Stokes Flow

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Abstract

Properties of Stokes flow were analyzed. Stokes equations were derived. The Papkovich-Neuber solution method for incompressible Stokes flow was stated, proved, and applied to the problem of creeping flow around a ball. Finally, a sample of five solutions to Stokes equations for zero pressure were visualized and compared.

1 Introduction

The viscosity of a fluid describes its resistance to flow[1]; the ability to transport and deform under shear stresses[2]. Often represented by the *kinematic viscosity coefficient*, ν, it is an experimentally measurable quantity and a critical factor in determining fluid behaviour. The physical interpretation of ν is the rate of shear stresses to tangential stresses per unit density between layers of fluid moving past each other[3]. Highly viscous media such as glucose has a kinematic viscosity of over 7 million[4] at room temperature and greatly resists shear flows while cool, distilled water has a kinematic viscosity of roughly 1[4] and flows with ease.

An important quantity when considering flow behaviour is the *Reynolds number*[5], $Re \equiv \frac{UL}{\nu}$, the ratio of velocity scale U and length scale L to the kinematic viscosity. U and L are determined by the geometry and initial conditions of the flow while ν is a material property[5].

Slow moving fluid flows of a viscous material on small length scales have a resultingly tiny Reynolds number, $Re \ll 1$, and are governed by the Stokes equations[6], a linearization of the Navier-Stokes equations. These flows mostly ignore inertial effects and are instead governed by viscous forces[7]. These flows are very laminar, hence easier to work with. Flows of this type are referred to as *Stokes flow*[3] or *creeping flow*[6].

A major benefit to incompressible Stokes flow is that the equation is linear, making it typically much easier to solve than the nonlinear Navier Stokes equation. The assumptions for these flows are very realistic for classical treatment of viscous fluids like honey and oil[6], or even less viscous fluids in particular geometries such as the flow of blood in tiny capillaries[6].

Due to the importance of boundary conditions for partial differential equations, the behaviour of fluid flow around objects has been rigorously studied. Many different methods to find analytic solutions to the incompressible Stokes equations have been found, such as the Papkovich-Neuber solution method where the velocity and pressure fields are decomposed into harmonic potential fields. This method can be applied to many different initial boundary value problems to determine exact solutions, such as the problem of Stokes flow around two spheres.

2 Equations

Governing the flow of viscous fluids, the Navier-Stokes equations for incompressible fluids can be written as[7]:

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v} \ , \tag{1}$$

$$\nabla \cdot \vec{v} = 0 \ , \tag{2}$$

where $\vec{v}$ is the velocity of the fluid, $\frac{D}{Dt}$ is the material derivative, p is the pressure field, ρ is the density, and ν is the kinematic viscosity. Being an incompressible fluid, the density is assumed to be constant. Neglecting external body forces, this can be linearized for highly viscous flows with small Reynolds numbers[7] $Re << 1$. This yields the equations of motions for Stokes flow, Stokes equations[7]:

$$\frac{\nabla p}{\rho} = \nu \nabla^2 \vec{v} \ , \tag{3}$$

$$\nabla \cdot \vec{v} = 0 \ , \tag{4}$$

which is the governing equation for Stokes flow.

3 Analysis

To tackle the problem of finding Stokes flows it is first important to validate the equations and find a solution method. Linearizing the Navier-Stokes and decomposing the solutions into harmonic potentials will be shown, along with an application of this method.

3.1 Derivation of Stokes Equations

Utilizing the scale factors introduced in the Reynolds number, the Navier-Stokes equations can be non-dimensionalized[7]. We define the following dimensionless quantities, denoted by asterisks:

$$\vec{v} = U\vec{v^*},$$
$$\frac{\partial}{\partial t} = \frac{U}{L}\frac{\partial}{\partial t^*},$$
$$\nabla = \frac{1}{L}\nabla^*,$$
$$p = \rho U^2 p^*.$$

Substituting into equation (1):

$$\frac{U^2}{L}\frac{\partial \vec{v^*}}{\partial t^*} + \frac{U^2}{L}(\vec{v^*}\cdot\nabla^*)\vec{v^*} = -\frac{U^2}{L}\nabla^* p^* + \frac{U}{L^2}\nu\nabla^{*2}\vec{v^*} \ .$$

