The Muffin Problem

By William Gasarch and Alex Zhang

1 Introduction

The following problem, and similar ones, appeared in the *Julia Robinson Mathematics Festival*. These problems were proposed by Alan Frank [1].

You have 5 muffins and 3 students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?

Here is a procedure:

1. Divide M_1 and M_2 into $(\frac{1}{3}, \frac{2}{3})$.
2. M_3, M_4, M_5 are not cut. We call them *1-sized pieces*.
3. S_1 and S_2 each get a 1-sized piece and a $\frac{2}{3}$-sized piece.
4. S_3 gets a 1-sized piece and two $\frac{1}{3}$-sized pieces.

The smallest piece in the above solution is $\frac{1}{3}$. Can we do better? Theorem 3.1 will show that we can. Here is the general muffin problem:

You have m muffins and s students. You want to divide the muffins evenly, but no student wants a sliver. Which division of the muffins maximized the smallest piece?

Def 1.1 Let $m, s \in \mathsf{N}$. An (m, s)*-procedure* is a procedure to cut m muffins into pieces and then distribute them to the s students so that each student gets m/s. An (m, s)-procedure is *optimal* if it has the largest smallest piece of any procedure. $f(m, s)$ be the smallest piece in an optimal (m, s)-procedure.

2 General Theorems

Theorem 2.1

1. *If* $m \equiv 0 \pmod{s}$ *then* $f(m,s) = 1$.

2. *If* $s \equiv 0 \pmod{2m}$ *and* $\frac{m}{s} \notin \mathsf{N}$ *then* $f(m,s) = \frac{1}{2}$.

3. *If* $m, s \in \mathsf{N}$ *then* $f(m,s) \geq \frac{1}{s}$.

4. $f(1,s) = \frac{1}{s}$.

5. *If* s *is odd then* $f(2,s) = \frac{1}{s}$.

6. *Let* $f(m,s) > \frac{1}{L}$ *and* $\frac{m}{s} \notin \mathsf{N}$. *There is a procedure with min piece of size* $f(m,s)$ *such (a) every muffin is cut into either 2 or 3 or* $\cdots$ *or* $L-1$ *pieces, (b) the number of pieces created is between* $2m$ *and* $(L-1)m$.

7. *ALEX- I SUSPECT THIS APPROACH WILL NEVER GIVE OPT BOUNDS.*

   If $m \geq Ls$ *then* $f(m,s) \geq f(m - Ls, s)$.

Proof:

1) No muffin is cut. Give everyone $\frac{m}{s}$ muffins.

2) $s \equiv 0 \pmod{2m}$. The following procedure shows $f(m,s) \leq \frac{1}{2}$.

1. Divide $M_1, \ldots, M_m$ into $(\frac{1}{2}, \frac{1}{2})$.

2. $S_1, \ldots, S_s$ each get $\frac{2m}{s}$ $\frac{1}{2}$-sized pieces.

   Since $\frac{m}{s} \notin \mathsf{N}$ some muffin is cut. Hence $f(m,s) \leq \frac{1}{2}$.

3) The following procedure shows $f(m,s) \geq \frac{1}{s}$.

1. Divide $M_1, \ldots, M_m$ into $(\frac{1}{s}, \ldots, \frac{1}{s})$.

2. $P_1, \ldots, P_s$ each get m $\frac{1}{s}$-sized pieces.

4) $f(1, s) = \frac{1}{s}$: By part 3 $f(1, s) \geq \frac{1}{s}$. Since any procedure will give each student $\frac{1}{s}$ muffins, $f(1, s) \leq \frac{1}{s}$.

5) By Part 3 $f(2, s) \geq \frac{1}{s}$. Assume there is a $(2, s)$-procedure. Let N be the size of the smallest piece produced.

Case 1: Some student gets ≥ 2 pieces. Then $N \leq \frac{2}{2s} = \frac{1}{s}$.

