}$. By the above chain of inclusions, clearly the face $\Phi((A^l)^{p-1} x_1)$ contains the vectors $x_1, A^l x_1, \ldots, (A^l)^{p-1} x_1$, which are linearly independent and hence form a basis for the said subspace; so $\Phi((A^l)^{p-1}x_1)$ includes, and hence is equal to, $\Phi(\operatorname{span}\{(A^l)^j x_1\colon\ j=0,1,\ldots\}\cap K)$. Note that the latter face is the smallest A^l-invariant face of K that contains x_1. If A^l is K-irreducible, the latter face is clearly K. On the other hand, if the circuit $\mathcal{C}$ contains a vertex $\Phi(u)$ for which $\gamma(A,u)$ is finite, then by Fact 2.3 $\gamma(A,x_1)$ is also finite. Hence $A^j x_1 \in \operatorname{int} K$ for all positive integers j sufficiently large and, as a consequence, the smallest A^l-invariant face of K that contains x_1 is K. In either case, we have, $\Phi((A^l)^{p-1}x_1) = K$ and so $(A^l)^{p-1}x_1 \in \operatorname{int} K$; hence $\gamma(A,x_1) \leqslant (p-1)l$. Then by Fact 2.3 we have

$$\gamma(A,x) \leqslant w + \gamma(A,x_1) \leqslant w + (p-1)l.$$

It is clear that $p \leqslant m_{A^l}$. But we also have $m_{A^l} \leqslant m_A \leqslant n$, so the desired inequalities follow. □

Lemma 3.2. *Let $K \in \mathcal{P}(m,n)$ and let A be a K-primitive matrix. If $\Phi(x)$ is a vertex of $(\mathcal{E},\mathcal{P}(A,K))$ which has access to a circuit of length l, then*

$$\gamma(A,x) \leqslant m + (m_A - 2)l \leqslant m + (n-2)l.$$

Proof. This follows from Lemma 3.1, as the distance from $\Phi(x)$ to $\mathcal{C}$ is at most $m-l$ and also A^l is K-irreducible. □

Using Lemma 3.1, one can also readily deduce the following result.

Corollary 3.3. *Let $K \in \mathcal{P}(m,n)$, and let $A \in \pi(K)$. Suppose that the digraph $(\mathcal{E},\mathcal{P}(A,K))$ is strongly connected. Let s be the shortest circuit length of the digraph. If A^s is K-irreducible, then A is K-primitive and $\gamma(A) \leqslant m + s(m_A - 2)$.*

It is known that if K is a polyhedral cone with m extreme rays, then a K-nonnegative matrix A is K-primitive if A^j are K-irreducible for $j = 1,\ldots,2^m - 1$ (see [1, Theorem 2]). The preceding corollary tells us that when the digraph $(\mathcal{E},\mathcal{P})$ is strongly connected, to show the K-primitivity of A, it suffices to check the K-irreducibility of only one positive power of A.

Clearly, the following result of Niu [17] is a consequence of Corollary 3.3:

Theorem B. *Let $K \in \mathcal{P}(m,n)$ and let A be K-primitive. If the digraph $(\mathcal{E},\mathcal{P}(A,K))$ is strongly connected and s is the length of the shortest circuit in $(\mathcal{E},\mathcal{P}(A,K))$, then $\gamma(A) \leqslant m + s(m-2)$.*

In Theorem B, by choosing $K = \mathbb{R}^n_+$ we recover Theorem A.

It is known (see [20]) that if A is K-irreducible, then $(I+A)^{n-1}$ is K-positive (where n is the dimension of K). Hartwig and Neumann [11] have shown that in the nonnegative matrix case the result can be strengthened by replacing n by m_A, the degree of the minimal polynomial of A. Now we can show that the latter improvement is also valid for a cone-preserving map on a proper cone.

Corollary 3.4. *If $A \in \pi(K)$ is K-irreducible, then $(I+A)^{m_A-1}$ is K-positive.*

Proof. If A is K-irreducible, then clearly $I+A$ is also K-irreducible and in the digraph $(\mathcal{E},\mathcal{P}(I+A,K))$ there is a loop at each vertex. By Lemma 3.1, $\gamma(I+A,x) \leqslant m_{I+A} - 1$ for every $x \in \operatorname{Ext} K$. But $m_{I+A} = m_A$, so $(I+A)^{m_A-1}$ is K-positive. □

It is also possible to provide a direct proof for Corollary 3.4, one that does not involve the digraph $(\mathcal{E},\mathcal{P})$.

We denote by $\mathcal{N}(A)$ the nullspace of A. It is easy to show that for any $A \in \pi(K)$, $\mathcal{N}(A) \cap K = \{0\}$ if and only if the outdegree of each vertex of $(\mathcal{E},\mathcal{P})$ is positive. As a consequence, for any K-primitive matrix A, the digraph $(\mathcal{E},\mathcal{P})$ has at least one circuit.

In contrast with the nonnegative matrix case, the digraph $(\mathcal{E},\mathcal{P})$ associated with a K-primitive matrix A (where K is polyhedral) need not be strongly connected. (Many such examples can be found in [22].) Nevertheless, every vertex of $(\mathcal{E},\mathcal{P})$ has access to some circuit of $(\mathcal{E},\mathcal{P})$. This makes it possible to apply Lemma 3.2 to obtain bounds for the exponents of K-primitive matrices.

As yet another application of Lemma 3.1, we obtain the following result, which is an extension of the corresponding result for a symmetric primitive matrix (cf. [5, Theorem 3.5.3]). Recall that a digraph D is said to be *symmetric* if for every pair of vertices u, v of D, (u,v) is an arc if and only if (v,u) is an arc.

Corollary 3.5. *Let $A \in \pi(K)$. If A is K-primitive and the digraph $(\mathcal{E},\mathcal{P}(A,K))$ is symmetric, then*

$$\gamma(A) \leqslant 2(m_{A^2} - 1) \leqslant 2(m_A - 1).$$

Proof. Since A is K-primitive, $\mathcal{N}(A) \cap K = \{0\}$; hence the digraph $(\mathcal{E},\mathcal{P})$ has an outgoing edge (possibly a loop) at each vertex. As the digraph $(\mathcal{E},\mathcal{P})$ is symmetric, it follows that $(\mathcal{E},\mathcal{P}(A^2,K))$ has a loop at each vertex. By Lemma 3.1 we have $\gamma(A^2) \leqslant m_{A^2} - 1$ and hence $\gamma(A) \leqslant 2(m_{A^2} - 1) \leqslant 2(m_A - 1)$. □

It is clear that for any K-primitive matrix $A, m_A \geqslant 2$. When $m_A = 2$, more can be said.

