Analyzing Paxos with Fault-Tolerant Multiparty Session Types

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1 Introduction

In distributed systems components on different computers coordinate and communicate via message passing to achieve a common goal. Sometimes, to achieve this goal, the individual components need to reach consensus, i.e., agree on the value of some data using a consensus algorithm. For example in state machine replication or when deciding which database transactions should be committed in what order. For such a distributed system to behave correctly the consensus algorithm needs to be correct. Thus, analyzing consensus algorithms is important.

To achieve consensus, consensus algorithms must satisfy the following properties: termination, validity, and agreement [4]. Proving these properties can be complicated. Model checking tools lead to big state-spaces so static analysis is preferable. For static analysis Multiparty Session Types are particularly interesting because session typing can ensure protocol conformance and the absence of communication errors and deadlocks [11].

Due to the presence of faulty processes and unreliable communication consensus algorithms are designed to be fault-tolerant. Modelling fault-tolerance is not possible using Multiparty Session Types, thus a fault-tolerant extension is necessary. Peters, Nestmann, and Wagner developed such an extension called Fault-Tolerant Multiparty Session Types.

In this work we will use Fault-Tolerant Multiparty Session Types to analyze the consensus algorithm Paxos, as described in [8].

2 Technical Preliminaries

2.1 Paxos

Lamport describes the Paxos algorithm in [8]. First, he outlines the problem and then the algorithm to solve that problem.

2.1.1 The Problem

Assume a collection of processes that can propose values. A consensus algorithm ensures that a single one among the proposed values is chosen. If no value is proposed, then no value should be chosen. If a value has been chosen, then processes should be able to learn the chosen value. The safety requirements for consensus are:

- Only a value that has been proposed may be chosen,
- Only a single value is chosen, and
- A process never learns that a value has been chosen unless it actually has been.

Assume that agents can communicate with one another by sending messages. We use the customary asynchronous, non-Byzantine model, in which:

- Agents operate at arbitrary speed, may fail by stopping, and may restart. Since all agents may fail after a value is chosen and then restart, a solution is impossible unless some information can be remembered by an agent that has failed and restarted.
- Messages can take arbitrarily long to be delivered, can be duplicated, and can be lost, but they are not corrupted.

We assume two classes of agents: proposers and acceptors.

A *quorum* is a set of acceptors that can choose a value proposed by a proposer [7]. For Paxos, a quorum is a majority of acceptors.

2.1.2 The Algorithm

Phase 1. (a) A proposer selects a proposal number n and sends a *prepare* request with number n to a majority of acceptors.

(b) If an acceptor receives a *prepare* request with number n greater than that of any *prepare* request to which it has already responded, then it responds to the request with a promise not to accept any more proposals numbered less than n and with the highest-numbered proposal (if any) that it has accepted.

Phase 2. (a) If the proposer receives a response to its *prepare* requests (numbered n) from a majority of acceptors, then it sends an *accept* request to each of those acceptors for a proposal numbered n with a value v, where v is the value of the highest-numbered proposal among the responses, or is any value if the responses reported no proposals.

(b) If an acceptor receives an *accept* request for a proposal numbered n, it accepts the proposal unless it has already responded to a *prepare* request having a number greater than n.

A proposer can make multiple proposals, so long as it follows the algorithm for each one. It can abandon a proposal in the middle of the protocol at any time. It is probably a good idea to abandon a proposal if some proposer has begun trying to issue a higher-numbered one. Therefore, if an acceptor ignores a *prepare* or *accept* request because it has already received a *prepare* request with a higher number, then it should probably inform the proposer, who should then abandon its proposal. This is a performance optimization that does not affect correctness.

To guarantee progress, a distinguished proposer must be selected as the only one to try issuing proposals. If the distinguished proposer can communicate successfully with a majority of acceptors, and if it uses a proposal with number greater than any already used, then it will succeed in issuing a proposal that is accepted. By abandoning a proposal and trying again if it learns about some request with a higher proposal number, the distinguished proposer will eventually choose a high enough proposal number.

If enough of the system (proposer, acceptors, and communication network) is working properly, liveness can therefore be achieved by electing a single distinguished proposer.

2.1.3 The Implementation

The Paxos algorithm [9] assumes a network of processes. In its consensus algorithm, each process plays the role of proposer, acceptor. The algorithm chooses a leader, which plays the roles of the distinguished proposer. The Paxos consensus algorithm is precisely the one described above, where requests and responses are sent as ordinary messages. Stable storage, preserved during failures, is used to maintain the information that the acceptor must remember. An acceptor records its intended response in stable storage before actually sending the response.

Different proposers choose their numbers from disjoint sets of numbers, so two different proposers never issue a proposal with the same number. Each proposer remembers (in stable storage) the highest-numbered proposal it has tried to issue, and begins phase 1 with a higher proposal number than any it has already used.

2.2 Fault-Tolerant Multiparty Session Types

Our model of the Paxos algorithm uses a fault-tolerant extension of Multiparty Session Types introduced by Peters, Nestmann, and Wagner in [10]. The following explanation of Fault-Tolerant Multiparty Session Types is from that same paper.

We consider three sources of failure in an unreliable communication: (1) the sender may crash before it releases the message, (2) the receiver may crash before it can consume the message, or (3) the communication medium may lose the message. Since types consider only static and predictable information, we do not distinguish between different kinds of failure or model their source in types. Instead, we only allow types, i.e., the specifications of systems, to distinguish between potentially faulty and reliable interactions.

We consider three levels of failures in interactions:

Strongly Reliable (r): Neither the sender nor the receiver can crash as long as they are involved in this interaction. The message cannot be lost by the communication medium. This form corresponds to reliable communication as it was described in [1] in the context of distributed algorithms. This is the standard, failure-free case.

Weakly Reliable (w): Both the sender and the receiver might crash at every possible point during this interaction. But the communication medium cannot lose the message.

Unreliable (u): Both the sender and the receiver might crash at every possible point during this interaction and the communication medium might lose the message. There are no guarantees that this interaction—or any part of it—takes place.

2.2.1 Fault-Tolerant Types and Processes

We assume that the sets $\mathcal{N}$ of names $a, s, x \ldots$; $\mathcal{R}$ of roles $\mathsf{n}, \mathsf{r}, \ldots$; $\mathcal{L}$ of labels $l, l_{\mathrm{d}}, \ldots$; $\mathcal{V}_{\mathrm{T}}$ of type variables t; and $\mathcal{V}_{\mathrm{P}}$ of process variables X are pairwise distinct. To simplify the reduction semantics of our session calculus, we use natural numbers as roles (compare to [6]). Sorts S range over $\mathbb{B}, \mathbb{N}, \ldots$. The set $\mathcal{E}$ of expressions $e, v, b, \ldots$ is constructed from the standard Boolean operations, natural numbers, names, and (in)equalities.

Global types specify the desired communication structure of systems from a global point of view. In local types this global view is projected to the specification of a single role/participant. We use standard MPST ([5, 6]) extended by operators for unreliable communication and weakly reliable as shown in Fig. 2.1.

The processes $\overline{a}[\mathsf{n}](s).P$ and $a[\mathsf{r}](s).P$ initialize a new session s with n roles via the shared channel a and then proceed as P. We identify sessions with their unique session channel.

The type $\mathsf{r}_1 \to_{\mathrm{r}} \mathsf{r}_2{:}\langle \mathrm{S} \rangle.G$ specifies a strongly reliable communication from role r_1 to role r_2 to transmit a value of the sort S and then continues with G. A system with this type will be guaranteed to perform a corresponding action. In a session s this communication is implemented by the sender $s[\mathsf{r}_1, \mathsf{r}_2]!_{\mathrm{r}}\langle e \rangle.P_1$ (specified as $[\mathsf{r}_2]!_{\mathrm{r}}\langle \mathrm{S} \rangle.T_1$) and the receiver $s[\mathsf{r}_2, \mathsf{r}_1]?_{\mathrm{r}}(x).P_2$ (specified as $[\mathsf{r}_1]?_{\mathrm{r}}\langle \mathrm{S} \rangle.T_2$). As result of the communication, the receiver instantiates x in its continuation P_2 with the received value.

The type $\mathsf{r}_1 \to_{\mathrm{u}} \mathsf{r}_2{:}l\langle \mathrm{S} \rangle.G$ specifies an unreliable communication from r_1 to r_2 transmitting (if successful) a label l and a value of type S and then continues (regardless of the success of this communication) with G. The unreliable counterparts of senders and receivers are $s[\mathsf{r}_1, \mathsf{r}_2]!_{\mathrm{u}}l\langle e \rangle.P_1$ (specified as $[\mathsf{r}_2]!_{\mathrm{u}}l\langle \mathrm{S} \rangle.T_1$) and

Global Types	Local Types	Processes
		$P ::= \overline{a}[\mathsf{n}](s).P$
		$\mid\ a[\mathsf{r}](s).P$
$G ::= \mathsf{r}_1 \to_{\mathrm{r}} \mathsf{r}_2{:}\langle \mathrm{S}\rangle.G$	$T ::= [\mathsf{r}_2]!_{\mathrm{r}}\langle \mathrm{S}\rangle.T$	$\mid\ s[\mathsf{r}_1,\mathsf{r}_2]!_{\mathrm{r}}\langle e\rangle.P$
	$\mid\ [\mathsf{r}_1]?_{\mathrm{r}}\langle \mathrm{S}\rangle.T$	$\mid\ s[\mathsf{r}_2,\mathsf{r}_1]?_{\mathrm{r}}(x).P$
$\mid\ \mathsf{r}_1 \to_{\mathrm{u}} \mathsf{r}_2{:}l\langle \mathrm{S}\rangle.G$	$\mid\ [\mathsf{r}_2]!_{\mathrm{u}}l\langle \mathrm{S}\rangle.T$	$\mid\ s[\mathsf{r}_1,\mathsf{r}_2]!_{\mathrm{u}}l\langle e\rangle.P$
	$\mid\ [\mathsf{r}_1]?_{\mathrm{u}}l\langle \mathrm{S}\rangle.T$	$\mid\ s[\mathsf{r}_2,\mathsf{r}_1]?_{\mathrm{u}}l\langle v\rangle(x).P$
$\mid\ \mathsf{r}_1 \to_{\mathrm{r}} \mathsf{r}_2{:}\{l_i.G_i\}_{i\in\mathrm{I}}$	$\mid\ [\mathsf{r}_2]!_{\mathrm{r}}\{l_i.T_i\}_{i\in\mathrm{I}}$	$\mid\ s[\mathsf{r}_1,\mathsf{r}_2]!_{\mathrm{r}}l.P$
	$\mid\ [\mathsf{r}_1]?_{\mathrm{r}}\{l_i.T_i\}_{i\in\mathrm{I}}$	$\mid\ s[\mathsf{r}_2,\mathsf{r}_1]?_{\mathrm{r}}\{l_i.P_i\}_{i\in\mathrm{I}}$
$\mid\ \mathsf{r} \to_{\mathrm{w}} \mathsf{R}{:}\{l_i.G_i\}_{i\in\mathrm{I},l_{\mathrm{d}}}$	$\mid\ [\mathsf{R}]!_{\mathrm{w}}\{l_i.T_i\}_{i\in\mathrm{I}}$	$\mid\ s[\mathsf{r},\mathsf{R}]!_{\mathrm{w}}l.P$
	$\mid\ [\mathsf{r}]?_{\mathrm{w}}\{l_i.T_i\}_{i\in\mathrm{I},l_{\mathrm{d}}}$	$\mid\ s[\mathsf{r}_j,\mathsf{r}]?_{\mathrm{w}}\{l_i.P_i\}_{i\in\mathrm{I},l_{\mathrm{d}}}$
$\mid\ G_1 \parallel G_2$		$\mid\ P_1 \mid P_2$
$\mid\ (\mu t)G \mid t \mid \mathtt{end}$	$\mid\ (\mu t)T \mid t \mid \mathtt{end}$	$\mid\ (\mu X)P \mid X \mid \mathbf{0}$
		$\mid\ \mathtt{if}\ b\ \mathtt{then}\ P_1\ \mathtt{else}\ P_2$
		$\mid\ (\nu x)P \mid \bot$
$\mid\ \mathsf{r}_1 \to \mathsf{r}_2{:}\langle s'[\mathsf{r}]{:}T\rangle.G$	$\mid\ [\mathsf{r}_2]!\langle s'[\mathsf{r}]{:}T\rangle.T'$	$\mid\ s[\mathsf{r}_1,\mathsf{r}_2]!\langle\!\langle s'[\mathsf{r}]\rangle\!\rangle.P$
	$\mid\ [\mathsf{r}_1]?\langle s'[\mathsf{r}]{:}T\rangle.T'$	$\mid\ s[\mathsf{r}_2,\mathsf{r}_1]?(\!(s'[\mathsf{r}])\!).P$
		$\mid\ s_{\mathsf{r}_1\to\mathsf{r}_2}{:}\mathrm{M}$

Message Types	Messages
$\mathrm{MT} ::= \langle \mathrm{S}\rangle^{\mathrm{r}} \mid l\langle \mathrm{S}\rangle^{\mathrm{u}} \mid l^{\mathrm{r}} \mid l^{\mathrm{w}} \mid s[\mathsf{r}]$	$\mathrm{M} ::= \langle v\rangle^{\mathrm{r}} \mid l\langle v\rangle^{\mathrm{u}} \mid l^{\mathrm{r}} \mid l^{\mathrm{w}} \mid s[\mathsf{r}]$

Figure 2.1: Syntax of Fault-Tolerant MPST

$s[\mathsf{r}_2,\mathsf{r}_1]?_{\mathrm{u}}l\langle v\rangle(x).P_2$ (specified as $[\mathsf{r}_1]?_{\mathrm{u}}l\langle \mathrm{S}\rangle.T_2$). The receiver $s[\mathsf{r}_2,\mathsf{r}_1]?_{\mathrm{u}}l\langle v\rangle(x).P_2$ declares a default value v that is used instead of a received value to instantiate x after a failure.

The strongly reliable branching $\mathsf{r}_1 \to_{\mathrm{r}} \mathsf{r}_2{:}\{l_i.G_i\}_{i\in\mathrm{I}}$ allows r_1 to pick one of the branches offered by r_2. We identify the branches with their respective label. Selection of a branch is implemented by $s[\mathsf{r}_1,\mathsf{r}_2]!_{\mathrm{r}}l.P$ (specified as $[\mathsf{r}_2]!_{\mathrm{r}}\{l_i.T_i\}_{i\in\mathrm{I}}$). Upon receiving branch l_j from r_1 the process $s[\mathsf{r}_2,\mathsf{r}_1]?_{\mathrm{r}}\{l_i.P_i\}_{i\in\mathrm{I}}$ (specified as $[\mathsf{r}_1]?_{\mathrm{r}}\{l_i.T_i\}_{i\in\mathrm{I}}$) continues with P_j.

The weakly reliable counterpart of branching is $\mathsf{r} \to_{\mathrm{w}} \mathsf{R}{:}\{l_i.G_i\}_{i\in\mathrm{I},l_{\mathrm{d}}}$, where $\mathsf{R} \subseteq \mathcal{R}$ and l_{d} with $\mathrm{d} \in \mathrm{I}$ is the default branch. We use a broadcast from r to all roles in R to ensure that the sender can influence several participants with its decision consistently as it is the case for strongly reliable branching. The type system will ensure that this branching construct is weakly reliable, i.e., the involved participants might crash, but no message is lost. Because of that, all processes that are not crashed will move to the same branch. We often abbreviate branching w.r.t. to a small set of branches by omitting the set brackets and instead separating the branches by $\oplus$, where the last branch is always the default branch. In contrast to the strongly reliable cases, the weakly reliable selection $s[\mathsf{r},\mathsf{R}]!_{\mathrm{w}}l.P$ (specified as $[\mathsf{R}]!_{\mathrm{w}}\{l_i.T_i\}_{i\in\mathrm{I}}$) allows to broadcast its decision to R and $s[\mathsf{r}_j,\mathsf{r}]?_{\mathrm{w}}\{l_i.P_i\}_{i\in\mathrm{I},l_{\mathrm{d}}}$ (specified as $[\mathsf{r}]?_{\mathrm{w}}\{l_i.T_i\}_{i\in\mathrm{I},l_{\mathrm{d}}}$) defines a default label l_{d}.

The $\bot$ denotes a process that crashed. Similar to [6], we use message queues to implement asynchrony in sessions. Therefore, session initialization introduces a directed and initially empty message queue $s_{\mathsf{r}_1\to\mathsf{r}_2}{:}[\,]$ for each pair of roles $\mathsf{r}_1 \neq \mathsf{r}_2$ of the session s. We have five kinds of messages and corresponding message types in Fig. 2.1—one for each kind of interaction.

