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The kth local exponent of doubly symmetric primitive matrices

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Abstract

Let $D = (V, E)$ be a primitive digraph. The local exponent of D at a vertex $u \in V$, denoted by $\gamma_D(u)$, is the least integer p such that there is an $u \to v$ walk of length p for each $v \in V$. Let $V = \{v_1, v_2, \ldots, v_n\}$. Following Brualdi and Liu, we order the vertices of V so that $\gamma_D(v_1) \leq \gamma_D(v_2) \leq \cdots \leq \gamma_D(v_n)$. Then $\gamma_D(v_k)$ is called the kth local exponent of D and is denoted by $\exp_D(k)$, $1 \leq k \leq n$. In this work we define $\exp(n, k) = \max\{\exp_G(k) | G = G(A) \text{ with } A \in DSP(n)\}$, where $DSP(n)$ is the set of all $n \times n$ doubly symmetric primitive matrices and $G(A)$ is the associated graph of matrix A. For $n \geq 3$, we determine that $\exp(n, k) = n - 1$ for all k with $1 \leq k \leq n$.

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1. Introduction

An $n \times n$ nonnegative matrix A is *primitive* if $A^k > 0$ for some positive integer k. The smallest such k is called the *exponent* of A and denoted by $\gamma(A)$. The *associated digraph* of matrix $A = (a_{ij})$, denoted by $D(A)$, is the digraph with a vertex set $V(D(A)) = \{v_1, v_2, \ldots, v_n\}$ such that there is an arc from v_i to v_j if and only if $a_{ij} \neq 0$ ($i, j = 1, \ldots, n$). A strongly connected digraph D is *primitive* provided the greatest common divisor of the lengths of its directed cycles equals 1. It is well known (see e.g. [1]) that

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D is primitive if and only if there exists a positive integer k such that for all ordered pairs of vertices x and y (not necessarily distinct), there is an $x \to y$ walk of length k. The smallest such k is called the *exponent* of D, denoted by $\gamma(D)$. Clearly, a matrix A is primitive if and only if its associated digraph $D(A)$ is primitive and in this case we have $\gamma(A) = \gamma(D(A))$.

In 1990, Brualdi and Liu [2] generalized the concept of exponent for a primitive digraph (primitive matrix). Let $D = (V, E)$ be a primitive digraph. If $x, y \in V$, the *exponent* from x to y, denoted by $\gamma_D(x, y)$, is the least integer p such that there exists an $x \to y$ walk of length t for all $t \geq p$. The *local exponent* of D at a vertex $x \in V$, denoted by $\gamma_D(x)$, is the least integer q such that for each $y \in V$, there is an $x \to y$ walk of length q. Clearly, $\gamma(D) = \max_{x\in V} \gamma_D(x) = \max_{x,y\in V} \gamma_D(x, y)$. Let the vertices of D be ordered as $v_1, v_2, \ldots, v_n$ so that $\gamma_D(v_1) \leq \gamma_D(v_2) \leq \cdots \leq \gamma_D(v_n)$. Then $\gamma_D(v_k)$ is called the *kth local exponent* of D and is denoted by $\exp_D(k)$, $1 \leq k \leq n$. We have $\exp_D(1) \leq \exp_D(2) \leq \cdots \leq \exp_D(n)$. Clearly, $\gamma(D) = \exp_D(n)$. So the local exponents of D are generalizations of the classical exponent of D. The kth local exponent of $D(A)$ is the smallest power of A for which there are k rows with no zero entry.

A graph G can be naturally associated with a symmetric digraph D_G by replacing each undirected edge $[x, y]$ of G by a pair of directed arcs (x, y) and (y, x) (a loop gets replaced by one arc). The primitivity and primitive exponent for a graph can be defined in a same way as for a digraph. It is well known (see e.g. [3,4]) that G is primitive if and only if G is connected and has at least one odd cycle; namely, G is a connected nonbipartite graph. The kth local exponent for a primitive graph can also be defined in the same way as for a primitive digraph. It is easy to see that a graph G is primitive if and only if the corresponding digraph D_G is primitive, and in this case we have $\exp_G(k) = \exp_{D_G}(k)$, $1 \leq k \leq n$.