We can then multiply through by $\frac{L^2}{U\nu}$ to get:

$$Re\left(\frac{D}{Dt}\right)^* \vec{v^*} = -Re\nabla^* p^* + \nabla^{*2}\vec{v^*} \ ,$$

where $\left(\frac{D}{Dt}\right)^*$ is the material derivative with respect to the dimensionless quantities t^* and $\vec{v^*}$. While the pressure field may vary from the velocity field by orders of magnitude, it is clear that for sufficiently small Reynolds numbers the material derivative term can be ignored[7]. The resulting dimensionless equation, with asterisks withheld for clarity, is then:

$$Re\nabla p = \nabla^2\vec{v} \ . \tag{5}$$

Reintroducing dimensional variables to equation (5) trivially produces equation (8).

3.2 Papkovich-Neuber Solution

The Papkovich-Neuber[10][11] method for solving partial differential equations involves performing Helmholtz decompositions of the variables in question to achieve a solution as a function of harmonic potentials. The Helmholtz decomposition of a square integrable vector field $\vec{v}$ on a bounded, simply connected, Lipschitz domain is given by[12]:

$$\vec{v} = \nabla\Phi + \nabla\times\vec{\Psi},$$

where Φ is an irrotational scalar potential and Ψ is a divergence free vector potential. Tran-Cong[9] shows that whenever the Helmholtz decomposition is possible, any harmonic function can be expressed as the divergence of another harmonic vector:

$$\nabla^2 f = 0 \implies f = \nabla\cdot\vec{A},$$

for a harmonic vector field $\vec{A}$. These properties lead Tran-Cong and Blake[8] to the conclusion that any solution to the Stokes equations is of the form:

$$\vec{v} = \nabla(\vec{r} \cdot \vec{\Phi} + \chi) - 2\vec{\Phi}, \tag{6}$$

$$\frac{p}{\nu\rho} = 2\nabla \cdot \vec{\Phi}, \tag{7}$$

for a harmonic vector potential $\vec{\Phi}$ and harmonic scalar potential χ, which is the Papkovich-Neuber solution for Stokes flow.

3.2.1 Proof of the Papkovich-Neuber Solution to Stokes Flow

Taking the divergence of equation (8) yields:

$$\frac{\nabla^2 p}{\nu\rho} = \nabla \cdot (\nabla^2 \vec{v}). \tag{8}$$

The vector identity (17) in the Appendix and the incompressibility condition, equation (4), yield that $\frac{p}{\nu\rho}$ is a harmonic function. The pressure can then be written in the form:

$$p = 2\nabla \cdot \vec{\Psi}, \tag{9}$$

for a harmonic vector potential $\vec{\Psi}$.Substituting this into equation (8) with vector identity (18) from the appendix, we get the equation for $\vec{v}$:

$$\nabla^2 \vec{v} = \nabla^2 \nabla(\vec{r} \cdot \vec{\Psi}).$$

Hence $\vec{v}$ is of the form:

$$\vec{v} = -2\vec{\Phi} + \nabla(\vec{r} \cdot \vec{\Psi}), \tag{10}$$

for another harmonic vector potential Φ. Since $\vec{v}$ is divergence free, taking the divergence of equation (10) and applying vector identity (19), we get that:

$$\nabla \cdot \vec{\Phi} = \nabla \cdot \vec{\Psi}, \tag{11}$$

$$\implies \vec{\Psi} = \vec{\Phi} + \vec{\eta}, \tag{12}$$

where η is harmonic and divergence free. We now note that for a divergence free $\vec{\eta}$, the quantity $\chi = \vec{r} \cdot \vec{\eta}$ is a harmonic scalar potential by identity (19). We can then substitute our expressions for χ and equation (12) into equation (10) to yield the desired form for $\vec{v}$:

$$\vec{v} = \nabla(\vec{r} \cdot \vec{\Phi} + \chi) - 2\vec{\Phi}.$$

Similarly, substituting equation (11) into (9) yields the desired form for p:

$$\frac{p}{\nu\rho} = 2\nabla \cdot \vec{\Phi}.$$

3.3 Stokes Flow Around a Sphere

To apply the Papkovich-Neuber solution, consider the case of Stokes flow around a sphere. Following the same setup as in Lautrup's section on creeping flow around a solid ball[6], we'll consider a sphere of radius a in a fluid with asymptotic flow $\vec{U} = U\hat{z}$. Due to the symmetry of the sphere, the flow and pressure will not depend on the altitude angle, ϕ, about the z-axis, only depending on the altitude angle, θ, and the radius, r, from the center of the sphere.

