The Muffin Problem

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1 Introduction

The following problem, and similar ones, appeared in the *Julia Robinson Mathematics Festival*. These problems were proposed by Alan Frank [1].

You have 5 muffins and 3 students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?

Here is a procedure:

1. Divide M_1 and M_2 into $(\frac{1}{3}, \frac{2}{3})$.

2. M_3, M_4, M_5 are not cut. We call them *1-sized pieces*.

3. S_1 and S_2 each get a 1-sized piece and a $\frac{2}{3}$-sized piece.

4. S_3 gets a 1-sized piece and two $\frac{1}{3}$-sized pieces.

The smallest piece in the above solution is $\frac{1}{3}$. Can we do better? Theorem 3.1 will show that we can. Here is the general muffin problem:
You have m muffins and s students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?

Def 1.1 Let $m, s \in \mathsf{N}$. An (m, s)*-procedure* is a procedure to cut m muffins into pieces and then distribute them to the s students so that each student gets m/s. An (m, s)-procedure is *optimal* if it has the largest smallest piece of any procedure. $f(m, s)$ be the smallest piece in an optimal (m, s)-procedure.

2 General Theorems

2.1 Easy Cases

Theorem 2.1

1. *If* $m \equiv 0 \pmod{s}$ *then* $f(m,s) = 1$.

2. *If* $s \equiv 0 \pmod{2m}$ *and* $\frac{m}{s} \notin \mathsf{N}$ *then* $f(m,s) = \frac{1}{2}$.

3. *If* $m, s \in \mathsf{N}$ *then* $f(m,s) \geq \frac{1}{s}$.

4. $f(1,s) = \frac{1}{s}$.

5. *If* s *is odd then* $f(2,s) = \frac{1}{s}$.

6. *ALEX- I SUSPECT THIS APPROACH WILL NEVER GIVE OPT BOUNDS. BUT MIGHT BE USEFUL IN INITIALL SHOWING* $f(m,s) > \frac{1}{3}$.

   If $m \geq Ls$ *then* $f(m,s) \geq f(m - Ls, s)$.

Proof:

1) No muffin is cut. Give everyone $\frac{m}{s}$ muffins.

2) $s \equiv 0 \pmod{2m}$. The following procedure shows $f(m,s) \leq \frac{1}{2}$.

1. Divide $M_1, \ldots, M_m$ into $(\frac{1}{2}, \frac{1}{2})$.

2. $S_1, \ldots, S_s$ each get $\frac{2m}{s}$ $\frac{1}{2}$-sized pieces.

Since $\frac{m}{s} \notin \mathsf{N}$ some muffin is cut. Hence $f(m,s) \leq \frac{1}{2}$.

3) The following procedure shows $f(m,s) \geq \frac{1}{s}$.

1. Divide $M_1, \ldots, M_m$ into $(\frac{1}{s}, \ldots, \frac{1}{s})$.

2. $P_1, \ldots, P_s$ each get m $\frac{1}{s}$-sized pieces.

4) $f(1,s) = \frac{1}{s}$: By part 3 $f(1,s) \geq \frac{1}{s}$. Since any procedure will give each student $\frac{1}{s}$ muffins, $f(1,s) \leq \frac{1}{s}$.

5) By Part 3 $f(2,s) \geq \frac{1}{s}$. Assume there is a $(2,s)$-procedure. Let N be the size of the smallest piece produced.

Case 1: Some student gets ≥ 2 pieces. Then $N \leq \frac{2}{2s} = \frac{1}{s}$.

Case 2: Every student gets 1 piece. Each piece must be of size $\frac{2}{s}$. Let M be a muffin. It is cut into x (note $x \in \mathsf{N}$) pieces of size $\frac{2}{s}$. Hence $2x/s = 1$ so $x = s/2$. Since s is odd $x \notin \mathsf{N}$. This cannot happen.

6) The following procedure shows $f(m,s) \geq f(m - Ls, s)$.

1. For $1 \leq i \leq s$ S_i gets L muffins. Note that there are $m - Ls$ muffins left

2. Apply the optimal (m,s)-procedure to divide the remaining $m - Ls$ muffins.

∎

2.2 Upper and Lower Bounds on $f(m,s)$ when $f(m,s)$ is Small

The following theorems seem to be useful when $f(m,s)$ is small. Also note that they can be applied just knowing m, s which will be a contrast to Theorems 2.10,2.11 later.

Theorem 2.2 *Let $m, s \in \mathsf{N}$ such that s does not divide m. Then*

$$f(m,s) \leq \max\Bigg\{$$

- $\frac{m}{3s}$,

- $\min\{\frac{m}{s}, \frac{s-m}{s\lceil m/2(s-m)\rceil}, \frac{m}{s} - \frac{s-m}{s\lceil m/2(s-m)\rceil}\}$
- $\min\{\frac{m}{s}, \frac{s-m}{s\lfloor m/2(s-m)\rfloor}, \frac{m}{s} - \frac{s-m}{s\lfloor m/2(s-m)\rfloor}\}$
- $\min\left\{\frac{1}{\lceil 2s/m\rceil}, \frac{m}{s} - \frac{1}{\lceil 2s/m\rceil}, \frac{m}{2s}\right\}$
- $\min\left\{\frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{s} - \frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{2s}\right\}$

$$\Bigg\}.$$

Proof: Assume there is an (m,s)-procedure. Let N be the size of the smallest piece produced.

Case 1: Some student gets ≥ 3 pieces. Then $N \le \frac{m}{3s}$.

Case 2: Some student gets 1 piece which we call P_1. P_1 is of size $\frac{m}{s}$. Say P_1 came from muffin M. Let $P_2 = M - P_1$. P_2 is of size $1 - \frac{m}{s}$. P_2 is cut into x pieces (x could be 1). There is a piece P_3 of size $\le \frac{1}{x} - \frac{m}{sx} = \frac{s-m}{sx}$. There is a piece of size $P_4 \ge \frac{s-m}{sx}$. Some student gets P_4 together with some other piece P_5 (its possible P_5 is size 0). P_5 has size $\le \frac{m}{s} - \frac{s-m}{sx}$.

