Kuratowski's Theorem

(Toán rời rạc)

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Tóm tắt nội dung

Đây là tóm tắt [1]

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1 Introduction

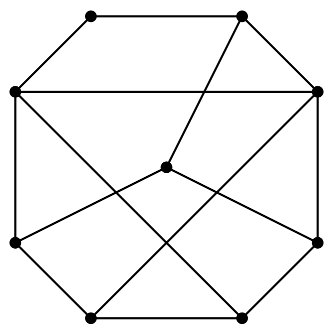
Đôi lời phát biểu, thêm sau

2 Defination

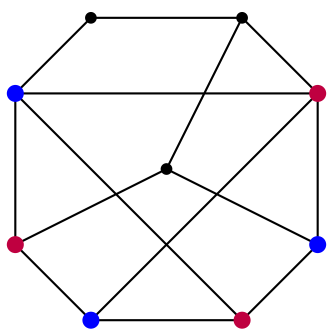
Vài cái định nghĩa cơ bản, thêm sau

3 Statement of the Theorem

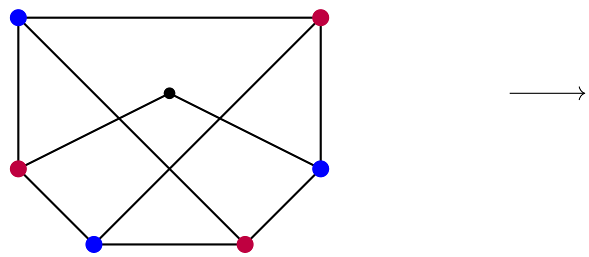
Định lý (Kuratowski)**.** *A graph is nonplanar if and only if it has a subgraph which is a subdivision of* K_5 *or* $K_{3,3}$



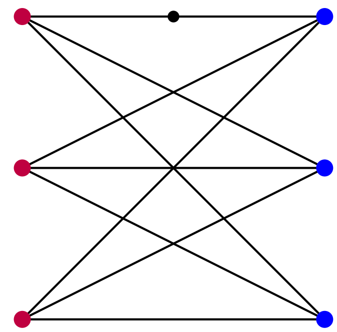
Nonplanar graph G



Nonplanar graph G



Subgraph of G

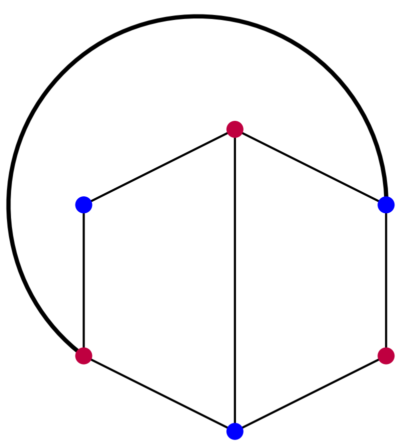


Subdivision of $K_{3,3}$

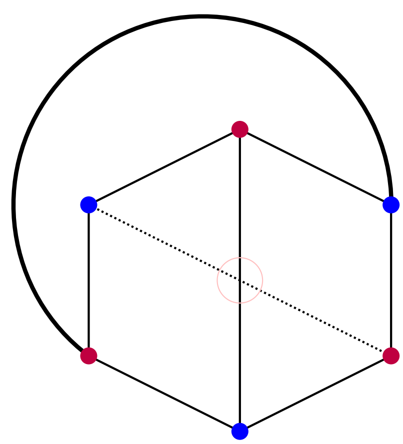
4 Preliminaries

4.1 Planar Graphs and their Properties

Định nghĩa 1 (Planarity)**.** *A graph is planar if some embedding of it onto the plane has no edge intersections.*

