Kuratowski's Theorem

(Toán rời rạc)

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Tóm tắt nội dung

Đây là tóm tắt [1]

*K64 ...
†K65 ...
‡K65 ...

Mục lục

1 Introduction

Đôi lời phát biểu, thêm sau

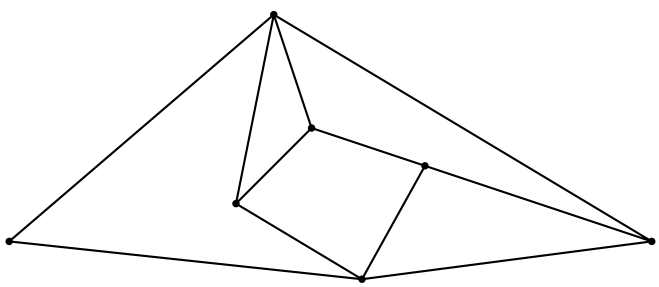
2 Defination

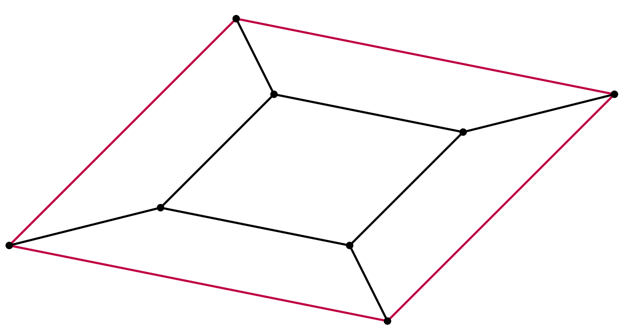
Vài cái định nghĩa cơ bản, thêm sau

Định lý (Euler's formula)**.** *$V - E + F = 2$ for convex polyhedra*

But we will generalize it to planar graphs by using stereographic projection

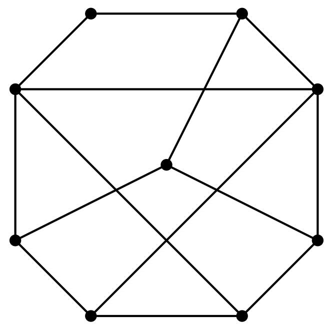
First, we see that we can take a graph embedded on the surface of a sphere and under stereographic projection get a planar graph



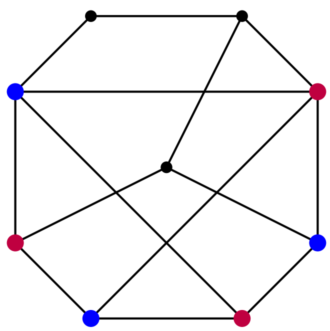


3 Statement of the Theorem

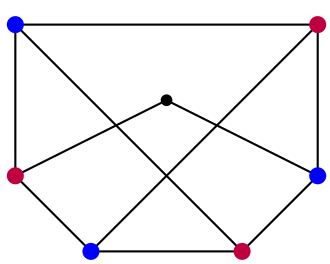
Định lý (Kuratowski)**.** *A graph is nonplanar if and only if it has a subgraph which is a subdivision of K_5 or $K_{3,3}$*



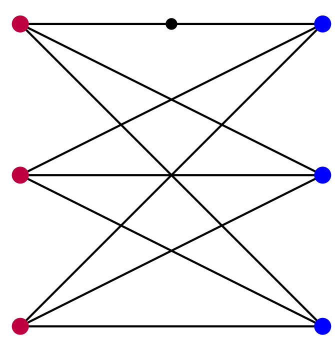
Nonplanar graph G



Nonplanar graph G



Subgraph of G

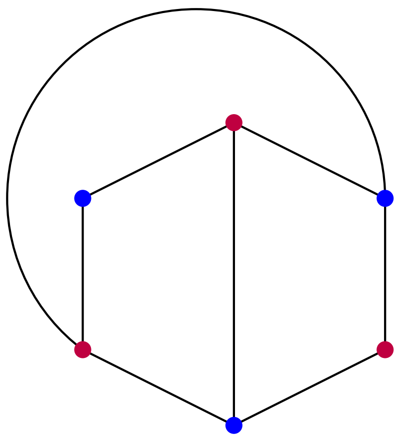


Subdivision of $K_{3,3}$

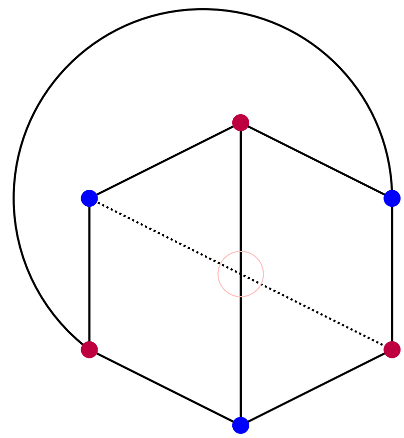
4 Preliminaries

4.1 Planar Graphs and their Properties

Định nghĩa 1 (Planarity). *A graph is planar if some embedding of it onto the plane has no edge intersections.*