The remaining operators for independence $G \parallel G'$; parallel composition $P \mid P'$; recursion $(\mu t)G$, $(\mu X)P$; inaction $\mathtt{end}$, $\mathbf{0}$; conditionals $\mathtt{if}\ b\ \mathtt{then}\ P_1\ \mathtt{else}\ P_2$; session delegation $\mathsf{r}_1 \to \mathsf{r}_2{:}\langle s'[\mathsf{r}]{:}T\rangle.G$, $s[\mathsf{r}_1,\mathsf{r}_2]!\langle\!\langle s'[\mathsf{r}]\rangle\!\rangle.P$,

$s[\mathsf{r}_2, \mathsf{r}_1]?(\!(s'[\mathsf{r}])\!).P$; and restriction $(\nu x)P$ are all standard.

Our type system verifies processes, i.e., implementations, against a specification that is a global type. Since processes implement local views, local types are used as a mediator between the global specification and the respective local end points. To ensure that the local types correspond to the global type, they are derived by *projection*. Instead of the projection function described in [6] we use a more relaxed variant of projection as introduced in [12].

Projection maps global types onto the respective local type for a given role p. The projections of the new global types are obtained straightforwardly from the projection of their respective strongly reliable counterparts:

$$(\mathsf{r}_1 \to_\diamond \mathsf{r}_2{:}\mathfrak{S}.G)\!\upharpoonright_{\mathsf{p}} \triangleq \begin{cases} [\mathsf{r}_2]!_\diamond\mathfrak{S}.G\!\upharpoonright_{\mathsf{p}} & \text{if } \mathsf{p} = \mathsf{r}_1 \\ [\mathsf{r}_1]?_\diamond\mathfrak{S}.G\!\upharpoonright_{\mathsf{p}} & \text{if } \mathsf{p} = \mathsf{r}_2 \\ G\!\upharpoonright_{\mathsf{p}} & \text{otherwise} \end{cases}$$

where either $\diamond = \mathrm{r}$, $\mathfrak{S} = \langle \mathrm{S} \rangle$ or $\diamond = \mathrm{u}$, $\mathfrak{S} = l\langle \mathrm{S} \rangle$ and

$$\left(\mathsf{r}_1 \to_\diamond \mathfrak{R}{:}\{l_i.G_i\}_{i\in \mathrm{I}\mathfrak{D}}\right)\!\upharpoonright_{\mathsf{p}} \triangleq \begin{cases} [\mathfrak{R}]!_\diamond\{l_i.G_i\!\upharpoonright_{\mathsf{p}}\}_{i\in\mathrm{I}} & \text{if } \mathsf{p} = \mathsf{r}_1 \\ [\mathsf{r}_1]?_\diamond\{l_i.G_i\!\upharpoonright_{\mathsf{p}}\}_{i\in\mathrm{I}\mathfrak{D}} & \text{if } \mathfrak{B} \\ \bigsqcup_{i\in\mathrm{I}}(G_i\!\upharpoonright_{\mathsf{p}}) & \text{otherwise} \end{cases}$$

where either $\diamond = \mathrm{r}$, $\mathfrak{R} = \mathsf{r}_2$, $\mathfrak{B}$ is $\mathsf{p} = \mathsf{r}_2$, $\mathfrak{D}$ is empty or $\diamond = \mathrm{w}$, $\mathfrak{R} = \mathsf{R}$, $\mathfrak{B}$ is $\mathsf{p} \in \mathsf{R}$, $\mathfrak{D}$ is $, l_{\mathrm{d}}$. In the last case of strongly reliable or weakly reliable branching—when projecting onto a role that does not participate in this branching—we map to $\bigsqcup_{i\in\mathrm{I}}(G_i\!\upharpoonright_{\mathsf{p}}) = (G_1\!\upharpoonright_{\mathsf{p}}) \sqcup \ldots \sqcup (G_n\!\upharpoonright_{\mathsf{p}})$. The operation $\sqcup$ is (similar to [12]) inductively defined as:

$$\begin{aligned} T \sqcup T &= T \\ ([\mathsf{r}]?_{\mathrm{r}}\mathrm{I}_1) \sqcup ([\mathsf{r}]?_{\mathrm{r}}\mathrm{I}_2) &= [\mathsf{r}]?_{\mathrm{r}}(\mathrm{I}_1 \sqcup \mathrm{I}_2) \\ ([\mathsf{r}]?_{\mathrm{w}}\mathrm{I}_1) \sqcup ([\mathsf{r}]?_{\mathrm{w}}\mathrm{I}_2) &= [\mathsf{r}]?_{\mathrm{w}}(\mathrm{I}_1 \sqcup \mathrm{I}_2) \quad \text{if } \mathrm{I}_1 \text{ and } \mathrm{I}_2 \text{ have the same default branch} \\ \mathrm{I} \sqcup \emptyset &= \mathrm{I} \\ \mathrm{I} \sqcup (\{l.T\} \cup \mathrm{J}) &= \begin{cases} \{l.\,(T' \sqcup T)\} \cup ((\mathrm{I} \setminus \{l.T'\}) \sqcup \mathrm{J}) & \text{if } l.T' \in \mathrm{I} \\ \{l.T\} \cup (\mathrm{I} \sqcup \mathrm{J}) & \text{if } l \notin \mathrm{I} \end{cases} \end{aligned}$$

where $T, T' \in \mathcal{T}$, $l \notin \mathrm{I}$ is short hand for $\nexists T'.\ l.T' \in \mathrm{I}$, and is undefined in all other cases. The mergeability relation $\sqcup$ states that two types are identical up to their branching types, where only branches with distinct labels are allowed to be different. This ensures that if the sender r_1 in $\mathsf{r}_1 \to_{\mathrm{r}} \mathsf{r}_2{:}\{l_i.G_i\}_{i\in\mathrm{I}}$ decides to branch then only processes that are informed about this decision can adapt their behavior accordingly; else projection is **not** defined.

The remaining global types are projected as follows:

$$(G_1 \parallel G_2)\!\upharpoonright_{\mathsf{p}} \triangleq \begin{cases} G_1\!\upharpoonright_{\mathsf{p}} & \text{if } \mathsf{p} \notin \mathrm{R}(G_2) \\ G_2\!\upharpoonright_{\mathsf{p}} & \text{if } \mathsf{p} \notin \mathrm{R}(G_1) \end{cases} \qquad ((\mu t)G)\!\upharpoonright_{\mathsf{p}} \triangleq \begin{cases} (\mu t)G\!\upharpoonright_{\mathsf{p}} & \text{if } \mathsf{p} \in \mathrm{R}(G) \\ \mathtt{end} & \text{otherwise} \end{cases}$$

$$t\!\upharpoonright_{\mathsf{p}} \triangleq t \qquad \mathtt{end}\!\upharpoonright_{\mathsf{p}} \triangleq \mathtt{end}$$

Projecting a recursive global type results in a recursive local type if p occurs in the body of the recursion or else in successful termination. Type variables and successful termination are mapped onto themselves. We denote a global type G as *projectable* if for all $\mathsf{r} \in \mathrm{R}(G)$ the projection $G\!\upharpoonright_{\mathsf{r}}$ is defined. We restrict our attention to projectable global types.

In types $(\mu t)G$ and $(\mu t)T$ the type variable t is *bound*. In processes $(\mu X)P$ the process variable X is bound. Similarly, all names in round brackets are bound in the remainder of the respective process, e.g. s is bound in P by $\overline{a}[\mathsf{n}](s).P$ and x is bound in P by $s[\mathsf{r}_1, \mathsf{r}_2]?_{\mathrm{r}}(x).P$. A variable or name is *free* if it is not bound. Let $\mathrm{FN}(P)$ return the free names of P.

We use '.' (as e.g. in $\overline{a}[\mathsf{r}](s).P$) to denote sequential composition. In all operators the *prefix* before '.' guards the *continuation* after the '.'. Let $\prod_{1\leq i\leq n} P_i$ abbreviate the parallel composition $P_1 \mid \ldots \mid P_n$.

We write $\mathrm{nsr}(G)$, $\mathrm{nsr}(T)$, and $\mathrm{nsr}(P)$, if none of the prefixes in G, T, and P is strongly reliable or for delegation and if P does not contain message queues.

The combination of a session channel and a role uniquely identifies a participant of a session, called an *actor*. A process has an actor $s[\mathsf{r}]$ if it has an action prefix on s, where r is the first role mentioned in the prefix. Let $\mathrm{A}(P)$ be the set of actors of P.

2.2.2 A Semantics with Failure Patterns

The application of a substitution $\{y/x\}$ on a term A, denoted as $A\{y/x\}$, is defined as the result of replacing all free occurrences of x in A by y, possibly applying alpha-conversion to avoid capture or name clashes. For all names $n \in \mathcal{N} \setminus \{x\}$ the substitution behaves as the identity mapping. We use substitution on types as well as processes and naturally extend substitution to the substitution of variables by terms (to unfold recursions) and names by expressions (to instantiate a bound name with a received value).

Labels allow us to distinguish between different branches. Our MPST variant will ensure that all occurrences of the same label are associated with the same sort. We assume a predicate $\doteq$ that compares two labels and is valid if the parts of the labels that do not refer to runtime information correspond. We require that $\doteq$ is unambiguous on labels used in types [10].

We use structural congruence to abstract from syntactically different processes with the same meaning, where $\equiv$ is the least congruence that satisfies alpha conversion and the rules:

$$\begin{gathered} P \mid \mathbf{0} \equiv P \qquad P_1 \mid P_2 \equiv P_2 \mid P_1 \qquad P_1 \mid (P_2 \mid P_3) \equiv (P_1 \mid P_2) \mid P_3 \\ (\mu X)\mathbf{0} \equiv \mathbf{0} \qquad (\nu x)\mathbf{0} \equiv \mathbf{0} \qquad (\nu x)(\nu y)P \equiv (\nu y)(\nu x)P \\ (\nu x)\,(P_1 \mid P_2) \equiv P_1 \mid (\nu x)P_2 \quad \text{if } x \notin \mathrm{FN}(P_1) \end{gathered}$$

The reduction semantics of the session calculus is defined in Fig. 2.2, where we follow [6]: we assume that session initialization is synchronous and communication within a session is asynchronous implemented using message queues.

Rule (Init) initializes a session with n roles. Session initialization introduces a fresh session channel and unguards the participants of the session. Finally, the message queues of this session are initialized with the empty list under the restriction of the session channel.

Rule (RSend) implements an asynchronous strongly reliable message transmission. As a result the value $\mathrm{eval}(y)$ is wrapped in a message and added to the end of the corresponding message queue and the continuation of the sender is unguarded. Rule (USend) is the counterpart of (RSend) for unreliable senders. (RGet) consumes

$$
\begin{array}{lll}
\text{(Init)} & \overline{a}[\mathsf{n}](s).P_{\mathsf{n}} \mid \prod_{1\leq \mathsf{i}\leq \mathsf{n}-1} a[\mathsf{i}](s).P_{\mathsf{i}} \longmapsto (\nu s)\left(\prod_{1\leq \mathsf{i}\leq \mathsf{n}} P_{\mathsf{i}} \mid \prod_{1\leq \mathsf{i},\mathsf{j}\leq \mathsf{n},\mathsf{i}\neq \mathsf{j}} s_{\mathsf{i}\to \mathsf{j}}{:}[\,]\right) & \\
 & & \text{if } a \neq s \\
\text{(RSend)} & s[\mathsf{r}_1,\mathsf{r}_2]!_{\mathrm{r}}\langle y\rangle.P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M} \longmapsto P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M}\#\langle v\rangle^{\mathrm{r}} & \text{if } \mathrm{eval}(y) = v \\
\text{(RGet)} & s[\mathsf{r}_1,\mathsf{r}_2]?_{\mathrm{r}}(x).P \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}\langle v\rangle^{\mathrm{r}}\#\mathrm{M} \longmapsto P\{{}^{v}/_{x}\} \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}\mathrm{M} & \\
\text{(USend)} & s[\mathsf{r}_1,\mathsf{r}_2]!_{\mathrm{u}}l\langle y\rangle.P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M} \longmapsto P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M}\#l\langle v\rangle^{\mathrm{u}} & \text{if } \mathrm{eval}(y) = v \\
\text{(UGet)} & s[\mathsf{r}_1,\mathsf{r}_2]?_{\mathrm{u}}l\langle dv\rangle(x).P \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}l'\langle v\rangle^{\mathrm{u}}\#\mathrm{M} \longmapsto P\{{}^{v}/_{x}\} \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}\mathrm{M} & \\
 & & \text{if } l \doteq l',\ \mathtt{FP}_{\mathtt{uget}}(s,\mathsf{r}_1,\mathsf{r}_2,l') \\
\text{(USkip)} & s[\mathsf{r}_1,\mathsf{r}_2]?_{\mathrm{u}}l\langle dv\rangle(x).P \longmapsto P\{{}^{dv}/_{x}\} & \text{if } \mathtt{FP}_{\mathtt{uskip}}(s,\mathsf{r}_1,\mathsf{r}_2,l) \\
\text{(ML)} & s_{\mathsf{r}_1\to \mathsf{r}_2}{:}l\langle v\rangle^{\mathrm{u}}\#\mathrm{M} \longmapsto s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M} & \text{if } \mathtt{FP}_{\mathtt{ml}}(s,\mathsf{r}_1,\mathsf{r}_2,l) \\
\text{(RSel)} & s[\mathsf{r}_1,\mathsf{r}_2]!_{\mathrm{r}}l.P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M} \longmapsto P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M}\#l^{\mathrm{r}} & \\
\text{(RBran)} & s[\mathsf{r}_1,\mathsf{r}_2]?_{\mathrm{r}}\{l_i.P_i\}_{i\in \mathrm{I}} \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}l^{\mathrm{r}}\#\mathrm{M} \longmapsto P_j \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}\mathrm{M} & \text{if } l \doteq l_j,\ j \in \mathrm{I} \\
\text{(WSel)} & s[\mathsf{r},\mathsf{R}]!_{\mathrm{w}}l.P \mid \prod_{\mathsf{r}_i\in \mathsf{R}} s_{\mathsf{r}\to \mathsf{r}_i}{:}\mathrm{M}_i \longmapsto P \mid \prod_{\mathsf{r}_i\in \mathsf{R}} s_{\mathsf{r}\to \mathsf{r}_i}{:}\mathrm{M}_i\#l^{\mathrm{w}} & \\
\text{(WBran)} & s[\mathsf{r}_1,\mathsf{r}_2]?_{\mathrm{w}}\{l_i.P_i\}_{i\in \mathrm{I},l_{\mathrm{d}}} \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}l^{\mathrm{w}}\#\mathrm{M} \longmapsto P_j \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}\mathrm{M} & \text{if } l \doteq l_j,\ j \in \mathrm{I} \\
\text{(WSkip)} & s[\mathsf{r}_1,\mathsf{r}_2]?_{\mathrm{w}}\{l_i.P_i\}_{i\in \mathrm{I},l_{\mathrm{d}}} \longmapsto P_{\mathrm{d}} & \text{if } \mathtt{FP}_{\mathtt{wskip}}(s,\mathsf{r}_1,\mathsf{r}_2) \\
\text{(Crash)} & P \longmapsto \bot & \text{if } \mathtt{FP}_{\mathtt{crash}}(P) \\
\text{(If-T)} & \mathtt{if}\ e\ \mathtt{then}\ P\ \mathtt{else}\ P' \longmapsto P & \text{if } e \text{ is true} \\
\text{(If-F)} & \mathtt{if}\ e\ \mathtt{then}\ P\ \mathtt{else}\ P' \longmapsto P' & \text{if } e \text{ is false} \\
\text{(Deleg)} & s[\mathsf{r}_1,\mathsf{r}_2]!\langle\!\langle s'[\mathsf{r}]\rangle\!\rangle.P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M} \longmapsto P \mid s_{\mathsf{r}_1\to \mathsf{r}_2}{:}\mathrm{M}\#s'[\mathsf{r}] & \\
\text{(SRecv)} & s[\mathsf{r}_1,\mathsf{r}_2]?(\!(s'[\mathsf{r}])\!).P \mid s_{\mathsf{r}_2\to \mathsf{r}_1}{:}s''[\mathsf{r}']\#\mathrm{M} \longmapsto P\{{}^{s''}/_{s'}\}\{{}^{\mathsf{r}'}/_{\mathsf{r}}\} \mid \mathrm{M} & \\
\text{(Par)} & P_1 \mid P_2 \longmapsto P_1' \mid P_2 & \text{if } P_1 \longmapsto P_1' \\
\text{(Res)} & (\nu x)P \longmapsto (\nu x)P' & \text{if } P \longmapsto P' \\
\text{(Rec)} & (\mu X)P \longmapsto P\{{}^{(\mu X)P}/_{X}\} & \\
\text{(Struc)} & P_1 \longmapsto P_1' & \text{if } P_1 \equiv P_2,\ P_2 \longmapsto P_2',\ P_2' \equiv P_1'
\end{array}
$$

Figure 2.2: Reduction Rules ($\longmapsto$) of Fault-Tolerant Processes.

a message that is marked as strongly reliable with the index r from the head of the respective message queue and replaces in the unguarded continuation of the receiver the bound variable x by the received value y.