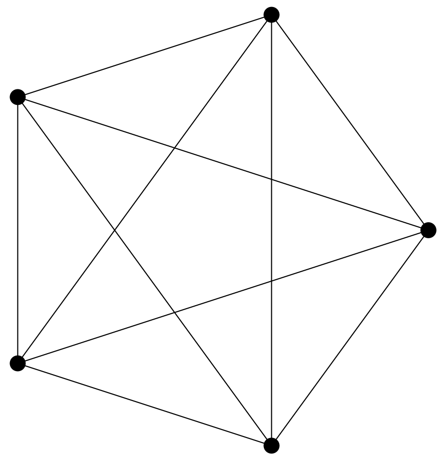


Planar graph

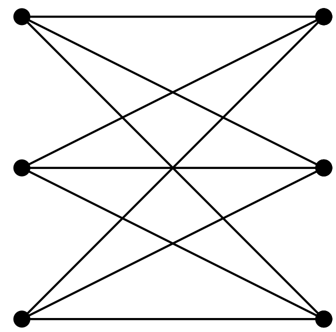


Nonplanar Embedding

4.2 Define K_5 and $K_{3,3}$



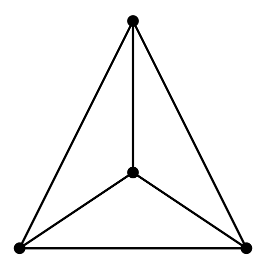
K_5



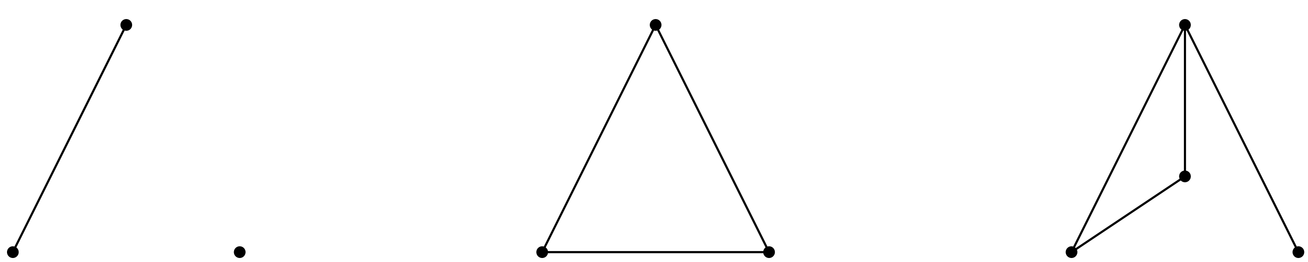
$K_{3,3}$

4.3 Subgraph and Subdivision

Định nghĩa 2. *Subgraphs are subsets of vertices and egdes of some original graphs*



Original graph



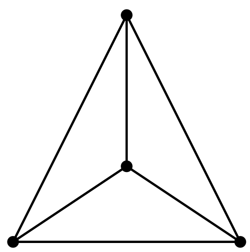
3 Subgraphs

Hệ quả 1. *If graph is planar then all subgraphs are planar*

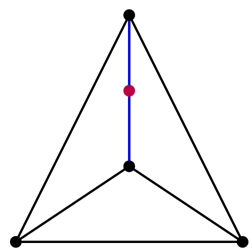
Chứng minh. Contradiction □

Định nghĩa 3. *Subdivisions are obtained by replacing an edge with 2 edges connected by a new vertex*

Chứng minh. □



Original graph



Subdivision graph

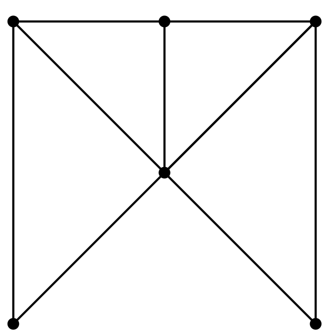
Hệ quả 2. *If some subdivision is planar then graph is planar*

Chứng minh. Ai biết đâu. □

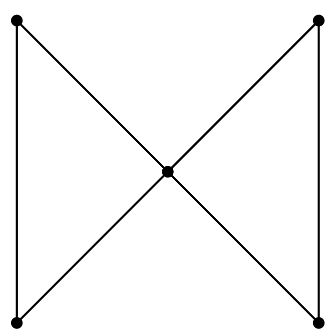
Bổ đề 1. *If graph is nonplanar then all subdivisions are nonplanar*

4.4 2-Connected Graphs and their Properties

Định nghĩa 4. *A graph is 2-connected if it cannot be separated into two components by removing a single vertex*



