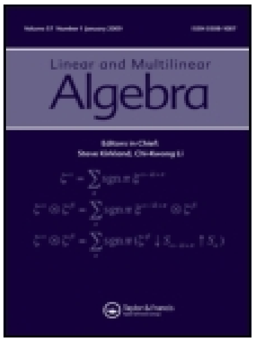

Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

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To cite this article: Bit-Shun Tam & George Phillip Barker (1992) Graphs and irreducible cone preserving maps, Linear and Multilinear Algebra, 31:1-4, 19-25, DOI: 10.1080/03081089208818118

To link to this article: https://doi.org/10.1080/03081089208818118

Published online: 01 Apr 2008.

Linear and Multilinear Algebra, 1992, Vol. 31, pp. 19–25
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Printed in the United States of America

Graphs and Irreducible Cone Preserving Maps

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(*Received October 25, 1990; in final form May 9, 1991*)

Let K be a full, pointed closed cone in a finite dimensional real vector space. For any linear map A for which $AK \subseteq K$, denote by $(E,P(A)[E,I(A))]$ the directed graph whose vertex set consists of all the extreme rays of K such that there is an edge from F to G iff $G \lhd \Phi[AF][G \lhd \Phi[(I + A)F]]$. It is proved that K is a simplicial cone if for any linear map A with $AK \subseteq K$, $(E,P(A))[(E,I(A))]$ is strongly connected whenever A is irreducible [provided, in addition, that K is 2-neighborly].

INTRODUCTION

There is a well known method for associating a directed graph with an entrywise nonnegative matrix (cf. [3] and [5]). It turns out that this graph is strongly connected iff the matrix A is irreducible. In an earlier paper [2] the present authors defined four similar graphs for a cone preserving map. In contrast with the entrywise nonnegative case the graph which connects with irreducibility has as vertices the set of all faces of the cone. In the entrywise nonnegative case where the cone is the nonnegative orthant, a simplicial cone, the set of vertices is the set of extreme rays. The question addressed here is, loosely speaking, whether this phenomenon of irreducibility of operators being determined by the extreme rays alone is characteristic of simplicial cones. The first main theorem gives an affirmative answer to this question.

1. DEFINITIONS AND NOTATION

The survey paper [1] is a general references for cones, faces, and cone preserving maps.

Let V be a finite dimensional real vector space with the unique topology which makes V a topological vector space. (If V is identified with $\mathbf{R}^n$, this is the usual topology.)

*This author's research work was partially supported by the National Science Council of the Republic of China.

DEFINITION 1 *Let K be a subset of V.*

(a) *K is a cone if it is nonempty and satisfies:*

$$x, y \in K, \alpha, \beta \geq 0 \quad (\alpha, \beta \text{ real}) \qquad \text{imply} \quad \alpha x + \beta y \in K.$$

(b) *K is closed if it is closed in the topology of V.*
(c) *K is full if* $\operatorname{int} K \neq \emptyset$ *(or equivalently,* $\operatorname{span} K = V$).
(d) *K is pointed if $K \cap (-K) = \{0\}$.*
(e) *A subcone F of a closed pointed full cone K is a face of K if $x \in K$, $y - x \in K$, and $y \in F$ imply $x \in F$.*

Notation. If K is a closed pointed full cone and if $S \subseteq K$, then the least face of K containing S is denoted by $\Phi(S)$. If $S = \{x\}$, we write $\Phi(x)$ for simplicity. If F is a face of K, then $\dim F$ is the dimension of the subspace spanned by F. If $\Phi(x)$ is a face of K and $\dim \Phi(x) = 1$, then $\Phi(x)$ is termed an *extreme ray*, and x an *extremal*, of K. The set $\mathcal{F}(K)$ of all faces is a lattice under the operations