There are two rules for the reception of a message in an unreliable communication that are guided by failure patterns. *Failure patterns* are predicates that we deliberately choose not to define here (see below). They allow us to provide information about the underlying communication medium and the reliability of processes. Rule (UGet) is similar to Rule (RGet), but specifies a failure pattern $\mathtt{FP}_{\mathtt{uget}}$ to decide whether this step is allowed. The Rule (USkip) allows to skip the reception of a message in an unreliable communication using a failure pattern $\mathtt{FP}_{\mathtt{uskip}}$ and instead substitutes the bound variable x in the continuation with the default value dv. The failure pattern $\mathtt{FP}_{\mathtt{uskip}}$ tells us whether a reception can be skipped.

Rule (RSel) puts the label l selected by r_1 at the end of the message queue towards r_2. Its weakly reliable counterpart (WSel) is similar, but puts the label at the end of all relevant message queues. With (RBran) a label is consumed from the top of a message queue and the receiver moves to the indicated branch. There are again two weakly reliable counterparts of (RBran). Rule (WBran) is similar to (RBran), whereas (WSkip) allows r_1 to skip the message and to move to its default branch if the failure pattern $\mathtt{FP}_{\mathtt{wskip}}$ holds.

The Rules (Crash) for *crash failures* and (ML) for *message loss*, describe failures of a system. With Rule (Crash) P can crash if $\mathtt{FP}_{\mathtt{crash}}$. (ML) allows to drop an unreliable message if the failure pattern $\mathtt{FP}_{\mathtt{ml}}$ is valid.

The remaining rules for conditionals, session delegation, parallel composition, restriction, recursion, and structural congruence in Fig. 2.2 are standard.

We augmented our reduction semantics in Fig. 2.2 by five different failure patterns that we deliberately do not specify, although we usually assume that the failure patterns $\mathtt{FP}_{\mathtt{uget}}$, $\mathtt{FP}_{\mathtt{uskip}}$, and $\mathtt{FP}_{\mathtt{wskip}}$ use only local information, whereas $\mathtt{FP}_{\mathtt{ml}}$ and $\mathtt{FP}_{\mathtt{crash}}$ may use global information of the system in the current run.

2.2.3 Typing Fault-Tolerant Processes

The type of a process P is checked in a *typed judgment*, i.e., triples $\Gamma \vdash P \triangleright \Delta$, where

$$\begin{aligned} \Gamma &::= \emptyset \mid \Gamma \cdot x{:}\mathrm{S} \mid \Gamma \cdot a{:}G \mid \Gamma \cdot l{:}\mathrm{S} \\ \Delta &::= \emptyset \mid \Delta \cdot s[\mathsf{r}]{:}T \mid \Delta \cdot s_{\mathsf{r}_1 \to \mathsf{r}_2}{:}\mathrm{MT}^* \end{aligned}$$

Assignments in Γ relate variables to their sort, shared channels to the type of the session they introduce, and connect labels with a sort. Session environments collect the local types of actors and the list of message types of queues, i.e., MT^* denotes a list of message types.

We write $x \sharp \Gamma$ and $x \sharp \Delta$ if the name x does not occur in Γ and Δ, respectively. We use $\cdot$ to add an assignment provided that the new assignment is not in conflict with the type environment, i.e., $\Gamma \cdot A$ implies that the respective name/variable/label in A is not contained in Γ and $\Delta \cdot A$ implies that the respective actor/queue in A is not contained in Δ. These conditions on $\cdot$ for global and session environments are referred to as *linearity*. We restrict in the following our attention to linear environments.

We write $\mathrm{nsr}(\Delta)$ if $\mathrm{nsr}(T)$ for all local types T in Δ and if Δ does not contain message queues. With $\Gamma \Vdash y{:}\mathrm{S}$ we check that y is an expression of the sort S if all names x in y are replaced by arbitrary values of sort S_x for $x{:}\mathrm{S}_x \in \Gamma$.

A process P is *well-typed* w.r.t. Γ and Δ if $\Gamma \vdash P \triangleright \Delta$ can be derived from the rules in the Fig. 2.3. We concentrate on the interaction cases, where we observe that all new cases are quite similar to their strongly reliable counterparts.

$$
(\mathsf{Req})\ \frac{\begin{array}{c} a{:}G \in \Gamma \quad |\mathrm{R}(G)| = \mathsf{n} \\ \Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{n}]{:}G{\upharpoonright}_{\mathsf{n}} \end{array}}{\Gamma \vdash \overline{a}[\mathsf{n}](s).P \triangleright \Delta}
\qquad
(\mathsf{Acc})\ \frac{\begin{array}{c} a{:}G \in \Gamma \quad 0 < \mathsf{r} < |\mathrm{R}(G)| \\ \Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{r}]{:}G{\upharpoonright}_{\mathsf{r}} \end{array}}{\Gamma \vdash a[\mathsf{r}](s).P \triangleright \Delta}
$$

$$
(\mathsf{RSend})\ \frac{\Gamma \Vdash y{:}\mathrm{S} \quad \Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]!_{\mathrm{r}}\langle y \rangle.P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]!_{\mathrm{r}}\langle \mathrm{S} \rangle.T}
$$

$$
(\mathsf{RGet})\ \frac{x \sharp (\Gamma, \Delta, s) \quad \Gamma \cdot x{:}\mathrm{S} \vdash P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]?_{\mathrm{r}}(x).P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]?_{\mathrm{r}}\langle \mathrm{S} \rangle.T}
$$

$$
(\mathsf{USend})\ \frac{\Gamma \Vdash y{:}\mathrm{S} \quad l \doteq l' \quad l'{:}\mathrm{S} \in \Gamma \quad \Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]!_{\mathrm{u}} l \langle y \rangle.P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]!_{\mathrm{u}} l' \langle \mathrm{S} \rangle.T}
$$

$$
(\mathsf{UGet})\ \frac{x \sharp (\Gamma, \Delta, s) \quad \Gamma \Vdash v{:}\mathrm{S} \quad l \doteq l' \quad l'{:}\mathrm{S} \in \Gamma \quad \Gamma \cdot x{:}\mathrm{S} \vdash P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]?_{\mathrm{u}} l \langle v \rangle (x).P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]?_{\mathrm{u}} l' \langle \mathrm{S} \rangle.T}
$$

$$
(\mathsf{RSel})\ \frac{j \in \mathrm{I} \quad l \doteq l_j \quad \Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T_j}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]!_{\mathrm{r}} l.P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]!_{\mathrm{r}} \{ l_i.T_i \}_{i \in \mathrm{I}}}
\qquad
(\mathsf{Var})\ \frac{}{\Gamma \cdot X{:}t \vdash X \triangleright s[\mathsf{r}]{:}t}
$$

$$
(\mathsf{RBran})\ \frac{\forall j \in \mathrm{I}_2.\ \exists i \in \mathrm{I}_1.\ l_i \doteq l_j \wedge \Gamma \vdash P_i \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T_j}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]?_{\mathrm{r}} \{ l_i.P_i \}_{i \in \mathrm{I}_1} \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]?_{\mathrm{r}} \{ l_i.T_i \}_{i \in \mathrm{I}_2}}
$$

$$
(\mathsf{WSel})\ \frac{j \in \mathrm{I} \quad l \doteq l_j \quad \Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{r}]{:}T_j}{\Gamma \vdash s[\mathsf{r}, \mathsf{R}]!_{\mathrm{w}} l.P \triangleright \Delta \cdot s[\mathsf{r}]{:}[\mathsf{R}]!_{\mathrm{w}} \{ l_i.T_i \}_{i \in \mathrm{I}}}
\qquad
(\mathsf{Crash})\ \frac{\mathrm{nsr}(\Delta)}{\Gamma \vdash \bot \triangleright \Delta}
$$

$$
(\mathsf{WBran})\ \frac{l_{\mathrm{d}} \doteq l'_{\mathrm{d}} \quad \forall j \in \mathrm{I}_2.\ \exists i \in \mathrm{I}_1.\ l_i \doteq l_j \wedge \Gamma \vdash P_i \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T_j}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]?_{\mathrm{w}} \{ l_i.P_i \}_{i \in \mathrm{I}_1, l_{\mathrm{d}}} \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]?_{\mathrm{w}} \{ l_i.T_i \}_{i \in \mathrm{I}_2, l'_{\mathrm{d}}}}
$$

$$
(\mathsf{Deleg})\ \frac{\Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]!\langle\!\langle s'[\mathsf{r}] \rangle\!\rangle.P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]!\langle s'[\mathsf{r}]{:}T' \rangle.T \cdot s'[\mathsf{r}]{:}T'}
$$

$$
(\mathsf{SRecv})\ \frac{\Gamma \vdash P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}T \cdot s'[\mathsf{r}]{:}T'}{\Gamma \vdash s[\mathsf{r}_1, \mathsf{r}_2]?(\!(s'[\mathsf{r}])\!).P \triangleright \Delta \cdot s[\mathsf{r}_1]{:}[\mathsf{r}_2]?\langle s'[\mathsf{r}]{:}T' \rangle.T}
\qquad
(\mathsf{End})\ \frac{}{\Gamma \vdash \mathbf{0} \triangleright \emptyset}
$$

$$
(\mathsf{If})\ \frac{\Gamma \Vdash e{:}\mathbb{B} \quad \Gamma \vdash P \triangleright \Delta \quad \Gamma \vdash P' \triangleright \Delta}{\Gamma \vdash \texttt{if}\ e\ \texttt{then}\ P\ \texttt{else}\ P' \triangleright \Delta}
\qquad
(\mathsf{Par})\ \frac{\Gamma \vdash P \triangleright \Delta \quad \Gamma \vdash P' \triangleright \Delta'}{\Gamma \vdash P \mid P' \triangleright \Delta \cdot \Delta'}
$$

$$
(\mathsf{Res1})\ \frac{x \sharp (\Gamma, \Delta) \quad \Gamma \cdot x{:}\mathrm{S} \vdash P \triangleright \Delta}{\Gamma \vdash (\nu x)P \triangleright \Delta}
\qquad
(\mathsf{Rec})\ \frac{\Gamma \cdot X{:}t \vdash P \triangleright \Delta \cdot s[\mathsf{r}]{:}T}{\Gamma \vdash (\mu X)P \triangleright \Delta \cdot s[\mathsf{r}]{:}(\mu t)T}
$$

Figure 2.3: Typing Rules for Fault-Tolerant Systems.

Rule (RSend) checks strongly reliable senders, i.e., requires a matching strongly reliable sending in the local type of the actor and compares the actor with this type. With $\Gamma \Vdash y{:}\mathrm{S}$ we check that y is an expression of the sort S if all names x in y are replaced by arbitrary values of sort S_x for $x{:}\mathrm{S}_x \in \Gamma$. Then the continuation of the process is checked against the continuation of the type. The unreliable case is very similar, but additionally checks that the label is assigned to the sort of the expression in Γ. Rule (RGet) type strongly reliable receivers, where again the prefix is checked against a corresponding type prefix and the assumption $x{:}\mathrm{S}$ is added to the type check of the continuation. Again the unreliable case is very similar, but apart from the label also checks the sort of the default value.

Rule (RSel) checks the strongly reliable selection prefix, that the selected label matches one of the specified labels, and that the process continuation is well-typed w.r.t. the type continuation following the selected label. The only difference in the weakly reliable case is the set of roles for the receivers. For strongly reliable branching we have to check the prefix and that for each branch in the type there is a matching branch in the process that is well-typed w.r.t. the respective branch in the type. For the weakly reliable case we have to additionally check that the default labels of the process and the type coincide.

Rule (Crash) for crashed processes checks that $\mathrm{nsr}(\Delta)$.

We have to fix conditions on failure patterns to ensure that subject reduction and progress hold in general.

Condition 1 (Failure Pattern).

1. *If* $\mathtt{FP}_{\mathtt{crash}}(P)$*, then* $\mathrm{nsr}(P)$.
2. *The failure pattern* $\mathtt{FP}_{\mathtt{uget}}(s, \mathsf{r}_1, \mathsf{r}_2, l)$ *is always valid.*
3. *The pattern* $\mathtt{FP}_{\mathtt{ml}}(s, \mathsf{r}_1, \mathsf{r}_2, l)$ *is valid iff* $\mathtt{FP}_{\mathtt{uskip}}(s, \mathsf{r}_2, \mathsf{r}_1, l)$ *is valid.*
4. *If* $\mathtt{FP}_{\mathtt{crash}}(P)$ *and* $s[\mathsf{r}] \in \mathrm{A}(P)$ *then eventually* $\mathtt{FP}_{\mathtt{uskip}}(s, \mathsf{r}_2, \mathsf{r}, l)$ *and also* $\mathtt{FP}_{\mathtt{wskip}}(s, \mathsf{r}_2, \mathsf{r}, l)$ *for all* r_2, l.
5. *If* $\mathtt{FP}_{\mathtt{crash}}(P)$ *and* $s[\mathsf{r}] \in \mathrm{A}(P)$ *then eventually* $\mathtt{FP}_{\mathtt{ml}}(s, \mathsf{r}_1, \mathsf{r}, l)$ *for all* r_1, l.
6. *If* $\mathtt{FP}_{\mathtt{wskip}}(s, \mathsf{r}_1, \mathsf{r}_2)$ *then* $s[\mathsf{r}_2]$ *is crashed, i.e., the system does no longer contain an actor* $s[\mathsf{r}_2]$ *and the message queue* $s_{\mathsf{r}_2 \to \mathsf{r}_1}$ *is empty.*

The crash of a process should not block strongly reliable actions, i.e., only processes with $\mathrm{nsr}(P)$ can crash (Condition 1.1). Condition 1.2 requires that no process can refuse to consume a message on its queue. This condition prevents deadlocks that may arise from refusing a message m that is never dropped from the message queue. Condition 1.3 requires that if a message can be dropped from a message queue then the corresponding receiver has to be able to skip this message and vice versa. Similarly, processes that wait for messages from a crashed process have to be able to skip (Condition 1.4) and all messages of a queue towards a crashed receiver can be dropped (Condition 1.5). Finally, weakly reliable branching requests should not be lost. To ensure that the receiver of such a branching request can proceed if the sender is crashed but is not allowed to skip the reception of the branching request before the sender crashed, we require that $\mathtt{FP}_{\mathtt{wskip}}(s, \mathsf{r}_1, \mathsf{r}_2)$ is false as long as $s[\mathsf{r}_2]$ is alive or messages on the respective queue are still in transit (Condition 1.6).

Coherence intuitively describes that a session environment captures all local endpoints of a collection of global types. Since we capture all relevant global types in the global environment, we define coherence on pairs of global and session environments.

Theorem 1 (Subject Reduction). *If* $\Gamma \vdash P \triangleright \Delta$, Γ, Δ *are coherent, and* $P \longmapsto P'$*, then there are some* Δ' *such that* $\Gamma \vdash P' \triangleright \Delta'$.

Progress states that no part of a well-typed and coherent system can block other parts, that eventually all matching communication partners of strongly reliable and weakly reliable communications (that are not crashed) are unguarded, and that there are no communication mismatches. Subject reduction and progress together then imply *session fidelity*, i.e., that processes behave as specified in their global types.

To ensure that the interleaving of sessions and session delegation cannot introduce deadlocks, we assume an interaction type system as introduced in [2, 6]. For this type system it does not matter whether the considered actions are strongly reliable, weakly reliable, or unreliable. More precisely, we can adapt the interaction type system of [2] in a straightforward way to the above session calculus, where unreliable communication and weakly reliable branching is treated in exactly the same way as strongly reliable communication/branching. We say that P *is free of cyclic dependencies between sessions* if this interaction type system does not detect any cyclic dependencies.

In the literature there are different formulations of progress. We are interested in a rather strict definition of progress that ensures that well-typed systems cannot block. Therefore, we need an additional assumption on session requests and acceptances. Coherence ensures the existence of communication partners within sessions only. If we want to avoid blocking, we need to be sure, that no participant of a session is missing during its initialization. Note that without action prefixes all participants either terminated or crashed.

Theorem 2 (Progress/Session Fidelity). *Let $\Gamma \vdash P \triangleright \Delta$, Γ, Δ be coherent, and let P be free of cyclic dependencies between sessions. Assume that in the derivation of $\Gamma \vdash P \triangleright \Delta$, whenever $\overline{a}[\mathsf{n}](s).Q$ or $a[\mathsf{r}](s).Q$ with $a{:}G$, then there are $\overline{a}[\mathsf{n}](s).Q_n$ or $a[\mathsf{r}_i](s).Q_i$ for all $1 \leq \mathsf{r}_i < \mathsf{n}$.*

1. *Then either P does not contain any action prefixes or $P \longmapsto P'$.*
2. *If P does not contain recursion, then there exists P' such that $P \longmapsto^* P'$ and P' does not contain any action prefixes.*

2.3 Additional Notation

2.3.1 Prefixes and Branches

In [10] the authors introduced notation for the construction of global types and processes.