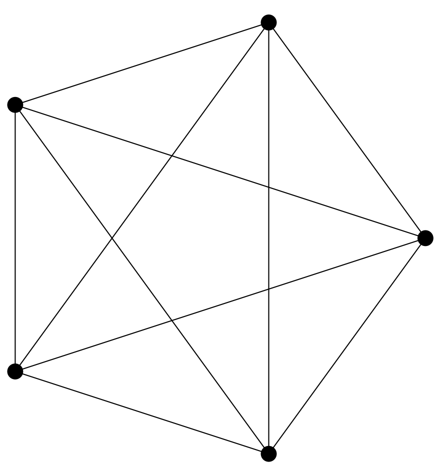


Planar graph

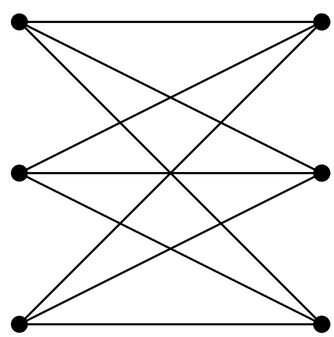


Nonplanar Embedding

4.2 Define K_5 and $K_{3,3}$



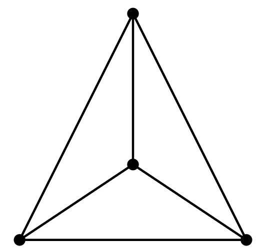
K_5



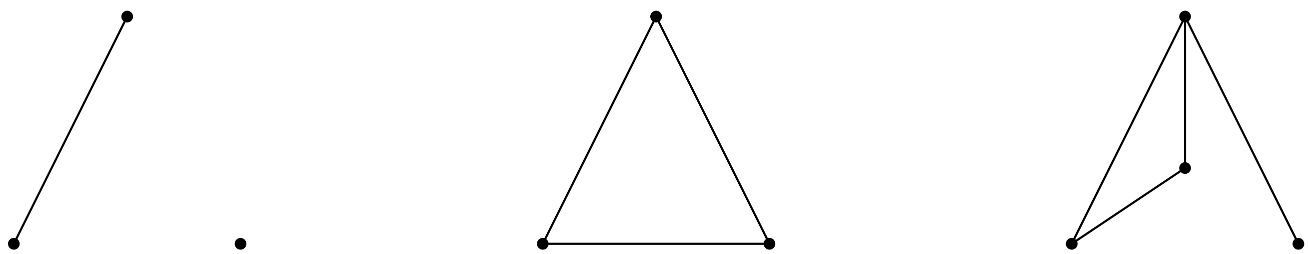
$K_{3,3}$

4.3 Subgraph and Subdivision

Định nghĩa 2. *Subgraphs are subsets of vertices and egdes of some original graphs*



Original graph



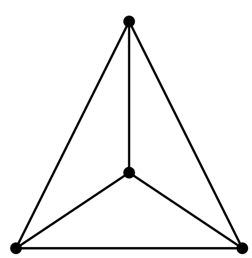
3 Subgraphs

Hệ quả 1. *If graph is planar then all subgraphs are planar*

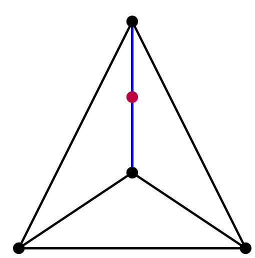
Chứng minh. Contradiction □

Định nghĩa 3. *Subdivisions are obtained by replacing an edge with 2 edges connected by a new vertex*

Chứng minh. □



Original graph



Subdivision graph

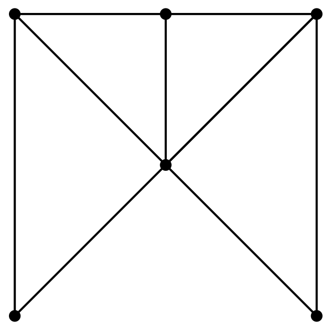
Hệ quả 2. *If some subdivision is planar then graph is planar*

