The Muffin Problem- Cheat Sheet

EASY CASES

THEOREM easy:

Theorem 0.1

1. *If* $m \equiv 0 \pmod{s}$ *then* $f(m,s) = 1$.

2. *If* $s \equiv 0 \pmod{2m}$ *and* $\frac{m}{s} \notin \mathsf{N}$ *then* $f(m,s) = \frac{1}{2}$.

3. *If* $m, s \in \mathsf{N}$ *then* $f(m,s) \geq \frac{1}{s}$.

4. $f(1,s) = \frac{1}{s}$.

5. *If* s *is odd then* $f(2,s) = \frac{1}{s}$.

6. *ALEX- I SUSPECT THIS APPROACH WILL NEVER GIVE OPT BOUNDS. BUT MIGHT BE USEFUL IN INITIALLY SHOWING* $f(m,s) > \frac{1}{3}$.

   If $m \geq Ls$ *then* $f(m,s) \geq f(m - Ls, s)$.

PROGRAM1: CHECK if any of the easy cases occur. If so then output $f(m,s)$ and also any one of the following:

- m *divides* s
- $m = 1$
- $m = 2$ *and* s *is odd*,

FUTURE- we may want to have a program that, given M, S computes $f(m,s)$ for every $1 \leq m \leq M$, $1 \leq s \leq S$. In this case Theorem 0.1.6 may be useful in getting a lower bound on $f(m,s)$.

Upper and Lower Bounds on $f(m,s)$ when $f(m,s)$ is Small

THEOREM msUB

Theorem 0.2 *Let $m, s \in \mathsf{N}$ such that s does not divide m. Then*

$$f(m,s) \le \max\Bigg\{$$

- $\frac{m}{3s}$,
- $\min\{\frac{m}{s}, \frac{s-m}{s\lceil m/2(s-m)\rceil}, \frac{m}{s} - \frac{s-m}{s\lceil m/2(s-m)\rceil}\}$
- $\min\{\frac{m}{s}, \frac{s-m}{s\lfloor m/2(s-m)\rfloor}, \frac{m}{s} - \frac{s-m}{s\lfloor m/2(s-m)\rfloor}\}$
- $\min\left\{\frac{1}{\lceil 2s/m\rceil}, \frac{m}{s} - \frac{1}{\lceil 2s/m\rceil}, \frac{m}{2s}\right\}$
- $\min\left\{\frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{s} - \frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{2s}\right\}$

$$\Bigg\}.$$

THEOREM msLB

Theorem 0.3 *Let $m, s \in \mathsf{N}$. For all $1 \le x, y \le s$ let $A(x,y)$ be the least A such that $(m-y)$ divides $A(s-xy)$. Let $x, y \in \mathsf{N}$ such that $xy \le s$ and $m-y$ divides xy. Then there is a procedure that shows*

$$f(x,s) \ge \min\left\{\frac{1}{x}, \frac{mx-s}{sx}, \frac{m}{A(x,y)s}\right\}$$

Hence

$$f(x,s) = \max_{\{(x,y): xy \le s \wedge xy \equiv 0 \pmod{m-y}\}} \min\left\{\frac{1}{x}, \frac{mx-s}{sx}, \frac{m}{A(x,y)s}\right\}$$

PROGRAM2: USE THEOREM msUB and msLB to get upper and lower bounds on $f(m, s)$.

If succeed at finding x, y then output:

MS: $XXX \le f(m, s) \le YYY$. x: , y:

(That is, also list the parameters x and y.)

If fail then output:

MS: $f(m, s) \le YYY$. FAILED to find x,y. :-(

Even if XXX=YYY keep going to PROGRAM3 and PROGRAM 4 since I am curious what they'll say. Will prob take that out in the future.

0.1 Upper and Lower Bounds on $f(m,s)$ when $f(m,s)$ is Large

THEOREM deltaUB

Theorem 0.4

1. *Let* $m, s, p \in \mathsf{N}$ *Let* p *be the number of pieces in an optimal* (m,s)*-procedure. Then*

$$f(m,s) \le \min\left\{\frac{m}{s\left\lceil p/s\right\rceil}, 1-\frac{m}{s\left\lfloor p/s\right\rfloor}\right\}.$$

2. *If* $f(m,s) > \frac{1}{L}$ *and* $\frac{m}{s} \notin \mathsf{N}$ *then* $f(m,s) \le \max_{2m \le p \le (L-1)m} \min\left\{\frac{m}{s\lceil p/s\rceil}, 1-\frac{m}{s\lfloor p/s\rfloor}\right\}$.