Let $\left(\bigodot_{1 \leq i \leq n} \pi_i\right).G$ abbreviate the sequence $\pi_1.\ldots.\pi_n.G$ to simplify the presentation, where $G \in \mathcal{G}$ is a global type and $\pi_1, \ldots, \pi_n$ are sequences of prefixes. More precisely, each π_i is of the form $\pi_{i,1}.\ldots.\pi_{i,m}$ and each $\pi_{i,j}$ is a type prefix of the form $\mathsf{r}_1 \to_{\mathrm{u}} \mathsf{r}_2{:}l\langle \mathrm{S} \rangle$ or $\mathsf{r} \to_{\mathrm{w}} \mathsf{R}{:}l_1.T_1 \oplus \ldots \oplus l_n.T_n \oplus l_{\mathrm{d}}$, where the latter case represents a weakly reliable branching prefix with the branches $l_1, \ldots, l_n, l_{\mathrm{d}}$, the default branch l_{d}, and where the next global type provides the missing specification for the default case.

Let $\left(\bigodot_{1 \leq i \leq n} \pi_i\right).P$ abbreviate the sequence $\pi_1.\ldots.\pi_n.P$, where $P \in \mathcal{P}$ is a process and $\pi_1, \ldots, \pi_n$ are sequences of prefixes.

2.3.2 Function Calls as Prefixes

We extend FTMPST by adding function calls as prefixes. These functions allow us to add side effects to our model.

Let $f : \tilde{X} \to \bot$ be a function with domain $\tilde{X} = X_1 \times \ldots \times X_n$ and $P \in \mathcal{P}$ a process. Then $f\left(x_1, \ldots, x_n\right).P$ is also a process in $\mathcal{P}$.

We introduce reduction rule (Func) as $f\left(x_1, \ldots, x_n\right).P \longmapsto P$. After the call to f, the process behaves like P.

Additionally, we define typing rule (Func). Let $F_{\tilde{X},\bot}$ be the set of functions $g : \tilde{X} \to \bot$.

$$(\mathrm{Func})\ \frac{f \in F_{\tilde{X},\bot} \qquad \Gamma \Vdash \left(x_1, \ldots, x_n\right) : \tilde{X} \qquad \Gamma \vdash P \rhd \Delta}{\Gamma \vdash f\left(x_1, \ldots, x_n\right).P \rhd \Delta}$$

In (Func) we require that f is a function from $\tilde{X}$ to $\bot$ and expression $(x_1, \ldots, x_n)$ is of the sort $\tilde{X}$. The latter implies that for $1 \leq i \leq n$ each sub-expression x_i is of the sort X_i if all names w in x_i are replaced by arbitrary values of sort S_w for $w : S_w \in \Gamma$. The sorts of the arguments given correspond to the sorts of the arguments expected in f. Applying (Func) does not affect the global environment Γ or the session environment Δ.

The addition of function calls as prefixes has no significant impact on any proofs in [10].

2.3.3 Type Parameters

We introduce a notation to define sorts using a grammar and type parameters.

Let a and b be any specified sorts. We define a sort $\mathrm{S\ a\ b}$.

$$\mathrm{S\ a\ b} = A\ \mathrm{a} \mid B\ \mathrm{b} \mid C\ \mathrm{a\ b} \mid D \mid E\ \mathbb{N}$$

The values in $\mathrm{S\ a\ b}$ are either of the form $A\ \mathrm{a}$, $B\ \mathrm{b}$, $C\ \mathrm{a\ b}$, D, or $E\ \mathbb{N}$. We say S of a and b is either an A of a, a B of b, a C of a and b, a D, or an E of $\mathbb{N}$. A, B, C, D, and E are constructors. A, B, and E each take one value, C takes two, and D takes none. Note that E does not take a value of a or b but still constructs a value in $\mathrm{S\ a\ b}$.

3 Model

First, we specify some sorts with which we can then define the global type. Afterwards, we define the processes for the proposer and the acceptor. Finally, we will study an example run of the model.

3.1 Sorts

Sorts are basic data types. We assume the following sorts.

First, we have Bool which we define as a set.

$$\mathrm{Bool} = \{\mathrm{true}, \mathrm{false}\}$$

Second, we assume a set of proposable values Value. This set contains at least two elements of any kind. For example, $\mathrm{Value} = \{\mathrm{abc}, \mathrm{def}, \ldots, \mathrm{vwx}, \mathrm{yz}\}$.

Then, we have some sorts which we define using a grammar. Each of these definitions contains a type parameter, which is a variable ranging over types. In this case the type parameter in each definition is called a.

$$\mathrm{Maybe}\ \ a = \mathrm{Just}\ \ a \mid \mathrm{Nothing}$$

A value of type $\mathrm{Maybe}\ \ a$ can have the form $\mathrm{Just}\ \ a$ or $\mathrm{Nothing}$. Some examples include $\mathrm{Just}\ \ 4$ of type $\mathrm{Maybe}\ \ \mathbb{N}$, $\mathrm{Just}\ \ \mathrm{false}$ of type $\mathrm{Maybe}\ \ \mathrm{Bool}$, and $\mathrm{Nothing}$. $\mathrm{Nothing}$ itself does not dictate an exact type because its definition does not include the type parameter a. The type is underspecified and is specified manually or through the context in which $\mathrm{Nothing}$ is used. It can be of type $\mathrm{Maybe}\ \ \mathbb{N}$, $\mathrm{Maybe}\ \ \mathrm{Bool}$, or $\mathrm{Maybe}\ \ b$ for any other type b. We use $\mathrm{Maybe}\ \ a$ where optional values are needed.

$$\mathrm{Proposal}\ \ a = \mathrm{Proposal}\ \ \mathbb{N}\ a$$

$\mathrm{Proposal}\ \ a$ only has one possible form, which is $\mathrm{Proposal}\ \ \mathbb{N}\ a$. A proposal contains its proposal number of type $\mathbb{N}$ and its value of type a. Again, a is a variable ranging over types. An example for a value of type $\mathrm{Proposal}\ \ \mathrm{Bool}$ could be $\mathrm{Proposal}\ \ 1\ \ \mathrm{true}$ and an example for a value of type $\mathrm{Proposal}\ \ (\mathrm{Maybe}\ \ \mathbb{N})$ could be $\mathrm{Proposal}\ \ 1\ \ (\mathrm{Just}\ \ 1)$. This sort models the proposals issued by the proposers in phase $2a$.

$$\mathrm{Promise}\ \ a = \mathrm{Promise}\ \ (\mathrm{Maybe}\ \ (\mathrm{Proposal}\ \ a)) \mid \mathrm{Nack}\ \ \mathbb{N}$$

$\mathrm{Promise}\ a$ has two possible forms. $\mathrm{Promise}\ (\mathrm{Maybe}\ (\mathrm{Proposal}\ a))$ and $\mathrm{Nack}\ \mathbb{N}$. Possible values include $\mathrm{Nack}\ 1$ and $\mathrm{Promise}\ (\mathrm{Just}\ (\mathrm{Proposal}\ 1\ \mathrm{false}))$ of type $\mathrm{Promise}\ \mathrm{Bool}$. The actual type of $\mathrm{Nack}\ 1$, much like that of $\mathrm{Nothing}$, is underspecified. Again, we have to specify the exact type manually or through context.

In phase $1b$ the acceptors respond to the proposers prepare request with a value of type $\mathrm{Promise}\ \mathrm{Value}$. The prepare request contains a number n. The acceptors may respond to the prepare request with a promise to not accept any proposal numbered less than n or with a rejection. In the first case the acceptor's response optionally includes the last proposal it accepted, if available, and is of the form $\mathrm{Promise}\ (\mathrm{Maybe}\ (\mathrm{Proposal}\ a))$. In the second case it includes the highest n that acceptor promised and is of the form $\mathrm{Nack}\ \mathbb{N}$.

3.2 Global Type

Since each proposer initiates its own session the global type can be defined for one proposer p and a quorum of acceptors A_Q.

The last phase of Paxos contains no inter-process communication, so it is not modeled in the global type.

$$\begin{aligned}\mathrm{G}_{\mathrm{p},\mathrm{A}_\mathrm{Q}} = (\mu t)\left(\bigodot_{a\in A_Q} p \to_u a : l1a\,\langle\mathbb{N}\rangle\right).\left(\bigodot_{a\in A_Q} a \to_u p : l1b\,\langle\mathrm{Promise}\ \mathrm{Value}\rangle\right).\\ p \to_w A_Q : \left\{\mathit{Accept}.\left(\bigodot_{a\in A_Q} p \to_u a : l2a\,\langle\mathrm{Proposal}\ \mathrm{Value}\rangle\right).0 \oplus \mathit{Restart}.t \oplus \mathit{Abort}.0\right\}\end{aligned}$$

We can distinguish the individual phases of the Paxos algorithm by the labels $l1a$, $l1b$, and $l2a$.

In the first two steps, $1a$ and $1b$, the proposer sends its proposal number to each acceptor in A_Q and listens for their responses. In step $2a$ the proposer decides whether to send an Accept or $\mathit{Restart}$ message to restart the algorithm. This decision is broadcast to all acceptors in A_Q. Should the proposer crash the algorithm ends for this particular proposer and quorum of acceptors.

3.3 Functions

We define some functions which we use in the next section to define the processes.

$$\mathrm{proposalNumber} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$\mathrm{proposalNumber}_p\,(n)$ returns a proposal number for proposer p when given a natural number n. It is used to pick a number for the prepare request in phase $1a$, which is also used in phase $2a$ in the actual proposal. We have two requirements for this function.

Let $\mathbb{P}$ be the set of proposers.

$$\forall p, q \in \mathbb{P}.\forall n, m \in \mathbb{N} : p \neq q \to \mathrm{proposalNumber}_p\,(n) \neq \mathrm{proposalNumber}_q\,(m)$$

Different proposers pick their proposal numbers from disjoint sets of numbers. This way different proposers never issue a proposal with the same proposal number.

$$\forall p \in \mathbb{P}.\forall n, m \in \mathbb{N} : n > m \rightarrow \mathrm{proposalNumber}_p\,(n) > \mathrm{proposalNumber}_p\,(m)$$

We require $\mathrm{proposalNumber}_p\,(n)$ to be strictly increasing for each proposer p so every proposer uses a higher proposal number than any it has already used.

$$\mathrm{promiseValue} : \texttt{list of}\ \mathrm{Promise}\ \ a \rightarrow a$$

$\mathrm{promiseValue}\,(ps)$ returns a fresh value if none of the promises in ps contain a value. Otherwise, the best value is returned. Usually, that means the value with the highest associated proposal number. A promise contains a value v if it is of the form $\mathrm{Promise}\ \ \mathrm{Just}\ \ v$. With this function we can model the picking of a value for a proposal in phase $2a$.

$$\begin{aligned}
&\mathrm{anyNack} : \texttt{list of}\ \mathrm{Promise}\ \ a \rightarrow \mathrm{Bool}\\
&\mathrm{anyNack}\,([]) = \mathrm{false}\\
&\mathrm{anyNack}\,((\mathrm{Nack}\ \ \_\#\_)) = \mathrm{true}\\
&\mathrm{anyNack}\,((\mathrm{Promise}\ \ \_\#xs)) = \mathrm{anyNack}\,(xs)
\end{aligned}$$

$\mathrm{anyNack}\,(ps)$ returns true if the list contains at least one promise of the form $\mathrm{Nack}\ \ n$. Otherwise, it returns false.

$$\begin{aligned}
&\mathrm{promiseCount} : \texttt{list of}\ \mathrm{Promise}\ \ a \rightarrow \mathbb{N}\\
&\mathrm{promiseCount}\,([]) = 0\\
&\mathrm{promiseCount}\,((\mathrm{Promise}\ \ \_\#xs)) = 1 + \mathrm{promiseCount}\,(xs)\\
&\mathrm{promiseCount}\,((\mathrm{Nack}\ \ \_\#xs)) = \mathrm{promiseCount}\,(xs)
\end{aligned}$$

$\mathrm{promiseCount}\,(ps)$ takes a list of promises ps and counts the number of promises that have the form $\mathrm{Promise}\ \ m$.

$\mathrm{anyNack}\,(ps)$ and $\mathrm{promiseCount}\,(ps)$ are used in the proposer to decide which branch to take in phase $2a$.

$$\begin{aligned}
&\mathrm{gt} : a \times \mathrm{Maybe}\ \ a \rightarrow \mathrm{Bool}\\
&\mathrm{gt}\,(\_, \mathrm{Nothing}) = \mathrm{true}\\
&\mathrm{gt}\,(a, \mathrm{Just}\ \ b) = a > b
\end{aligned}$$

$$\begin{aligned}
&\mathrm{ge} : a \times \mathrm{Maybe}\ \ a \rightarrow \mathrm{Bool}\\
&\mathrm{ge}\,(\_, \mathrm{Nothing}) = \mathrm{true}\\
&\mathrm{ge}\,(a, \mathrm{Just}\ \ b) = a \geq b
\end{aligned}$$

$$\mathrm{nFromProposal} : \mathrm{Proposal}\ \ a \rightarrow \mathbb{N}$$
$$\mathrm{nFromProposal}\,(\mathrm{Proposal}\ \ n\ \_) = n$$

$\mathrm{nFromProposal}\,(p)$ retrieves the proposal number n inside proposal p, which has the form $\mathrm{Proposal}\ \ n\ pr$.

$\mathrm{nFromProposal}\,(p)$, $\mathrm{gt}\,(a, ma)$, and $\mathrm{ge}\,(a, ma)$ are used to extract and compare proposal numbers in phase $2b$ of the acceptor.

$$\mathrm{genA_Q} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \texttt{list of } \mathbb{N}$$

$\mathrm{genA_Q}\,(p, c_A, c_P)$ returns a randomly selected set A_Q with $A_Q \subseteq A = \{1, \ldots, c_A\}$ and $|A_Q| > \frac{|A|}{2}$. A_Q consists of any majority of acceptors. In Paxos a majority of acceptors forms a quorum, i.e. an accepting set with which a value can be chosen [7]. We use this function when initiating the proposers to give them a quorum of acceptors with which they communicate.

3.4 Processes

3.4.1 System Initialization

$$\begin{aligned}
&\mathrm{Sys}\,(c_A, c_P) = \overline{o}\,[2]\,(z)\,.\,\mathrm{P^P_{init}}\,(c_A + 1, \mathrm{genA_Q}\,(c_A + c_P, c_A, c_P)\,, c_A + c_P, c_A + c_P, [])\\
&\mid o\,[1]\,(z)\,.\Pi_{c_A < k < c_A + c_P}\ \mathrm{P^P_{init}}\,(c_A + 1, \mathrm{genA_Q}\,(k, c_A, c_P)\,, k, k, [])\\
&\mid \Pi_{1 \le j \le c_A}\ \mathrm{P^A_{init}}\,(j, c_A + 1, c_A, c_P, n_a, pr_a)
\end{aligned}$$

$$\begin{aligned}
&\mathrm{P^P_{init}}\left(p, A_Q, n, m, \overrightarrow{V}\right) = \overline{b_n}\,[p]\,(s)\,.\,\mathrm{P^P}\\
&\mathrm{P^A_{init}}\,(a, p, c_A, c_P, n, pr) = \Pi_{c_A < k \le c_A + c_P}\ b_k\,[a]\,(s)\,.\,\mathrm{P^A_1}
\end{aligned}$$

$\mathrm{Sys}\,(c_A, c_P)$, $\mathrm{P^P_{init}}\left(p, A_Q, n, m, \overrightarrow{V}\right)$, and $\mathrm{P^A_{init}}\,(a, p, c_A, c_P, n, pr)$ describe the system initialization. c_A and c_P are the number of acceptors and proposers respectively.

An outer session is created through shared channel o. This outer session is not strictly necessary but was left in to allow for easier extension of the model. The acceptors are initialized using indices from 1 to c_A and the proposers are initialized using indices from $c_A + 1$ to $c_A + c_P$.

$\mathrm{P^P_{init}}\left(p, A_Q, n, m, \overrightarrow{V}\right)$ is initialized with the proposer's role in its own session p, which is always $c_A + 1$, a quorum of acceptors A_Q, an index n, and a vector $\overrightarrow{V}$. Each proposer has the same role $p = c_A + 1$ but uses a different shared channel b_n according to its index n. m is initialized to the same value as n but is never updated. $\overrightarrow{V}$ is used in the proposer to collect and evaluate the responses from the acceptors. It is always initialized with an empty list $[]$. Shared channel b_n is used to initiate a session. Afterwards, the process behaves like $\mathrm{P^P}$. We assume a mechanism for electing a distinguished proposer, which acts as the leader [8]. The leader is the only proposer that can try issuing proposers. A new leader is elected via the same mechanism when the previous leader terminates or crashes.