Example 2-connected graph

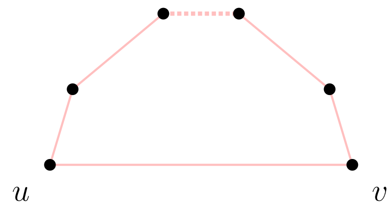


Not 2-connected

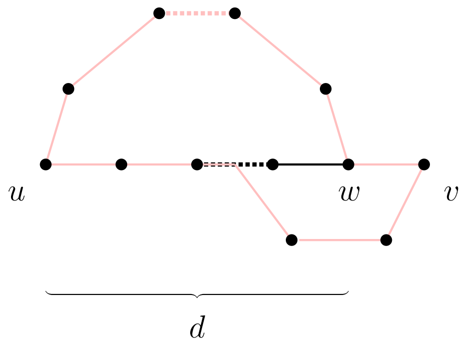
Định lý. *In a 2-connected graph, any pair of vertices is contained in a cycle*

Chứng minh. Quy nạp:

Trường hợp cơ bản: u kề v



Quy nạp: u, v có khoảng cách $d+1$



$\square$

5 Graph Theory Background

6 Proof the Theorem

The first direction of Kuratowski's theorem states: If graph G contains a subdivision of K_5 or $K_{3,3}$ then G is nonplanar

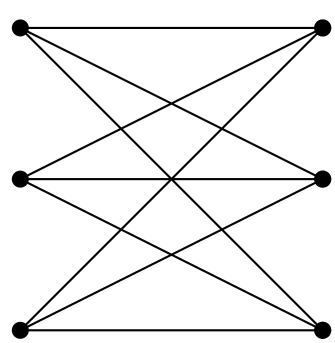
Subdivision of Nonplanar is Nonplanar

If a Subgraph is nonplanar then graph is nonplanar

If a subgraph of graph G is a subdivision of nonplanar then G is nonplanar

Bổ đề 2. $K_{3,3}$ *is nonplanar*

$$V - E + F = 2$$
$$6 - E + F = 2$$
$$6 - 9 + F = 2$$
$$F = 5$$



No 3 edge faces

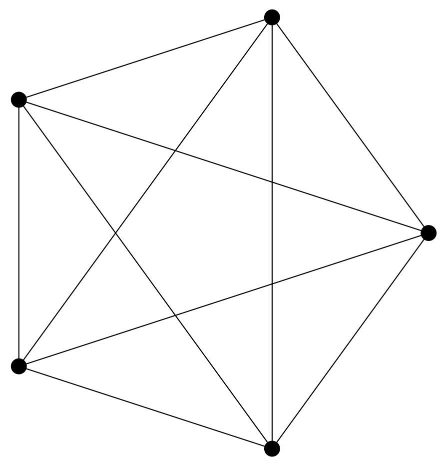
$$4F \leq 2E$$
$$4F \leq 2 \times 9$$
$$F \leq 4.5$$

$$5 \leq 4.5$$

Chứng minh. □

Bổ đề 3. *K_5 is nonplanar*

$$V - E + F = 2$$
$$5 - E + F = 2$$
$$5 - 10 + F = 2$$
$$F = 7$$



$$3F \leq 2E$$
$$3F \leq 2 \times 10$$
$$F \leq \frac{20}{3}$$

$$7 \leq \tfrac{20}{3}$$

Chứng minh. □

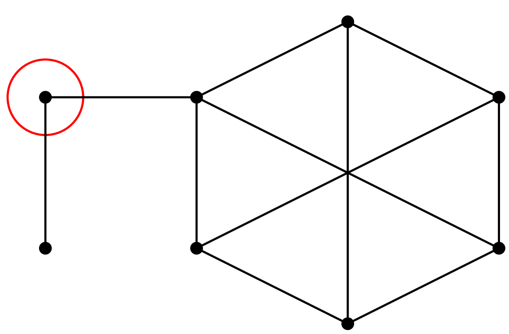
Tóm lại. *K_5 và $K_{3,3}$ are nonplanar*
$\Rightarrow$ *All of their subdivisions are nonplanar*
$\Rightarrow$ *If graph G contains a subdivision of K_5 or $K_{3,3}$ then G is nonplanar*

The second direction of Kuratowski's theorem states: If graph G is nonplanar then G contains a subdivision of K_5 or $K_{3,3}$

Chứng minh. Assume there exist nonplanar graphs which have no subdivisions of K_5 or $K_{3,3}$ as subgraphs.