Looking at P_1 P_3, and P_5 we have that

$$N \le \min\{\frac{m}{s}, \frac{s-m}{sx}, \frac{m}{s} - \frac{s-m}{sx}\}$$

This is minimized when

$$\frac{s-m}{sx} = \frac{m}{s} - \frac{s-m}{sx}$$

$$\frac{2(s-m)}{sx} = \frac{m}{s}$$

$$\frac{2(s-m)}{x} = m$$

$$2(s-m) = mx$$

$$x = \frac{m}{2(s-m)}.$$

Alas, this value of x is not an integer! Hence we take both its floor and its ceiling.

Case 2a:

$$N \le \min\{\frac{m}{s}, \frac{s-m}{s\lceil m/2(s-m)\rceil}, \frac{m}{s} - \frac{s-m}{s\lceil m/2(s-m)\rceil}\}$$

Case 2b:

$$N \le \min\{\frac{m}{s}, \frac{s-m}{s\lfloor m/2(s-m)\rfloor}, \frac{m}{s} - \frac{s-m}{s\lfloor m/2(s-m)\rfloor}\}$$

Case 3: Every student gets exactly two pieces. Hence every student has a piece P of size $\le \frac{m}{2s}$. Since s does not divide m there is an $2 \le x \le s$ such that some muffin is cut into x pieces. Hence there is a piece P_1 of size $\le \frac{1}{x}$ and a piece P_2 of size $\ge \frac{1}{x}$. Some student gets P_2 along with at least one piece P_3 of size $\le \frac{m}{s} - \frac{1}{x}$. Looking at P_1, P_3 to get

$$N \le \min\left\{\frac{1}{x}, \frac{m}{s} - \frac{1}{x}, \frac{m}{2s}\right\}$$

This is maximized when

$\frac{1}{x} = \frac{m}{s} - \frac{1}{x}$

$\frac{2}{x} = \frac{m}{s}$

$\frac{x}{2} = \frac{s}{m}$

$x = \frac{2s}{m}$

Alas, this value of x is not an integer! Hence we take both its floor and its ceiling.

Case 3a:

$$N \le \min\left\{\frac{1}{\lceil 2s/m\rceil}, \frac{m}{s} - \frac{1}{\lceil 2s/m\rceil}, \frac{m}{2s}\right\}$$

Case 3b:

$$N \le \min\left\{\frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{s} - \frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{2s}\right\}$$

∎

We now prove that if the lower bounds from Theorem 2.2 is also a lower bound then $f(m,s) = f(am,as)$.

Def 2.3 $g(m,s)$ be the upper bound on $f(m,s)$ from Theorem 2.2.

We leave the proof of the following theorem to the reader.

Theorem 2.4 *Let $a,m,s \in \mathsf{N}$.*

1. *$f(m,s) \le g(m,s) = g(am,as)$.*
2. *$f(m,s) \le f(am,as)$.*
3. *If $f(m,s) = g(m,s)$ then $f(m,s) = f(am,as)$. So if $f(m,s)$ matches the lower bound from Theorem 2.2 then $f(m,s) = f(am,as)$.*

Theorem 2.5 *Let $m,s \in \mathsf{N}$. For all $1 \le x,y \le s$ let $A(x,y)$ be the least A such that $(m-y)$ divides $A(s-xy)$. Let $x,y \in \mathsf{N}$ such that $xy \le s$ and $m-y$ divides xy. Then there is a procedure that shows*

$$f(x,s) \ge \min\left\{\frac{1}{x}, \frac{mx-s}{sx}, \frac{m}{A(x,y)s}\right\}$$

Hence $f(x,s)$ is the max over all such x,y of this quantity.

Proof: Let x, y be as in the premise. Let $A = A(x, y)$.

Consider the following procedure.

1. Divide $M_1, \ldots, M_y$ into $(\frac{1}{x}, \ldots, \frac{1}{x})$

   (There are xy pieces of size $\frac{1}{x}$.)

2. Divide each of $M_{y+1}, \ldots, M_m$ into $\frac{xy}{m-y}$ pieces of size $\frac{mx-s}{sx}$ and $\frac{A(s-xy)}{m-y}$ pieces of size $\frac{m}{As}$. (There are $(m-y)\frac{xy}{m-y} = xy$ pieces of size $\frac{mx-s}{sx}$ and $(m-y)\frac{A(s-xy)}{m-y} = A(s-xy)$ pieces of size $\frac{m}{As}$.)

3. $S_1, \ldots, S_{xy}$ each get one $\frac{1}{x}$-sized piece and one $\frac{mx-s}{sx}$-sized piece.

4. $S_{xy+1}, \ldots, S_s$ each get A $\frac{m}{As}$-sized piece.

Clearly

$$f(m, s) \geq \min\left\{\frac{1}{x}, \frac{mx-s}{sx}, \frac{m}{As}\right\}.$$

∎

2.3 Upper and Lower Bounds on $f(m, s)$ when $f(m, s)$ is Large

The following theorems seem to be useful when $f(m, s)$ is large.

The following theorem gives an upper bound on $f(m, s)$. It has the drawback of needing to know the number of pieces in the optimal (m, s)-procedure. We deal with this by first getting a bound on the number of pieces p and then going through all possible p. This bound on p needs a lower bound on $f(m, s)$. In practice we cut this gordian knot by using Theorem 2.5 to get an initial lower bound on $f(m, s)$.