$\mathrm{P}^{\mathrm{A}}_{\mathrm{init}}\left(a, p, c_A, c_P, n, pr\right)$ is initialized with the acceptor's index a, the proposer index p, which is always $c_A + 1$, c_A, c_P, initial knowledge for the highest promised proposal number $n = n_a$, if available, and initial knowledge for the most recently accepted proposal $pr = pr_a$, if available. n is of type $\mathrm{Maybe}\ \mathbb{N}$ and pr is of type $\mathrm{Maybe}\ (\mathrm{Proposal}\ \mathrm{Value})$ thus both can be $\mathrm{Nothing}$. Each of the proposers' session requests are accepted in a separate subprocess. These subprocesses run parallel to each other but still access the same values for n and pr. We observe that each subprocess in an acceptor accesses a different channel s, since it is generated by the proposer when its session request is accepted. Afterwards, each subprocess behaves like $\mathrm{P}^{\mathrm{A}}_{1}$.

3.4.2 Proposer

To define the proposer and the acceptor we introduce a function $\mathrm{update}\,(n, m)$ which replaces the value inside n with the value of m. We use this function to update the local variables of the processes.

$$
\begin{aligned}
\mathrm{P}^{\mathrm{P}} = (\mu X)\ &\mathrm{update}\,(n, n+1)\,.\\
&\left(\bigodot_{a\in A_Q}\ s\,[p,a]!_u l1a\,\langle \mathrm{proposalNumber}_m\,(n)\rangle\right).\\
&\left(\bigodot_{a\in A_Q}\ s\,[a,p]?_u l1b\,\langle \bot\rangle\,(v_a)\right).\\
&\text{if anyNack}\left(\overrightarrow{V}\right)\ \text{or}\ \left(\mathrm{promiseCount}\left(\overrightarrow{V}\right) < \left\lceil \frac{p}{2}\right\rceil\right)\\
&\quad \text{then}\ s\,[p, A_Q]!_w \mathit{Restart}.X\\
&\quad \text{else}\ s\,[p, A_Q]!_w \mathit{Accept}.\\
&\qquad \left(\bigodot_{a\in A_Q}\ s\,[p,a]!_u l2a\left\langle \mathrm{Proposal}\ \left(\mathrm{proposalNumber}_m\,(n)\right)\ \left(\mathrm{promiseValue}\left(\overrightarrow{V}\right)\right)\right\rangle\right).0
\end{aligned}
$$

At the start of the recursion n is incremented to make sure every run of the recursion uses a different n and thus a different proposal number. The proposal number is sent to every acceptor in A_Q and their replies are gathered in $\overrightarrow{V}$ through v_a. Because $p = c_A + 1$, the minimum number of acceptors needed to form a majority is $\left\lceil \frac{p}{2}\right\rceil = \left\lceil \frac{c_A+1}{2}\right\rceil$. If any $\mathrm{Nack}\ x$ was received or the number of $\mathrm{Promise}\ y$ received is less than that needed for the majority the proposer restarts the algorithm. Otherwise, the proposer sends its proposal to the acceptors and terminates.

3.4.3 Acceptor

$$
\begin{aligned}
\mathrm{P}_1^{\mathrm{A}} = {} & (\mu X)\, s\,[p,a]?_u l1a\,\langle\bot\rangle\,(n')\,.\\
& \big(\,\text{if } n' = \bot \text{ then } s\,[a,p]!_u l1b\,\langle\bot\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\\
& \quad \text{else}\,\big(\,\text{if gt}\,(n',n)\\
& \qquad \text{then update}\,(n,n')\,.s\,[a,p]!_u l1b\,\langle\text{Promise}\;\; pr\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\\
& \qquad \text{else}\, s\,[a,p]!_u l1b\,\langle\text{Nack}\;\; n\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\,)\big)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}_2^{\mathrm{A}} = {} & s\,[p,a]?_w\big\{ \mathit{Accept}.s\,[p,a]?_u l2a\,\langle\bot\rangle\,(pr')\,.\\
& \quad \big(\,\text{if } pr' = \bot \text{ then } 0\\
& \qquad \text{else}\,\big(\,\text{if ge}\,(\text{nFromProposal}\,(pr')\,,n)\\
& \qquad\quad \text{then update}\,(pr,pr')\,.\,\text{update}\,(n,\text{Just}\;\;\text{nFromProposal}\,(pr'))\,.0\\
& \qquad\quad \text{else}\,0)\big)\\
& \quad \oplus\, \mathit{Restart}.X\\
& \quad \oplus\, \mathit{Abort}.0\big\}
\end{aligned}
$$

For each proposer an acceptor has a corresponding subprocess, which behaves like $\mathrm{P}_1^{\mathrm{A}}$. These subprocesses access the same values for n and pr. This means that updating these values with $\text{update}\,(n,m)$ updates them for all subprocesses of an acceptor.

Each subprocess can communicate with one proposer. Thus, if that proposer does not or can not communicate with a particular subprocess of an acceptor then there is no need for that subprocess. It is possible that an acceptor participates in a proposer's session but is not contained in the proposer's quorum of acceptors A_Q, in which case the proposer does not communicate with that acceptor. It is also possible for a proposer to crash or otherwise terminate, in which case the proposer can not communicate with that acceptor.

Each subprocess begins by potentially receiving a proposal number n' from the corresponding proposer. If the acceptor does receive a proposal number n' it responds with either $\text{Promise}\;\; pr$ or $\text{Nack}\;\; n$, depending on the values of n' and n. If the acceptor does not receive a proposal number then it sends $\bot$ to the proposer. Sending $\bot$ to the proposer is only necessary to maintain the global type. In either case the subprocess moves on to receive the proposers' decision in phase $2a$.

Since the proposers' decision broadcast is weakly reliable, there are two cases in which the acceptor receives no decision. The proposer might have terminated or this particular acceptor is not in the proposers' quorum of acceptors A_Q. In either case this particular subprocess of the acceptor is no longer needed, because each subprocess of the acceptor exclusively communicates with one proposer. Thus, the subprocess terminates in the default branch Abort.

In the $\mathit{Restart}$ branch this particular subprocess of the acceptor restarts the algorithm to match the corresponding proposer.

In the Accept branch the acceptor potentially receives a proposal pr' from the corresponding proposer. The acceptor updates n and pr if the proposal number in pr' is greater or equal to n. Then the subprocess terminates. If the acceptor does not receive a proposal or the proposal number of pr' is less than n the subprocess terminates without updating n or pr.

3.5 Failure Patterns

Chandra and Toueg introduce a class of failure detectors $\Diamond\mathscr{S}$, which is called *eventually strong* in [3]. Failure detectors in $\Diamond\mathscr{S}$ satisfy the following properties: (1) eventually every process that crashes is permanently suspected by every correct process and (2) eventually some correct process is never suspected by any correct process.

In all three phases modeled in the global type it is possible to suspect senders. In phases $1a$ and $2a$, with labels $l1a$ and $l2a$ respectively, the acceptors may suspect some proposers. The proposers may suspect some acceptors in phase $1b$ with label $l1b$. Accordingly, $\mathtt{FP}_{\mathtt{uskip}}$ is implemented with a failure detector in $\Diamond\mathscr{S}$ for phases $1a$, $1b$, and $2a$.

Similarly, message loss is possible in all phases modeled in the global type. Thus, $\mathtt{FP}_{\mathtt{ml}}$ is also implemented with a failure detector in $\Diamond\mathscr{S}$ with one exception. $\mathtt{FP}_{\mathtt{ml}}\ (s, p, a, l)$ returns true if p is a proposer, a is an acceptor that is not contained in the proposers' quorum of acceptors, and $l = l1b$. Since any proposer only communicates with the acceptors in its quorum, we can discard any messages from acceptors outside it.

For the weakly reliable broadcast in phase $2a$, the failure pattern $\mathtt{FP}_{\mathtt{wskip}}$ returns true for subprocesses of acceptors if the corresponding proposer crashed, otherwise terminated, or the corresponding proposer's quorum does not include that particular acceptor.

For Paxos to work a majority of acceptors needs to be alive. That means that the number of failed acceptors f needs to satisfy $n > 2f$ where n is the total number of acceptors, except in one case where there are 2 acceptors. Then, at most one acceptor may crash [7]. For acceptors $\mathtt{FP}_{\mathtt{crash}}$ returns true if at least one more acceptor may crash, i.e. $n > 2(f + 1)$ is satisfied. Let $\mathbb{F}$ be the set of processes permanently suspected by a failure detector in $\Diamond\mathscr{S}$. For proposers $\mathtt{FP}_{\mathtt{crash}}$ returns true if $A_Q \setminus \mathbb{F}$ is not a quorum, i.e. if the set of acceptors in A_Q that are not permanently suspected is not enough to form a majority of acceptors.

In Paxos there is no need to reject outdated messages so $\mathtt{FP}_{\mathtt{uget}}$ is implemented with a constant true.

3.6 Example

In this section we will study an example run of the model with 3 acceptors and 2 proposers. First, we will take a look at the example scenario. Then we will examine the scenario using reduction rules starting at system initialization.

3.6.1 Scenario

Figure 3.1 provides an overview where A_1, A_2, and A_3 are the acceptors and P_4 and P_5 are the proposers. P_5 is elected to be the leader. In steps (1) to (5), P_5 completes the Paxos algorithm with A_2 and A_3 and terminates.

At this point A_2 has promised not to accept any proposal numbered less than 10 and has accepted the value abc. So, when P_4 tries to use 5 as its proposal number (6), it receives $\mathrm{Nack}\ 10$ from A_2 (8) and has to restart the algorithm (9).

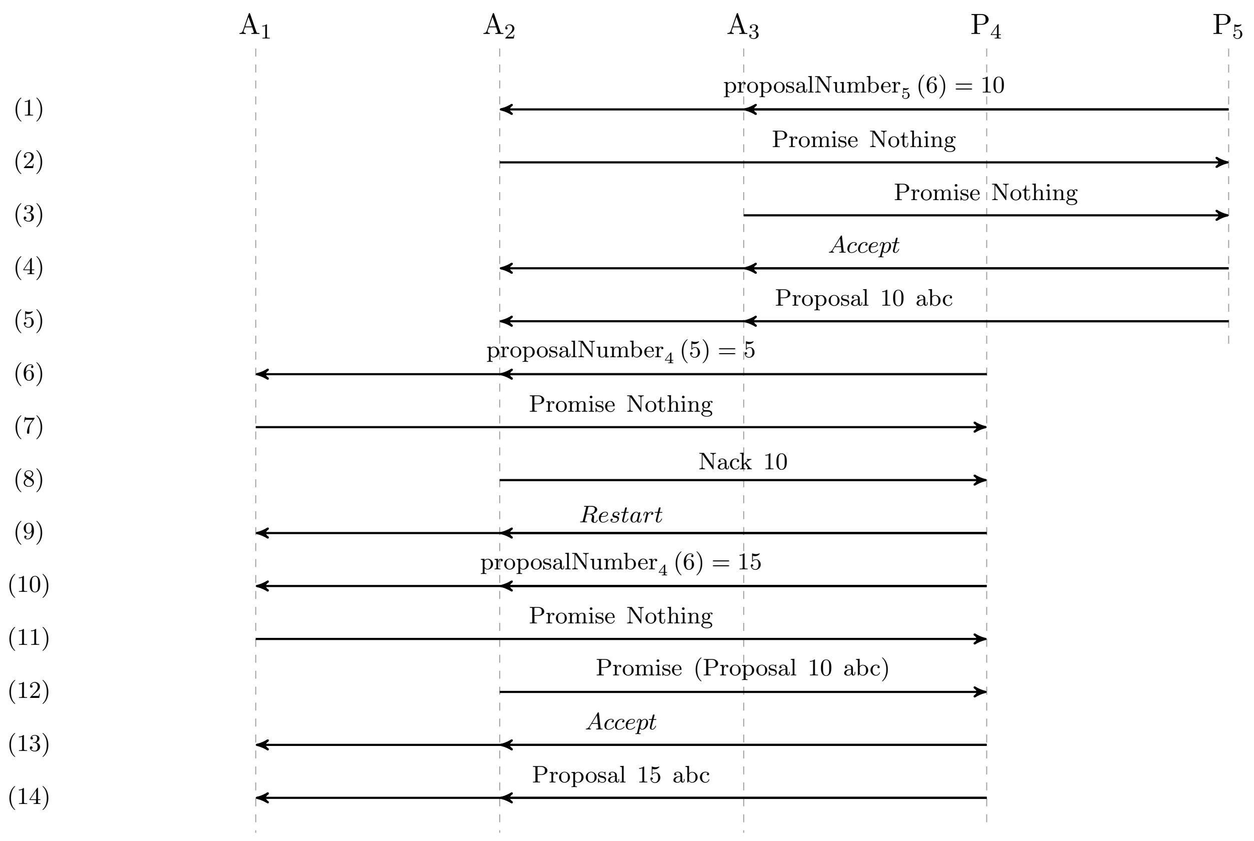

Figure 3.1: Example scenario with 3 acceptors and 2 proposers.

P_4 then runs through the Paxos algorithm with A_1 and A_2 starting with a new prepare request (10) with a higher proposal number. In step (12) P_4 learns that value abc with proposal number 10 has already been accepted by A_2. Later, in step (14), P_4 issues a proposal with the value of the highest-numbered proposal that it receives as a response to its prepare request. In this case there is only one such proposal, which is $\mathrm{Proposal}\ 10\ \mathrm{abc}$.

In the end all 3 acceptors have accepted the value abc. A_1 and A_2 have accepted $\mathrm{Proposal}\ 15\ \mathrm{abc}$ and A_3 has accepted $\mathrm{Proposal}\ 10\ \mathrm{abc}$.

3.6.2 Formulae

We set $c_A = 3$, $c_P = 2$, and $\mathrm{Value} = \{\mathrm{abc}, \mathrm{def}, \ldots, \mathrm{vwx}, \mathrm{yz}\}$.

System Initialization

After inserting c_A and c_P and applying (Init) once for shared channel a we have:

$$\begin{aligned}
&\mathrm{Sys}\,(c_A, c_P) = \mathrm{Sys}\,(3,2) = \\
&o\,[1]\,(z)\,.\Pi_{3<k<5}\ \mathrm{P}^{\mathrm{P}}_{\text{init}}\,(4, \mathrm{genA}_{\mathrm{Q}}\,(k,3,2)\,, k, k, [])\\
&\mid \overline{o}\,[2]\,(z)\,.\,\mathrm{P}^{\mathrm{P}}_{\text{init}}\,(4, \mathrm{genA}_{\mathrm{Q}}\,(5,3,2)\,, 5, 5, [])\\
&\mid \Pi_{1\leq a\leq 3}\ \mathrm{P}^{\mathrm{A}}_{\text{init}}\,(a, 4, 3, 2, n_a, pr_a)\\
&\\
&\longmapsto^* (\nu z)\ \big(\overline{b_4}\,[4]\,(s)\,.\,\mathrm{P}^{\mathrm{P}}\ = \mathrm{P}_4\\
&\mid \overline{b_5}\,[4]\,(r)\,.\,\mathrm{P}^{\mathrm{P}}\ = \mathrm{P}_5\\
&\mid\ \big(b_4\,[1]\,(s)\,.\,\mathrm{P}^{\mathrm{A}}_1\ \mid b_5\,[1]\,(r)\,.\,\mathrm{P}^{\mathrm{A}}_1\big)\ = \mathrm{A}_1\\
&\mid\ \big(b_4\,[2]\,(s)\,.\,\mathrm{P}^{\mathrm{A}}_1\ \mid b_5\,[2]\,(r)\,.\,\mathrm{P}^{\mathrm{A}}_1\big)\ = \mathrm{A}_2\\
&\mid\ \big(b_4\,[3]\,(s)\,.\,\mathrm{P}^{\mathrm{A}}_1\ \mid b_5\,[3]\,(r)\,.\,\mathrm{P}^{\mathrm{A}}_1\big)\ = \mathrm{A}_3\\
&\mid \Pi_{1\leq k,l\leq 2,k\neq l}\ t_{k\to l} : []\big)
\end{aligned}$$

Note that the outer session created via shared channel o is not strictly necessary in the model. We apply (Init) once for shared channel b_4 and once again for shared channel b_5 to obtain:

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\ \big(\\
&(\mu X)\,\mathrm{update}\,(n,5)\,.\,\Big(\bigodot_{a\in\{1,2\}}\ s\,[4,a]!_u l1a\,\langle \mathrm{proposalNumber}_4\,(n)\rangle\Big)\ldots\ = \mathrm{P}_4\\
&\mid\ (\mu X)\,\mathrm{update}\,(n,6)\,.\,\Big(\bigodot_{a\in\{2,3\}}\ r\,[4,a]!_u l1a\,\langle \mathrm{proposalNumber}_5\,(n)\rangle\Big)\ldots\ = \mathrm{P}_5\\
&\mid\ \big((\mu X)\,s\,[4,1]?_u l1a\,\langle\perp\rangle\,\big(n'\big)\,.\,\mathrm{if}\ldots\ \mid\ (\mu X)\,r\,[4,1]?_u l1a\,\langle\perp\rangle\,\big(n'\big)\,.\,\mathrm{if}\ldots\big)\ = \mathrm{A}_1\\
&\mid\ \big((\mu X)\,s\,[4,2]?_u l1a\,\langle\perp\rangle\,\big(n'\big)\,.\,\mathrm{if}\ldots\ \mid\ (\mu X)\,r\,[4,2]?_u l1a\,\langle\perp\rangle\,\big(n'\big)\,.\,\mathrm{if}\ldots\big)\ = \mathrm{A}_2\\
&\mid\ \big((\mu X)\,s\,[4,3]?_u l1a\,\langle\perp\rangle\,\big(n'\big)\,.\,\mathrm{if}\ldots\ \mid\ (\mu X)\,r\,[4,3]?_u l1a\,\langle\perp\rangle\,\big(n'\big)\,.\,\mathrm{if}\ldots\big)\ = \mathrm{A}_3\\
&\mid \Pi_{1\leq k,l\leq 4,k\neq l}\ s_{k\to l} : []\ \mid \Pi_{1\leq k,l\leq 4,k\neq l}\ r_{k\to l} : []\ \mid \Pi_{1\leq k,l\leq 2,k\neq l}\ t_{k\to l} : []\big)
\end{aligned}$$

Note that each process is shortened to only show the next few steps instead of the entire process.