Chứng minh. Ai biết đâu. □

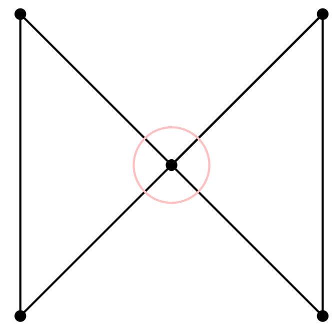
Bổ đề 1. *If graph is nonplanar then all subdivisions are nonplanar*

4.4 2-Connected Graphs and their Properties

Định nghĩa 4. *A graph is 2-connected if it cannot be separated into two components by removing a single vertex*



Example 2-connected graph

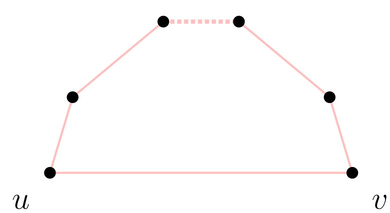


Not 2-connected

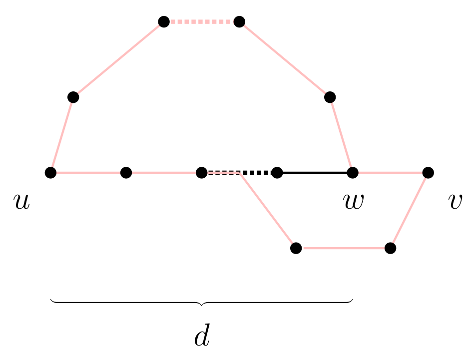
Định lý. *In a 2-connected graph, any pair of vertices is contained in a cycle*

Chứng minh. Quy nạp:

Trường hợp cơ bản: u kề v



Quy nạp: u, v có khoảng cách $d + 1$



□

5 Graph Theory Background

6 Proof the Theorem

The first direction of Kuratowski's theorem states: If graph G contains a subdivision of K_5 or $K_{3,3}$ then G is nonplanar

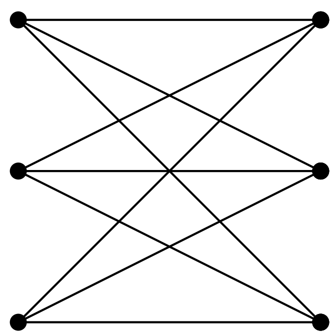
Subdivision of Nonplanar is Nonplanar

If a Subgraph is nonplanar then graph is nonplanar

If a subgraph of graph G is a subdivision of nonplanar then G is nonplanar

Bổ đề 2. *$K_{3,3}$ is nonplanar*

$$V - E + F = 2$$
$$6 - E + F = 2$$
$$6 - 9 + F = 2$$
$$F = 5$$



No 3 edge faces

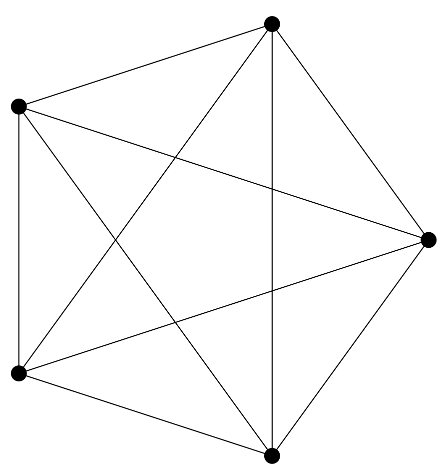
$$4F \leq 2E$$
$$4F \leq 2 \times 9$$
$$F \leq 4.5$$

$$5 \leq 4.5$$

Chứng minh. □

Bổ đề 3. *K_5 is nonplanar*

$$V - E + F = 2$$
$$5 - E + F = 2$$
$$5 - 10 + F = 2$$
$$F = 7$$



$$3F \leq 2E$$
$$3F \leq 2 \times 10$$
$$F \leq \frac{20}{3}$$

$$7 \leq \tfrac{20}{3}$$

Chứng minh. □

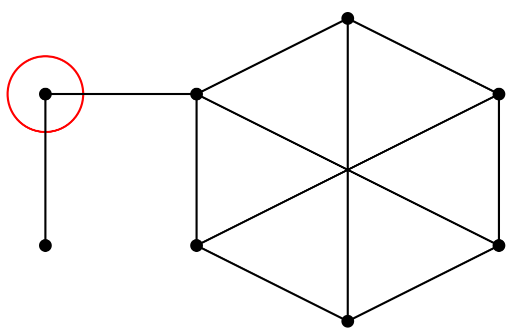
Tóm lại. *K_5 và $K_{3,3}$ are nonplanar*
$\Rightarrow$ *All of their subdivisions are nonplanar*
$\Rightarrow$ *If graph G contains a subdivision of K_5 or $K_{3,3}$ then G is nonplanar*