Case 2: Every student gets 1 piece. Each piece must be of size $\frac{2}{s}$. Let M be a muffin. It is cut into x (note $x \in \mathsf{N}$) pieces of size $\frac{2}{s}$. Hence $2x/s = 1$ so $x = s/2$. Since s is odd $x \notin \mathsf{N}$. This cannot happen.

6a) Since $\frac{m}{s} \notin \mathsf{N}$ some muffin is cut. Hence $f(m, s) \leq \frac{1}{2}$. If in the original procedure there is an uncut muffin that goes to a student then modify the procedure by dividing it $(\frac{1}{2}, \frac{1}{2})$ and giving both halves to that student. If in the original procedure there is a muffin cut into $\geq L$ pieces then there will be some piece of size $\frac{1}{L}$ which contradicts $f(m, s) > \frac{1}{L}$.

7) The following procedure shows $f(m, s) \geq f(m - Ls, s)$.

1. For $1 \leq i \leq s$ S_i gets L muffins. Note that there are $m - Ls$ muffins left

2. Apply the optimal (m, s)-procedure to divide the remaining $m - Ls$ muffins.

∎

Theorem 2.2

1. *Let $m, s, p \in \mathsf{N}$ Let p be the number of pieces in an optimal (m, s)-procedure. Then*

$$f(m, s) \leq \min\left\{ \frac{m}{s \lceil p/s \rceil}, 1 - \frac{m}{s \lfloor p/s \rfloor} \right\}.$$

2. *If* $f(m,s) > \frac{1}{3}$ *and* $\frac{m}{s} \notin \mathsf{N}$ *then*

$$f(m,s) \le \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}$$

3. *ALEX- WE NEVER SEEM TO USE THIS GENERALITY.*

   If $f(m,s) > \frac{1}{L}$ *and* $\frac{m}{s} \notin \mathsf{N}$ *then*

$$f(m,s) \le \max_{2m \le p \le (L-1)m} \min\left\{\frac{m}{s\lceil p/s\rceil}, 1 - \frac{m}{s\lfloor p/s\rfloor}\right\}.$$

Proof: Parts 2 and 3 follow from part 1 so we just prove part 1.

1) Since the smallest piece is of size $f(m,s)$, the largest piece is of size $\le 1 - f(m,s)$.

Since there p pieces and s students we can assume that (1) S_1 gets $\ge \lceil p/s\rceil$ pieces so he gets least $f(m,s)\lceil p/s\rceil$ muffins; hence $f(m,s)\lceil p/s\rceil \le \frac{m}{s}$, and (2) S_2 gets $\le \lfloor p/s\rfloor$ pieces so he gets at most $(1-f(m,s))\lceil p/s\rceil$ muffins; hence $(1-f(m,s))\lceil p/s\rceil \ge \frac{m}{s}$. The inequalities $f(m,s) \le \frac{m}{s\lceil p/s\rceil}$ and $f(m,s) \le 1 - \frac{m}{s\lfloor p/s\rfloor}$ follow. ∎

Def 2.3 Let

$$g(m,s) = \left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}$$

Theorem 2.4 *Let $m, s \in \mathsf{N}$ such that s does not divide m. Then*

$$f(m,s) \le \max\Bigg\{$$

- $\frac{m}{3s}$,
- $\min\{\frac{m}{s}, \frac{s-m}{s\lceil m/2(s-m)\rceil}, \frac{m}{s} - \frac{s-m}{s\lceil m/2(s-m)\rceil}\}$
- $\min\{\frac{m}{s}, \frac{s-m}{s\lfloor m/2(s-m)\rfloor}, \frac{m}{s} - \frac{s-m}{s\lfloor m/2(s-m)\rfloor}\}$
- $\min\left\{\frac{1}{\lceil 2s/m\rceil}, \frac{m}{s} - \frac{1}{\lceil 2s/m\rceil}, \frac{m}{2s}\right\}$
- $\min\left\{\frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{s} - \frac{1}{\lfloor 2s/m\rfloor}, \frac{m}{2s}\right\}$

$$\Bigg\}.$$

Proof: Assume there is an (m,s)-procedure. Let N be the size of the smallest piece produced.