An $n \times n$ symmetric matrix $A = (a_{ij})$ is said to be a *doubly symmetric matrix* if $a_{ij} = a_{n+1-i,n+1-j}$ $(i, j = 1, 2, \ldots, n)$. An $n \times n$ matrix A is called a *doubly symmetric primitive matrix* provided A is a doubly symmetric matrix and is primitive. If A is an $n \times n$ doubly symmetric matrix, then for any vertices v_i and v_j of the associated graph $G(A)$, $[v_i, v_j]$ is an edge if and only if $[v_{n+1-i}, v_{n+1-j}]$ is an edge. The vertex v_{n+1-i} is called the doubly symmetric vertex of v_i, denoted by v_i^d. Note that if $n \equiv 1 (\mathrm{mod}\ 2)$ and the vertex $v = v_{\frac{n+1}{2}}$, then $v = v^d$; otherwise, we always have $v \neq v^d$ for $v \in V(G(A))$. If $W = v_{i_1} v_{i_2} \cdots v_{i_m}$ is a walk from a vertex v_{i_1} to a vertex v_{i_m} in $G(A)$, then $v_{i_1}^d v_{i_2}^d \cdots v_{i_m}^d$ is a walk from $v_{i_1}^d$ to $v_{i_m}^d$ in $G(A)$, and denoted by W^d.

In this work, we use $DSP(n)$ to denote the class of all $n \times n$ doubly symmetric primitive matrices. Let $\exp(n, k)$ be the largest value of $\exp_G(k)$ for $G = G(A)$ with $A \in DSP(n)$, that is

$$\exp(n, k) = \max \left\{ \exp_G(k) | G = G(A) \text{ with } A \in DSP(n) \right\}. \tag{1}$$

It is easy to see that

$$\exp(n, 1) \leq \exp(n, 2) \leq \cdots \leq \exp(n, n). \tag{2}$$

Using the graph theoretical method we determine the number $\exp(n, k)$.

2. Preliminary results

In this section we discuss some lemmas which will be useful in obtaining our main results. Since $DSP(1)$ and $DSP(2)$ are sets of simple points, in the following we always assume that $n \geq 3$ for $DSP(n)$.

Lemma 2.1 ([4]). *If G is a primitive graph, then $\gamma(G) = \max_{x,\, y \in V(G)} \gamma_G(x, y)$.*

Lemma 2.2 ([5]). *Let G be a primitive graph, and let $x, y \in V(G)$. If there are walks joining x and y with lengths k_1 and k_2, where $k_1 + k_2 \equiv 1 (\mathrm{mod}\ 2)$, then $\gamma_G(x, y) \leq \max\{k_1, k_2\} - 1$.*

We will make use of the following notation. Let G be a graph. If W is a walk in G, then $|W|$ denotes the length of W. If P is a path (i.e. a walk without repeated vertices) in G and $x, y \in V(P)$, then xPy denotes the subpath of P joining x and y. In particular, $|xPx| = 0$. If C is an odd cycle and $x, y \in V(C)$, then C contains two walks joining x and y, and these walks are of different length since $|C|$ is odd. We denote these walks by $xC'y$ and $xC''y$ where $|xC'y| < |xC''y|$. Note that if $x = y$, then $|xC'y| = 0$ and $xC''y = C$. The concatenation of a walk W_1 from a vertex x to a vertex t, and a walk W_2 from t to a vertex y is denoted by $W_1 + W_2$. We denote the distance between two vertices x and y of G by $d(x, y)$. If G' and G'' are two subgraphs of G, then a shortest walk between G' and G'' has length $d(G', G'') = \min\{d(x, y) : x \in V(G'),\ y \in V(G'')\}$. Clearly, $d(G', G'') \geq 1$ if and only if $V(G') \cap V(G'') = \phi$.

Lemma 2.3. *Let x and y be two vertices of $G = G(A)$ with $A \in DSP(n)$, and let P_{xy} be a shortest path joining x and y. If $V(P_{xy}^d) = V(P_{xy})$, and $v^d \neq v$ for each vertex $v \in V(P_{xy})$, then $|zP_{xy}z^d|$ is odd, where z is any vertex in $V(P_{xy})$.*

Proof. Suppose that $y \neq x^d$. Then $x \neq y^d$. Since $V(P_{xy}^d) = V(P_{xy})$, we obtain the contradiction $d(x^d, y^d) = |x^d P_{xy} y^d| < |P_{xy}| = d(x, y)$. Hence $y = x^d$.