The Happy Path

After applying the updates in P_4 and P_5 the first inter-process communication can take place. In this case P_5 communicates with A_2 and A_3. We apply (USend) and (UGet) twice to send $\mathrm{proposalNumber}_5\,(6) = 10$ to A_2 and A_3.

$$
\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[4,1]!_u l1a\,\langle \mathrm{proposalNumber}_4\,(5)\rangle\,.s\,[4,2]!_u l1a\,\langle \mathrm{proposalNumber}_4\,(5)\rangle \ldots \;= \mathrm{P}_4\\
&\mid\; (\mu X)\, r\,[2,4]?_u l1b\,\langle\bot\rangle\,(v_2)\,.r\,[3,4]?_u l1b\,\langle\bot\rangle\,(v_3)\ldots \;= \mathrm{P}_5\\
&\mid\; \big((\mu X)\, s\,[4,1]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\, r\,[4,1]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots\big) \;= \mathrm{A}_1\\
&\mid\; \big((\mu X)\, s\,[4,2]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\,\mathrm{if}\;10 = \bot\;\mathrm{then}\; s\,[2,4]!_u l1b\,\langle\bot\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\;\mathrm{else}\ldots\big) \;= \mathrm{A}_2\\
&\mid\; \big((\mu X)\, s\,[4,3]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\,\mathrm{if}\;10 = \bot\;\mathrm{then}\; s\,[3,4]!_u l1b\,\langle\bot\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\;\mathrm{else}\ldots\big) \;= \mathrm{A}_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : [\,]\big)
\end{aligned}
$$

Since $10 \neq \bot$ both A_2 and A_3 move into their respective else branches by applying (If-F).

$$
\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[4,1]!_u l1a\,\langle \mathrm{proposalNumber}_4\,(5)\rangle\,.s\,[4,2]!_u l1a\,\langle \mathrm{proposalNumber}_4\,(5)\rangle \ldots \;= \mathrm{P}_4\\
&\mid\; (\mu X)\, r\,[2,4]?_u l1b\,\langle\bot\rangle\,(v_2)\,.r\,[3,4]?_u l1b\,\langle\bot\rangle\,(v_3)\ldots \;= \mathrm{P}_5\\
&\mid\; \big((\mu X)\, s\,[4,1]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\, r\,[4,1]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots\big) \;= \mathrm{A}_1\\
&\mid\; \big((\mu X)\, s\,[4,2]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\,\mathrm{if}\;\mathrm{gt}\,(10,\mathrm{Nothing})\;\mathrm{then}\ldots\mathrm{else}\ldots\big) \;= \mathrm{A}_2\\
&\mid\; \big((\mu X)\, s\,[4,3]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\,\mathrm{if}\;\mathrm{gt}\,(10,\mathrm{Nothing})\;\mathrm{then}\ldots\mathrm{else}\ldots\big) \;= \mathrm{A}_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : [\,]\big)
\end{aligned}
$$

Because $\mathrm{gt}\,(10,\mathrm{Nothing})$ returns true, A_2 and A_3 move into their respective then branches by applying (If-T). After executing $\mathrm{update}\,(n,10)$ with (Func), A_2 and A_3 are ready to send their responses to P_5.

$$
\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[4,1]!_u l1a\,\langle \mathrm{proposalNumber}_4\,(5)\rangle\,.s\,[4,2]!_u l1a\,\langle \mathrm{proposalNumber}_4\,(5)\rangle \ldots \;= \mathrm{P}_4\\
&\mid\; (\mu X)\, r\,[2,4]?_u l1b\,\langle\bot\rangle\,(v_2)\,.r\,[3,4]?_u l1b\,\langle\bot\rangle\,(v_3)\ldots \;= \mathrm{P}_5\\
&\mid\; \big((\mu X)\, s\,[4,1]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\, r\,[4,1]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots\big) \;= \mathrm{A}_1\\
&\mid\; \big((\mu X)\, s\,[4,2]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\, r\,[2,4]!_u l1b\,\langle\mathrm{Promise\;\;Nothing}\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\big) \;= \mathrm{A}_2\\
&\mid\; \big((\mu X)\, s\,[4,3]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\ldots \mid (\mu X)\, r\,[3,4]!_u l1b\,\langle\mathrm{Promise\;\;Nothing}\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\big) \;= \mathrm{A}_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : [\,]\big)
\end{aligned}
$$

We apply (USend) and (UGet) twice to do just that. Note that we also apply (USkip), (If-T), and then (USend) to A_1 and then (ML) to P_5 to discard the dummy message $\bot$. All three acceptors move into P_2^A.

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\,s\,[4,1]!_u l1a\,\langle \text{proposalNumber}_4\,(5)\rangle\,.s\,[4,2]!_u l1a\,\langle \text{proposalNumber}_4\,(5)\rangle \ldots\; = P_4\\
&\mid\; (\mu X)\,r\,[4,\{2,3\}]!_w \mathit{Accept}.r\,[4,2]!_u l2a\,\langle \text{Proposal } 10 \text{ abc}\rangle \ldots\; = P_5\\
&\mid\; \big((\mu X)\ldots \mid (\mu X)\,r\,[4,1]?_w\big\{\mathit{Accept}\cdots \oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\big)\; = A_1\\
&\mid\; \big((\mu X)\ldots \mid (\mu X)\,r\,[4,2]?_w\big\{\mathit{Accept}\cdots \oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\big)\; = A_2\\
&\mid\; \big((\mu X)\ldots \mid (\mu X)\,r\,[4,3]?_w\big\{\mathit{Accept}\cdots \oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\big)\; = A_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : []\big)
\end{aligned}$$

P_5 broadcasts its decision *Accept* to A_2 and A_3. By applying (WSel) once, (WBran) twice we obtain:

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\,s\,[4,1]!_u l1a\,\langle \text{proposalNumber}_4\,(5)\rangle\,.s\,[4,2]!_u l1a\,\langle \text{proposalNumber}_4\,(5)\rangle \ldots\; = P_4\\
&\mid\; (\mu X)\,r\,[4,2]!_u l2a\,\langle \text{Proposal } 10 \text{ abc}\rangle\,.r\,[4,3]!_u l2a\,\langle \text{Proposal } 10 \text{ abc}\rangle \ldots\; = P_5\\
&\mid\; \big((\mu X)\ldots \mid (\mu X)\,r\,[4,1]?_w\big\{\mathit{Accept}\cdots \oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\big)\; = A_1\\
&\mid\; \big((\mu X)\ldots \mid (\mu X)\,r\,[4,2]?_u l2a\,\langle\bot\rangle\,\big(pr'\big)\,.\,\text{if}\ldots\big)\; = A_2\\
&\mid\; \big((\mu X)\ldots \mid (\mu X)\,r\,[4,3]?_u l2a\,\langle\bot\rangle\,\big(pr'\big)\,.\,\text{if}\ldots\big)\; = A_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : []\big)
\end{aligned}$$

Now P_5 can send its proposal to A_2 and A_3 and terminate. P_4 will be the new leader. To do so we apply (USend) and (UGet) twice. A_2 and A_3 accept the proposal and the respective subprocesses terminate. Note that we apply (WSkip) in A_1 and terminate that subprocess as well.

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\,s\,[4,1]!_u l1a\,\langle \text{proposalNumber}_4\,(5)\rangle\,.s\,[4,2]!_u l1a\,\langle \text{proposalNumber}_4\,(5)\rangle \ldots\; = P_4\\
&\mid\; (\mu X)\,s\,[4,1]?_u l1a\,\langle\bot\rangle\,\big(n'\big)\,.\,\text{if}\ldots\; = A_1\\
&\mid\; (\mu X)\,s\,[4,2]?_u l1a\,\langle\bot\rangle\,\big(n'\big)\,.\,\text{if}\ldots\; = A_2\\
&\mid\; (\mu X)\,s\,[4,3]?_u l1a\,\langle\bot\rangle\,\big(n'\big)\,.\,\text{if}\ldots\; = A_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : []\big)
\end{aligned}$$

At this point the local variables of A_2 and A_3 are $n = 10$ and $pr = \text{Proposal } 10 \text{ abc}$. A_1 has not updated its local variables $n = \text{Nothing}$ and $pr = \text{Nothing}$.

Restarting the Algorithm

Next, P_4 sends prepare requests with a proposal number less than 10, which is rejected by A_2. P_4 then decides to restart the algorithm. We apply (USend) and (UGet) twice. We also apply (USkip) once in A_3.

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[1,4]?_u l1b\,\langle\bot\rangle\,(v_1)\,.s\,[2,4]?_u l1b\,\langle\bot\rangle\,(v_2)\ldots\ = \mathrm{P}_4\\
&\mid\ (\mu X)\,\text{if } 5=\bot \text{ then } \mathrm{P}_2^{\mathrm{A}} \text{ else}\ldots\ = \mathrm{A}_1\\
&\mid\ (\mu X)\,\text{if } 5=\bot \text{ then } \mathrm{P}_2^{\mathrm{A}} \text{ else}\ldots\ = \mathrm{A}_2\\
&\mid\ (\mu X)\,\text{if } \bot=\bot \text{ then } \mathrm{P}_2^{\mathrm{A}} \text{ else}\ldots\ = \mathrm{A}_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\ s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\ r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\ t_{k\to l} : [\,]\big)
\end{aligned}$$

We can apply (If-T) to A_3 and (If-F) to A_1 and A_2. A_3 moves directly to $\mathrm{P}_2^{\mathrm{A}}$ whereas A_1 and A_2 send their responses to P_4 before moving to $\mathrm{P}_2^{\mathrm{A}}$. A_1 also updates its local variable $n = 5$ via (Func).

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[1,4]?_u l1b\,\langle\bot\rangle\,(v_1)\,.s\,[2,4]?_u l1b\,\langle\bot\rangle\,(v_2)\ldots\ = \mathrm{P}_4\\
&\mid\ (\mu X)\, s\,[1,4]!_u l1b\,\langle\mathrm{Promise\ \ Nothing}\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\ = \mathrm{A}_1\\
&\mid\ (\mu X)\, s\,[2,4]!_u l1b\,\langle\mathrm{Nack\ \ 10}\rangle\,.\,\mathrm{P}_2^{\mathrm{A}}\ = \mathrm{A}_2\\
&\mid\ (\mu X)\, r\,[4,3]?_w \big\{\mathit{Accept}\cdots\oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\ = \mathrm{A}_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\ s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\ r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\ t_{k\to l} : [\,]\big)
\end{aligned}$$

Applying (USend) and (UGet) twice and (If-T) in P_4 yields:

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[4,\{1,2\}]!_w \mathit{Restart}.X\ = \mathrm{P}_4\\
&\mid\ (\mu X)\, s\,[4,1]?_w \big\{\mathit{Accept}\cdots\oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\ = \mathrm{A}_1\\
&\mid\ (\mu X)\, s\,[4,1]?_w \big\{\mathit{Accept}\cdots\oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\ = \mathrm{A}_2\\
&\mid\ (\mu X)\, r\,[4,3]?_w \big\{\mathit{Accept}\cdots\oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\big\}\ = \mathrm{A}_3\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\ s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\ r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\ t_{k\to l} : [\,]\big)
\end{aligned}$$

P_4 sends its decision to restart the algorithm to A_1 and A_2 by applying (WSel) once and (WBran) twice. A_3 terminates after applying (WSkip).

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[4,1]!_u l1a\,\langle 15\rangle\,.s\,[4,2]!_u l1a\,\langle 15\rangle\ldots\ = \mathrm{P}_4\\
&\mid\ (\mu X)\, s\,[4,1]?_u l1a\,\langle\bot\rangle\,\big(n'\big)\,.\,\text{if}\ldots\ = \mathrm{A}_1\\
&\mid\ (\mu X)\, s\,[4,2]?_u l1a\,\langle\bot\rangle\,\big(n'\big)\,.\,\text{if}\ldots\ = \mathrm{A}_2\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\ s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\ r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\ t_{k\to l} : [\,]\big)
\end{aligned}$$

The Happy Path, Again

This time P_4 uses a high enough proposal number so that A_1 and A_2 both promise not to accept any proposal numbered less than that. By applying (USend) in the remaining proposer and (UGet), (If-F), (If-T), (Func) in the remaining acceptors we arrive at:

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[1,4]?_u l1b\,\langle\perp\rangle\,(v_1)\,.s\,[2,4]?_u l1b\,\langle\perp\rangle\,(v_2)\,.\,\mathrm{if}\ldots \;=\mathrm{P}_4\\
&\mid\;(\mu X)\, s\,[1,4]!_u l1b\,\langle\mathrm{Promise\ \ Nothing}\rangle\,.\,\mathrm{P}_2^{\mathrm{A}} \;=\mathrm{A}_1\\
&\mid\;(\mu X)\, s\,[2,4]!_u l1b\,\langle\mathrm{Promise\ \ (Proposal\ \ 10\ } abc)\rangle\,.\,\mathrm{P}_2^{\mathrm{A}} \;=\mathrm{A}_2\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : [\,]\big)
\end{aligned}$$

Note that, at this point, A_1 and A_2 have updated their respective n to 15.

Because A_2 has already accepted a proposal, it responds to P_4's prepare request with that proposal. Twice more we apply (USend), (UGet), and (If-F) in P_4 to obtain:

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[4,\{1,2\}]!_w Accept.\ldots \;=\mathrm{P}_4\\
&\mid\;(\mu X)\, s\,[4,1]?_w\big\{Accept\cdots\oplus Restart.X\oplus Abort.0\big\} \;=\mathrm{A}_1\\
&\mid\;(\mu X)\, s\,[4,1]?_w\big\{Accept\cdots\oplus Restart.X\oplus Abort.0\big\} \;=\mathrm{A}_2\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : [\,]\big)
\end{aligned}$$

P_4 has received enough promises to send its own proposal. The value for that proposal is abc because that is the value of the highest-numbered proposal P_4 received as a response to its prepare request. First, we apply (WSel) and (WBran).

$$\begin{aligned}
&\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\\
&(\mu X)\, s\,[4,1]!_u l2a\,\langle\mathrm{Proposal\ \ 15\ \ abc}\rangle\,.s\,[4,2]!_u l2a\,\langle\mathrm{Proposal\ \ 15\ \ abc}\rangle\,.0 =\mathrm{P}_4\\
&\mid\;(\mu X)\, s\,[4,1]?_u l2a\,\langle\perp\rangle\,\big(pr'\big)\,.\,\mathrm{if}\ldots \;=\mathrm{A}_1\\
&\mid\;(\mu X)\, s\,[4,2]?_u l2a\,\langle\perp\rangle\,\big(pr'\big)\,.\,\mathrm{if}\ldots \;=\mathrm{A}_2\\
&\mid \Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : [\,]\big)
\end{aligned}$$

Then we apply (USend) and (UGet) to send the proposal from P_4 to the acceptors. P_4 terminates. We apply (If-F), (If-T), and (Func) twice to the acceptors. With that, A_1 and A_2 have accepted P_4's proposal.

$$\longmapsto^* (\nu z)\,(\nu s)\,(\nu r)\,\big(\Pi_{1\le k,l\le 4,k\ne l}\; s_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 4,k\ne l}\; r_{k\to l} : [\,] \mid \Pi_{1\le k,l\le 2,k\ne l}\; t_{k\to l} : [\,]\big)$$

Afterwards A_1 and A_2 have $n = 15$ and $pr = \mathrm{Proposal\ \ 15\ \ abc}$ and A_3 has $n = 10$ and $pr = \mathrm{Proposal\ \ 10\ \ abc}$. All acceptors have accepted the value abc.