Case 1: Some student gets ≥ 3 pieces. Then $N \le \frac{m}{3s}$.

Case 2: Some student gets 1 piece which we call P_1. P_1 is of size $\frac{m}{s}$. Say P_1 came from muffin M. Let $P_2 = M - P_1$. P_2 is of size $1 - \frac{m}{s}$. P_2 is cut into x pieces (x could be 1). There is a piece P_3 of size $\le \frac{1}{x} - \frac{m}{sx} = \frac{s-m}{sx}$. There is a piece of size $P_4 \ge \frac{s-m}{sx}$. Some student gets P_4 together with some other piece P_5 (its possible P_5 is size 0). P_5 has size $\le \frac{m}{s} - \frac{s-m}{sx}$.

Looking at P_1 P_3, and P_5 we have that

$$N \le \min\{\frac{m}{s}, \frac{s-m}{sx}, \frac{m}{s} - \frac{s-m}{sx}\}$$

This is minimized when

$$\frac{s-m}{sx} = \frac{m}{s} - \frac{s-m}{sx}$$

$$\frac{2(s-m)}{sx} = \frac{m}{s}$$

$$\frac{2(s-m)}{x} = m$$

$$2(s-m) = mx$$

$$x = \frac{m}{2(s-m)}.$$

Alas, this value of x is not an integer! Hence we take both its floor and its ceiling.

Case 2a:

$$N \le \min\{\frac{m}{s}, \frac{s-m}{s\lceil m/2(s-m)\rceil}, \frac{m}{s} - \frac{s-m}{s\lceil m/2(s-m)\rceil}\}$$

Case 2b:

$$N \le \min\{\frac{m}{s}, \frac{s-m}{s\lfloor m/2(s-m)\rfloor}, \frac{m}{s} - \frac{s-m}{s\lfloor m/2(s-m)\rfloor}\}$$

Case 3: Every student gets exactly two pieces. Hence every student has a piece P of size $\le \frac{m}{2s}$. Since s does not divide m there is an $2 \le x \le s$ such that some muffin is cut into x pieces. Hence there is a piece P_1 of size $\le \frac{1}{x}$ and a piece P_2 of size $\ge \frac{1}{x}$. Some student gets P_2 along with at least one piece P_3 of size $\le \frac{m}{s} - \frac{1}{x}$. Looking at P_1, P_3 to get

$$N \le \min\left\{\frac{1}{x}, \frac{m}{s} - \frac{1}{x}, \frac{m}{2s}\right\}$$

This is maximized when

$\frac{1}{x} = \frac{m}{s} - \frac{1}{x}$

$\frac{2}{x} = \frac{m}{s}$

$\frac{x}{2} = \frac{s}{m}$

$x = \frac{2s}{m}$

Alas, this value of x is not an integer! Hence we take both its floor and its ceiling.

Case 3a:

$$N \le \min\left\{\frac{1}{\lceil 2s/m \rceil}, \frac{m}{s} - \frac{1}{\lceil 2s/m \rceil}, \frac{m}{2s}\right\}$$

Case 3b:

$$N \le \min\left\{\frac{1}{\lfloor 2s/m \rfloor}, \frac{m}{s} - \frac{1}{\lfloor 2s/m \rfloor}, \frac{m}{2s}\right\}$$

▮

We now prove that if either of our lower bounds are also upper bounds then $f(m, s) = f(am, as)$.

Def 2.5 Let $g(m, s)$ be the upper bound on $f(m, s)$ from Theorem 2.2. Let $g(m, s)$ be the upper bound on $f(m, s)$ from Theorem 2.4.

We leave the proof of the following theorem to the reader.

Theorem 2.6 *Let $a, m, s \in \mathsf{N}$ such that $\frac{m}{s} > \frac{1}{3}$.*

1. $f(m, s) \le g(m, s) = g(am, as)$.

2. $f(m, s) \le h(m, s) = h(am, as)$.