Lemma 3.6. *Let A be K-primitive. If $m_A = 2$ then $\gamma(A) = 1$ or 2.*

Proof. Since $m_A = 2$ (and A is a real matrix), there exist real numbers a, b such that $A^2 + aA + bI = 0$. Clearly, a, b cannot be both zero, as A is not nilpotent. By the pointedness of the cone $\pi(K)$, at least one of a, b is negative. If $b < 0$ and $a \geqslant 0$, then A^2 belongs to the face $\Phi(I)$ (of $\pi(K)$) and so it must be K-reducible, which is a contradiction. If $a < 0$ and $b \geqslant 0$, then $A^2 \in \Phi(A)$ or $A^2 \leqslant \alpha A$ for some $\alpha > 0$, which implies that all positive powers of A lie in $\Phi(A)$. But A^p is K-positive or, equivalently, belongs to $\operatorname{int}\pi(K)$ for p sufficiently large, it follows that in this case we must have $\Phi(A) = \pi(K)$, or in other words, $\gamma(A) = 1$. In the remaining case when a, b are both negative, A^2 is a positive linear combination of A and I and hence lies in $\operatorname{ri}\Phi(A + I)$. Then one readily shows that all positive powers of A also lie in $\operatorname{ri}\Phi(A + I)$. By the K-primitivity of A, A^p belongs to $\operatorname{int}\pi(K)$ for p sufficiently large. This implies that $\Phi(A + I) = K$. As A^2 is a positive linear combination of A and I, it follows that A^2 also belongs to $\operatorname{int}\pi(K)$; so we have $\gamma(A) \leqslant 2$. This completes the proof. □

4. Two special digraphs for K-primitive matrices

The results of Section 3 may suggest that for a K-primitive matrix A the longer the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is, the more likely it is that $\gamma(A)$ has a larger value. Given a pair of positive integers m, n with $3 \leqslant n \leqslant m$, what can we say about digraphs on m vertices, with the length of the shortest circuit equal to $m - 1$, which can be identified (up to graph isomorphism) with $(\mathcal{E}, \mathcal{P}(A, K))$ for some pair (A, K) where $K \in \mathcal{P}(m, n)$ and A is a K-primitive matrix ? It turns out that such digraphs must be primitive. When $m \geqslant 4$, apart from the labelling of its vertices (or, in other words, up to graph isomorphism), there are two such digraphs, which are given by Fig. 1 and Fig. 2. When $m = 3$, there is one more digraph. (See Lemma 4.1 below.)

Note that if K is a polyhedral cone with m extreme rays then for any K-primitive matrix A, the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is at most $m - 1$. This is because, if the length of the shortest circuit is m, then the digraph must be a circuit of length m and, as a consequence, A^m is K-reducible, which is impossible.

In what follows when we say the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 (or by other figures), we mean the digraph is given either by the figure up to graph isomorphism or by the figure as a labelled digraph. In most instances, we mean it in the former sense but in a few instances we mean it in the latter sense. It should be clear from the context in what sense we mean. (For instance, in parts (i) and (iii) of Lemma 4.2 we mean the former sense, but in part (ii) we mean the latter sense.)

We will obtain certain geometric properties of K when K is a non-simplicial polyhedral cone with m ($\geqslant 4$) extreme rays for which there exists a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2. More precisely, we show that if $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 then K is indecomposable; if $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2 and if K is decomposable, then m is odd and K is the direct sum of a ray and an indecomposable minimal cone with a balanced relation on its extreme vectors. We also obtain some properties on the corresponding K-primitive matrix A.

Lemma 4.1. *Let $K \in \mathcal{P}(m, n)$ ($3 \leqslant n \leqslant m$) and let A be a K-primitive matrix. Then the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ equals $m - 1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is, apart from the labelling of its vertices, given by Fig.* 1 *or Fig.* 2, *or* (*in case $m = n = 3$*) *by the digraph of order* 3 *whose arcs are precisely all possible arcs between every pair of distinct vertices*:

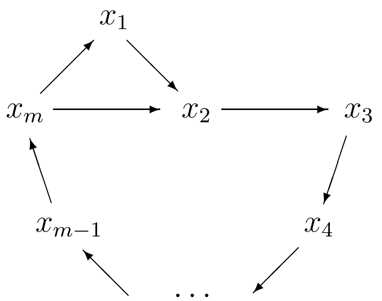

Fig. 1.

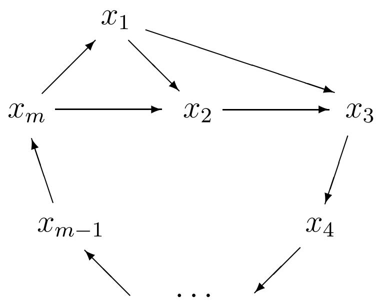

Fig. 2.

(*For simplicity, we label the vertex $\Phi(x_i)$ simply by x_i.*)

Proof. We treat only the "only if" part, the "if" part being obvious.

It is not difficult to show that there are precisely three non-isomorphic primitive digraphs of order three with shortest circuit length two, namely, the digraphs given by Fig. 1, Fig. 2 and the one with all possible arcs between every pair of distinct vertices. So there is no problem when $m = 3$ ($= n$). Hereafter, we assume that $m \geqslant 4$.

Let $x_1, \ldots, x_m$ denote the pairwise distinct extreme vectors of K. Let A be a K-primitive matrix such that the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P})$ is $m - 1$. Without loss of generality, we may assume that the digraph $(\mathcal{E}, \mathcal{P})$ contains the circuit $\mathcal{C}$ (of length $m - 1$) that is made up of the arcs $(\Phi(x_m), \Phi(x_2))$ and $(\Phi(x_j), \Phi(x_{j+1}))$ for $j = 2, 3, \ldots, m - 1$.

Being a circuit of shortest length, $\mathcal{C}$ cannot contain any chord, nor can it have loops at its vertices. If there is no arc from a vertex of $\mathcal{C}$ to the remaining vertex $\Phi(x_1)$, then we have $A\Phi(x_m) = \Phi(x_2)$ and $A\Phi(x_j) = \Phi(x_{j+1})$ for $j = 2, 3, \ldots, m-1$ and it will follow that $A^{m-1}x_m$ is a positive multiple of x_m, hence A^{m-1} is K-reducible, which contradicts the assumption that A is K-primitive. So there is at least one arc from a vertex of $\mathcal{C}$ to $\Phi(x_1)$, say, $(\Phi(x_m), \Phi(x_1))$ is one such arc. Similarly, there is also an arc from $\Phi(x_1)$ to a vertex of $\mathcal{C}$. Since the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P})$ is $m-1$, there cannot be an arc of the form $(\Phi(x_1), \Phi(x_j))$ with $4 \leqslant j \leqslant m$. So $(\Phi(x_1), \Phi(x_2))$ and $(\Phi(x_1), \Phi(x_3))$ are the only possible arcs with initial vertex $\Phi(x_1)$, and at least one of them must be present.

We treat the case when the arc $(\Phi(x_1), \Phi(x_2))$ is present first. Clearly, none of the arcs $(\Phi(x_j), \Phi(x_1))$, for $j = 2, \ldots, m-2$, can be present, as the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is $m-1$. However, it is possible that $(\Phi(x_{m-1}), \Phi(x_1))$ is an arc, provided that $(\Phi(x_1), \Phi(x_3))$ is not an arc. Note that the digraph that consists of the circuit $\mathcal{C}$ and the arcs $(\Phi(x_m), \Phi(x_1)), (\Phi(x_1), \Phi(x_2)), (\Phi(x_{m-1}), \Phi(x_1))$ is isomorphic with the one given by Fig. 2. So, in this case, up to graph isomorphism, the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2.

Now we consider the case when the arc $(\Phi(x_1), \Phi(x_2))$ is absent. Then the arc $(\Phi(x_1), \Phi(x_3))$ is present and the digraph $(\mathcal{E}, \mathcal{P})$ contains two circuits of length $m-1$. As each of these two circuits cannot contain a chord, $(\Phi(x_2), \Phi(x_1))$ is the only possible remaining arc. If the arc $(\Phi(x_2), \Phi(x_1))$ is absent, then a little calculation shows that $A^{m-1}x_m$ is a positive multiple of x_m, which violates the assumption that A is K-primitive. On the other hand, if the arc $(\Phi(x_2), \Phi(x_1))$ is present, then the digraph $(\mathcal{E}, \mathcal{P})$ is isomorphic with the one given by Fig. 2. This completes the proof. □

Note that Fig. 1 is the same as the (unique) digraph associated with an $m \times m$ primitive matrix whose exponent attains Wielandt's bound $m^2 - 2m + 2$ (see [5]).