4 Analysis

We take the model from the previous chapter, type check it, and discuss what the type check means for agreement, validity, and termination of the Paxos algorithm. To execute the type check we project the global type to local types and use the typing rules given in [10].

4.1 Local Types

Because no communication occurs in the outer session, the outer session's type is $\mathrm{G} = 0$. The projection of G to a local type is $\mathrm{G} \upharpoonright_k = 0$ for every k.

For $1 \leq a \leq c_A$ and $c_A + 1 \leq p \leq c_A + c_P$ we project the global type $\mathrm{G_{p,A_Q}}$ to local types $\mathrm{G_{p,A_Q}} \upharpoonright_p$ and $\mathrm{G_{p,A_Q}} \upharpoonright_a$.

$$\begin{aligned}\mathrm{G_{p,A_Q}} \upharpoonright_p = (\mu t) \left(\bigodot_{a \in A_Q} [a]!_u l1a \langle \mathbb{N} \rangle\right) . \left(\bigodot_{a \in A_Q} [a]?_u l1b \langle \mathrm{Promise\ \ Value} \rangle\right) . \\ [A_Q]!_w \left\{ Accept. \left(\bigodot_{a \in A_Q} [a]!_u l2a \langle \mathrm{Proposal\ \ Value} \rangle\right) .0, Restart.t, Abort.0 \right\}\end{aligned}$$

$\mathrm{G_{p,A_Q}} \upharpoonright_p$ defines the local type for proposers. First, the proposer sends a proposal number to all acceptors in its quorum in phase $1a$. It receives their responses in phase $1b$ and then branches in phase $2a$. We can see that the proposer communicates with all acceptors in its quorum in every phase.

$$\begin{aligned}\mathrm{G_{p,A_Q}} \upharpoonright_a = (\mu t) [p]?_u l1a \langle \mathbb{N} \rangle . [p]!_u l1b \langle \mathrm{Promise\ \ Value} \rangle . \\ [p]?_w \{ Accept. [p]?_u l2a \langle \mathrm{Proposal\ \ Value} \rangle .0, Restart.t, Abort.0 \}_{Abort}\end{aligned}$$

$\mathrm{G_{p,A_Q}} \upharpoonright_a$ defines the local type for acceptors, assuming a proposer p. Since Paxos defines two roles that communicate with each other, their local types complement each other. An acceptor first receives a proposal number, then it responds with a $\mathrm{Promise\ \ Value}$. Finally, it receives the proposer's branching choice.

4.2 Type Check

$$\Gamma = \left(o : \mathrm{G} \cdot b_{c_A+1} : \mathrm{G}_{\mathrm{p},\mathrm{A}_\mathrm{Q}} \cdot \ldots \cdot b_{c_A+c_P} : \mathrm{G}_{\mathrm{p},\mathrm{A}_\mathrm{Q}}\right) \cdot (1 : \mathbb{N} \cdot \ldots \cdot c_A : \mathbb{N} \cdot c_A + 1 : \mathbb{N} \cdot \ldots \cdot c_A + c_P : \mathbb{N} \cdot c_P : \mathbb{N})$$

Γ contains the type for our shared channels o and b_n where $c_A + 1 \leq n \leq c_A + c_P$.

We start the type check with the global environment Γ and the entry point of the model $\mathrm{Sys}\,(c_A, c_P)$. Then, we apply the typing rules in [10] in a proof tree and show that the model can be derived from the axioms (Var) and (End).

4.2.1 System Initialization

$$\dfrac{\overset{(S_1)}{\Gamma \vdash \bar{o}\,[2]\,(z)\ldots \triangleright \emptyset} \qquad \dfrac{\overset{(S_2)}{\Gamma \vdash o\,[1]\,(z)\ldots \triangleright \emptyset} \qquad \overset{(S_3)}{\Gamma \vdash \Pi_{1\leq a\leq c_A}\ \mathrm{P}^{\mathrm{A}}_{\mathrm{init}}\,(\ldots) \triangleright \emptyset}}{\Gamma \vdash o\,[1]\,(z)\ldots \mid \Pi_{1\leq a\leq c_A}\ \mathrm{P}^{\mathrm{A}}_{\mathrm{init}}\,(\ldots) \triangleright \emptyset}\,(\mathrm{Par})}{\Gamma \vdash \bar{o}\,[2]\,(z)\,.\,\mathrm{P}^{\mathrm{P}}_{\mathrm{init}}\,(\ldots) \mid o\,[1]\,(z)\ldots \mid \Pi_{1\leq a\leq c_A}\ \mathrm{P}^{\mathrm{A}}_{\mathrm{init}}\,(\ldots) \triangleright \emptyset}\,(\mathrm{Par})$$

We apply (Par) twice and split off into the proof trees (S_1), (S_2), and (S_3).

$$(S_1) = \dfrac{\overset{(P)}{\Gamma \vdash \overline{b_{c_A+c_P}}\,[c_A+1]\,(s)\,.\,\mathrm{P}^{\mathrm{P}} \triangleright z\,[2] : \mathrm{G}\restriction_2}}{\Gamma \vdash \bar{o}\,[2]\,(z)\,.\,\mathrm{P}^{\mathrm{P}}_{\mathrm{init}}\,(c_A+1, \mathrm{genA}_\mathrm{Q}\,(c_A+c_P, c_A, c_P)\,, c_A+c_P, c_A+c_P, [])\triangleright \emptyset}\,(\mathrm{Rec})$$

In (S_1) we apply (Rec) once. The rest of the proof tree is in (P). Note that, since $\mathrm{G}\restriction_2 = 0$, $z\,[2] : \mathrm{G}\restriction_2 = \emptyset$. This is relevant later when continuing (P).

$$(S_2) = \dfrac{\dfrac{\overset{(P)}{\Gamma \vdash \overline{b_{c_A+1}}\,[c_A+1]\,(s)\,.\,\mathrm{P}^{\mathrm{P}} \triangleright \emptyset} \qquad \ldots \qquad \overset{(P)}{\Gamma \vdash \overline{b_{c_A+c_P-1}}\,[c_A+1]\,(s)\,.\,\mathrm{P}^{\mathrm{P}} \triangleright \emptyset}}{\Gamma \vdash \Pi_{c_A<k<c_A+c_P}\ \mathrm{P}^{\mathrm{P}}_{\mathrm{init}}\,(c_A+1, \mathrm{genA}_\mathrm{Q}\,(k, c_A, c_P)\,, k, k, []) \triangleright z\,[1] : G\restriction_1}\,(\mathrm{Par})^{c_P-1}}{\Gamma \vdash o\,[1]\,(z)\,.\Pi_{c_A<k<c_A+c_P}\ \mathrm{P}^{\mathrm{P}}_{\mathrm{init}}\,(\ldots) \triangleright \emptyset}\,(\mathrm{Acc})$$

Applying (Acc) in (S_2) requires that $o : \mathrm{G} \in \Gamma$. (Par) is applied $c_P - 1$ times to separate all the proposer processes. Each individual proposer can be type checked with the same proof tree (P). Because $\mathrm{G}\restriction_1 = 0$, $z\,[1] : \mathrm{G}\restriction_1 = \emptyset$. The session environment Δ in (P) is empty for every proposer.

$$(S_3) = \dfrac{\dfrac{\dfrac{\dfrac{(A_1)}{\Gamma \vdash \mathrm{P}^{\mathrm{A}}_1 \rhd s\,[a] : \mathrm{G}_{\mathrm{p},\mathrm{A_Q}} \restriction_a}}{\Gamma \vdash b_k\,[a]\,(s)\,.\,\mathrm{P}^{\mathrm{A}}_1 \rhd \emptyset}\,(\mathrm{Acc}) \quad \ldots}{\Gamma \vdash \Pi_{c_A<k\leq c_A+c_P}\ b_k\,[c_A]\,(s)\,.\,\mathrm{P}^{\mathrm{A}}_1 \rhd \emptyset}\,(\mathrm{Par})^{c_P} \quad \ldots}{\Gamma \vdash \Pi_{1\leq j\leq c_A}\ \left(\Pi_{c_A<k\leq c_A+c_P}\ b_k\,[j]\,(s)\,.\,\mathrm{P}^{\mathrm{A}}_1\right) \rhd \emptyset}\,(\mathrm{Par})^{c_A}$$

(Par) is applied c_A times to separate the individual acceptors and then c_P times for each acceptor to separate the individual subprocesses. Since every subprocess of every acceptor behaves like $\mathrm{P}^{\mathrm{A}}_1$ and has the same local type, the same proof tree (A_1) can be applied. Applying (Acc) to every subprocess of every acceptor requires $\forall k \in \mathbb{N} : (c_A + 1 \leq k \wedge k \leq c_A + c_P) \rightarrow b_k : \mathrm{G}_{\mathrm{p},\mathrm{A_Q}} \in \Gamma$.

Note that only one acceptor and one of its subprocesses is shown in (S_3). The other subprocesses and acceptors only differ in their values for the iteration variables k and j. The rest has been left out to improve readability.

4.2.2 Proposer

Let $p = c_A + 1, A_Q = \mathrm{genA_Q}\,(k, c_A, c_P)\,, n = k, m = k, \overrightarrow{V} = []$ where $c_A < k \leq c_A + c_P$. This gives us the values for the arguments of $\mathrm{P}^{\mathrm{P}}_{\mathrm{init}}$. We observe that $\Gamma \Vdash p : \mathbb{N}$, $\Gamma \Vdash k : \mathbb{N}$, $\Gamma \Vdash n : \mathbb{N}$, $\Gamma \Vdash m : \mathbb{N}$, and $\Gamma \Vdash A_Q : \texttt{list of } \mathbb{N}$. p, k, n, and m are natural numbers and A_Q is a list of natural numbers under global environment Γ.

To abbreviate the proposer's local type in the following proof trees we define the following sub-formulae.

$$\begin{aligned}
\mathrm{T}^{\mathrm{P}}_{\mathrm{acc}} &= \left(\bigodot\nolimits_{a\in A_Q}\ [a]!_u l2a\,\langle \mathrm{Proposal\ \ Value}\rangle\right).0 \\
\mathrm{T}^{\mathrm{P}}_{\mathrm{branch}} &= [A_Q]!_w \left\{ \mathit{Accept}.\,\mathrm{T}^{\mathrm{P}}_{\mathrm{acc}} \oplus \mathit{Restart}.t \oplus \mathit{Abort}.0 \right\}
\end{aligned}$$

Note that $\mathrm{G}_{\mathrm{p},\mathrm{A_Q}} \restriction_p = (\mu t)\left(\bigodot\nolimits_{a\in A_Q}\ [a]!_u l1a\,\langle \mathbb{N}\rangle\right).\left(\bigodot\nolimits_{a\in A_Q}\ [a]?_u l1b\,\langle \mathrm{Promise\ \ Value}\rangle\right).\mathrm{T}^{\mathrm{P}}_{\mathrm{branch}}$.

In order to shorten the proposer's process we define some variables.

$$\begin{aligned}
e &= \mathrm{anyNack}\left(\overrightarrow{V}\right)\ \text{ or }\ \left(\mathrm{promiseCount}\left(\overrightarrow{V}\right) < \left\lceil \frac{p}{2} \right\rceil\right) \\
pn &= \mathrm{proposalNumber}_m\,(n) \\
\mathit{prop} &= \mathrm{Proposal}\ \left(\mathrm{proposalNumber}_m\,(n)\right)\ \left(\mathrm{promiseValue}\left(\overrightarrow{V}\right)\right)
\end{aligned}$$

The actual values of e, pn, and prop are not relevant for the type check. We observe that $\Gamma \Vdash e : \mathrm{Bool}$, $\Gamma \Vdash pn : \mathbb{N}$, and $\Gamma \Vdash \mathit{prop} : \mathrm{Proposal\ \ Value}$.

To further abbreviate the terms in the proof trees we define two global environments Γ' and Γ''.

$$\Gamma' = \Gamma \cdot X : t$$

Γ' contains Γ and a type for the recursion variable X.

$$\Gamma'' = \Gamma' \cdot v_a : \text{Promise Value}, \forall a \in A_Q$$

Γ'' contains Γ' and types for the entries of $\overrightarrow{V}$. These are added to the global environment when applying (UGet) in phase $1b$.

$$(P) = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(P_t)}{\Gamma'' \vdash s\,[p, A_Q]!_w \mathit{Restart}.X \,\triangleright\, s\,[p] : \mathrm{T}^{\mathrm{P}}_{\text{branch}}} \qquad \overset{(P_f)}{\Gamma'' \vdash s\,[p, A_Q]!_w \mathit{Accept}.\dots \,\triangleright\, s\,[p] : \mathrm{T}^{\mathrm{P}}_{\text{branch}}}}{\Gamma'' \vdash \text{if } e \text{ then } s\,[p, A_Q]!_w \mathit{Restart}.X \text{ else } s\,[p, A_Q]!_w \mathit{Accept}.\dots \,\triangleright\, s\,[p] : \mathrm{T}^{\mathrm{P}}_{\text{branch}}}\,(\text{If})}{\Gamma' \vdash \left(\bigodot_{a \in A_Q} \; s\,[a, p]?_u l1b\,\langle \bot \rangle\,(v_a)\right)\dots \,\triangleright\, s\,[p] : \left(\bigodot_{a \in A_Q} \; [a]?_u l1b\,\langle \text{Promise Value} \rangle\right)}\,(\text{UGet})^{|A_Q|}}{\Gamma' \vdash \left(\bigodot_{a \in A_Q} \; s\,[p, a]!_u l1a\,\langle pn \rangle\right)\dots \,\triangleright\, s\,[p] : \left(\bigodot_{a \in A_Q} \; [a]!_u l1a\,\langle \mathbb{N} \rangle\right)\dots}\,(\text{USend})^{|A_Q|}}{\Gamma' \vdash \text{update}\,(n, n+1)\dots \,\triangleright\, s\,[p] : \left(\bigodot_{a \in A_Q} \; [a]!_u l1a\,\langle \mathbb{N} \rangle\right)\dots}\,(\text{Func})}{\Gamma \vdash (\mu X)\,\text{update}\,(n, n+1)\dots \,\triangleright\, s\,[p] : (\mu t)\left(\bigodot_{a \in A_Q} \; [a]!_u l1a\,\langle \mathbb{N} \rangle\right)\dots}\,(\text{Rec})}{\Gamma \vdash \overline{b_n}\,[p]\,(s)\,.\,\mathrm{P}^{\mathrm{P}} \,\triangleright\, \emptyset}\,(\text{Req})$$

(P) is the continuation of (S_1) and (S_2). In both proof trees the session environment Δ was empty. Here, we apply (Req) and add $s\,[p] : \mathrm{G}_{\mathrm{p}, A_Q} \upharpoonright_p$ to the session environment. Applying (Rec) changes the global environment from Γ to Γ'. (Func) advances the process. First (USend) and then (UGet) is applied for every acceptor in A_Q. (UGet) expands the session environment to Γ''. (If) splits the proof tree into (P_t) and (P_f).

$$(P_t) = \dfrac{\dfrac{}{\Gamma'' \vdash X \,\triangleright\, s\,[p] : t}\,(\text{Var})}{\Gamma'' \vdash s\,[p, A_Q]!_w \mathit{Restart}.X \,\triangleright\, s\,[p] : [A_Q]!_w \left\{\mathit{Accept}.\,\mathrm{T}^{\mathrm{P}}_{\text{acc}} \oplus \mathit{Restart}.t \oplus \mathit{Abort}.0\right\}}\,(\text{WSel})$$

We apply (WSel) and then (Var) to finish (P_t).

$$(P_f) = \dfrac{\dfrac{\dfrac{}{\Gamma'' \vdash 0 \,\triangleright\, s\,[p] : 0}\,(\text{End})}{\Gamma'' \vdash \left(\bigodot_{a \in A_Q} \; s\,[p, a]!_u l2a\,\langle \mathit{prop} \rangle\right).0 \,\triangleright\, s\,[p] : \left(\bigodot_{a \in A_Q} \; [a]!_u l2a\,\langle \text{Proposal Value} \rangle\right).0}\,(\text{USend})^{|A_Q|}}{\Gamma'' \vdash s\,[p, A_Q]!_w \mathit{Accept}.\dots.\,\triangleright\, s\,[p] : [A_Q]!_w \left\{\mathit{Accept}.\,\mathrm{T}^{\mathrm{P}}_{\text{acc}} \oplus \mathit{Restart}.t \oplus \mathit{Abort}.0\right\}}\,(\text{WSel})$$

After applying (USend) we can apply (USend) once for every acceptor in A_Q. Finally, we can finish (P_f) — and with it (P) — by applying (End).