$$F \vee G = \Phi(F \cup G), \qquad F \wedge G = F \cap G$$

where $F, G \in \mathcal{F}(K)$. In addition, the cone K is *simplicial* iff it has exactly $n = \dim V$ extreme rays. In this case K is the image of the nonnegative orthant under a nonsingular linear transformation. If there is no confusion about the cone K we shall write $\mathcal{F}$ for $\mathcal{F}(K)$. Note that the set $\mathcal{F}$ contains the trivial faces K and $\{0\}$. We shall use $\mathcal{F}'$ to denote the set of all nontrivial faces. We let $\mathcal{E}$ denote the set of all extreme rays. Thus $\mathcal{E} \subseteq \mathcal{F}'$. If $F \in \mathcal{F}$ we also write $F \triangleleft K$. If $F, G \in \mathcal{F}$ and $F \subseteq G$ we write $F \triangleleft G \triangleleft K$. This notation is easily checked to be consistent, that is, F is a face of G where G is considered to be a cone in its span.

DEFINITION 2 *Let K be a full pointed closed cone and let $L(V)$ be the set of all linear transformations on V.*

(a) *Let $\Pi(K) = \{A \in L(V) : AK \subseteq K\}$. A linear map $A \in \Pi(K)$ will be called nonnegative.*
(b) *$A \in \Pi(K)$ is irreducible iff A leaves no nontrivial face of K invariant.*
(c) *$A \in \Pi(K)$ is primitive iff for all nonzero $x \in K$ there is a positive integer k such that $A^k x \in \operatorname{int} K$.*

Remark Since V is finite dimensional, we have, A is primitive iff there is an integer $k > 0$ such that for all nonzero $x \in K$, $A^k x \in \operatorname{int} K$.

2. GRAPHS ASSOCIATED WITH A NONNEGATIVE OPERATOR

If $A \in \Pi(K)$ we may associate with A four directed graphs. If $F, G \in \mathcal{F}'$, we say there is an $\mathcal{I}$-edge from F to G, $\mathcal{I}(F, G)$, iff $G \triangleleft \Phi[(I + A)F]$. We say there is a $\mathcal{P}$-edge from F to $G, \mathcal{P}(F, G)$, iff $G \triangleleft \Phi[AF]$. Note that every $\mathcal{P}$-edge corresponds to an $\mathcal{I}$-edge since $\Phi[AF] \triangleleft \Phi[(I + A)F]$. Let $\mathcal{I}$ and $\mathcal{P}$ denote the set of $\mathcal{I}$-edges and $\mathcal{P}$-edges respectively. If necessary we write $\mathcal{I}(A)$ or $\mathcal{P}(A)$ to indicate the dependence upon A. Then $(\mathcal{F}',\mathcal{I})$ denotes the directed graph with vertex set $\mathcal{F}'$ and edge set $\mathcal{I}$. The other pairings are defined analogously. These definitions differ from those in [2] by the use of $\mathcal{F}'$ instead of $\mathcal{F}$.

The strong connectedness of the directed graphs $(\mathcal{F}',\mathcal{I})$ and $(\mathcal{F}',\mathcal{P})$ are defined in the standard way. In other words, we say that $(\mathcal{F}',\mathcal{I})$ [respectively $(\mathcal{F}',\mathcal{P})$] is *strongly connected* iff for any $F,G \in \mathcal{F}'$ there is a path of $\mathcal{I}$-edges [respectively $\mathcal{P}$-edges] from F to G. It is readily checked that the strong connectedness of $(\mathcal{F}',\mathcal{I})$ [respectively $(\mathcal{F}',\mathcal{P})$] is equivalent to that of $(\mathcal{F},\mathcal{I})$ [respectively $(\mathcal{F},\mathcal{P})$] as defined in [2]. So the results in [2] remain valid using $\mathcal{F}'$ in place of $\mathcal{F}$.

In [2] we proved that $A \in \Pi(K)$ is irreducible iff $(\mathcal{F}',\mathcal{I})$ is strongly connected and that A is primitive iff $(\mathcal{F}',\mathcal{P})$ is strongly connected. Note that the statement of Lemma 2 there needs to be modified by adding the condition $\mathcal{N}(A) \cap K = \{0\}$. However, the result still holds. If K is a simplicial cone, in particular if K is the nonnegative orthant, then $A \in \Pi(K)$ is irreducible iff $(\mathcal{E},\mathcal{P})$ is strongly connected. For general cones K if $(\mathcal{E},\mathcal{P})$ is strongly connected, then A is irreducible. The converse is false as we will show shortly. The question addressed by the main results is whether the truth of the converse for all $A \in \Pi(K)$ entails that K is simplicial. We give an affirmative answer.