The second direction of Kuratowski's theorem states: If graph G is nonplanar then G contains a subdivision of K_5 or $K_{3,3}$

Chứng minh. Assume there exist nonplanar graphs which have no subdivisions of K_5 or $K_{3,3}$ as subgraphs.

Let G be the graph of this kind with the $fewest$ edges. Then removing any edge from G gives a *planar* graph

1. G is 2-connected



2. $deg(v) \geq 3$ for all vertex v in G

   Chứng minh phản chứng: assume some vertex $v \in G$ has $deg(v) \leq 2$

3. for some $uv \in G$, $G - uv$ is 2-connected

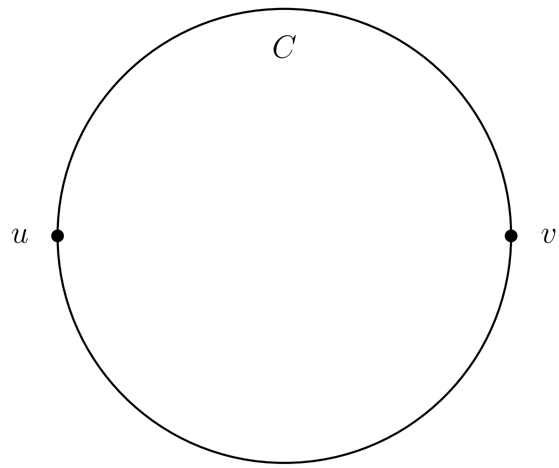
□

Take the egde uv from the previous statement, and consider the graph $G - uv$ obtained by removing it

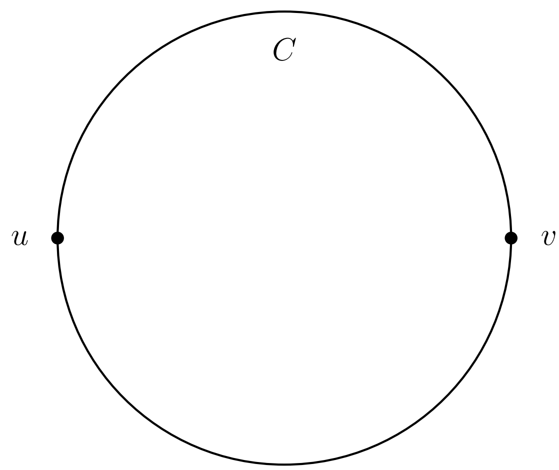
$G - uv$ is planar by minimality

$G - uv$ is 2-connected, so there is a cycle contains u, v.

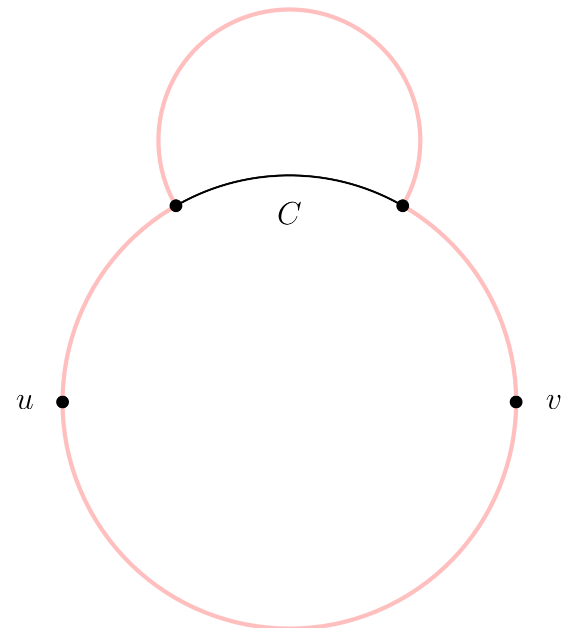
Nhận xét. *Note that the edges we are drawing here are really paths in graph*