Lemma 2.6 *Let $f(m,s) > \frac{1}{L}$ and $\frac{m}{s} \notin \mathsf{N}$. There is a procedure with min piece of size $f(m,s)$ such (a) every muffin is cut into either 2 or 3 or $\cdots$ or $L-1$ pieces, (b) the number of pieces created is between $2m$ and $(L-1)m$.*

Proof: Since $\frac{m}{s} \notin \mathsf{N}$ some muffin is cut. Hence $f(m,s) \le \frac{1}{2}$. If in the original procedure there is an uncut muffin that goes to a student then modify the procedure by dividing it $(\frac{1}{2}, \frac{1}{2})$ and giving both halves to that student. If in the original procedure there is a muffin cut into $\ge L$ pieces then there will be some piece of size $\frac{1}{L}$ which contradicts $f(m,s) > \frac{1}{L}$. ∎

Theorem 2.7

1. *Let $m, s, p \in \mathsf{N}$ Let p be the number of pieces in an optimal (m,s)-procedure. Then*

$$f(m,s) \le \min\left\{\frac{m}{s\lceil p/s\rceil}, 1 - \frac{m}{s\lfloor p/s\rfloor}\right\}.$$

2. *If $f(m,s) > \frac{1}{L}$ and $\frac{m}{s} \notin \mathsf{N}$ then*

$$f(m,s) \le \max_{2m \le p \le (L-1)m} \min\left\{\frac{m}{s\lceil p/s\rceil}, 1 - \frac{m}{s\lfloor p/s\rfloor}\right\}.$$

3. *If $f(m,s) > \frac{1}{3}$ and $\frac{m}{s} \notin \mathsf{N}$ then*

$$f(m,s) \le \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}$$

Proof: Parts 2 and 3 follow from part 1 so we just prove part 1.

1) Since the smallest piece is of size $f(m,s)$, the largest piece is of size $\le 1 - f(m,s)$.

Since there p pieces and s students we can assume that (1) S_1 gets $\ge \lceil p/s\rceil$ pieces so he gets least $f(m,s)\lceil p/s\rceil$ muffins; hence $f(m,s)\lceil p/s\rceil \le \frac{m}{s}$, and (2) S_2 gets $\le \lfloor p/s\rfloor$ pieces so he gets at most $(1 - f(m,s))\lceil p/s \rceil$ muffins; hence $(1 - f(m,s))\lceil p/s \rceil \geq \frac{m}{s}$. The inequalities $f(m,s) \leq \frac{m}{s\lceil p/s \rceil}$ and $f(m,s) \leq 1 - \frac{m}{s\lfloor p/s \rfloor}$ follow. ▮

We now prove that if either of our lower bounds are also upper bounds then $f(m,s) = f(am,as)$.

Def 2.8 Let $h(m,s)$ be the upper bound on $f(m,s)$ from Theorem 2.7.2.

We leave the proof of the following theorem to the reader.

Theorem 2.9 *Let $a,m,s \in \mathsf{N}$.*

1. $f(m,s) \leq h(m,s) = h(am,as)$.
2. $f(m,s) \leq f(am,as)$.
3. *If $f(m,s) = h(m,s)$ then $f(m,s) = f(am,as)$. So if $f(m,s)$ matches the lower bound from Theorem 2.7.2 then $f(m,s) = f(am,as)$.*

Theorem 2.10 *Let $m,s \in \mathsf{N}$ and $0 < \delta < \frac{1}{2} < 1-\delta < 1$. Assume there exists $x_1, y_1, x_2, y_2, z_1, z_2 \in \mathsf{N}$ such that the following hold:*

1. $x_1 y_1 = x_2 y_2 \leq m$
2. $x_1 + x_2 = s$
3. $z_1 x_1 + z_2 x_2 = 2(m - x_1 y_1) = 2(m - x_2 y_2)$.
4. $y_1 \delta + \frac{z_1}{2} = \frac{m}{s}$
5. $y_2 \delta + \frac{z_2}{2} = \frac{m}{s}$

Then $f(m,s) \geq \delta$.

Proof:

The idea is that

- x_1 people will get y_1 δ-sized pieces and z_1 $\frac{1}{2}$-sized pieces.
- x_2 people will get y_2 $(1-\delta)$-sized pieces and z_2 $\frac{1}{2}$-sized pieces.

The following procedure show $f(m, s) \geq \delta$.

1. Divide $M_1, \ldots, M_{x_1 y_1}$ into $(\delta, 1-\delta)$.

   (There are $x_1 y_1$ δ-sized pieces, and $x_2 y_2$ $(1-\delta)$-sized pieces.)

2. Divide $M_{x_1 y_1+1}, \ldots, M_m$ into $(\frac{1}{2}, \frac{1}{2})$.

   (There are $2(m - x_1 y_1) = z_1 x_1 + z_2 x_2$ $\frac{1}{2}$-pieces.)

3. $S_1, \ldots, S_{x_1}$ each get y_1 δ-sized pieces and z_1 $\frac{1}{2}$-sized pieces.

   (Each get $y_1 \delta + \frac{z_1}{2} = \frac{m}{s}$.)

4. $S_{x_1+1}, \ldots, S_{x_1+x_2}$ each get y_2 $(1-\delta)$-sized pieces and z_2 $\frac{1}{2}$-sized pieces.

   (Each get $y_2 \delta + \frac{z_2}{2} = \frac{m}{s}$.)

∎

Theorem 2.11 *Let $m, s \in \mathsf{N}$ and $0 < \delta_1 < \delta_2 < \frac{1}{s} < 1 - \delta_2 < 1 - \delta_1 < 1$. Assume there exists: for $1 \leq i \leq 4$, x_i; for $1 \leq i \leq 4$, for $1 \leq j \leq 2$, y_{ij}, for $1 \leq i \leq 4$, z_i: such that the following hold:*

1. $x_1 y_{11} + x_2 y_{21} = x_3 y_{31} + x_4 y_{41}$. *(Number of δ_1-pieces equals the number of $(1-\delta_1)$ pieces.)*

2. $x_1 y_{12} + x_3 y_{32} = x_2 y_{22} + x_4 y_{42}$. *(Number of δ_2-pieces equals the number of $(1-\delta_2)$ pieces.)*

3. $2(m - x_1 y_1) = z_1 x_1 + z_2 x_2$.