EXAMPLE Let $K \subseteq R^3$ be defined by

$$K = \{x : (x_1^2 + x_2^2)^{1/2} < x_3\},$$

and let A be a rotation about the x_3-axis through an angle which is not a multiple of 2π. Then A is irreducible, $(\mathcal{E},\mathcal{I})$ is strongly connected, but $(\mathcal{E},\mathcal{P})$ is not strongly connected.

3. MAIN RESULTS

THEOREM 1 *Let K be a closed, pointed full cone in a finite dimensional real vector space V. Then K is simplicial if for any $A \in \Pi(K)$, $(\mathcal{E},\mathcal{P}(A))$ is strongly connected whenever A is irreducible.*

Notation. If $x_1,\ldots,x_r$ are vectors we denote by $\mathrm{pos}\{x_1,\ldots,x_r\}$ the cone generated by $x_1,\ldots,x_r$, that is, $\mathrm{pos}\{x_1,\ldots,x_r\} = \{\Sigma\alpha_j x_j : \alpha_j \geq 0,\ j = 1,\ldots,r\}$.

Before giving the proof we make the following

Observation. If C is any n-dimensional closed, pointed full non-simplicial cone, then there exist linearly independent extreme vectors $x_1,\ldots,x_r$ with $r < n$ such that $x_1 + \cdots + x_r \in \mathrm{int}K$.

Indeed, first choose any n linearly independent extreme vectors $x_1,\ldots,x_n$ of C. Since C is not simplicial, there exists some vector $y \in C\backslash\mathrm{pos}\{x_1,\ldots,x_n\}$. As

$$\sum_{i=1}^{n} x_i \in \mathrm{int\,pos}\{x_1,\ldots,x_n\},$$

there exists $\alpha > 0$ such that

$$\sum_{i=1}^{n} x_i + y \in bdy\mathrm{pos}\{x_1,\dots,x_n\}.$$

Call this vector w. Then clearly w can be expressed as the positive linear combination of fewer than n vectors from $\{x_1,\dots,x_n\}$. But $w \in \mathrm{int}\, C$ as $y \in C$ and

$$\sum_{i=1}^{n} x_i \in \mathrm{int}\,\mathrm{pos}\{x_1,\dots,x_n\} \subseteq \mathrm{int}\, C.$$

Proof Suppose that K is not simplicial. Among all finite sets $\{e_1,\dots,e_p\}$ of extreme vectors of K with $e_1 + \cdots + e_p \in \mathrm{int} K$, choose one with minimal cardinality and call it $\{x_1,\dots,x_r\}$. (From the above observation, we have $r < n$, where $n = \dim K$.) Since r is minimal, it is clear that, for any j, $1 \le j \le r$, $x_1 + \cdots + \hat{x}_j + \cdots + x_r \in bdy K$, where $\hat{x}_j$ indicates that the term x_j is to be omitted. In fact, $\mathrm{pos}\{x_i,\dots,x_r\}$ is a simplicial cone, and for each j, we have $\Phi(x_1 + \cdots + \hat{x}_j + \cdots + x_r) = \mathrm{pos}\{x_1,\dots,\hat{x}_j,\dots,x_r\}$, where the left side denotes a face of K. This follows from the beginning observation. For instance, if $\Phi(x_2 + \cdots + x_r)$ is not equal to $\mathrm{pos}\{x_2,\dots,x_r\}$, then $\Phi(x_2 + \cdots + x_r)$ is not simplicial (since $x_2,\dots,x_r$ are extreme vectors forming a basis of $\mathrm{span}\,\Phi(x_1 + \cdots + x_r)$), and so from the beginning observation (and its proof), there exist $2 \le i_1 < i_2 < \cdots < i_t \le r$ with $t < r - 1$ such that $\sum_{k=1}^{t} x_{i_k} \in \mathrm{relint}\,\Phi(x_2 + \cdots + x_r)$. But then since