To proceed further, we need to manipulate with the relations on the extreme vectors of a polyhedral cone. We now explain the relevant terminology.

Let R be a relation on the extreme vectors of a polyhedral cone K. Suppose that the vectors that appear in R come from p ($\geqslant 2$) different indecomposable summands of K, say, $K_1, \ldots, K_p$. To be specific, let R be given by: $\sum_{i\in M} \alpha_i x_i = \sum_{j\in N} \beta_j y_j$, where M, N are finite index sets, each with at least two elements and the α_is, β_js are all positive real numbers. For each $r = 1, \ldots, p$, let $M_r = \{i \in M\colon\ x_i \in K_r\}$ and $N_r = \{j \in N\colon\ y_j \in K_r\}$. Then for each fixed r, rewriting relation R, we obtain

$$\sum_{i\in M_r} \alpha_i x_i - \sum_{j\in N_r} \beta_j y_j = \sum_{j\in N\setminus N_r} \beta_j y_j - \sum_{i\in M\setminus M_r} \alpha_i x_i.$$

Now the vector on the left side of the above relation belongs to $\operatorname{span} K_r$, whereas the one on the right side belongs to $\sum_{s\neq r} \operatorname{span} K_s$. But $\operatorname{span} K_r \cap \sum_{s\neq r} \operatorname{span} K_s = \{0\}$ (as $K_1, \ldots, K_p$ are pairwise distinct indecomposable summands of K), so it follows that we have the relation

$$\sum_{i\in M_r} \alpha_i x_i = \sum_{j\in N_r} \beta_j y_j,$$

which we denote by R_r. This is true for each r. It is clear that relation R can be obtained by adding up relations $R_1, \ldots, R_p$. In this case, we say relation R *splits into the subrelations* $R_1, \ldots, R_p$. Note that each R_r has at least four (nonzero) terms. So when we pass from the relation R to one of its subrelations R_r, the number of terms involved in the relation decreases by at least four.

Recall that an $n \times n$ complex matrix A is said to be *non-derogatory* if every eigenvalue of A has geometric multiplicity 1 or, equivalently, if the minimal and characteristic polynomials of A are identical. (See, for instance, [10, Theorem 3.3.15].)

Lemma 4.2. *Let $K \in \mathcal{P}(m, n)$ ($3 \leqslant n \leqslant m$). Let A be a K-nonnegative matrix. Suppose that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig.* 1 *or Fig.* 2. *Then*:

(i) *A is K-primitive, nonsingular, non-derogatory, and has a unique annihilating polynomial of the form $t^m - ct - d$, where $c, d > 0$.*
(ii) *$\gamma(A)$ equals $\gamma(A, x_1)$ or $\gamma(A, x_2)$ depending on whether the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig.* 1 *or Fig.* 2. *In either case, $\max_{1\leqslant i\leqslant m} \gamma(A, x_i)$ is attained at precisely one i.*
(iii) *Assume, in addition, that K is non-simplicial. If $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig.* 1 *then K must be indecomposable. If the digraph is given by Fig.* 2 *then either K is indecomposable or m is odd and K is the direct sum of a ray and an indecomposable minimal cone with a balanced relation for its extreme vectors.*

Proof. (i) Since the digraphs given by Fig. 1 and Fig. 2 are primitive, by Fact 2.2 A is K-primitive. To show that A is nonsingular, we treat the case when the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1, the argument for the other case being similar. Then for $j = 1, \ldots, m-1$, Ax_j is a positive multiple of x_{j+1}. So $x_2, x_3, \ldots, x_m$ all belong to $\mathcal{R}(A)$, the range space of A. On the other hand, since $x_2 \in \mathcal{R}(A)$ and Ax_m is a positive linear combination of x_1 and x_2, we also have $x_1 \in \mathcal{R}(A)$. Therefore, regarded as a linear map A is onto and hence is nonsingular.

To establish the second half of this part, we may assume that the spectral radius $\rho(A)$ of A equals 1 as $\rho(A) > 0$, A being K-primitive. We first deal with the case when $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1. Since A is K-primitive, A^T is K^*-primitive. Let v denote the Perron vector of A^T. As $v \in \operatorname{int} K^*$, $C := \{x \in K\colon\ \langle x, v\rangle = 1\}$ is a complete (and hence compact) cross-section of K and indeed it is a polytope with m extreme points. Replacing the extreme vectors $x_1, \dots, x_m$ of K by suitable positive multiples, we may assume that $x_1, \dots, x_m$ are precisely all the extreme points of C. It is clear that $AC \subseteq C$, as $A \in \pi(K)$ and $\rho(A) = 1$. Since the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1, we have $Ax_j = x_{j+1}$ for $j = 1, \dots, m-1$ and also $Ax_m = (1-c)x_1 + cx_2$ for some $c \in (0,1)$. The latter condition can be rewritten as $(A^m - cA - (1-c)I_n)x_1 = 0$. It is clear that the A-invariant subspace of $\mathbb{R}^n$ generated by x_1 is $\mathbb{R}^n$ itself; so A is non-derogatory and also it follows that $t^m - ct - (1-c)$ is an annihilating polynomial for A. Therefore, A has an annihilating polynomial of the desired form.

When $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 2, we proceed in a similar way. In this case we have

$$Ax_i = x_{i+1} \quad \text{for } i = 2, \dots, m-1,$$

and

$$Ax_m = (1-a)x_1 + ax_2 \quad \text{and} \quad Ax_1 = (1-b)x_2 + bx_3$$

for some $a, b \in (0,1)$. Then after a little calculation we obtain

$$\left[A^m - \big((1-a)b + a\big)A - (1-a)(1-b)I\right]x_2 = 0.$$

Since the A-invariant subspace of $\mathbb{R}^n$ generated by x_2 is $\mathbb{R}^n$ itself, it follows that A is non-derogatory and $t^m - ((1-a)b + a)t - (1-a)(1-b)$ is an annihilating polynomial for A. The latter polynomial can be rewritten as $t^m - ct - (1-c)$, where $c = (1-a)b + a \in (0,1)$.

The uniqueness of the annihilating polynomial for A of the desired form is obvious, because $\{A, I_n\}$ is a linearly independent set.

(ii) Note that for any $0 \neq x \in K$ and $j = 0, 1, \dots, \gamma(A, x) - 1$, $\gamma(A, x) = \gamma(A, A^j x) + j$.

First, consider the case when the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1. Since $A^{j-1}x_1$ is a positive multiple of x_j for $j = 2, \dots, m$, we have $\gamma(A, x_1) > m$ and

$$\gamma(A, x_j) = \gamma\big(A, A^{j-1}x_1\big) = \gamma(A, x_1) - j + 1$$

for $j = 2, \dots, m$; hence

$$\gamma(A) = \max_{1 \leqslant j \leqslant m} \gamma(A, x_j) = \gamma(A, x_1).$$

When the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2, we apply a similar but slightly more elaborate argument. For $j = 3, \dots, m$, we have

$$\gamma(A, x_2) = \gamma\big(A, A^{j-2}x_2\big) + j - 2 = \gamma(A, x_j) + j - 2;$$

hence $\gamma(A, x_2) > \gamma(A, x_j)$ for each such j. A little calculation shows that $A^m x_2$ is a positive linear combination of x_2 and x_3. But Ax_1 is also a positive linear combination of x_2 and x_3, hence $\Phi(Ax_1) = \Phi(A^m x_2)$. So we have

$$\gamma(A, x_2) = \gamma\big(A, A^m x_2\big) + m = \gamma(A, Ax_1) + m = \gamma(A, x_1) - 1 + m,$$

which implies $\gamma(A, x_2) > \gamma(A, x_1)$. Therefore, we have $\gamma(A) = \gamma(A, x_2)$.