Assuming the continuity and spherical symmetry carries over from $\vec{v}$ to the harmonic potentials $\vec{\Phi}$ and χ, we get the well studied equation:

$$\nabla^2 u(r,\theta) = 0, \tag{13}$$
$$u(r,\pi) = u(r,-\pi) \tag{14}$$
$$\frac{\partial u}{\partial \theta}(r,\pi) = \frac{\partial u}{\partial \theta}(r,-\pi) \tag{15}$$

for each χ and each component of ϕ. This partial differential equation has spherical solution:

$$u(r,\theta) = \sum_{k=0}^{\infty} \left[A_k r^k + B_k r^{-k}\right]\left[C_k cos(k\theta) + D_k sin(k\theta)\right], \tag{16}$$

We now notice that since all integer powers of r and all integer frequencies of $sin(k\theta)$ and $cos(k\theta)$ are summed over with arbitrary Fourier coefficients in front, any derivative of u and any linear combination of u will be of the same general form as u but with different Fourier coefficients. We can then conclude from the Papkovich-Neuber solution for v, equation (6), that the components of $\vec{v}$ are of the same form as u in equation (16). Since $\vec{v}$ must satisfy the no-slip and asymptotic flow boundary conditions[7] as well, we get the equations:

$$\begin{aligned}
v_r(r,\theta) &= \sum_{k=0}^{\infty} \left[A_k r^k + B_k r^{-k}\right]\left[C_k cos(k\theta) + D_k sin(k\theta)\right], \\
v_\theta(r,\theta) &= \sum_{k=0}^{\infty} \left[a_k r^k + b_k r^{-k}\right]\left[c_k cos(k\theta) + d_k sin(k\theta)\right], \\
\nabla \cdot \vec{v} &= 0, \\
\lim_{r\to 0} \vec{v} &= U\hat{z} = U cos\theta \hat{r} - U sin\theta\hat{\theta}, \\
\vec{v}(a,\theta) &= 0.
\end{aligned}$$

Applying the asymptotic limit, we see that $D_k = c_k = 0$, $C_k = -D_k = U\delta_{k,1}$,

$A_k = a_k = 0$, and $B_1 = b_1 = 1$. Our system then becomes:

$$
\begin{aligned}
v_r(r,\theta) &= Ucos(\theta)\left[1 + \sum_{k=1}^{\infty} B_k r^{-k}\right], \\
v_\theta(r,\theta) &= -Usin(\theta)\left[1 + \sum_{k=1}^{\infty} b_k r^{-k}\right], \\
\nabla \cdot \vec{v} &= 0, \\
\vec{v}(a,\theta) &= 0.
\end{aligned}
$$

Applying $\nabla \cdot \vec{v} = \vec{0}$:

$$
\begin{aligned}
\nabla \cdot \vec{v} &= Ucos(\theta) \sum_{k=1}^{\infty} r^{-k-1}\big[B_k(2-k) - 2b_k\big], \\
0 &= B_k(2-k) - 2b_k, \\
b_k &= \frac{1}{2} B_k(2-k).
\end{aligned}
$$

Following Lautrup's steps[6] or Landau and Lifshitz' derivation[7], we make the ansatz that B_k terms vanish for $k \notin \{0,1,3\}$. This allows us to utilize the final boundary condition, that $\vec{v}(a,\theta) = 0$, and the formula for b_k:

$$
\begin{aligned}
0 &= Ucos(\theta)\left[1 + \sum_{k=1}^{\infty} B_k r^{-k}\right] &&\implies \quad 1 + B_1 a^{-1} + B_3 a^{-3} = 0, \\
0 &= -Usin(\theta)\left[1 + \sum_{k=1}^{\infty} b_k r^{-k}\right] &&\implies 1 + \frac{1}{2}B_1 a^{-1} - \frac{1}{2}B_3 a^{-3} = 0.
\end{aligned}
$$