$$\begin{aligned} K &= \Phi(x_1 + \cdots + x_r) = \Phi(x_1) \vee \Phi(x_2 + \cdots + x_r) \\ &= \Phi(x_1) \vee \Phi(x_{i_1} + \cdots + x_{i_t}) = \Phi\left(x_1 + \sum_{k=1}^{t} x_{i_k}\right), \end{aligned}$$

we have $x_1 + \sum_{k=1}^{t} x_{i_k} \in \mathrm{int} K$, contradicting the minimality of r.

Let us recall the definitions of dual cone and dual face. The dual cone K^* of K is the cone in the space V^* of linear functionals on V given by

$$K^* = \{f \in V^* : f(x) \ge 0, \text{ all } x \in K\}.$$

If F is a face of K, its dual face F^D is the face of K^* given by

$$F^D = \{f \in K^* : f(x) = 0, \text{ all } x \in F\}.$$

Since we identify V^{**} with V, we may talk about the dual face of a face of K^*. In particular, we have $F^{DD} \supset F$. We say that F is an *exposed* face if $F^{DD} = F$ (cf. [1] for a discussion).

Proceeding with the proof we choose for each j, $1 \le j \le r$, an $f_j \in K^*$ such that $\Phi(f_j) = \Phi(x_1 + \cdots + \hat{x}_j + \cdots + x_r)^D$. Then $f_j(x_k) = 0$ if $j \ne k$. But since $x_1 + \cdots + x_r \in \mathrm{int} K$, it is clear that $f_j(x_j) \ne 0$. Normalizing f_j, if necessary, we may assume that $f_j(x_j) = 1$.

Claim. $f_1 + \cdots + f_r \in \mathrm{int} K^*$.

First assume that every face of K is exposed. Then

$$\begin{aligned}\Phi(f_1+\cdots+f_r)^D &= \left[\bigvee_{j=1}^{r}\Phi(f_j)\right]^D = \bigwedge_{j=1}^{r}\Phi(f_j)^D\\ &= \bigwedge_{j=1}^{r}\Phi(x_1,\ldots,\hat{x}_j,\ldots,x_r)^{DD}\\ &= \bigwedge_{j=1}^{r}\Phi(x_1,\ldots,\hat{x}_j,\ldots,x_r)\\ &= \bigwedge_{j=1}^{r}\operatorname{pos}\{x_1,\ldots,\hat{x}_j,\ldots,x_r\} = \{0\},\end{aligned}$$

where $f_1+\cdots+f_r \in \operatorname{int}K^*$.

For the general case we may need to modify the choice of the x_j. For the previous argument to work we need only require, say, that $\Phi(x_2,\ldots,x_r)^{DD} = \Phi(x_2,\ldots,x_r)$. For then $\Phi(f_1+\cdots+f_r)^D \subseteq \Phi(x_2,\ldots,x_r)$. But the only vector in $\Phi(x_2,\ldots,x_r)$ which is orthogonal to $f_1+\cdots+f_r$ is the zero vector since $f_j(x_i) = \delta_{ij}$. So we have $\Phi(f_1+\cdots+f_r)^D = \{0\}$. In order to choose the x_j so that $\Phi(x_2,\ldots,x_r)$ is exposed we use the following lemma (Lemma 4.1 in [4]).