Let G be the graph of this kind with the *fewest* edges. Then removing any edge from G gives a *planar* graph

1. G is 2-connected



2. $deg(v) \geq 3$ for all vertex v in G

   Chứng minh phản chứng: assume some vertex $v \in G$ has $deg(v) \leq 2$

3. for some $uv \in G$, $G - uv$ is 2-connected

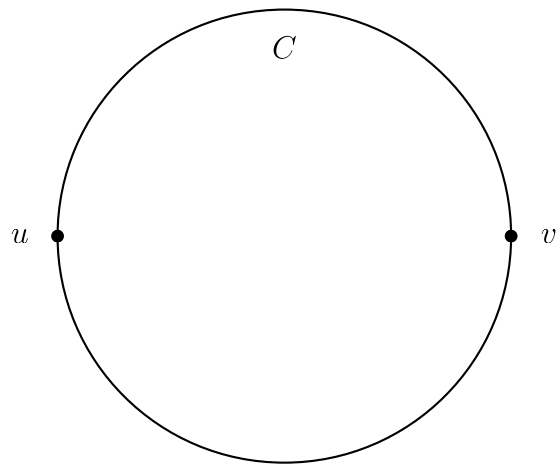
□

Take the egde uv from the previous statement, and consider the graph $G - uv$ obtained by removing it

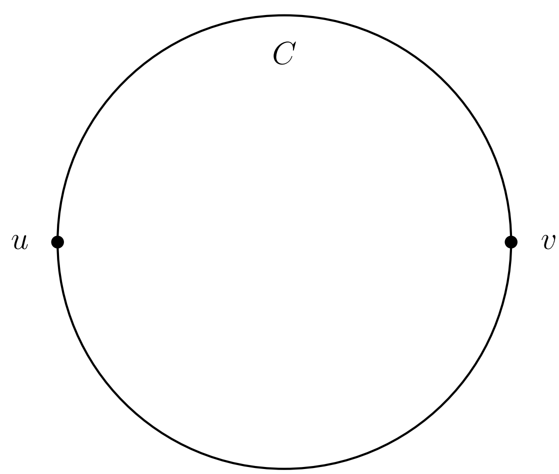
$G - uv$ is planar by minimality

$G - uv$ is 2-connected, so there is a cycle contains u, v.

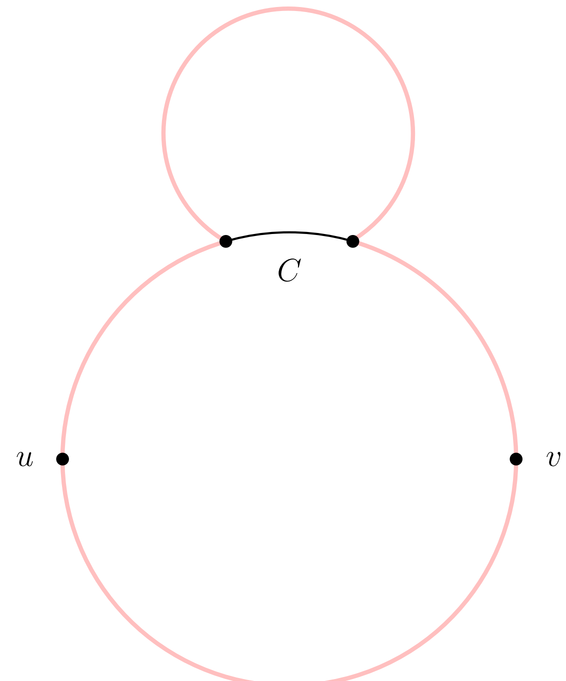
Nhận xét. *Note that the edges we are drawing here are really paths in graph*



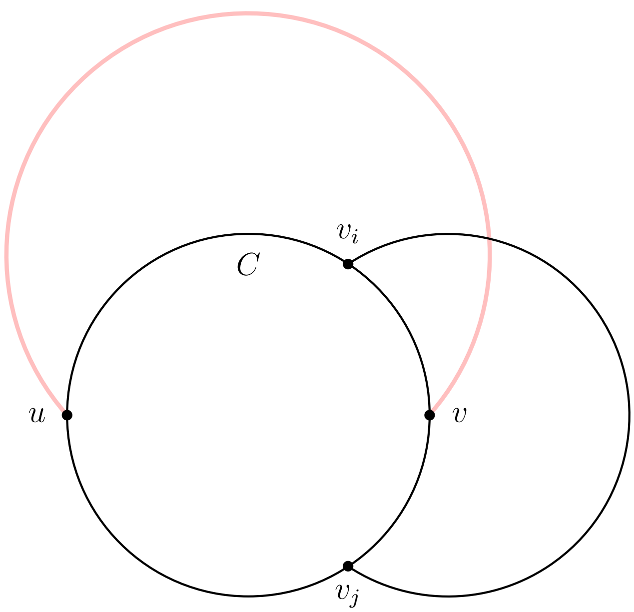
Embeded maximal cycle C containing u, v



We embeded $G - uv$ so that C enclose more regions of the graph than any other cycle containing u and v could