(iii) In the following argument, unless specified otherwise, we assume that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2. First, we show that each of the extreme vectors $x_1, \dots, x_m$, except possibly x_2, is involved in at least one relation on $\operatorname{Ext} K$. For the purpose, it suffices to show that x_3 is involved in one such relation; because, by applying A or its positive powers to a relation on $\operatorname{Ext} K$ involving x_3, we can obtain for each of the vectors $x_4, \dots, x_{m-1}, x_m, x_1$ a relation that involves the vector. Suppose that x_3 is not involved in any (nonzero) relation on $\operatorname{Ext} K$. Take any relation S on $\operatorname{Ext} K$; as K is non-simplicial, such relation certainly exists. Note that, since x_3 does not appear in S, x_4 (and also x_3) cannot appear in the (necessarily nonzero) relation obtained from S by applying A. Similarly, the vectors x_3, x_4, x_5 all do not appear in the relation obtained from S by applying A^2. Continuing the argument, we can show that the only vectors that can appear in the nonzero relation obtained from S by applying A^{m-3} are x_1, x_2. This contradicts the hypothesis that x_1, x_2 are distinct nonzero extreme vectors of K.

Next, we note that if x_2 is not involved in any relation on $\operatorname{Ext} K$ and if $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 then, by applying A repeatedly to a nonzero relation on $\operatorname{Ext} K$ sufficiently many times, we would obtain a nonzero relation on $\operatorname{Ext} K$ that involves less than four vectors, which is a contradiction. So if x_2 is not involved in any relation on $\operatorname{Ext} K$ then $(\mathcal{E}, \mathcal{P})$ must be given by Fig. 2.

Now we contend that K is either indecomposable or is the direct sum of a ray and an indecomposable cone. By what we have done above, each of the extreme vectors $x_1, \dots, x_m$, except possibly x_2, belongs to an indecomposable summand of K that is not a ray. Let K_1 be the indecomposable summand of K that contains x_m. To establish our contention, it remains to show that K has no indecomposable summand which is not a ray and is different from K_1. It is known that every indecomposable polyhedral cone of dimension greater than 1 has a full relation for its extreme vectors (see [8, p. 37, (2.14)]). So it suffices to show that there is no relation on $\operatorname{Ext} K$ that involves vectors not belonging to K_1. Assume to the contrary that there are such relations. Let T_0 be one such shortest relation (i.e., one having the minimum number of terms). Note that, since x_m is not involved in T_0, the relation obtained from T_0 by applying A has the same number of terms as T_0, unless x_1 appears in T_0 but x_2 does not and $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2, in which case the said relation may have one term more than T_0. Suppose that the extreme vectors that appear in T_0 are $x_{k_1}, x_{k_2}, \dots, x_{k_s}$, where $1 \leqslant k_1 < k_2 < \dots < k_s \leqslant m-1$.

It is readily seen that the relations obtained from T_0 by applying A^i for $i = 1, \dots, m - k_s$ all have the same number of terms, as x_1 and x_m are both not involved in the first $m - k_s - 1$ of these relations.

Let q denote the least positive integer such that $x_{m-q} \notin K_1$. Certainly, we have $k_s \leqslant m - q$. Denote by $\tilde{T}_0$ the relation obtained from T_0 by applying A^{m-q-k_s}. (If $k_s = m - q$, we take $\tilde{T}_0$ to be T_0.) Then $\tilde{T}_0$ either has the same number of terms as T_0 or has one term more. Note that now x_{m-q} is involved in $\tilde{T}_0$ and, by our choice of q, $x_{m-q} \notin K_1$. If $\tilde{T}_0$ involves also extreme vectors of K_1, then $\tilde{T}_0$ splits and we would obtain a relation for extreme vectors not belonging to K_1, which has at least four terms fewer than that of $\tilde{T}_0$ and hence is a relation shorter than the shortest relation T_0, which is a contradiction. So we assume that all the vectors appearing in $\tilde{T}_0$ do not belong to K_1 (and in fact they all lie in the same indecomposable summand of K).

For $j = 1, 2, \dots$, let T_j denote the relation obtained from $\tilde{T}_0$ by applying A^j. By considering the cases when $\tilde{T}_0 = T_0$ and $\tilde{T}_0 \neq T_0$ separately, one readily sees that relation T_1 has at most one term more than T_0. Also, T_1 involves x_{m-q+1} which, by the definition of q, belongs to K_1. If T_1 involves also extreme vectors not belonging to K_1, then T_1 splits and we would obtain a contradiction. So T_1 is a relation on $\operatorname{Ext} K_1$ and $x_{\tilde{k}_1+1}, \dots, x_{\tilde{k}_s+1}$ all belong to K_1, where $\tilde{k}_j = (m - q - k_s) + k_j$. By the same argument we may assume that for $j = 1, \dots, q$, T_j is a relation on $\operatorname{Ext} K_1$ with at most one term more than T_0. So, we have $x_{\tilde{k}_j+r} \in K_1$ for $r = 1, \dots, q$ and $j = 1, \dots, s$. Note that x_m is involved in T_q but x_1 is not (as x_m is not involved in T_{q-1}). So x_1, x_2 are both involved in T_{q+1} and lie on the same side of it. As a consequence, x_2, x_3 are both involved in T_{q+2}, x_3, x_4 are both involved in T_{q+3}, and so forth. Clearly, T_{q+1} has one term more than T_q and hence at most two terms more than T_0.

If T_{q+1} involves vectors not belonging to the same indecomposable summand of K, then the relation splits and the minimality of T_0 would be violated. So we assume that T_{q+1} is a relation on $\operatorname{Ext} K_2$ — here K_2 may or may not be the same as K_1. Note that now we have $x_1, x_2, x_{\tilde{k}_j+q+1} \in K_2$ for $j = 1, \dots, s-1$. Let l denote the smallest positive integer such that at least one of the vectors $A^l x_2, A^l x_{\tilde{k}_j+q+1}$, $j = 1, \dots, s-1$ does not belong to K_2. It is readily seen that $A^l x_2$ is always a positive multiple of x_{2+l} and $A^l x_1$ is a positive multiple of x_{1+l} or a positive linear combination of x_{1+l} and x_{2+l}, depending on whether $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1 or Fig. 2. If $A^l x_2 \notin K_2$, then, from the definition of l, necessarily we have $x_{2+l} \notin K_2$ and $x_{1+l} \in K_2$. On the other hand, if $A^l x_2 \in K_2$, then there must exist $j = 1, \dots, s-1$ such that $A^l x_{\tilde{k}_j+q+1} \notin K_2$. In any case, the relation obtained from T_{q+1} by applying A^l involves at least one extreme vector in K_2 and at least one extreme vector not in K_2. Then the relation splits and we would obtain a relation for extreme vectors not belonging to K_1, which is shorter than the shortest such relation, which is a contradiction. (A cautious reader may wonder whether there is a positive integer $r < l$ such that $A^r x_{\tilde{k}_{s-1}+q+1} = x_m$. If this is so, then at one of the steps in applying A l times to relation T_{q+1} the number of terms in the relation is increased by one. However, it is possible to show that such positive integer r does not exist by analyzing carefully the cases $\tilde{k}_{s-1} + q + 1 < \tilde{k}_s$ and $\tilde{k}_{s-1} + q + 1 = \tilde{k}_s$ separately.)