Now let $P_{xy} = v_0 v_1 \cdots v_m$, where $x = v_0$ and $y = v_m$. Let $z = v_t$ be any vertex in $V(P_{xy})$. It follows from $V(P_{xy}^d) = V(P_{xy})$ that there exists some vertex $v_k \in V(P_{xy})$ such that $v_k = z^d$ (and thus $v_k^d = v_t = z$). Hence $|xP_{xy}v_t| = d(v_0, v_t) = d(v_0^d, v_t^d) = |v_0^d P_{xy}^d v_t^d| = |v_k P_{xy} y|$, and hence $t + k = m$. Suppose that $|P_{xy}| = m$ is even. Then $v_{\frac{m}{2}}^d = v_{\frac{m}{2}}$, a contradiction. We conclude that m is odd, and so $|zP_{xy}z^d| = d(v_t, v_k) = |t - k| = |m - 2k|$ is odd. $\square$

Lemma 2.4. *Let x be any vertex of $G = G(A)$ with $A \in DSP(n)$. Let C_x be an odd cycle such that $d(x, C_x) = \min\{d(x, C) : C$ is an odd cycle in $G\}$, and let P be a shortest path between x and C_x. Assume that $|P| > 0$ and $V(P^d) \cap V(P) \neq \phi$. Then $V(P^d) \cap V(P) = \{v\} = \{v^d\}$.*

Proof. Let $|P| = m$ and $P = x_0 x_1 \cdots x_m$, where $x_0 = x$ and $x_m \in V(C_x)$. Let $v = x_t^d = x_k$ be any vertex in $V(P^d) \cap V(P)$, where $0 \leq t,\ k \leq m$. Suppose that $k \neq t$. Notice that $x_k^d = x_t$. If $k > t$, then $d(x, C_x^d) \leq |x_0 P x_t| + |x_k^d P^d x_m^d| = m - (k - t) < m$. If $k < t$, then $d(x, C_x^d) \leq |x_0 P x_k| + |x_t^d P^d x_m^d| = m - (t - k) < m$. Thus in any case we have $d(x, C_x^d) < m = d(x, C_x)$, contradicting the definition of C_x. Hence $k = t$; that is $v^d = v \in V(P^d) \cap V(P)$. Since the vertex v is unique if there exists some vertex v of G such that $v^d = v$, we have that $V(P^d) \cap V(P) = \{v\} = \{v^d\}$. $\square$

Lemma 2.5. *Let x and y be two vertices of $G = G(A)$ with $A \in DSP(n)$. Let P_{xy} be a shortest path joining x and y, and let $C_{P_{xy}}$ be an odd cycle such that $d(P_{xy}, C_{P_{xy}}) = \min\{d(P_{xy}, C) : C$ is an odd cycle in $G\}$. Assume that $d(P_{xy}, C_{P_{xy}}) > 0$. Then*

$$|P_{xy}| + |C_{P_{xy}}| + 2d(P_{xy}, C_{P_{xy}}) \leq n. \tag{3}$$

Proof. Let $P = x_0 x_1 \cdots x_m$ be a shortest path between P_{xy} and $C_{P_{xy}}$, where $x_0 \in V(P_{xy})$ and $x_m \in V(C_{P_{xy}})$. Clearly, $n \geq |V(P^d) \cup (V(P_{xy}) \cup V(P) \cup V(C_{P_{xy}}))|$, and $|V(P_{xy}) \cup V(P) \cup V(C_{P_{xy}})| = |P_{xy}| + |P| + |C_{P_{xy}}|$. It follows that

$$|P_{xy}| + |C_{P_{xy}}| + 2|P| \leq n - 1 + |V(P^d) \cap (V(P_{xy}) \cup V(P) \cup V(C_{P_{xy}}))|. \tag{4}$$

By the definition of $C_{P_{xy}}$, we have that $V(P^d) \cap V(C_{P_{xy}}) = \{x_m^d\}$ if $V(P^d) \cap V(C_{P_{xy}}) \neq \phi$, and that $V(P^d) \cap V(P_{xy}) = \{x_0^d\}$ if $V(P^d) \cap V(P_{xy}) \neq \phi$. We consider two cases.