LEMMA *Let K be a closed pointed full cone. If $y \in K$ is such that* $\dim\Phi(y) = k$, *then there is a sequence of vectors $\{y_j\}_{j\in\mathbb{N}}$ in K converging to y such that each $\Phi(y_j)$ is an exposed face of K of dimension at most k.*

If $\Phi(x_2,\ldots,x_r)$ (which is of dimension $r-1$) is not an exposed face of K, then according to the lemma we can choose some vector $w \in K$ sufficiently close to $x_2+\cdots+x_r$ such that $x_1+w \in \operatorname{int}K$ and $\Phi(w)$ is an exposed face of K of dimension at most $r-1$. By the minimality of r and the observation preceding the proof of Theorem 1, $\Phi(w)$ is a simplicial face of dimension $r-1$. Choose extreme vectors $z_2,\ldots,z_r$ of $\Phi(w)$ in place of $x_2,\ldots,x_r$ and choose the f_j as before. So in all cases we can guarantee that $f_1+\cdots+f_r \in \operatorname{int}K^*$.

We next construct an irreducible $A \in \Pi(K)$. Recall that for any $z \in V$, $g \in V^*$ the rank one operator $z\otimes g$ is defined by $(z\otimes g)(x) = (g(x))z$ for all $x \in V$. Now put $A = \sum_{j=1}^{r} x_j \otimes f_{j+1}$, where f_{r+1} stands for t_1. Clearly $A \in \Pi(K)$. Note that the index of A is one, since $\operatorname{Im}A = \operatorname{span}\{x_1,\ldots,x_r\}$ $(r<n)$ and A takes the space $\operatorname{span}\{x_1,\ldots,x_r\}$ onto itself. Since $x_1+\cdots+x_r \in \operatorname{int}K$ we have that $\operatorname{Im}A\cap K \not\subseteq bdyK$. As $f_1+\cdots+f_r \in \operatorname{int}K^*$ it is also clear that $\mathcal{N}(A)\cap K = \{0\}$. Also, $\operatorname{Im}A\cap K$ is the simplicial cone $\operatorname{pos}\{x_1,\ldots,x_r\}$; otherwise the minimality of r will be contradicted. As A permutes the extreme vectors $x_1,\ldots,x_r$ cyclically (in the backwards order), it is clear that $A|_{\operatorname{Im}A}$ is irreducible with respect to $\operatorname{Im}A\cap K$. So by Lemma 2 of [2] (as corrected in the previous section), A is irreducible.

If x_{r+1} is an extreme vector of K distinct from $x_1,\ldots,x_r$, it is clear that there is no path in $(\mathcal{E},\mathcal{P}(A))$ from x_1 to x_{r+1}. So the graph $(\mathcal{E},\mathcal{P}(A))$ is not strongly connected. ∎

The definition and general properties of k-neighborly polytopes are well known in the theory of convex sets. We can extend these ideas to (not necessarily polyhedral) cones. In particular we say that K is *2-neighborly* iff for any two extreme rays $\Phi(x_1)$ and $\Phi(x_2)$ the face $\Phi(x_1) \vee \Phi(x_2)$ is in the boundary of K.

THEOREM 2 *Let K be closed, pointed full cone in a finite dimensional real vector space. Suppose that K is 2-neighborly. Then K is simplicial if, for any $A \in \Pi(K)$, $(\mathcal{E},\mathcal{I}(A))$ is strongly connected whenever A is irreducible.*

Proof Suppose that K is not simplicial. We are going to construct some $A \in \Pi(K)$ which is K-irreducible, but for which $(\mathcal{E},\mathcal{I}(A))$ is not strongly connected.

The proof is almost exactly the same as that for the previous theorem. We work with the same minimal set of extreme vectors $\{x_1,\ldots,x_r\}$ with the property that $\sum_{i=1}^{r} x_i \in \operatorname{int} K$. Construct the same K-irreducible matrix A as before. It remains to show that the directed graph $(\mathcal{E},\mathcal{I}(A))$ is not strongly connected.