It remains to show that if $(\mathcal{E}, \mathcal{P})$ is given by Fig. 2 and K is the direct sum of the ray $\operatorname{pos}\{x_2\}$ and the indecomposable polyhedral cone $\operatorname{pos}\{x_1, x_3, x_4, \dots, x_m\}$, then m is odd and the latter cone is an indecomposable minimal cone with a balanced relation for its extreme vectors. Note that the assumption that x_2 does not appear in any relation on $\operatorname{Ext} K$ guarantees that every relation obtained from a shortest relation on $\operatorname{Ext} K$ by applying A or its positive powers is still a shortest relation. We contend every shortest relation involves each of the vectors $x_1, x_3, \dots, x_m$. Assume that the contrary holds. Take a shortest relation R. Since R has at least four terms, one of the vectors $x_3, x_4, \dots, x_m$ must appear in R. On the other hand, R cannot involve all of these vectors; otherwise, x_1 does not appear in R, and so the relation obtained from R by applying A involves the vector x_2, which is a contradiction. Thus we can find an i, $4 \leqslant i \leqslant m$, such that x_i appears in R and x_{i-1} does not or the other way round. Then the relation obtained from R by applying A^{m-i+1} involves one of the vectors x_m, x_1 but not both, and so the relation obtained from R by applying A^{m-i+2} must involve the vector x_2, which is a contradiction. This proves our contention. Since every shortest relation on $\{x_1, x_3, \dots, x_m\}$ is a full relation, it is clear that any two relations on the latter set are multiples of each other; else, by subtracting an appropriate multiple of one relation from another we would obtain a shorter nonzero relation. This proves that the cone $\operatorname{pos}\{x_1, x_3, \dots, x_m\}$ is minimal. Let R denote the relation on $\operatorname{Ext} K$. Since x_2 does not appear in any relation on $\operatorname{Ext} K$, x_m, x_1 must appear on opposite sides of R. So x_1, x_3 also appear on opposite sides of the relation obtained from R by applying A and hence on opposite sides of relation R. Continuing the argument, we infer that for $j = 3, \dots, m-1$, x_j and x_{j+1} lie on opposite sides of R. It follows that m is odd and R has the same number of terms on its two sides, i.e., R is a balanced relation. $\square$

Lemma 4.2(i) tells us, in particular, that if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ of a K-nonnegative matrix A is given by Fig. 1 or Fig. 2, and if $\rho(A) = 1$, then A has an annihilating polynomial of the form $t^m - ct - (1-c)$, where $c \in (0, 1)$. Polynomials of such form will play a role in our future paper [16]. It is of interest to note that polynomials of the more general form $t^m - ct^{m-p} - (1-c)$, where $c \in [0, 1]$ and $m, p \in \mathbb{N}$ with $m > p \geqslant 1$ have also been considered by Kirkland [12,13] in his study of primitive stochastic matrices. In particular, in [13] ordinary primitive matrices with large exponents are considered.

5. Further upper bounds for exponents

In this section we apply the results of the previous two sections to obtain further upper bounds for exponents of K-primitive matrices. It will be shown that if K is a polyhedral cone with m extreme rays, then for any K-primitive matrix A,

$\gamma(A) \leqslant (m_A - 1)(m - 1) + 1$, where the equality holds only if the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 1. As a consequence, $(n - 1)(m - 1) + 1$ is an upper bound for $\gamma(K)$ when $K \in \mathcal{P}(m, n)$.

It is easy to show the following:

Remark 5.1. For any real numbers p, l with $p \geqslant 3$, we have

$$(p - 1)(l - 2) + 2 \leqslant (p - 1)(l - 1),$$

where the inequality becomes equality if and only if $p = 3$.

Theorem 5.2. *Let $K \in \mathcal{P}(m, n)$, where $m \geqslant 4$, and let A be a K-primitive matrix. Then*:

(i) *$\gamma(A) \leqslant (m_A - 1)(m - 1) + 1$, where the equality holds only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig.* 1, *in which case $\gamma(A) = (n - 1)(m - 1) + 1$.*

(ii) *$\gamma(A) = (m_A - 1)(m - 1)$ only if either $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig.* 1 *or Fig.* 2, *in which case $\gamma(A) = (n - 1)(m - 1)$, or $m_A = 3$.*

(iii) *$\gamma(A) = (m_A - 1)(m - 2) + 2$ only if $m_A \geqslant 3$ and either $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig.* 1, *Fig.* 2, *Fig.* 3, *Fig.* 4 *or Fig.* 5, *or $(\mathcal{E}, \mathcal{P}(A, K))$ is obtained from Fig.* 5 *by deleting any one or two of the three arcs $(\Phi(x_{m-1}), \Phi(x_1))$, $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$, or from Fig.* 3 *with $m = 4$ by adding the arc $(\Phi(x_3), \Phi(x_1))$ or substituting it for the arc $(\Phi(x_4), \Phi(x_1))$.*

(iv) *If $m_A = 2$ or $(\mathcal{E}, \mathcal{P}(A, K))$ is not given by Figs.* 1–5, *nor is derived from Fig.* 5 *or from Fig.* 3 (*with $m = 4$*) *in the way as described in part* (iii), *then*

$$\gamma(A) \leqslant (m_A - 1)(m - 2) + 1.$$

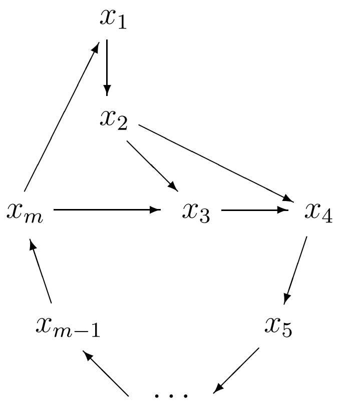

Fig. 3.

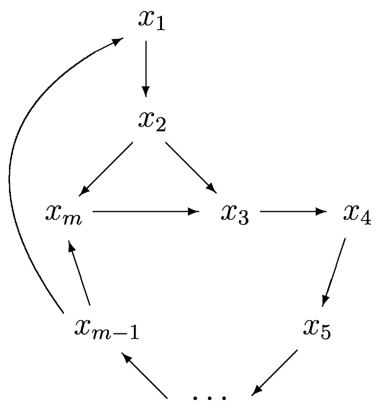

Fig. 4.

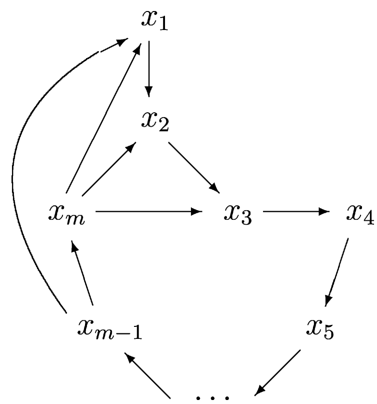

Fig. 5.

Proof. When $m_A = 2$, by Lemma 3.6 we have $\gamma(A) \leqslant 2$. As $m \geqslant 4$, in this case, the inequality $\gamma(A) \leqslant (m_A - 1)(m - 2) + 1$ is clearly satisfied and none of the equalities $\gamma(A) = (m_A - 1)(m - 1)$ or $\gamma(A) = (m_A - 1)(m - 2) + 2$ can be attained. Hereafter, we assume that $m_A \geqslant 3$.