Case 1: $V(P) \cap V(P^d) = \phi$. Suppose that $V(P^d) \cap V(C_{P_{xy}}) \neq \phi$ and $V(P^d) \cap V(P_{xy}) \neq \phi$. Then $x_m \neq x_m^d \in V(C_{P_{xy}})$ and $x_0 \neq x_0^d \in V(P_{xy})$. So the walks $x_0 P x_m + x_m C'_{P_{xy}} x_m^d + x_m^d P^d x_0^d + x_0^d P_{xy} x_0$ and $x_0 P x_m + x_m C''_{P_{xy}} x_m^d + x_m^d P^d x_0^d + x_0^d P_{xy} x_0$ are two cycles, and one of these cycles is an odd cycle denoted by C^*. Hence, $d(P_{xy}, C^*) = 0 < m = d(P_{xy}, C_{P_{xy}})$, contradicting the definition of $C_{P_{xy}}$. We conclude that $V(P^d) \cap V(C_{P_{xy}}) = \phi$ or $V(P^d) \cap V(P_{xy}) = \phi$. Thus

$$|V(P^d) \cap (V(P) \cup V(P_{xy}) \cup V(C_{P_{xy}}))| \leq 1. \tag{5}$$

Combining (4) and (5) we obtain (3).

Case 2: $V(P) \cap V(P^d) \neq \phi$. Then by Lemma 2.4 we have that $V(P) \cap V(P^d) = \{x_t\} = \{x_t^d\}$ $(0 \leq t \leq m)$, and hence by the definition of G we have $v \neq v^d$ for each vertex $v \in V(G) \backslash \{x_t\}$. We consider two subcases.

Subcase 2.1: $t = m$. Clearly, $V(P^d) \cap (V(P) \cup V(C_{P_{xy}})) = \{x_m\} = \{x_m^d\}$.

If $V(P^d) \cap V(P_{xy}) = \phi$, then

$$|V(P^d) \cap (V(P) \cup V(P_{xy}) \cup V(C_{P_{xy}}))| = 1. \tag{6}$$

Hence (3) follows from (4) and (6).

If $V(P^d) \cap V(P_{xy}) \neq \phi$, then $V(P^d) \cap V(P_{xy}) = \{x_0^d\}$, and hence

$$|V(P^d) \cap (V(P) \cup V(P_{xy}) \cup V(C_{P_{xy}}))| = 2. \tag{7}$$

Suppose that $V(P_{xy}^d) = V(P_{xy})$. Since $v \neq v^d$ for each vertex $v \in V(P_{xy})$, it follows from Lemma 2.3 that $|x_0 P_{xy} x_0^d|$ is odd. Hence the cycle $x_0 P x_m + x_m^d P^d x_0^d + x_0^d P_{xy} x_0$ (since $x_m = x_m^d$) is an odd cycle, contradicting the definition of $C_{P_{xy}}$. Thus $V(P_{xy}^d) \neq V(P_{xy})$, implying that there exists at least one vertex $w \in V(P_{xy})$ such that $w^d \in V(G) \backslash V(P_{xy})$. We conclude that the vertex $w^d \in V(G) \backslash (V(P_{xy}) \cup V(C_{P_{xy}}) \cup V(P) \cup V(P^d))$. Hence $|V(P^d) \cup (V(P_{xy}) \cup V(P) \cup V(C_{P_{xy}}))| \leq n - 1$, and hence

$$|P_{xy}| + |C_{P_{xy}}| + 2|P| \leq n - 2 + |V(P^d) \cap (V(P_{xy}) \cup V(P) \cup V(C_{P_{xy}}))|. \tag{8}$$

Now (3) follows from (7) and (8).

Subcase 2.2: $t \leq m - 1$. It is easy to see that $V(P^d) \cap V(C_{P_{xy}}) = \phi$. So $V(P^d) \cap (V(P) \cup V(C_{P_{xy}})) = \{x_t\} = \{x_t^d\}$.

If $t = 0$ or $V(P^d) \cap V(P_{xy}) = \phi$. Then we also have (6), and hence (3) also holds.

If $t \neq 0$ and $V(P^d) \cap V(P_{xy}) \neq \phi$. Then $V(P^d) \cap V(P_{xy}) = \{x_0^d\} \neq \{x_0\}$. Similar to the treatment in Subcase 2.1, we have (7) and (8), and hence (3) holds.

Now the proof of Lemma 2.5 is complete. □

Lemma 2.6. *Let x be any vertex of $G = G(A)$ with $A \in DSP(n)$, and let C_x be an odd cycle such that $d(x, C_x) = \min\{d(x, C) : C$ is an odd cycle in $G\}$. Assume that $d(x, C_x) > 0$. Then $|C_x| + 2d(x, C_x) \leq n$.*

Proof. The proof is similar to that in the previous lemma. □

Lemma 2.7. *Let x be any vertex of $G = G(A)$ with $A \in DSP(n)$. Then $\gamma_G(x, x) \leq n - 1$.*

Proof. Clearly, there is a walk joining x and x with length 2. Let C_x be an odd cycle such that $d(x, C_x) = \min\{d(x, C) : C$ is an odd cycle in $G\}$.