4. $x_1 + x_2 + x_3 + x_4 = s.$

5. $y_{11}\delta_1 + y_{12}\delta_2 + \frac{z_1}{2} = \frac{m}{s}$

6. $y_{21}\delta_1 + y_{22}(1-\delta_2) + \frac{z_2}{2} = \frac{m}{s}$

7. $y_{31}(1-\delta_1) + y_{32}\delta_2 + \frac{z_3}{2} = \frac{m}{s}$

8. $y_{41}(1-\delta_1) + y_{42}(1-\delta_2) + \frac{z_4}{2} = \frac{m}{s}$

Then $f(m,s) \geq \delta_1$.

Proof:

The idea is:

- x_1 people will get y_{11} δ_1-sized pieces, y_{12} δ_2-sized pieces, and z_1 $\frac{1}{2}$-sized pieces.
- x_2 people will get y_{21} δ_1-sized pieces, y_{22} $(1-\delta_2)$-sized pieces, and z_2 $\frac{1}{2}$-sized pieces.
- x_3 people will get y_{31} $(1-\delta_1)$-sized pieces, y_{32} δ_2-sized pieces, and z_3 $\frac{1}{2}$-sized pieces.
- x_4 people will get y_{41} $(1-\delta_1)$-sized pieces, y_{42} $(1-\delta_2)$-sized pieces, and z_4 $\frac{1}{2}$-sized pieces.

The following procedure show $f(m,s) \geq \delta$.

1. Divide $M_1, \ldots, M_{x_1y_1}$ into $(\delta, 1-\delta)$.

   (There are x_1y_1 δ-sized pieces, and x_2y_2 $(1-\delta)$-sized pieces.)

2. Divide $M_{x_1y_1+1}, \ldots, M_m$ into $(\frac{1}{2}, \frac{1}{2})$.

   (There are $2(m - x_1y_1) = z_1x_1 + z_2x_2$ $\frac{1}{2}$-pieces.)

3. $S_1, \ldots, S_{x_1}$ each get y_{11} δ_1-sized pieces, y_{12} δ_2-sized pieces, z_1 $\frac{1}{2}$-sized pieces. (Each get $y_{11}\delta_1 + y_{12}\delta_2 + \frac{z_1}{2} = \frac{m}{s}$.)

4. $S_{x_1+1}, \ldots, S_{x_1+x_2}$ each get y_{21} δ_1-sized pieces and y_{22} $(1-\delta_2)$-sized z_2 $\frac{1}{2}$-sized pieces. (Each get $y_{21}\delta_1 + y_{22}(1 = \delta_2) + \frac{z_2}{2} = \frac{m}{s}$.)

5. $S_{x_1+x_2+1}, \ldots, S_{x_1+x_2+x_3}$ each get y_{31} $(1-\delta_1)$-sized pieces and y_{32} δ_2–sized pieces, and z_3 $\frac{1}{2}$-sized pieces. (Each get $y_{31}(1-\delta_1) + y_{32}(\delta_2) + \frac{z_2}{2} = \frac{m}{s}$.)

6. $S_{x_1+x_2+x_3+1}, \ldots, S_{x_1+x_2+x_3+x_4}$ each get y_{41} $(1-\delta_1)$-sized pieces and y_{42} $(1-\delta_2)$–sized pieces, and z_4 $\frac{1}{2}$-sized pieces. (Each get $y_{41}(1-\delta_1) + y_{42}(1-\delta_2) + \frac{z_2}{2} = \frac{m}{s}$.)

∎

3 Five Muffins, Three Students

In the introduction we showed that $f(5,3) \geq \frac{1}{3}$. We show that $f(5,3) = \frac{5}{12}$.

Theorem 3.1 $f(5,3) = \frac{5}{12}$.

Proof:

The following procedure shows $f(5,3) \geq \frac{5}{12}$.

1. Divide M_1, M_2, M_3, M_4 into $(\frac{5}{12}, \frac{7}{12})$.
2. Divide M_5 into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1 and S_2 each get two of the $\frac{7}{12}$-sized pieces and one of the $\frac{1}{2}$-sized pieces.
4. S_3 gets four $\frac{5}{12}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{5}{12} > \frac{1}{3}$ so it applies,

$$f(5,3) \leq \min\left\{\frac{5}{3 \times \lceil 10/3 \rceil}, 1 - \frac{5}{3 \times \lfloor 10/3 \rfloor}\right\} = \min\left\{\frac{5}{12}, 1 - \frac{5}{9}\right\} = \frac{5}{12}$$

∎

BILL - WRITE A NOTE HOW YOU COULD GET THIS FROM THE DELTA THEOREMS

4 m Muffins, Three Students

Theorem 4.1

0) If $m \equiv 0 \pmod 3$ *then* $f(m,3) = 1$.

1) $f(1,3) = \frac{1}{3}$. *If* $m \equiv 1 \pmod 3$ *and* $m = 3k+1$, *with* $k \geq 1$, *then* $f(m,3) = \frac{3k-1}{6k}$.

2) If $m \equiv 2 \pmod 3$ *and* $m = 3k+2$, *with* $k \geq 0$, *then* $f(m,3) = \frac{3k+2}{6k+6}$.

Proof:

For parts 2 and 3 we use Theorem 2.7.2 to obtain an upper bound on $f(m,s)$. To apply this we need that $f(m,s) > \frac{1}{3}$. This is the case for every $f(m,s)$ except in part 2 with $k = 0$. In this case we have $f(m,s) \geq \frac{1}{3}$; therefore we could structure the proof as a proof by contradiction.

0) This follows from Theorem 2.1.1.

1a) $f(1,3) = \frac{1}{3}$ by Theorem 2.1.4

1b) $m = 3k+1$ with $k \geq 1$. The following procedure shows $f(m,3) \geq \frac{3k-1}{6k}$.