3. $f(m, s) \le f(am, as)$.

4. *If* $f(m,s) = g(m,s)$ *then* $f(m,s) = f(am,as)$*. So if* $f(m,s)$ *matches the lower bound from Theorem 2.2.2 then* $f(m,s) = f(am,as)$*.*

5. *If* $f(m,s) = h(m,s)$ *then* $f(m,s) = f(am,as)$*. So if* $f(m,s)$ *matches the lower bound from Theorem 2.4 then* $f(m,s) = f(am,as)$*.*

ALEX: I AM KEEPING THIS NEXT THEOREM BUT IT WILL BE DELETED LATER. THE THEOREM AFTER IT IS ONE IMPROVEMENT, AND THEN I HAVE A SECOND IMPROVEMENT.

Theorem 2.7 *Let* $m,s \in \mathsf{N}$ *and* $0 < \delta \le \frac{1}{2}$*. Assume there exists* $x_1, y_1, x_2, y_2, y_3, z_1, z_2 \in \mathsf{N}$ *such that the following hold:*

1. $x_1y_1 + x_2y_2 = x_1y_3 \le m$
2. $x_1 + x_2 = s$
3. $z_1 + z_2 = 2(m - x_1y_3)$
4. x_1 *divides* z_1*; and* x_2 *divides* z_2
5. $y_1\delta + y_3(1-\delta) + \frac{z_1}{2x_1} = \frac{m}{s}$
6. $y_2\delta + \frac{z_2}{2x_2} = \frac{m}{s}$

Then $f(m,s) \ge \min\{\delta, 1-\delta\}$*.*

Proof:

The following procedure show $f(m,s) \ge \delta$.

1. Divide $M_1, \ldots, M_{x_1y_3}$ into $(\delta, 1-\delta)$.

   (There are $x_1y_1 + x_2y_2$ δ-sized pieces, and x_1y_3 $(1-\delta)$-sized pieces.)

2. Divide $M_{x_1y_3+1},\ldots,M_m$ into $(\frac{1}{2},\frac{1}{2})$.

   (There are $2(m-x_1y_3)=(z_1+z_2)$ $\frac{1}{2}$-pieces.)

3. $S_1,\ldots,S_{x_1}$ each get y_1 δ-sized pieces, y_3 $(1-\delta)$-sized pieces, and $\frac{z_1}{x_1}$ $\frac{1}{2}$-sized pieces.

   (Each get $y_1\delta+y_3(1-\delta)+\frac{z_1}{2x_1}=\frac{m}{s}$.)

4. $S_{x_1+1},\ldots,S_{s=x_1+x_2}$ each get y_2 δ-sized pieces and $\frac{z_2}{x_2}$ $\frac{1}{2}$-sized pieces.

   (Each get $y_2\delta+\frac{z_2}{2x_2}=\frac{m}{s}$.)

∎

ALEX- THE PROBLEM WITH THEOREM 2.7 IS THAT WE ALLOW A STUDENT TO GET δ AND $1-\delta$ WHICH IS SILLY SINCE JUST GIVE THEM ONE MUFFIN. IN THAT CASE. I GOT RID OF y_3. ALSO ITS AWKWARD HAVING x_1 HAVE TO DIVIDE z_1 SO I''VE CHANGED HOW I DO THINGS.

Theorem 2.8 *Let $m,s\in\mathsf{N}$ and $0<\delta<\frac{1}{2}<1-\delta<1$. Assume there exists $x_1,y_1,x_2,y_2,z_1,z_2\in\mathsf{N}$ such that the following hold:*

1. $x_1y_1=x_2y_2\le m$
2. $x_1+x_2=s$
3. $z_1x_1+z_2x_2=2(m-x_1y_1)=2(m-x_2y_2)$.
4. $y_1\delta+\frac{z_1}{2}=\frac{m}{s}$
5. $y_2\delta+\frac{z_2}{2}=\frac{m}{s}$

Then $f(m,s)\ge\delta$.