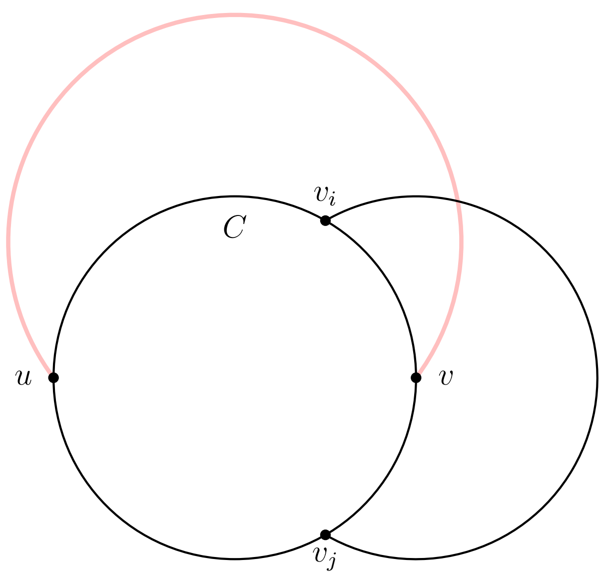
Embeded maximal cycle C containing u, v



We embeded $G - uv$ so that C enclose more regions of the graph than any other cycle containing u and v could



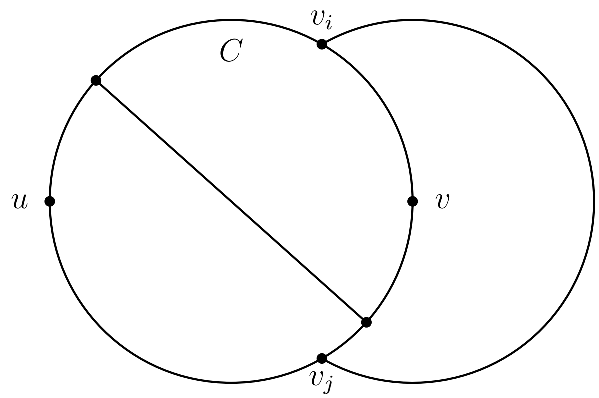
Loop along upper or lower part of C?
We can't have any extra paths on the upper or lower part of C since then there would be a cycle which contains more regions
Larger cycle $\Rightarrow$ Contradiction



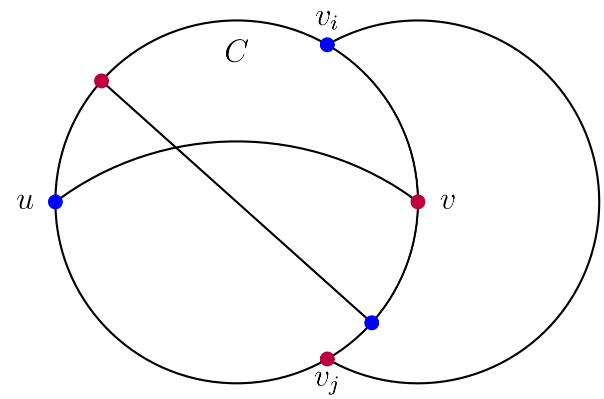
G is nonplanar, so we need an osbtruction to uv on the outside of C.
There must exist a path v_iv_j that blocks uv

The inside of C must contain an obstruction. This obstruction also has to block v_iv_j from being draw inside of C since otherwise we could just draw it inside and draw uv on the outside.

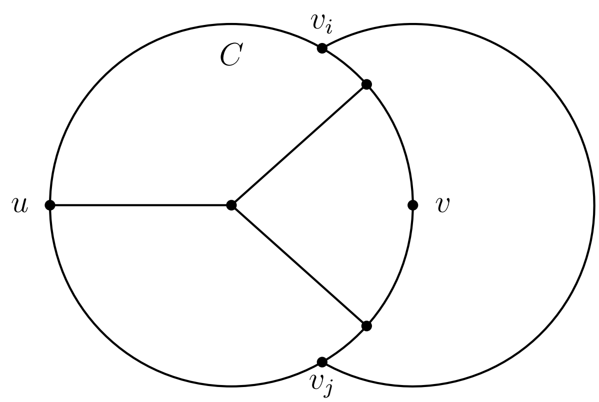
Up to equivalence there are only four types of obstructions we could draw here.