1. Divide $M_1, \ldots, M_{2k}$ into $(\frac{3k-1}{6k}, \frac{3k+1}{6k})$.
2. Divide $M_{2k+1}, \ldots, M_{3k+1}$ into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1 gets $2k$ of the $\frac{3k+1}{6k}$-sized pieces.
4. S_2 and S_3 each get k of the $\frac{3k-1}{6k}$-sized pieces and $k+1$ of the $\frac{1}{2}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{3} = \frac{6k+2}{3} = 2k + \frac{2}{3}$.

$$f(3k+1,3) \leq \min\left\{\frac{3k+1}{3(2k+1)}, 1 - \frac{3k+1}{3(2k)}\right\} = \frac{3k-1}{6k}.$$

2) $m = 3k+2$. The following procedure shows $f(m,3) \geq \frac{3k+2}{6k+6}$.

1. Divide $M_1, \ldots, M_{2k+2}$ into $(\frac{3k+2}{6k+6}, \frac{3k+4}{6k+6})$.

2. Divide $M_{2k+3}, \ldots, M_{3k+2}$ into $(\frac{1}{2}, \frac{1}{2})$.

3. S_1 gets $2k+2$ of the $\frac{3k+2}{6k+6}$-sized pieces.

4. S_2 and S_3 each get $k+1$ of the $\frac{3k+4}{6k+6}$-sized pieces and k $\frac{1}{2}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{3} = \frac{6k+4}{3} = 2k + \frac{4}{3}$,

$$f(3k+2, 3) \leq \min\left\{ \frac{3k+2}{3(2k+2)}, 1 - \frac{3k+2}{3(2k+1)} \right\} = \frac{3k+2}{6k+6}$$

■

5 m Muffins, Four Students

Theorem 5.1

0) If $m \equiv 0 \pmod 4$ *then* $f(m, 4) = 1$.

1) $f(1, 4) = \frac{1}{4}$. *If* $m \equiv 1 \pmod 4$ *and* $m = 4k+1$, *with* $k \geq 1$, *then* $f(m, 4) = \frac{4k-1}{8k}$.

2) If $m \equiv 2 \pmod 4$ *then* $f(m, 4) = \frac{1}{2}$.

3) If $m \equiv 3 \pmod 4$ *and* $m = 4k+3$ *then* $f(m, 4) = \frac{4k+1}{8k+4}$.

Proof:

For parts 2 and 3 we use Theorem 2.7.2 to obtain an upper bound on $f(m, s)$. To apply this we need that $f(m, s) > \frac{1}{3}$. There is one case where it does not apply. We mention that when it happens.

0) This follows from Theorem 2.1.1.

0) This follows from Theorem 2.1.1.

1a) $f(1, 4) = \frac{1}{4}$ by Theorem 2.1.4.

1b) 1) $m = 4k+1$. The following procedure shows $f(m, 4) \geq \frac{4k-1}{8k}$.

1. Divide $M_1, \ldots, M_{4k}$ into $(\frac{4k-1}{8k}, \frac{4k+1}{8k})$.

2. Divide M_{4k+1} into $(\frac{1}{2}, \frac{1}{2})$.

3. S_1 and S_2 each get $2k$ of the $\frac{4k+1}{8k}$-sized pieces.

4. S_3 and S_4 each get $2k$ of the $\frac{4k-1}{8k}$-sized pieces and one of the $\frac{1}{2}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{4} = \frac{8k+2}{4} = 2k + \frac{1}{2}$,

$$f(4k+1, 4) \leq \min\left\{\frac{4k+1}{4(2k+1)}, 1 - \frac{4k+1}{4(2k)}\right\} = \frac{4k-1}{8k}$$

2) This follows from Theorem 2.1.2.

3) $m = 4k+3$. The following procedure shows $f(m, 4) \geq \frac{4k+1}{8k+4}$.

1. Divide $M_1, \ldots, M_{4k+2}$ into $(\frac{4k+1}{8k+4}, \frac{4k+3}{8k+4})$.

2. Divide M_{4k+3} into $(\frac{1}{2}, \frac{1}{2})$.

3. S_1 and S_2 each get $2k+1$ of the $\frac{4k+3}{8k+4}$-sized pieces.

4. S_3 and S_4 each get $2k+1$ of the $\frac{4k+1}{8k+4}$-sized pieces and one of the $\frac{1}{2}$-sized pieces.

If $k \geq 1$ then $f(4k+3) > \frac{1}{3}$ then, by Theorem 2.7.2, noting that $\frac{2m}{4} = \frac{8k+6}{4} = 2k+1+\frac{1}{2}$,

$$f(4k+3, 4) \leq \min\left\{\frac{4k+3}{4(2k+2)}, 1 - \frac{4k+3}{4(2k+1)}\right\} = \frac{4k+1}{8k+4}$$

If $k = 0$ then by Theorem 2.2

$$f(3, 4) \leq \max\left\{1 - \frac{3}{4}, \frac{3}{3 * 4}, \frac{3}{4} - \frac{1}{2}\right\} = \frac{1}{4}$$

∎

6 m Muffins, Five Students

We first prove a general theorem that covers many but not all cases.

Theorem 6.1 *The following table gives the value of $f(m,5)$ depending on what m is mod 30.*

1. *If 5 divides m then the table says 1 and nothing is put in the other columns.*
2. *If Theorem 2.10 applies then we give the upper bound and the parameters $x_1, x_2, y_1, y_2, z_1, z_2$. In all of these cases the upper bound equals the lower bound.*
3. *If Theorem 2.11 applies then we give the upper bound and the parameters $1 \le i \le 4$, x_i; for $1 \le i \le 4$, for $1 \le j \le 2$, y_{ij}, for $1 \le i \le 4$, z_i: In all of these cases the upper bound equals the lower bound.*
4. *The table is mod 30. Some of the results need a finer subdivision. In these cases we note this and give the finer subdivision.*
5. *If 5 does not divide s and Theorem 2.11 does not apply we just note that its hard.*

BILL- CHECK EDGE CASES, $L = 0$ OR $L = 1$.