Solving this linear system yields $B_1 = -\frac{a3}{2}$, $b_1 = -\frac{a3}{4}$, $B_3 = \frac{a^3}{2}$, $b_2 = -\frac{a^3}{4}$. We arrive at the solution:

$$
\vec{v} = \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)Ucos(\theta)\hat{r} - \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)Usin(\theta)\hat{\theta}
$$

4 Computations

A simple but interesting solution family of Stokes flows arise from the case $\vec{\Phi} = 0$, Φ being the vector potential given in the Papkovich-Neuber - equation (6). The resulting fields are:

$$
\begin{aligned}
\vec{v} &= \nabla \chi, \\
p &= 0.
\end{aligned}
$$

The constant pressure case then allows χ to be any harmonic function. Five example cases are Shown in the following table for comparison:

$\chi(x, y, z)$	$\vec{v}(x, y, z) = [v_x, v_y, v_z]$	Error	Figure(s)
$e^x sin(y)$	$\left[e^x sin(y), e^x cos(y), 0\right]$	None	1
$-ln(x^2 + y^2)$	$\left[\frac{-2x}{x^2+y^2}, \frac{-2y}{x^2+y^2}, 0\right]$	$x = y = 0$	2
$-ln(r + z)$	$\left[\frac{-x}{r(z+r)}, \frac{-y}{r(z+r)}, \frac{-1}{r}\right]$	$z \leq 0 \ \ \& \ \ r = 0$	3(a), 3(b)
$\frac{x}{x^2+y^2}$	$\left[\frac{y^2-x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2}, 0\right]$	$x = y = 0$	4
$\frac{x}{r(r+z)}$	$\left[\frac{(z^2+y^2)(z+r)-x^2r}{r^3(r+z)^2}, \frac{-2xy}{r^3(r+z)^2}, \frac{-2xy}{r^3}\right]$	$r = 0$	5(a), 5(b)

Column one states the form of the scalar potential, χ, in equation (6). The resulting velocity field is in the second column. Any areas where the velocity is undefined are listed in column three. Simply connected domains must not include the subsets listed in this column. The final column gives the figure numbers of the velocity field plots for reference. In all equations, the spherical coordinate radius is denoted $r = \sqrt{x^2 + y^2 + z^2}$ for simplicity, and subsequent plots are shown in Cartesian coordinates. The asymptotic behaviour and possible domains for the given Stokes flows will be discussed based on the resulting velocity fields and their plots.

All sample flows discussed are pressure free solutions to Stokes equations, and will both exist and satisfy the assumptions required for Stokes flow on a subset of $\Re^2$ or $\Re^3$. By linearity of the gradient operator, any linear combination of these flows will also be pressure free and satisfy the incompressible Stokes equations on some domain. Despite having diverging fluid speeds in some regions, the pressure of these flows is still zero. Realistically however the model will break before the speed gets too large due to the increasing Reynolds number[6] and the advective acceleration will no longer be negligible[7], changing the flow behaviour completely. As such, the domain of interest will require small velocities and break near to singularities.

Figure 1 displays the only flow that is well behaved on the entire real plane. The assumption of low Reynolds number will be broken however at large values of x as the mean fluid speed diverges.

Figures 2 and 4 display flows that will break the assumption of low Reynolds numbers for values far enough away from the origin. Both flows asymptotically approach zero velocity as the distance from the origin increases and will thus be able to maintain Stokes flow for domains sufficiently far from the origin.

Figures 3(a) and 5(a) display well behaved cross sections of otherwise poorly behaved fluids. Both show the (x, y) plane when $z = 2$, hence $r \geq 2$. Both flows asymptotically approach zero, leading us to believe that the assumptions required for Stokes flow will hold. However both functions diverge at the origin, with the flow in figure 3(b) diverging on the entire negative z-axis. As such, domains of interest must be sufficiently far from the negative z-axis to maintain a low Reynolds number, while the flow in figure 5(b) only requires flows sufficiently far from the origin.