3. *If* $f(m,s) > \frac{1}{3}$ *and* $\frac{m}{s} \notin \mathsf{N}$ *then* $f(m,s) \le \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}$

THEOREM delta1LB

Theorem 0.5 *Let* $m, s \in \mathsf{N}$ *and* $0 < \delta < \frac{1}{2} < 1-\delta < 1$. *Assume there exists* $x_1, y_1, x_2, y_2, z_1, z_2 \in \mathsf{N}$ *such that the following hold:*

1. $x_1 y_1 = x_2 y_2 \le m$

2. $x_1 + x_2 = s$

3. $z_1 x_1 + z_2 x_2 = 2(m - x_1 y_1) = 2(m - x_2 y_2)$.

4. $y_1 \delta + \frac{z_1}{2} = \frac{m}{s}$

5. $y_2 \delta + \frac{z_2}{2} = \frac{m}{s}$

Then $f(m,s) \ge \delta$.

THEOREM deltatwoLB

Theorem 0.6 *Let $m, s \in \mathsf{N}$ and $0 < \delta_1 < \delta_2 < \frac{1}{s} < 1 - \delta_2 < 1 - \delta_1 < 1$. Assume there exists: for $1 \le i \le 4$, x_i; for $1 \le i \le 4$, for $1 \le j \le 2$, y_{ij}, for $1 \le i \le 4$, z_i: such that the following hold:*

1. $x_1y_{11} + x_2y_{21} = x_3y_{31} + x_4y_{41}$. *(Number of δ_1-pieces equals the number of $(1-\delta_1)$ pieces.)*
2. $x_1y_{12} + x_3y_{32} = x_2y_{22} + x_4y_{42}$. *(Number of δ_2-pieces equals the number of $(1-\delta_2)$ pieces.)*
3. $2(m - x_1y_1) = z_1x_1 + z_2x_2$.
4. $x_1 + x_2 + x_3 + x_4 = s$.
5. $y_{11}\delta_1 + y_{12}\delta_2 + \frac{z_1}{2} = \frac{m}{s}$
6. $y_{21}\delta_1 + y_{22}(1-\delta_2) + \frac{z_2}{2} = \frac{m}{s}$
7. $y_{31}(1-\delta_1) + y_{32}\delta_2 + \frac{z_3}{2} = \frac{m}{s}$
8. $y_{41}(1-\delta_1) + y_{42}(1-\delta_2) + \frac{z_4}{2} = \frac{m}{s}$

Then $f(m, s) \ge \delta_1$.

PROGRAM HONEST 3: By PROGRAM2 we have $f(m,s) > YYY$. Find smallest L such that $YYY > 1/L$. Use that L to properly use THEOREM deltaUB.2 to find $f(m,s) > \delta$. (In the past I think you were just assuing $L = 3$ and that was probably always correct but lets do this right and see what happens.) I call this the honest lower bound approach. Then use Theorem deltaUB with this value of δ.

If succeed then output:

HONEST DELTA: $f(m,s) = \delta 1$. x1: x2: y1: y2: z1: z2: (THEN YOU ARE DONE)

If it fails output

HONEST DELTA: $f(m,s) > \delta 1$. FAILED TO FIND xi,yi,zi :-(

PROGRAM HONEST 4: If PROGRAM HONEST 3 failes then try to find procedure using THEOREM deltatwoUB. To use THEOREM deltatwoUB you need to go over all possible value of $\delta 2$ with same denom as $\delta 1$.

If succeed then output (and this includes stuff from PROGRAM 3)

HONEST DELTATWO: $f(m,s) = \delta 1$. x1: x2: x3: x4: y11: y12: y13: y14: y21: y22: y23: y24: z1: z2: z3: z4:

(WE might look into making this more compact later)

If it fails output

DELTATWO: $f(m,s) > \delta 1$. FAILED TO FIND xi,yij,zi. YOU ARE A BIG FAT FAILURE!!!

TWO THOUGHTS FOR LATER:

1) We currently do the following: If $delta1 = 11/30$ then we look at $delta2 = 12/30, 13/30/14/30$. BUT if we instead took $delta1 = 22/60$ then could look at $delta2 = 23/60, \ldots, 29/60$.

2) We could look at using $\delta 1, \delta 2, \delta 3$.

BOTH (1) and (2) would be easy to do, but lets first see how far we get as is for $s = 5$.

PROGRAM DISHONEST 5: If $L = 3$ in the above then DO NOT DO THIS STEP, you have already failed. If $L \geq 4$ then Calculate δ AS IF $L = 3$ and then do PROGRAM HONEST 3, PROGRAM HONEST 4 from there, though with a dishonest value for δ.