All of the bounds below hold when $L \ge 1$.

m	$f(m,s)$	x_1	x_2	y_1	y_2	y_3	z_1	z_2	*Comment*
$30L$	1								$m \equiv 0 \pmod{s}$
$30L+1$									*Hard*
$30L+2$		2	3	0	$8L$	$12L$	0	$12L+4$	
$30L+3$									*Hard*
$30L+4$									*Hard*
$30L+5$	1								$m \equiv 0 \pmod{s}$
$30L+6$									*3 cases*
$90L+6$		3	2	$6L$	$36L+3$	$36L+2$	0	0	
$90L+36$		3	2	$6L+2$	$36L+15$	$30L+12$	0	0	
$90L+66$		2	3	$6L+4$	$16L+12$	$30L+22$	0	$60L+44$	
$30L+7$									*Hard*
$30L+8$									*2 cases*
$60L+8$		4	1	$9L+1$	$24L+4$	$15L+2$	0	0	
$60L+38$		3	2	$9L+5$	$9L+6$	$15L+9$	3	$30L+19$	
$30L+9$		2	3	0	$8L+2$	$12L+3$	0	$12L+6$	
$30L+10$	1								$m \equiv 0 \pmod{s}$
$30L+11$									*Hard*
$30L+12$									*4 cases*
$120L+12$		4	1	$18L+2$	$48L+4$	$30L+3$	0	0	
$120L+42$		2	3	$15L+5$	$12L+4$	$33L+11$	1	$108L+39$	
$120L+72$		4	1	$18L+11$	$48L+28$	$30L+18$	0	0	
$120L+102$		2	3	$15L+12$	$12L+10$	$33L+27$	3	$108L+93$	

m	$f(m,s)$	x_1	x_2	y_1	y_2	y_3	z_1	z_2	*Comment*
$30L+13$		3	2	$2L$	$12L+6$	$10L+4$	2	0	
$30L+14$									*Hard*
$30L+15$	1								$m \equiv 0 \pmod{s}$
$30L+16$									*Hard*
$30L+17$		2	3	0	$8L+4$	$12L+6$	0	$12L+10$	
$30L+18$									*2 cases*
$60L+18$		3	2	$9L+2$	$9L+3$	$15L+4$	3	$30L+9$	
$60L+48$		4	1	$9L+7$	$24L+20$	$15L+12$	0	0	
$30L+19$									*Hard*
$30L+20$	1								$m \equiv 0 \pmod{s}$
$30L+21$									*3 cases*
$90L+21$		2	3	$6L+1$	$16L+4$	$30L+7$	0	$60L+14$	
$90L+51$		3	2	$6L+3$	$36L+21$	$30L+17$	0	0	*Check*
$90L+81$		3	2	$6L+5$	$36L+33$	$30L+27$	0	6	*Check*
$30L+22$									
$120L+22$									
$120L+52$									
$120L+82$									
$120L+112$									
$30L+23$									*need data*
$30L+24$		2	3	0	$8L+6$	$12L+9$	0	$12L+12$	
$30L+25$	1								$m \equiv 0 \pmod{s}$
$30L+26$									*Hard*
$30L+27$									*Hard*
$30L+28$									*need data*
$30L+29$									*Hard*

Theorem 6.2

1. *If* $m \equiv 0 \pmod 5$ *then* $f(m, s) = 1$.
2. $f(1, 5) = \frac{1}{5}$.
3. $f(2, 5) = \frac{1}{5}$
4. $f(3, 5) = \frac{1}{4}$
5. $f(4, 5) = \frac{3}{10}$
6. $f(6, 5) = \frac{2}{5}$
7. $f(7, 5) = \frac{1}{3}$
8. $f(8, 5) = \frac{2}{5}$
9. $f(9, 5) = \frac{2}{5}$
10. $\frac{13}{30} \le f(11, 5) \le \frac{11}{25}$.
11. $f(12, 5) = \frac{2}{5}$
12. $f(13, 5) = \frac{13}{30}$
13. $f(14, 5) = \frac{11}{25}$
14. $f(16, 5) = \frac{16}{35}$
15. $f(17, 5) = \frac{13}{30}$
16. $f(18, 5) = \frac{9}{20}$
17. $f(19, 5) = \frac{16}{35}$
18. $f(21, 5) = \frac{7}{15}$

19. $f(22,5) = \frac{9}{20}$

Proof:

1) This follows from Theorem 2.1.1.

2) This follows from Theorem 2.1.5.

3) The following procedure shows $f(3,5) \geq \frac{1}{4}$.

1. Divide M_1 into $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.
2. Divide M_2, M_3 into $(\frac{3}{10}, \frac{7}{20}, \frac{7}{20})$.
3. S_1, S_2, S_3, S_4 each get one of the $\frac{1}{4}$-sized pieces and one of the $\frac{7}{20}$-sized pieces.
4. S_5 gets two of the $\frac{3}{10}$-sized pieces.

$f(3,5) \leq \frac{1}{4}$ by Theorem 2.2.

4) The following procedure shows $f(4,5) \geq \frac{3}{10}$.

1. Divide M_1, M_2 into $(\frac{3}{10}, \frac{3}{10}, \frac{2}{5})$.
2. Divide M_3, M_4 into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1, S_2, S_3, S_4 each get a $\frac{3}{10}$-sized piece and a $\frac{1}{2}$-sized piece.
4. S_5 gets two of the $\frac{2}{5}$-sized pieces.

By Theorem 2.2 $f(4,5) \leq \frac{3}{10}$.

5) This follows from Theorem 2.10 with $m = 6$, $s = 5$, $\delta = \frac{2}{5}$, $x_1 = 2$, $y_1 = 3$, $x_2 = 3$, $y_2 = 2$.