5 Conclusions

For incompressible flows with sufficiently small Reynolds number, the Stokes equation of motion was presented. The derivation was shown to be a direct linearization of the dimensionless Navier-Stokes equation for appropriate length and velocity scales.

The Papkovich-Neuber method for obtaining analytic solutions was presented and proven. The decomposition of the velocity and pressure fields into harmonic potential fields was shown to be possible and powerful. The problem of Stokes flow around a sphere was solved with this method.

A handful of pressure free Stokes flows were presented and compared. All flows were found to satisfy the assumption of small velocities on some domain. All flows were also shown to diverge in some limit, either point-wise at the origin, asymptotically as the distance from the origin approached infinity, or along an axis.
Many interesting results of Stokes flow can be analyzed in future works. The time-reversibility property and phenomena of un-mixing is worth exploring. For such a problem, Taylor-Couette flow could be analyzed and solved with the Papkovich-Neuber method discussed in this work. Other phenomena such as Stokes' paradox or the behaviour of the Stokeslet are promising topics to explore.

6 Figures

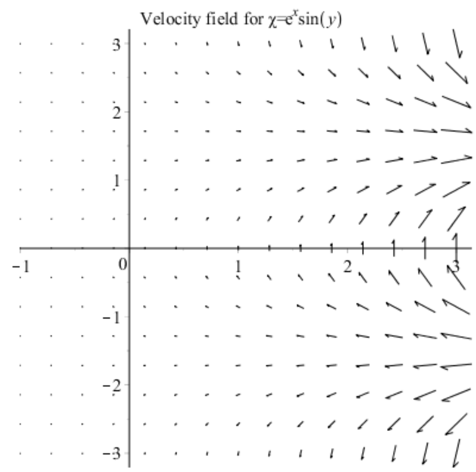


Figure 1: This figure displays the Stokes flow for the scalar potential $\chi = e^x sin(y)$ and a zero vector potential in the Papkovich-Neuber solution. The velocity is well defined on $\Re^2$.

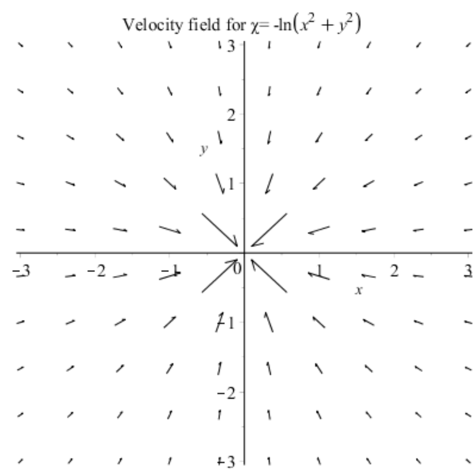
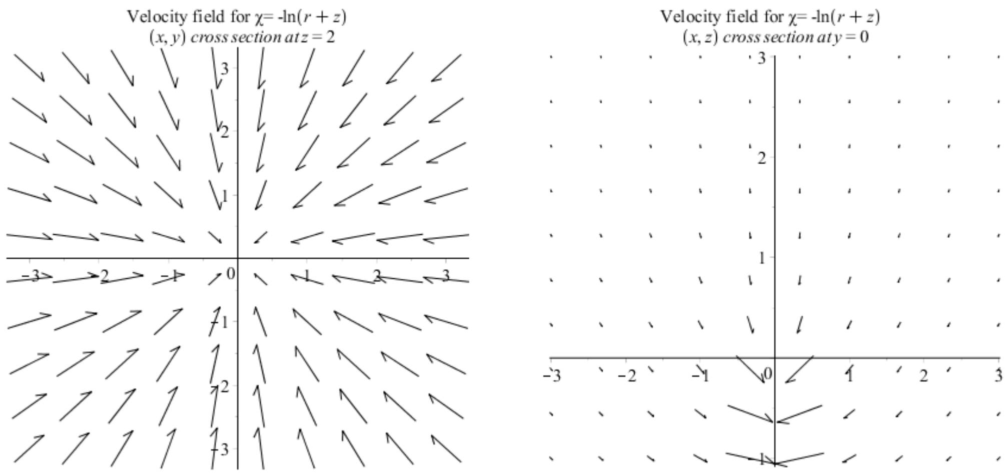


Figure 2: This figure displays the Stokes flow for the scalar potential $\chi = -ln(x^2 + y^2)$ and a zero vector potential in the Papkovich-Neuber solution. The velocity is well defined on $\Re^2 \backslash \{(0,0)\}$.