4.2.3 Acceptor

First, we define the arguments of $\mathrm{P}^{\mathrm{A}}_{\mathrm{init}}$ and $\mathrm{P}^{\mathrm{A}}_{1}$. Let $a = j$ and $p = c_A + 1$ where $1 \leq j \leq c_A$. With session environment Γ we have $\Gamma \Vdash a : \mathbb{N}$ and $\Gamma \Vdash p : \mathbb{N}$.

To improve readability of the proof trees we break down the acceptor's process and local type.

$$\begin{aligned}
&\mathrm{P}^{\mathrm{A}}_{\mathrm{acc}} = s\,[p,a]?_u l2a\,\langle\bot\rangle\,(pr')\,.\,\mathrm{if}\;pr' = \bot\\
&\quad \mathrm{then}\;0\\
&\quad \mathrm{else\;if\;ge}\,(\mathrm{nFromProposal}\,(pr')\,,n)\\
&\qquad \mathrm{then\;update}\,(pr,pr')\,.\,\mathrm{update}\,(n,\mathrm{Just}\;\;\mathrm{nFromProposal}\,(pr'))\,.0\\
&\qquad \mathrm{else}\;0
\end{aligned}$$

We can see that $\mathrm{P}^{\mathrm{A}}_{2}$ contains $\mathrm{P}^{\mathrm{A}}_{\mathrm{acc}}$ as $\mathrm{P}^{\mathrm{A}}_{2} = s\,[p,a]?_w\{\mathit{Accept}.\,\mathrm{P}^{\mathrm{A}}_{\mathrm{acc}} \oplus \mathit{Restart}.X \oplus \mathit{Abort}.0\}$.

$$\mathrm{P}^{\mathrm{A}}_{\mathrm{t}} = \mathrm{update}\,(n,n')\,.s\,[a,p]!_u l1b\,\langle\mathrm{Promise}\;\;pr\rangle\,.\,\mathrm{P}^{\mathrm{A}}_{2}$$

$$\mathrm{P}^{\mathrm{A}}_{\mathrm{f}} = s\,[a,p]!_u l1b\,\langle\mathrm{Nack}\;\;n\rangle\,.\,\mathrm{P}^{\mathrm{A}}_{2}$$

$$\mathrm{P}^{\mathrm{A}}_{\mathrm{gt}} = \mathrm{if\;gt}\,(n',n)\;\mathrm{then}\;\mathrm{P}^{\mathrm{A}}_{\mathrm{t}}\;\mathrm{else}\;\mathrm{P}^{\mathrm{A}}_{\mathrm{f}}$$

With $\mathrm{P}^{\mathrm{A}}_{\mathrm{gt}}$, $\mathrm{P}^{\mathrm{A}}_{1}$ can be written as $\mathrm{P}^{\mathrm{A}}_{1} = (\mu X)\,s\,[p,a]?_u l1a\,\langle\bot\rangle\,(n')\,.\,\mathrm{if}\;n' = \bot\;\mathrm{then}\;s\,[a,p]!_u l1b\,\langle\bot\rangle\,.\,\mathrm{P}^{\mathrm{A}}_{2}\;\mathrm{else}\;\mathrm{P}^{\mathrm{A}}_{\mathrm{gt}}$.

$$\mathrm{T}^{\mathrm{A}}_{\mathrm{acc}} = [p]?_u l2a\,\langle\mathrm{Proposal}\;\;\mathrm{Value}\rangle\,.0$$

$$\mathrm{T}^{\mathrm{A}}_{\mathrm{branch}} = [p]?_w\,\{\mathit{Accept}.\,\mathrm{T}^{\mathrm{A}}_{\mathrm{acc}}, \mathit{Restart}.t, \mathit{Abort}.0\}_{\mathit{Abort}}$$

$$\mathrm{T}^{\mathrm{A}}_{\mathrm{1b}} = [p]!_u l1b\,\langle\mathrm{Promise}\;\;\mathrm{Value}\rangle\,.\,\mathrm{T}^{\mathrm{A}}_{\mathrm{branch}}$$

The acceptor's local type $\mathrm{G}_{\mathrm{p,A_Q}} \upharpoonright_a$ can be written as $\mathrm{G}_{\mathrm{p,A_Q}} \upharpoonright_a = (\mu t)\,[p]?_u l1a\,\langle\mathbb{N}\rangle\,.\,\mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}$.

Finally, we define the global environments Γ', Γ'', and Γ'''.

$$\begin{aligned}
\Gamma' &= \Gamma \cdot X : t\\
\Gamma'' &= \Gamma' \cdot n' : \mathbb{N}\\
\Gamma''' &= \Gamma'' \cdot pr' : \mathrm{Proposal}\;\;\mathrm{Value}
\end{aligned}$$

Γ' contains Γ and assigns the type t to X. Γ'' additionally maps n' to type $\mathbb{N}$. Γ''' adds type $\mathrm{Proposal}\;\;\mathrm{Value}$ for pr'.

$$(A_1) = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A_2)}{\Gamma'' \vdash \mathrm{P}_2^{\mathrm{A}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{branch}}}}{\Gamma'' \vdash s\,[a,p]!_u l1b\,\langle \bot \rangle\,.\,\mathrm{P}_2^{\mathrm{A}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}}\,(\mathrm{USend}) \qquad \dfrac{(A_{gt})}{\Gamma'' \vdash \mathrm{P}^{\mathrm{A}}_{\mathrm{gt}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}}}{\Gamma'' \vdash \text{if}\; n' = \bot \;\text{then}\; s\,[a,p]!_u l1b\,\langle \bot \rangle\,.\,\mathrm{P}_2^{\mathrm{A}} \;\text{else if gt}\,(n',n) \ldots \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}}\,(\mathrm{If})}{\Gamma' \vdash s\,[p,a]?_u l1a\,\langle \bot \rangle\,(n')\,.\,\text{if}\; n' = \bot \ldots \rhd s\,[a] : [p]?_u l1a\,\langle \mathbb{N} \rangle\,.\,\mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}}\,(\mathrm{UGet})}{\Gamma \vdash (\mu X)\, s\,[p,a]?_u l1a\,\langle \bot \rangle\,(n') \ldots . \rhd s\,[a] : (\mu t)\,[p]?_u l1a\,\langle \mathbb{N} \rangle \ldots .}\,(\mathrm{Rec})$$

After applying (Acc) in (S_3) the session environment contains the acceptor's local type $\mathrm{G}_{\mathrm{p,A_Q}} \upharpoonright_a$. We apply (Rec) and (UGet) and the split the proof tree with (If). By applying (Rec) and (UGet) the global environment expands from Γ to Γ' to Γ''. On the left branch we apply (USend) and continue in (A_2). The right branch continues in (A_{gt}).

We examine (A_{gt}) first, because it contains the proof tree of the left branch (A_2).

$$(A_{gt}) = \dfrac{\dfrac{(A_t)}{\Gamma'' \vdash \mathrm{P}^{\mathrm{A}}_{\mathrm{t}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}} \qquad \dfrac{(A_f)}{\Gamma'' \vdash \mathrm{P}^{\mathrm{A}}_{\mathrm{f}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}}}{\Gamma'' \vdash \text{if gt}\,(n',n)\;\text{then}\;\mathrm{P}^{\mathrm{A}}_{\mathrm{t}}\;\text{else}\;\mathrm{P}^{\mathrm{A}}_{\mathrm{f}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{1b}}}\,(\mathrm{If})$$

First, we split the proof tree with (If). We continue the resulting branches in separate proof trees (A_t) and (A_f).

$$(A_t) = \dfrac{\dfrac{(A_2)}{\Gamma'' \vdash \mathrm{P}_2^{\mathrm{A}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{branch}}}}{\Gamma'' \vdash s\,[a,p]!_u l1b\,\langle \text{Nack}\;\; n \rangle\,.\,\mathrm{P}_2^{\mathrm{A}} \rhd s\,[a] : [p]!_u l1b\,\langle \text{Promise}\;\; \text{Value} \rangle\,.\,\mathrm{T}^{\mathrm{A}}_{\mathrm{branch}}}\,(\mathrm{USend})$$

$$(A_f) = \dfrac{\dfrac{(A_2)}{\Gamma'' \vdash \mathrm{P}_2^{\mathrm{A}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{branch}}}}{\Gamma'' \vdash s\,[a,p]!_u l1b\,\langle \text{Nack}\;\; n \rangle\,.\,\mathrm{P}_2^{\mathrm{A}} \rhd s\,[a] : [p]!_u l1b\,\langle \text{Promise}\;\; \text{Value} \rangle\,.\,\mathrm{T}^{\mathrm{A}}_{\mathrm{branch}}}\,(\mathrm{USend})$$

In both, (A_t) and (A_f), we apply (USend). Now we can examine (A_2), which is the proof tree for $\mathrm{P}_2^{\mathrm{A}}$.

$$(A_2) = \dfrac{\dfrac{(A_{Accept})}{\Gamma'' \vdash \mathrm{P}^{\mathrm{A}}_{\mathrm{acc}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{acc}}} \qquad \dfrac{}{\Gamma'' \vdash X \rhd s\,[a] : t}\,(\mathrm{Var}) \qquad \dfrac{}{\Gamma'' \vdash 0 \rhd s\,[a] : 0}\,(\mathrm{End})}{\Gamma'' \vdash \mathrm{P}_2^{\mathrm{A}} \rhd s\,[a] : \mathrm{T}^{\mathrm{A}}_{\mathrm{branch}}}\,(\mathrm{WBran})$$

By applying (WBran) we separate the three branches. From left to right we get an *Accept*, a *Restart*, and an *Abort* branch. The proof tree for the *Accept* branch is in (A_{Accept}). The *Restart* branch can be finished by applying (Var) and the *Abort* branch by applying (End).

$$(A_{Accept}) = \dfrac{\dfrac{\dfrac{}{\Gamma''' \vdash 0 \rhd s\,[a] : 0}\,(\text{End}) \quad \dfrac{\dfrac{(A_{update})}{\Gamma''' \vdash \text{update}\,(pr, pr') \ldots . \rhd s\,[a] : 0} \quad \dfrac{}{\Gamma''' \vdash 0 \rhd s\,[a] : 0}\,(\text{End})}{\Gamma''' \vdash \text{if ge}\,(\text{nFromProposal}\,(pr')\,, n)\,\text{then} \ldots \text{else}\,0 \rhd s\,[a] : 0}\,(\text{If})}{\Gamma''' \vdash \text{if}\,pr' = \bot\,\text{then}\,0\,\text{else} \ldots \rhd s\,[a] : 0}\,(\text{If})}{\Gamma'' \vdash s\,[p,a]?_u l2a\,\langle\bot\rangle\,(pr') \ldots . \rhd s\,[a] : [p]?_u l2a\,\langle\text{Proposal Value}\rangle\,.0}\,(\text{UGet})$$

We apply (UGet) and expand the global session to Γ'''. The proof tree is split twice by applying (If) twice. The proof trees on the left and on the right are finished by applying (End). To keep this proof tree readable we continue the proof of the remaining branch in (A_{update}).

$$(A_{update}) = \dfrac{\dfrac{\dfrac{}{\Gamma''' \vdash 0 \rhd s\,[a] : 0}\,(\text{End})}{\Gamma''' \vdash \text{update}\,(n, \text{Just nFromProposal}\,(pr'))\,.0 \rhd s\,[a] : 0}\,(\text{Func})}{\Gamma''' \vdash \text{update}\,(pr, pr') \ldots . \rhd s\,[a] : 0}\,(\text{Func})$$

Finally, we apply (Func) twice and (End) once. This concludes the type check and proves that the model is well-typed.

4.3 Termination, Agreement, Validity

4.3.1 Termination

The global type and well-typedness ensure the absence of deadlocks. This means that the processes either loop forever or terminate. Acceptors terminate if all of their subprocesses terminate. Each subprocess of an acceptor corresponds to one proposer. A subprocess can only terminate via the weakly reliable broadcast in P_2^{A}, which depends on the corresponding proposer. If that proposer crashes or its quorum does not include the acceptor, the subprocess terminates because $\texttt{FP}_{\texttt{wskip}}$ returns true and the default branch is $Abort$, which terminates immediately. The termination of a subprocess with a correct proposer requires the termination of that proposer. Thus, we need to prove that correct proposers terminate to prove termination for our model.

If the set of acceptors in a proposer's quorum A_Q that are correct is not enough to form a majority of acceptors, that proposer repeatedly restarts the algorithm. In this case the proposer will be unable to issue a valid proposal. Because $\texttt{FP}_{\texttt{crash}}$ returns true if $A_Q \setminus \mathbb{F}$, where $\mathbb{F}$ is the set of processes permanently suspected by a failure detector in $\Diamond\mathscr{S}$, is not a quorum, the proposer eventually crashes. Proposers either complete the Paxos algorithm after phase $2a$ or crash.

In [8] Lamport describes a scenario in which two proposers loop endlessly, never having their proposals accepted: Proposer p completes phase 1 for a proposal number n_1. Another proposer q then completes phase 1 for a proposal number $n_2 > n_1$. Proposer p's phase 2 accept requests for a proposal numbered n_1 are ignored because the acceptors have all promised not to accept any new proposal numbered less than n_2. So, proposer p then begins and completes phase 1 for a new proposal number $n_3 > n_2$, causing the second phase 2 accept requests of proposer q to be ignored. And so on.

From [8] we know that this problem is solved by electing a single distinguished proposer to be the leader. The leader eventually picks a proposal number high enough for its proposal to be accepted. The model assumes some sort of leader selection. A new leader is elected when the previous leader terminates.

4.3.2 Agreement

Any proposer p requires that the set of correct acceptors in its quorum of acceptors is itself a quorum, i.e. an accepting set with which a value can be chosen [7]. Should message loss occur in labels $l1a$ or $l1b$, p restarts the algorithm. This broadcast is weakly reliable and thus only fails when p crashes or terminates because $\texttt{FP}_{\texttt{wskip}}$ disallows suspicion of correct proposers by acceptors. Given the definition of $\mathrm{promiseValue}$, p will only propose a fresh value if none of the acceptors have accepted a proposal yet. At least one acceptor in every other proposer's quorum is contained in p's quorum. Thus, if a majority of acceptors accept p's proposal it is sent to every other proposer when they reach phase $1b$. These proposers then propagate the accepted value by proposing it again but with a higher proposal number. This way all correct acceptors accept the same value.

4.3.3 Validity

To prove validity for our model we examine the communication structure and the origins of the accepted values. Because the model is well-typed we know the communication structure is as specified in global type $\mathrm{G}_{\mathrm{p},\mathrm{A}_{\mathrm{Q}}}$. From [10] we know that validity then holds globally if it holds for each local process.

Labels $l1b$ and $l2a$ are used to send values that can be accepted.

Label $l1b$ is used to send messages of sort $\mathrm{Promise}\ \ \mathrm{Value}$. These messages are sent from the acceptors to a proposer and may contain the acceptors' accepted proposal. Should an acceptor previously have accepted a proposal, that proposal then contains the accepted value. The accepted proposal pr is either the acceptor's already accepted proposal pr_a or a proposal that was previously proposed by a proposer. pr is sent over $l1b$ without alteration. The proposer receiving these messages stores them in $\overrightarrow{V}$ without changing their values.

Proposers send a message of sort $\mathrm{Proposal}\ \ \mathrm{Value}$ to their quorum of acceptors over label $l2a$. To do so, proposers pick the best value from a proposal in $\overrightarrow{V}$, if any is available, with $\mathrm{promiseValue}$. An entry in $\overrightarrow{V}$ contains a proposal $prop$ if it is of the form $\mathrm{Promise}\ \ \mathrm{Just}\ \ prop$. These proposals are either some acceptor's initial accepted proposal or a proposal proposed by a proposer. Not all entries in $\overrightarrow{V}$ contain a proposal but if at least one does, $\mathrm{promiseValue}$ returns the value of one of them. If no entry in $\overrightarrow{V}$ contains a proposal a fresh value is chosen and returned. In both cases the return value of $\mathrm{promiseValue}$ is not altered before being sent over label $l2a$, which constitutes proposing that value. Acceptors that receive and accept this proposal store it without alteration.

Since label $l1a$ is not used to transmit values that can be accepted, we conclude that validity holds for each local process and thus globally.

5 Conclusion

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