BILL- CHECK THIS, THOUGH WILL LIKELY DELETE ANYWAY.

6) The following procedure shows $f(7,5) \geq \frac{1}{3}$.

1. Divide $M_1, \ldots, M_6$ into $(\frac{7}{15}, \frac{8}{15})$.

2. Divide M_7 into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

3. S_1, S_2 each get three $\frac{7}{15}$-sized pieces.

4. $S_3, \ldots, S_7$ each get two $\frac{1}{3}$-sized pieces and two $\frac{8}{15}$-sized pieces.

To show $f(7,5) \le \frac{1}{3}$ there are two cases.

Case 1: An optimal division cuts some muffin into ≥ 3 pieces. One of those pieces will be of size $\le \frac{1}{3}$, hence $f(7,5) \le \frac{1}{3}$.

Case 2: An optimal division cuts all muffins into ≤ 2 pieces. We can assume that every muffin is in exactly 2 pieces since if a muffin is not cut we can still cut it $(\frac{1}{2}, \frac{1}{2})$. Hence there are 10 pieces. By Theorem 2.7.2, noting that $\frac{2m}{5} = \frac{14}{5} = 2 + \frac{4}{5}$,

$$f(7,5) \le \min\left\{\frac{7}{5 \times 3}, 1 - \frac{7}{5 \times 2}\right\} = \frac{3}{10}$$

Since we have a division where the smallest piece is $\frac{1}{3}$ this case does not happen.

7) The following procedure shows $f(8,5) \ge \frac{2}{5}$.

ALEX- I CAN GET A GENERAL THEOREM FROM WHICH THIS FALLS OUT.

1. Divide $M_1, \ldots, M_8$ into $(\frac{2}{5}, \frac{3}{5})$.

2. S_1, S_2, S_3, S_4 each get one $\frac{2}{5}$-sized pieces and two $\frac{3}{5}$-sized piece.

3. S_5 gets four $\frac{2}{5}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{5} = \frac{16}{5} = 3 + \frac{1}{5}$,

$$f(8,5) \le \min\left\{\frac{8}{5 \times 4}, 1 - \frac{8}{5 \times 3}\right\} = \frac{2}{5}.$$

8) The following procedure shows $f(9,5) \geq \frac{2}{5}$.

ALEX- I WILL HAVE A GEN THEOREM FROM WHICH THIS FALLS OUT.

1. Divide $M_1, \ldots, M_6$ into $(\frac{2}{5}, \frac{3}{5})$.
2. Divide M_7, M_8, M_9 into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1, S_2 each get three $\frac{3}{5}$-sized piece.
4. $S_3, S_4, S_5, S_6, S_7, S_8, S_9$ each get two $\frac{1}{2}$-sized pieces and two $\frac{2}{5}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{5} = \frac{18}{5} = 3 + \frac{3}{5}$,

$$f(9,5) \leq \min\left\{\frac{9}{5 \times 4}, 1 - \frac{9}{5 \times 3}\right\} = \frac{2}{5}.$$

9) The following procedure shows $f(11,5) \geq \frac{13}{30}$.

1. Divide $M_1, \ldots, M_6$ into $(\frac{13}{30}, \frac{17}{30})$.
2. Divide $M_7, \ldots, M_{10}$ into $(\frac{9}{20}, \frac{11}{20})$.
3. Divide M_{11} into $(\frac{1}{2}, \frac{1}{2})$.
4. S_1 and S_2 each get three of the $\frac{17}{30}$-sized pieces and one $\frac{1}{2}$-sized piece.
5. S_3 and S_4 each get three of the $\frac{13}{30}$-sized pieces and two $\frac{9}{20}$-sized pieces.
6. S_5 gets four $\frac{11}{30}$-sized pieces.

By Theorem 2.7.2 $f(11,5) \leq \frac{11}{25}$.

10) The following procedure shows $f(12,5) \geq \frac{2}{5}$.

1. Divide $M_1, \ldots, M_{12}$ into $(\frac{2}{5}, \frac{3}{5})$.

2. S_1 and S_2 each get six of the $\frac{2}{5}$-sized pieces.

3. S_3, S_4, S_5 each get four of the $\frac{3}{5}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{24}{5} = 4 + \frac{4}{5}$,

$$f(12,5) \leq 1 - \frac{12}{5 \times 4} = \frac{2}{5}$$

11) The following procedure shows $f(13,5) \geq \frac{13}{30}$.

1. Divide $M_1, \ldots, M_6$ into $(\frac{13}{30}, \frac{17}{30})$.

2. Divide M_7, M_8, M_9 into $(\frac{7}{15}, \frac{8}{15})$.

3. Do not divide $M_{10}, M_{11}, M_{12}, M_{13}$.

4. S_1 gets six of the $\frac{13}{30}$-sized pieces.

5. S_2, S_3, S_4 each get one of the $\frac{7}{15}$-sized pieces, two of the $\frac{17}{30}$-sized pieces, and one of the 1-sized pieces,

6. S_5 gets three of the $\frac{8}{15}$-sized pieces and one of the 1-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{26}{5} = 5 + \frac{1}{5}$,

$$f(13,5) \leq \frac{13}{5 \times 6} = \frac{13}{30}$$

12) The following procedure shows $f(14,5) \geq \frac{11}{25}$.

1. Divide $M_1, \ldots, M_{10}$ into $(\frac{11}{25}, \frac{14}{25})$.

2. Divide $M_{11}, \ldots, M_{14}$ into $(\frac{12}{25}, \frac{13}{25})$.

3. S_1, S_2 each get five of the $\frac{14}{25}$-sized pieces.

4. S_3, S_4 each get four of the $\frac{11}{25}$-sized pieces, and two of the $\frac{11}{25}$-sized pieces.

5. S_5 gets two of the $\frac{11}{25}$-sized pieces and four of the $\frac{12}{25}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{28}{5} = 5 + \frac{3}{5}$,

$$f(14,5) \leq 1 - \frac{14}{5 \times 5} = \frac{11}{25}$$

13) The following procedure shows $f(16,5) \geq \frac{16}{35}$.