As explained before, the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is at most $m - 1$.

(i) Since A is K-primitive, A is non-nilpotent. So the outdegree of each vertex of $(\mathcal{E}, \mathcal{P})$ is positive. Consider any vertex $\Phi(x)$ of the digraph $(\mathcal{E}, \mathcal{P})$. It is clear that $\Phi(x)$ lies on or has access to a circuit of length $l \leqslant m - 1$. By Lemma 3.2 we have

$$\gamma(A, x) \leqslant (m_A - 2)l + m \leqslant (m_A - 2)(m - 1) + m = (m_A - 1)(m - 1) + 1.$$

Since this is true for every nonzero extreme vector x of K, the inequality $\gamma(A) \leqslant (m_A - 1)(m - 1) + 1$ follows.

To establish the desired necessary condition for the inequality to become equality, first we dispense with the case when the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is less than or equal to $m - 2$. We contend that in this case every vertex of the digraph lies on or has access to a circuit of length less than or equal to $m - 2$. Consider any vertex $\Phi(x)$ of the digraph. As we have explained before, $\Phi(x)$ lies on or has access to some circuit, say $\mathcal{C}$. Choose such a $\mathcal{C}$ of shortest length, say length l. By the definition of l, C contains no chords or loops (unless $\mathcal{C}$ is itself a loop). If $l = m$ then A is not primitive, contradiction. If $l = m - 1$ let z be the unique vertex of the digraph not on $\mathcal{C}$. Then, there is an access from $\mathcal{C}$ to z and vice versa, or else A is not primitive. Hence the graph is strongly connected, so $\Phi(x)$ has access to a circuit of length $m - 2$ or less, contradicting $l = m - 1$. Hence, necessarily, $l \leqslant m - 2$. This proves our contention. So, in this case, we have

$$\gamma(A) \leqslant (m_A - 2)(m - 2) + m = (m_A - 1)(m - 2) + 2 \leqslant (m_A - 1)(m - 1), \tag{5.1}$$

where the first inequality holds by Lemma 3.2 (with $l \leqslant m - 2$) and the second inequality follows from Remark 5.1 (with $p = m_A$ and $l = m$).

When the length of the shortest circuit in $(\mathcal{E},\mathcal{P})$ is $m-1$, by Lemma 4.1 the digraph is given by Fig. 1 or Fig. 2. If the digraph is given by Fig. 2, then each vertex lies on a circuit of length $m-1$ and by Lemma 3.1 we obtain $\gamma(A) \leqslant (m_A-1)(m-1)$. So when the equality $\gamma(A)=(m_A-1)(m-1)+1$ holds, the digraph $(\mathcal{E},\mathcal{P})$ must be given by Fig. 1. In that case, by Lemma 4.2(i) A is non-derogatory; thus we have $m_A=n$ and hence $\gamma(A)=(n-1)(m-1)+1$.

(ii) Suppose that the equality $\gamma(A)=(m_A-1)(m-1)$ holds. The length of the shortest circuit in $(\mathcal{E},\mathcal{P})$ is either $m-1$ or less. In the former case, by Lemma 4.1 the digraph is given by Fig. 1 or Fig. 2; then by Lemma 4.2(i), as $m_A=n$ the said equality becomes $\gamma(A)=(n-1)(m-1)$. In the latter case, by the proof of part (i) the inequalities in (5.1) both hold as equality, and by Remark 5.1 we have $m_A=3$ and hence also $\gamma(A)=2(m-1)$.

(iii) Suppose that $\gamma(A)=(m_A-1)(m-2)+2$. If $(\mathcal{E},\mathcal{P})$ is given by Fig. 1 or Fig. 2 we are done, so assume that this is not the case. Note that it is not possible that every vertex of $(\mathcal{E},\mathcal{P})$ lies on or has access to a circuit of length $\leqslant m-3$ or is at a distance at most 1 to a circuit of length $m-2$, because then by Lemma 3.2 (with $l=m-3$) or by Lemma 3.1 (with $w=1$ and $l=m-2$) it will follow that $\gamma(A) \leqslant (m_A-1)(m-2)+1$. It remains to show that if $(\mathcal{E},\mathcal{P})$ has a vertex which is at a distance 2 to a circuit of length $m-2$ and which does not lie on or has access to a circuit of length $m-3$ or less, nor is it at a distance at most 1 to another circuit of length $m-2$, then the digraph is given by Fig. 3, Fig. 4 or Fig. 5, or is derived from them in the manner as described in the theorem.

To treat the remaining case we assume that the digraph $(\mathcal{E},\mathcal{P})$ contains the circuit $\mathcal{C}: \Phi(x_3) \to \Phi(x_4) \to \cdots \to \Phi(x_{m-1}) \to \Phi(x_m) \to \Phi(x_3)$ and also the path $\Phi(x_1) \to \Phi(x_2) \to \Phi(x_3)$. For $i=2,3,\ldots,m$, since $\Phi(x_i)$ is at a distance at most 1 to the circuit $\mathcal{C}$, which is of length $m-2$, by Lemma 3.1 we have $\gamma(A,x_i) \leqslant (m_A-1)(m-2)+1$. This forces $\gamma(A,x_1)=\gamma(A)=(m_A-1)(m-2)+2$, which, in turn, implies that $\Phi(x_1)$ does not lie on or has access to a circuit of length $m-3$ or less, nor is $\Phi(x_1)$ at a distance at most 1 to a circuit of length $m-2$. Therefore, $\mathcal{C}$ does not contain any chords or loops. Besides the arcs on the circuit $\mathcal{C}$ and the above-mentioned path, $(\mathcal{E},\mathcal{P})$ certainly has other arcs. We want to find out what possible additional arcs there can be.

Note that there is at least one arc from a vertex of $\mathcal{C}$ to either one of the vertices $\Phi(x_1)$ or $\Phi(x_2)$; else, A^{m-2} maps the extreme ray $\Phi(x_3)$ of K onto itself, which contradicts the K-primitivity of A. Since $\Phi(x_1)$ is not allowed to lie on a circuit of length $m-2$ or less, none of the arcs $(\Phi(x_j),\Phi(x_1))$, for $j=2,\ldots,m-2$, can be present. Similarly, since $\Phi(x_1)$ is not allowed to be at a distance 1 to a circuit of length $m-2$ or less, the arcs $(\Phi(x_j),\Phi(x_2))$, for $j=3,\ldots,m-1$, also cannot be present. So $(\Phi(x_{m-1}),\Phi(x_1))$, $(\Phi(x_m),\Phi(x_1))$ and $(\Phi(x_m),\Phi(x_2))$ are the only possible arcs from a vertex of $\mathcal{C}$ to either $\Phi(x_1)$ or $\Phi(x_2)$; also, at least one of these three arcs is present.

There cannot exist an arc from $\Phi(x_1)$ to a vertex of $\mathcal{C}$, because in the presence of any such arc the distance from $\Phi(x_1)$ to the circuit $\mathcal{C}$ becomes 1. Similarly, for $m \geqslant 6$, each of the arcs $(\Phi(x_2),\Phi(x_j))$, $j=5,\ldots,m-1$, also cannot exist, because the arc $(\Phi(x_2),\Phi(x_j))$ and one of the arcs $(\Phi(x_{m-1}),\Phi(x_1))$, $(\Phi(x_m),\Phi(x_1))$ and $(\Phi(x_m),\Phi(x_2))$ (which must be present), together with some of the arcs in $\mathcal{C}$, form either a circuit of length $m-3$ or less, which is at a distance 1 from $\Phi(x_1)$, or a circuit of length $m-2$ or less that contains $\Phi(x_1)$, but this is not allowed. So $(\Phi(x_2),\Phi(x_4))$ and $(\Phi(x_2),\Phi(x_m))$ are the only possible arcs from $\Phi(x_1)$ or $\Phi(x_2)$ to a vertex of $\mathcal{C}$ when $m \geqslant 6$. The preceding argument does not cover the cases when $m=4$ or 5. However, we have not ruled out the possibility of the existence of the arcs $(\Phi(x_2),\Phi(x_4))$ and $(\Phi(x_2),\Phi(x_m))$ in these cases.