(a) (x,y) cross section at z=2.

(b) (x,z) cross section at y=0.

Figure 3: This figure displays the Stokes flow for the scalar potential $\chi = -ln(r+z)$ and a zero vector potential in the Papkovich-Neuber solution. The velocity is well defined on $\Re^3\backslash\{(x,y,z) : x = 0, y = 0, z \leq 0\}$.

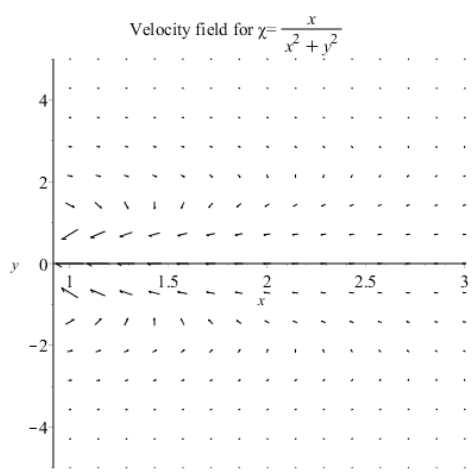
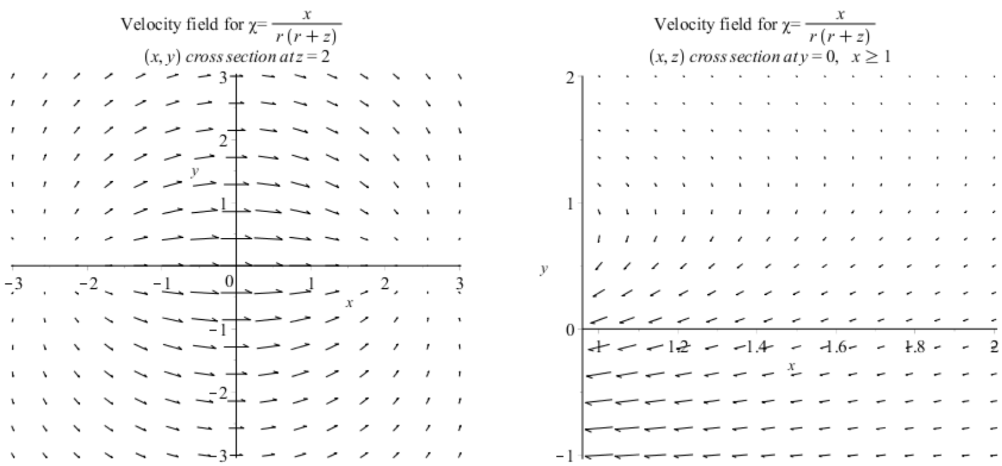


Figure 4: This figure displays the Stokes flow for the scalar potential $\chi = \frac{x}{x^2+y^2}$ and a zero vector potential in the Papkovich-Neuber solution. The velocity is well defined on $\Re^2$.



(a) (x,y) cross section at z=2.

(b) (x,z) cross section at y=0.

Figure 5: This figure displays the Stokes flow for the scalar potential $\chi = \frac{x}{r(r+z)}$ and a zero vector potential in the Papkovich-Neuber solution. The velocity is well defined on $\Re^2$.

7 Appendix - Vector Identities

$$\nabla \cdot (\nabla^2 \vec{A}) = \nabla^2 (\nabla \cdot \vec{A}) \tag{17}$$

$$2\nabla(\nabla \cdot \vec{A}) = \nabla^2 \nabla(\vec{r} \cdot \vec{A}) \text{ for harmonic } \vec{A} \; ^{[8]} \tag{18}$$

$$2\nabla \cdot \vec{A} = \nabla \cdot \nabla(\vec{r} \cdot \vec{A}) \text{ for harmonic } \vec{A} \; ^{[8]} \tag{19}$$

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