If $d(x, C_x) > 0$. Let P be a shortest path between x and C_x. Then $P + C_x + P$ is a walk joining x and x with odd length $|C_x| + 2|P|$. Hence by Lemmas 2.2 and 2.6 we have that $\gamma_G(x, x) \leq \max\{|C_x| + 2|P|,\ 2\} - 1 \leq \max\{n,\ 2\} - 1 = n - 1$.

If $d(x, C_x) = 0$. Then C_x is a walk joining x and x with odd length $|C_x| \leq n$. So by Lemma 2.2 we have $\gamma_G(x, x) \leq \max\{|C_x|,\ 2\} - 1 \leq n - 1$. □

Lemma 2.8. *Let x and y be two vertices of $G = G(A)$ with $A \in DSP(n)$. Then $\gamma_G(x, y) \leq n - 1$.*

Proof. Let P_{xy} be a shortest path joining x and y, and let $C_{P_{xy}}$ be an odd cycle such that $d(P_{xy}, C_{P_{xy}}) = \min\{d(P_{xy}, C) : C$ is an odd cycle in $G\}$.

If $d(P_{xy}, C_{P_{xy}}) = 0$. Let $u, v \in V(P_{xy}) \cap V(C_{P_{xy}})$ (perhaps $u = v$), where u (v) is the first (last) vertex on $C_{P_{xy}}$ along P_{xy}. Then the lengths of walks $xP_{xy}u + uC'_{P_{xy}}v + vP_{xy}y$ and $xP_{xy}u + uC''_{P_{xy}}v + vP_{xy}y$ have different parity and are not greater than n. So by Lemma 2.2 we have that $\gamma_G(x, y) \leq n - 1$.

If $d(P_{xy}, C_{P_{xy}}) > 0$. Let $P = x_0x_1 \cdots x_m$ be a shortest path between P_{xy} and $C_{P_{xy}}$, where $x_0 \in V(P_{xy})$ and $x_m \in V(C_{P_{xy}})$, and let the walk $W = xP_{xy}x_0 + P + C_{P_{xy}} + P + x_0P_{xy}y$. Then by Lemma 2.5 we have $|W| = |P_{xy}| + |C_{P_{xy}}| + 2|P| \leq n$. It follows from Lemma 2.2 that $\gamma_G(x, y) \leq \max\{|W|,\ |P_{xy}|\} - 1 = |W| - 1 \leq n - 1$. □

3. The number $\exp(n, k)$

In this section we determine the number $\exp(n, k)$.

Theorem 3.1. *Let n and k be integers with $n \geq 3$ and $1 \leq k \leq n$. Then we have*

$$\exp(n, k) = n - 1. \tag{9}$$

Proof. Let A be any matrix in $DSP(n)$, and let $x, y \in V(G(A))$, where $G(A)$ is the associated graph of A. Then by Lemmas 2.1, 2.7 and 2.8, we have $\gamma(G(A)) = \max\{\gamma_{G(A)}(x, y) | x, y \in V(G(A))\} \leq n - 1$, that is $\exp_{G(A)}(n) \leq n - 1$. But A is also an arbitrary matrix in $DSP(n)$, so we have by the definition of $\exp(n, k)$ that

$$\exp(n, k) \leq \exp(n, n) = \max\left\{\exp_{G(A)}(n) | A \in DSP(n)\right\} \leq n - 1.$$

On the other hand, let $G^* = (V, E)$ be a graph, where $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{[v_i, v_{i+1}] : 1 \leq i \leq n-1\} \cup \{[v_1, v_1],\ [v_n, v_n]\}$, and let $A(G^*)$ be the adjacency matrix of G^*. Clearly, $A(G^*) \in DSP(n)$. For each vertex $v_i \in V$, there is no walk of length $n - 2$ from v_i to v_{n+1-i}. Hence $\exp_{G^*}(v_i) \geq n - 1$, and hence $\exp_{G^*}(k) \geq n - 1$ for any k with $1 \leq k \leq n$. So we have that

$$\exp(n, k) \geq \exp_{G^*}(k) \geq n - 1.$$

Combining the above two relations, we obtain $\exp(n, k) = n - 1$, as desired. □

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