1. Divide $M_1, \ldots, M_{14}$ into $(\frac{16}{35}, \frac{19}{35})$

2. Divide M_{15}, M_{16} into $(\frac{17}{35}, \frac{18}{35})$

3. S_1, S_2 each get seven of the $\frac{16}{35}$-sized pieces.

4. S_3, S_4 each get five of the $\frac{19}{35}$-sized pieces and one of the $\frac{17}{35}$-sized pieces.

5. S_5 gets two of the $\frac{18}{35}$-sized pieces and four of the $\frac{19}{35}$-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{32}{5} = 6 + \frac{2}{5}$,

$$f(16,5) \leq \frac{16}{5 \times 7} = \frac{16}{35}$$

14) The following procedure shows $f(17,5) \geq \frac{13}{30}$.

1. Divide $M_1, \ldots, M_6$ into $(\frac{13}{30}, \frac{17}{30})$

2. Divide M_7, M_8, M_9 into $(\frac{9}{15}, \frac{8}{15})$

3. $M_{10}, \ldots, M_{17}$ are uncut.

4. S_1 gets six of the $\frac{17}{30}$-sized pieces.

5. S_2, S_3, S_4 each get two of the $\frac{13}{30}$-sized pieces, one of the $\frac{16}{30}$-sized pieces, and two 1-sized pieces.

6. S_5 gets three of the $\frac{14}{30}$-sized pieces and two 1-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{34}{5} = 6 + \frac{4}{5}$,

$$f(17,5) \leq 1 - \frac{17}{5 \times 6} = \frac{13}{30}$$

15) The following procedure shows $f(18,5) \geq \frac{9}{20}$.

1. Divide $M_1, \ldots, M_8$ into $(\frac{9}{20}, \frac{11}{20})$

2. Divide $M_9, M_{10}, M_{11}, M_{12}$ into $(\frac{1}{2}, \frac{1}{2})$

3. $M_{13}, \ldots, M_{18}$ are uncut.

4. S_1 gets eight of the $\frac{9}{20}$-sized pieces.

5. $S_2, \ldots, S_5$ each get one $\frac{1}{2}$-sized piece, two $\frac{11}{20}$-sized pieces, and two 1-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{36}{5} = 7 + \frac{1}{5}$,

$$f(18,5) \leq \frac{18}{5 \times 8} = \frac{9}{20}$$

16) The following procedure shows $f(19,5) \geq \frac{16}{35}$.

1. Divide $M_1, \ldots, M_{14}$ into $(\frac{16}{35}, \frac{19}{35})$

2. Divide M_{15}, M_{16}, M_{17} into $(\frac{17}{35}, \frac{18}{35})$

3. Do not divide M_{18}, M_{19}.

4. S_1, S_2 each get seven $\frac{19}{35}$-sized pieces.

5. S_3 gets five $\frac{16}{35}$-sized piece, one $\frac{17}{35}$ piece and two $\frac{18}{35}$-sized pieces.

6. S_4 gets five $\frac{16}{35}$-sized pieces, one $\frac{18}{35}$-sized piece, and one 1-piece.

7. S_5 gets four $\frac{16}{35}$-sized pieces, two $\frac{17}{35}$-sized pieces, and one 1-piece.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{38}{5} = 7 + \frac{3}{5}$,

$f(19, 5) \leq 1 - \frac{19}{5*7} = \frac{16}{35}$

17) The following procedure shows $f(21, 5) \geq \frac{7}{15}$.

1. Divide $M_1, \ldots, M_{18}$ into $(\frac{7}{15}, \frac{8}{15})$

2. M_{19}, M_{20}, M_{21} are uncut.

3. S_1, S_2 each get nine $\frac{7}{15}$-sized pieces.

4. S_3, S_4, S_5 gets six $\frac{8}{15}$-sized piece and one 1-sized piece.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{42}{5} = 8 + \frac{2}{5}$,

$f(21, 5) \leq \frac{21}{5*9} = \frac{7}{15}$

18) The following procedure shows $f(22, 5) \geq \frac{9}{20}$.

1. Divide $M_1, \ldots, M_{18}$ into $(\frac{9}{20}, \frac{11}{20})$

2. M_{19}, M_{20}, M_{21} are uncut.

3. S_1 gets eight $\frac{11}{20}$-sized pieces.

4. S_2, S_3, S_4, S_5 each get two $\frac{9}{20}$-sized piece, one $\frac{1}{2}$-sized piece, and three 1-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{44}{5} = 8 + \frac{4}{5}$,

$f(22, 5) \leq \frac{21}{5*9} = \frac{9}{20}$

19) $f(23, 5)$

20) The following procedure shows $f(24,5) \geq \frac{7}{15}$.

STILL WORKING ON IT

$24/5 = 216/45$

1. Divide $M_1, \ldots, M_{18}$ into $(\frac{7}{15}, \frac{8}{15})$

2. $M_{19}, \ldots, M_{24}$ are uncut.

3. S_1, S_2 each get nine $\frac{7}{15}$-sized pieces.

4. S_2, S_3, S_4, S_5 each get two $\frac{9}{20}$-sized piece, one $\frac{1}{2}$-sized piece, and three 1-sized pieces.

By Theorem 2.7.2, noting that $\frac{2m}{s} = \frac{48}{5} = 9 + \frac{3}{5}$,

$f(24,5) \leq 1 - \frac{24}{5*9} = \frac{21}{45}$

∎

References

[1] A. Frank. The muffin problem, 2013. Described to Jeremy Copeland and in the New York Times Numberplay Online Blog `wordplay.blogs.nytimes.com/2013/08/19/cake`.