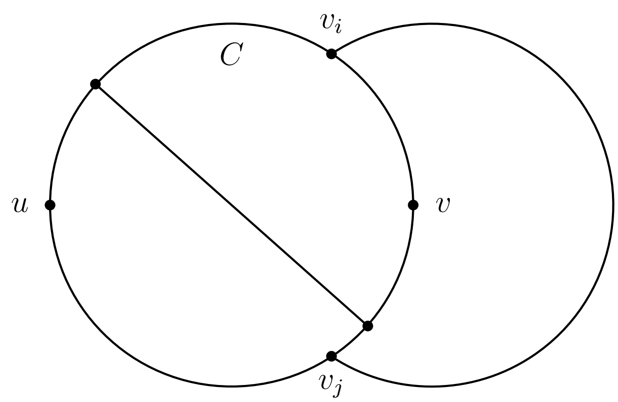
Loop along upper or lower part of C?
We can't have any extra paths on the upper or lower part of C since then there would be a cycle which contains more regions
Larger cycle $\Rightarrow$ Contradiction



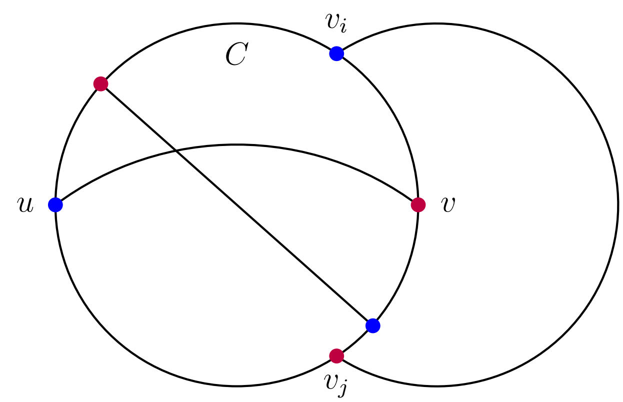
G is nonplanar, so we need an osbtruction to uv on the outside of C. There must exist a path v_iv_j that blocks uv

The inside of C must contain an obstruction. This obstruction also has to block v_iv_j from being draw inside of C since otherwise we could just draw it inside and draw uv on the outside.

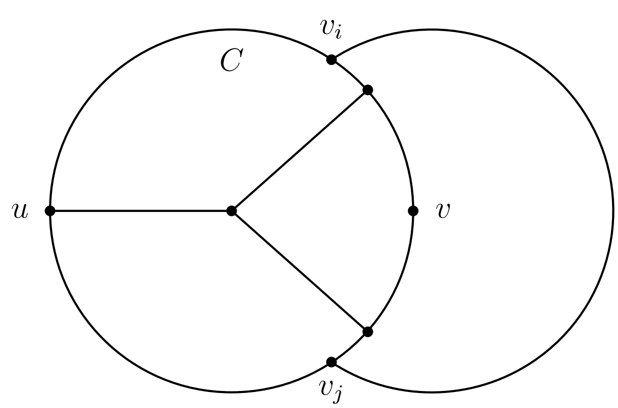
Up to equivalence there are only four types of obstructions we could draw here.