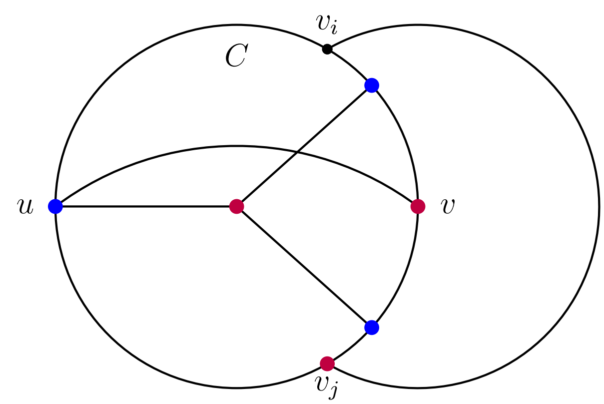
Obstruction 1



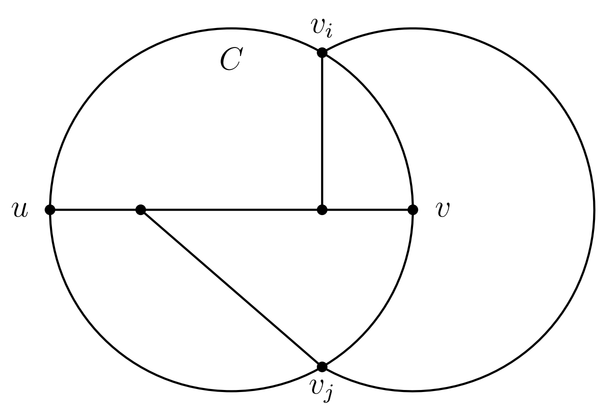
G contains a subdivision of $K_{3,3}$



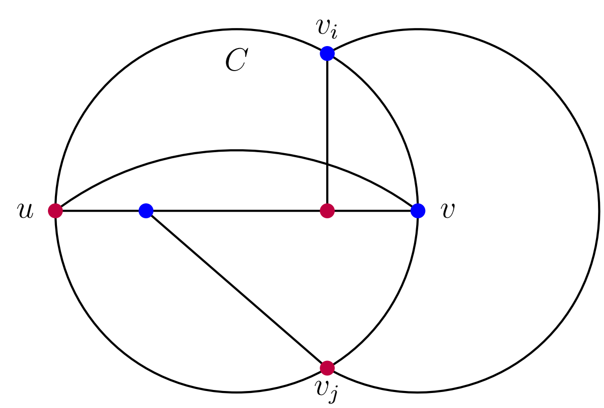
Obstruction 2



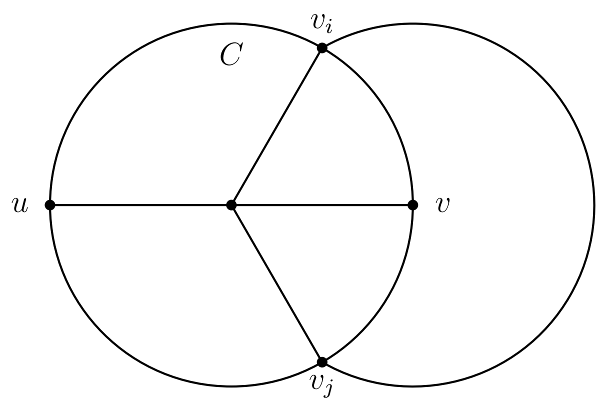
G contains a subdivision of $K_{3,3}$



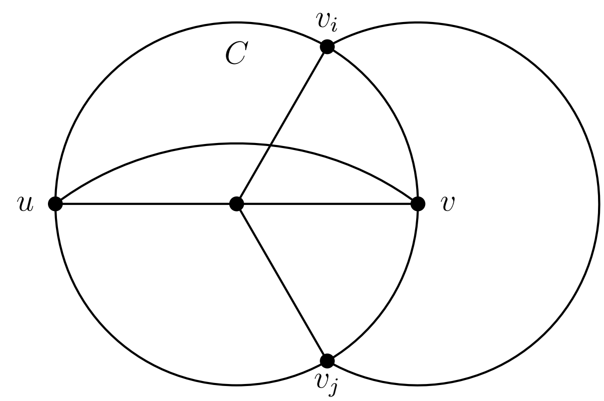
Obstruction 3



G contains a subdivision of $K_{3,3}$



Obstruction 4



G contains a subdivision of K_5

Nhận xét. *G always contains a subgraph which is subdivisionof K_5 or $K_{3,3}$*

In all of the four cases the result is a 4 cases, the result is a contradiction. We assume that G contained at neither of the two graphs. With this, we are left to conclude that there are no nonplanar graphs like G. This proves the theorem.

Acknowledgement

We would like to thank:

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- Professor "" for organizing the HUS.