Note that since K is 2-neighborly, we have $r \geq 3$. For any j, $1 \leq j \leq r$, $(\mathcal{I} + A)x_j = x_j + x_{j-1}$, where x_0 stands for x_r and x_{r+1} stands for x_1. But $\Phi(x_j + x_{j-1}) \subseteq \Phi(x_1 + \cdots + \hat{x}_{j+1} + \cdots + x_r)$, and as mentioned in the beginning part of the proof of Theorem 1 $\Phi(x_1 + \cdots + \hat{x}_{j+1} + \cdots + x_r)$ is equal to the simplicial cone $\operatorname{pos}\{x_1,\ldots,\hat{x}_{j+1},\ldots,x_r\}$. Hence, $\Phi(x_j + x_{j-1}) = \operatorname{pos}\{x_j, x_{j-1}\}$. Thus in the directed graph $(\mathcal{E},\mathcal{I}(A))$, the only $\mathcal{I}$-edges with initial vertex $\Phi(x_j)$ are the loop at $\Phi(x_j)$ and the edge from $\Phi(x_j)$ to $\Phi(x_{j-1})$. Take any extreme ray F of K, distinct from $\Phi(x_i)$, $1 \leq i \leq r$. It is now quite clear that in $(\mathcal{E},\mathcal{I}(A))$ there is no path from $\Phi(x_i)$ to F. So $(\mathcal{E},\mathcal{I}(A))$ is not strongly connected. ■

Remark A 2-neighborly cone may not be polyhedral. For instance, consider the 4-dimensional proper cone which comes from the following 3-dimensional compact convex set C in the usual way: $C = \operatorname{conv}[\{(0,0,1)\} \cup \{(\xi_1,\xi_2,0) : \xi_1^2 + \xi_2^2 \leq 1]$.

Remark In the preceding theorem, if we drop the assumption that K is 2-neighborly, then the result no longer holds. For a counter-example, consider the following.

EXAMPLE Let K be a minimal cone in R^3 generated by extreme vectors x_1, x_2, x_3 and x_4 such that $x_1 + x_3 = x_2 + x_4$. Suppose that there exists some $A \in \Pi(K)$ such that A is irreducible but $(\mathcal{E},\mathcal{I}(A))$ is not strongly connected. We are going to show that this will lead to a contradiction. As $(\mathcal{E},\mathcal{I}(A))$ is not strongly connected, without loss of generality, we may assume that in this directed graph there is no directed path leading from $\Phi(x_1)$ to some extreme ray of K. Note that $(I + A)x_1 \in \operatorname{int} K$, if Ax_1 belongs to $\Phi(x_3)$, relint $\Phi(x_4 + x_3)$ or relint $\Phi(x_2 + x_3)$; but then there are $\mathcal{I}$-edges from $\Phi(x_1)$ to every extreme ray of K. So we have, $Ax_1 \in \Phi(x_1 + x_2)$ or $\Phi(x_1 + x_4)$. By symmetry, we need only consider the case when $Ax_1 \in \Phi(x_1 + x_2)$. Note that the irreducibility of A requires that $Ax_1 \notin \Phi(x_1)$. It follows that there is an $\mathcal{I}$-edge from $\Phi(x_1)$ to $\Phi(x_2)$. The irreducibility of A also requires that $Ax_2 \notin \Phi(x_1 + x_2)$; otherwise A leaves the face $\Phi(x_1 + x_2)$ invariant. If $(I + A)x_2 \in \operatorname{int} K$, then there are $\mathcal{I}$-edges from $\Phi(x_2)$ to every extreme ray, and hence there are paths from $\Phi(x_1)$ to every extreme ray. In the remaining case we have, $Ax_2 \in \Phi(x_2 + x_3)$, but $Ax_2 \notin \Phi(x_2)$. Then there is an $\mathcal{I}$-edge from $\Phi(x_2)$ to $\Phi(x_3)$. If $Ax_3 \in \operatorname{int} K$, again we conclude that there are paths from $\Phi(x_1)$ to every extreme ray of K. Otherwise, $Ax_3 \in \Phi(x_3 + x_4)$ and $Ax_3 \notin \Phi(x_3)$. Then there is an edge from $\Phi(x_3)$ to $\Phi(x_4)$. So in all cases, we can conclude that there are paths in $(\mathcal{E}, \mathcal{I}(A))$ from $\Phi(x_1)$ to every extreme rays of K. This contradicts our initial hypothesis on $\Phi(x_1)$.

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