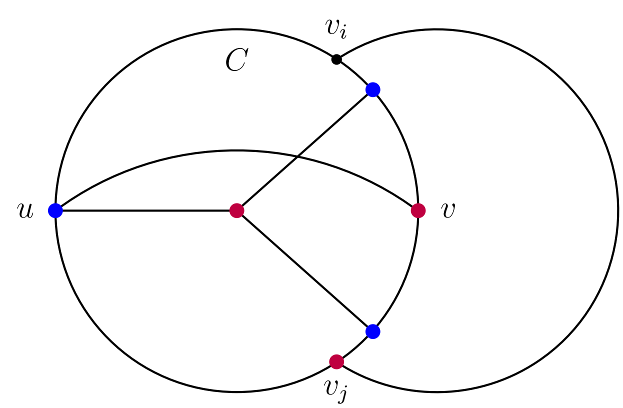
Obstruction 1



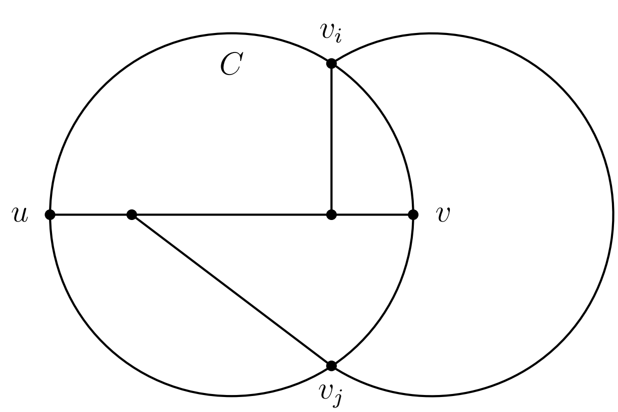
G contains a subdivision of $K_{3,3}$



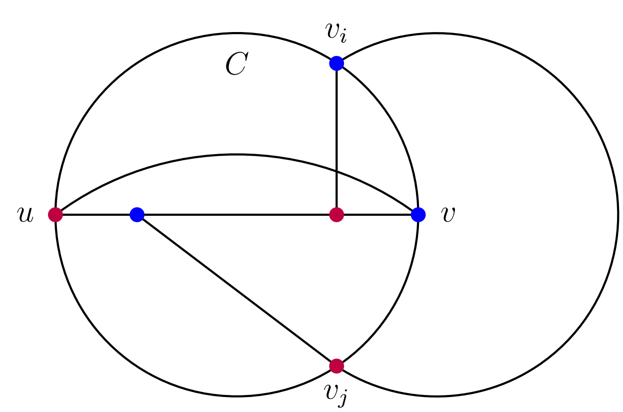
Obstruction 2



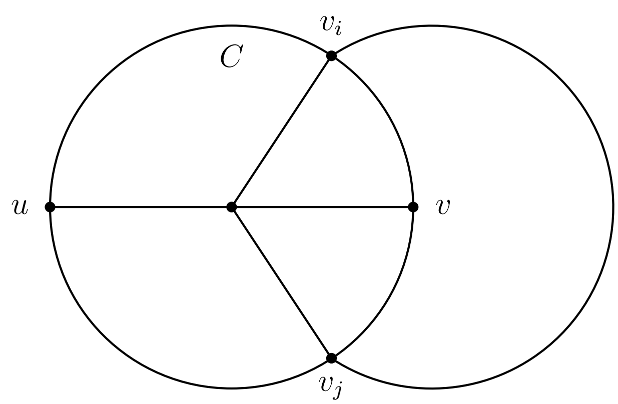
G contains a subdivision of $K_{3,3}$



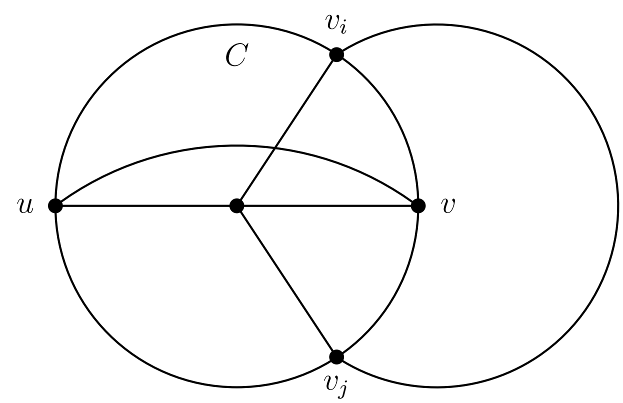
Obstruction 3



G contains a subdivision of $K_{3,3}$



Obstruction 4



G contains a subdivision of K_5

Nhận xét. *G always contains a subgraph which is subdivisionof K_5 or $K_{3,3}$*

In all of the four cases the result is a 4 cases, the result is a contradiction. We assume that G contained at neither of the two graphs. With this, we are left to conclude that there are no nonplanar graphs like G. This proves the theorem.

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