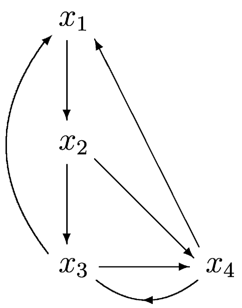

Fig. 3′.

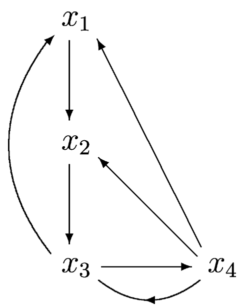

Fig. 5′.

Consider the case when the arc $(\Phi(x_2),\Phi(x_4))$ is present. If $m \geqslant 5$, then the last two of the three arcs $(\Phi(x_m),\Phi(x_1))$, $(\Phi(x_m),\Phi(x_2))$ and $(\Phi(x_{m-1}),\Phi(x_1))$ cannot be present, else $\Phi(x_1)$ is at a distance at most 1 to a circuit of length $m-2$, which is not allowed. So in this case the arc $(\Phi(x_m),\Phi(x_1))$ must be present and, furthermore, the arc $(\Phi(x_2),\Phi(x_m))$ also cannot be present (otherwise, we have a circuit of length three containing $\Phi(x_1)$). Therefore, the digraph $(\mathcal{E},\mathcal{P})$ is given by Fig. 3. If $m=4$, we find that the arc $(\Phi(x_4),\Phi(x_2))$ cannot be present, but the arcs $(\Phi(x_3),\Phi(x_1))$ and $(\Phi(x_4),\Phi(x_1))$ may be present and, indeed, at least one of them must be present. So the digraph $(\mathcal{E},\mathcal{P})$ is given by Fig. 3′ or is obtained from it by deleting one of the arcs $(\Phi(x_3),\Phi(x_1))$, $(\Phi(x_4),\Phi(x_1))$. In other words, the digraph is given by Fig. 3 (with $m=4$) or is derived from it in the manner as described in the theorem.

Now suppose that the arc $(\Phi(x_2),\Phi(x_m))$ is present. Using the same kind of argument as before, for $m \geqslant 5$, one readily rules out the presence of the arcs $(\Phi(x_m),\Phi(x_1))$ and $(\Phi(x_m),\Phi(x_2))$. So in this case the arc $(\Phi(x_{m-1}),\Phi(x_1))$ must be present. Then we can show that the arc $(\Phi(x_2),\Phi(x_4))$ cannot be present. Therefore, the digraph $(\mathcal{E},\mathcal{P})$ is given by Fig. 4.

For $m = 4$, we are dealing with the situation when $(\Phi(x_2), \Phi(x_4))$ is an arc, but this has already been treated above (for arbitrary $m \geqslant 4$).

It remains to consider the case when the arcs $(\Phi(x_2), \Phi(x_4))$ and $(\Phi(x_2), \Phi(x_m))$ are both absent. Then the presence of any one, two or three of the arcs

$$\big(\Phi(x_{m-1}), \Phi(x_1)\big),\ \big(\Phi(x_m), \Phi(x_1)\big),\ \big(\Phi(x_m), \Phi(x_2)\big)$$

will produce only circuits of length at least $m-1$, but that causes no problem. Then the digraph $(\mathcal{E}, \mathcal{P})$ is given by Fig. 5 (which becomes Fig. 5′ when $m = 4$) or is obtained from it by deleting any one or two of the above-mentioned three arcs.

(iv) Now this is obvious. □

Remark 5.3. Let $K \in \mathcal{P}(3,3)$, and let A be a K-primitive matrix. Then $\gamma(A) \leqslant 2m_A - 1$, where the equality holds only if $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig. 1, in which case $\gamma(A) = 5$.

The preceding remark, in fact, says that part (i) of Theorem 5.2 still holds when $m = n = 3$. It holds by what is known in the 3×3 nonnegative matrix case. However, parts (ii)–(iv) of Theorem 5.2 cannot be extended to the case $m = n = 3$. This is mainly because in that case the equality in part (ii) or (iii) can hold when $m_A = 2$.

By Lemma 4.1, for any polyhedral cone K with $m \geqslant 3$ extreme rays and any K-primitive matrix A, the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is $m-2$ or less if and only if the digraph $(\mathcal{E}, \mathcal{P})$ is not given by Fig. 1 or Fig. 2 or by a digraph of order 3 whose arc set consists of all possible arcs between every pair of distinct vertices. The proof of Theorem 5.2 (i) shows that in this case $(m_A - 1)(m-2) + 2$ is an upper bound for $\gamma(A)$. (The case $m = n = 3$ can be treated separately.) So we have the following

Corollary 5.4. *For any $K \in \mathcal{P}(m,n)$ and any K-primitive matrix A, if the digraph of $(\mathcal{E}, \mathcal{P}(A,K))$ is not given by Fig.* 1 *or Fig.* 2 *or by a digraph of order* 3 *whose arc set consists of all possible arcs between every pair of distinct vertices, then $\gamma(A) \leqslant (n-1)(m-2) + 2$.*

In below we give another bound for $\gamma(A)$ in terms of m_A, m and s, where s is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$. Before we do that, we need to obtain a general result on a digraph first.

Remark 5.5. Let D be a digraph on $m \geqslant 3$ vertices, each of which has positive out-degree. If the length of the shortest circuit in D is greater than $\lfloor \frac{m-1}{2} \rfloor$, then every vertex of D lies on or has access to a circuit of D of shortest length.

Proof. Since each vertex of D has positive out-degree, each vertex lies on or has access to a circuit. Denote by $s(D)$ the length of the shortest circuit in D. If there is a vertex that does not lie on or has access to a circuit of length $s(D)$, then such vertex must lie on or has access to a circuit, say $\mathcal{C}$, of length $s(D)+1$ or more. It is clear that the circuit $\mathcal{C}$ is vertex disjoint from every circuit of shortest length. Consequently, we have $m \geqslant s(D) + (s(D)+1)$ or $\lfloor \frac{m-1}{2} \rfloor \geqslant s(D)$, which is a contradiction. □

By Lemma 3.2 and Remark 5.5 we have

Remark 5.6. Let $K \in \mathcal{P}(m,n)$ and let A be a K-primitive matrix. Let s be the length of the shortest circuit in $(\mathcal{E}, \mathcal{P}(A,K))$. If $s > \lfloor \frac{m-1}{2} \rfloor$, then $\gamma(A) \leqslant s(m_A - 2) + m$.

It is interesting to note that the digraphs given by Fig. 3, Fig. 4 and Fig. 5 are all primitive, like the digraphs given by Fig. 1 and Fig. 2. Moreover, if A is a K-primitive matrix such that $(\mathcal{E}, \mathcal{P})$ is given by Fig. 3, Fig. 4 or Fig. 5 then A is necessarily nonsingular—this can be proved using the argument given in the proof of Lemma 4.2(i). However, the digraph obtained from Fig. 5′ (i.e., Fig. 5 with $m = 4$) by removing the arcs (x_4, x_2) and (x_3, x_1) is strongly connected but not primitive, whereas the one obtained from Fig. 3′ (i.e., Fig. 3 with $m = 4$) by removing the arcs (x_4, x_1) and (x_3, x_1) is not even strongly connected. Also, A is singular if it is a K-primitive matrix such that its digraph $(\mathcal{E}, \mathcal{P})$ is derived from Fig. 5 by deleting the arcs $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$.

Corollary 5.7. *For any $K \in \mathcal{P}(m,n)$ with $m = n + k$, we have $\gamma(K) \leqslant (n-1)(m-1) + 1 = m^2 - (k+2)m + k + 2$. The equality holds only if there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A,K))$ is given by Fig.* 1.

Proof. Follows from Theorem 5.2(i) for $m \geqslant 4$ and from Remark 5.3 for $m = 3$ $(= n)$. □

By Corollary 5.7 the answer to Kirkland's conjecture mentioned at the beginning of Section 1 is in the affirmative.

Corollary 5.8. *For any positive integer $m \geqslant 3$,*

$$\max\big\{\gamma(K):\ K \text{ is a polyhedral cone with } m \text{ extreme rays}\big\} = m^2 - 2m + 2.$$

Proof. Let K be an n-dimensional polyhedral cone with m extreme rays. Since $n \leqslant m$, by Corollary 5.7, $\gamma(K) \leqslant (m-1)^2 + 1$. So we have

$$\max\{\gamma(K)\colon\ K \text{ is a polyhedral cone with } m \text{ extreme rays}\} \leqslant m^2 - 2m + 2.$$

On the other hand, by Wielandt's bound we also have $\gamma(\mathbb{R}^m_+) = m^2 - 2m + 2$. Hence, the desired equality follows. $\square$

We would like to emphasize that in Corollary 5.8 the number of extreme rays (i.e., m) for the polyhedral cones K under consideration is fixed but there is no restriction on their dimensions (i.e., n).

6. An example of a cone with infinite exponent

A positive integer κ is called the *critical exponent* of a normed space E (or of the norm on E) if the equalities $\|A^\kappa\| = \|A\| = 1$ imply that $\rho(A) = 1$, and if κ is the smallest number with the indicated property. It is known that not every norm in a finite-dimensional space has a critical exponent. An example of one such norm can be found in [3]. Borrowing the latter example, we are going to show that there exists a proper cone which does not have finite exponent.

Example 6.1. Let $\|\cdot\|$ denote the norm of $\mathbb{R}^2$ whose unit closed ball is defined by the inequalities:

$$3\xi_1 - 2 \leqslant \xi_2 \leqslant \xi_1^3, \quad \text{if } -2 \leqslant \xi_1 \leqslant -1,$$
$$3\xi_1 - 2 \leqslant \xi_2 \leqslant 3\xi_1 + 2, \quad \text{if } |\xi_1| \leqslant 1,$$

and

$$\xi_1^3 \leqslant \xi_2 \leqslant 3\xi_1 + 2, \quad \text{if } 1 \leqslant \xi_1 \leqslant 2.$$

(See Fig. 6.) Let K be the proper cone in $\mathbb{R}^3$ given by: $K = \{\alpha\binom{x}{1}\colon\ \alpha \geqslant 0 \text{ and } \|x\| \leqslant 1\}$. For every positive integer k, let B_k denote the 2×2 diagonal matrix $\operatorname{diag}(2^{-1/k}, 2^{-3/k})$. As shown in [3, p. 67], B_k has the property that $\|B_k\| = \|B_k^k\| = 1$ but $\|B_k^{k+1}\| < 1$. Let $A_k = B_k \oplus (1)$. Then it is easy to see that A_k is K-primitive and $\gamma(A_k) = k + 1$. Since k can be arbitrarily large, this shows that for this K we have $\gamma(K) = \infty$. It is also of interest to note that the K-primitive matrices A_k obtained in this example are, in fact, all extreme matrices of the cone $\pi(K)$. The point is, K is an indecomposable cone and each of the A_k's maps infinitely many extreme rays of K onto extreme rays.

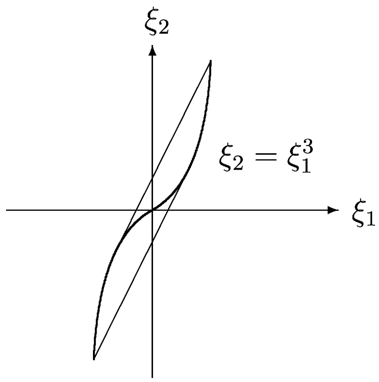

Fig. 6.

7. An open question

For further work on maximal exponents of K-primitive matrices, the following is a relevant question (cf. Corollary 5.4):

Question. Given positive integers m, n with $3 \leqslant n < m$, characterize the $n \times n$ real matrices A with the property that there exists $K \in \mathcal{P}(m, n)$ such that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1.

It is known (see, for instance, [4, Chapter 1]) that in order that a real square matrix A is K-primitive (or K-positive) for some proper cone K it is necessary and sufficient that the spectral radius $\rho(A)$ is an eigenvalue of A with modulus strictly greater than the moduli of all other eigenvalues of A. By Lemma 4.2, if A possesses the property given in the question necessarily A is also nonsingular, non-derogatory and has a unique annihilating polynomial of the form $t^m - ct - d$, where $c, d > 0$.

Note that in the above we do not pose the same question for Fig. 2, because the two questions are equivalent; that is, the existence of a pair (K, A) with $(\mathcal{E}, \mathcal{P}(A, K))$ given by Fig. 1 guarantees the existence of a pair (K, A) with $(\mathcal{E}, \mathcal{P}(A, K))$ given by Fig. 2, and vice versa.

To see this, suppose that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 2. Then $\Phi(x_1 + x_2)$ is a 2-dimensional face of K and Ax_m lies in its relative interior. Let $\hat{K}$ denote the polyhedral cone generated by $Ax_m, x_2, x_3, \ldots, x_m$. It is readily shown that $Ax_m, x_2, \ldots, x_m$ are precisely (up to multiples) all the extreme vectors of $\hat{K}$. Furthermore, A is $\hat{K}$-nonnegative and $(\mathcal{E}, \mathcal{P}(A, \hat{K}))$ is isomorphic to Fig. 1 (under the isomorphism given by: $\Phi(Ax_m) \mapsto \Phi(x_m), \Phi(x_j) \mapsto \Phi(x_{j-1})$ for $j = 2, \ldots, m$). Conversely, suppose that A is K-nonnegative and $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Fig. 1. Let $\tilde{K} = \mathrm{pos}\{(1-\alpha)x_1 + \alpha x_m, x_1, x_2, \ldots, x_{m-1}\}$. It is not difficult to show that for $\alpha > 1$, sufficiently close to 1, $(1-\alpha)x_1 + \alpha x_m, x_1, x_2, \ldots, x_{m-1}$ are precisely all the extreme vectors of $\tilde{K}$. Furthermore, A is $\tilde{K}$-nonnegative and $(\mathcal{E}, \mathcal{P}(A, \tilde{K}))$ is isomorphic to Fig. 2 (under the isomorphism given by: $\Phi((1-\alpha)x_1 + \alpha x_m) \mapsto \Phi(x_1), \Phi(x_j) \mapsto \Phi(x_{j+1})$ for $j = 1 \ldots, m-1$).

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