Proof:

The idea is that

- x_1 people will get y_1 δ-sized pieces and z_1 $\frac{1}{2}$-sized pieces.

- x_2 people will get y_2 $(1-\delta)$-sized pieces and z_2 $\frac{1}{2}$-sized pieces.

The following procedure show $f(m,s) \ge \delta$.

1. Divide $M_1, \ldots, M_{x_1 y_1}$ into $(\delta, 1-\delta)$.

   (There are $x_1 y_1$ δ-sized pieces, and $x_2 y_2$ $(1-\delta)$-sized pieces.)

2. Divide $M_{x_1 y_1 + 1}, \ldots, M_m$ into $(\frac{1}{2}, \frac{1}{2})$.

   (There are $2(m - x_1 y_1) = z_1 x_1 + z_2 x_2$ $\frac{1}{2}$-pieces.)

3. $S_1, \ldots, S_{x_1}$ each get y_1 δ-sized pieces and z_1 $\frac{1}{2}$-sized pieces.

   (Each get $y_1 \delta + \frac{z_1}{2} = \frac{m}{s}$.)

4. $S_{x_1+1}, \ldots, S_{x_1+x_2}$ each get y_2 $(1-\delta)$-sized pieces and z_2 $\frac{1}{2}$-sized pieces.

   (Each get $y_2 \delta + \frac{z_2}{2} = \frac{m}{s}$.)

∎

ALEX- ALL OF THE $s = 5$ CASES THAT WE HAVE SOLVED, THAT HAVE $f(m,s) > 1/3$ HAVE FIT THEOREM 2.8 OR USE TWO δ's AND THE SEARCH FOR THE SECOND δ IS EASY. FOR EXAMPLE, IF THE LOWER BOUNDS IS $11/30$ SO YOU WILL BE USING $(11/30, 19/30)$ and $(1/2, 1/2)$ THE OTHER TYPE OF SPLIT TO USE IS EITHER $(12/30, 18/30)$, OR $(13/30, 17/30)$, OR $(14/30, 16/30)$. NOT THAT MANY TO TRY, AND IN FACT THE ABOVE IS MADE UP, USUALLY THERE ARE EVEN LESS TO TRY. HERE IS A THEOREM THAT I HOPE COVES MANY MORE CASES AND IS EASY FOR YOU TO CODE UP, WHICH I WILL DISCUSS AFTER IT

Theorem 2.9 *Let $m, s \in \mathsf{N}$ and $0 < \delta_1 < \delta_2 < \frac{1}{s} < 1 - \delta_2 < 1 - \delta_1 < 1$. Assume there exists: for $1 \le i \le 4$, x_i; for $1 \le i \le 4$, for $1 \le j \le 2$, y_{ij}, for $1 \le i \le 4$, z_i: such that the following hold:*

1. $x_1y_{11} + x_2y_{21} = x_3y_{31} + x_4y_{41}$. *(Number of δ_1-pieces equals the number of $(1-\delta_1)$ pieces.)*

2. $x_1y_{12} + x_3y_{32} = x_2y_{22} + x_4y_{42}$. *(Number of δ_2-pieces equals the number of $(1-\delta_2)$ pieces.)*

3. $2(m - x_1y_1) = z_1x_1 + z_2x_2$.

4. $x_1 + x_2 + x_3 + x_4 = s$.

5. $y_{11}\delta_1 + y_{12}\delta_2 + \frac{z_1}{2} = \frac{m}{s}$

6. $y_{21}\delta_1 + y_{22}(1-\delta_2) + \frac{z_2}{2} = \frac{m}{s}$

7. $y_{31}(1-\delta_1) + y_{32}\delta_2 + \frac{z_3}{2} = \frac{m}{s}$

8. $y_{41}(1-\delta_1) + y_{42}(1-\delta_2) + \frac{z_4}{2} = \frac{m}{s}$

Then $f(m,s) \geq \delta_1$.

Proof:

The idea is:

- x_1 people will get y_{11} δ_1-sized pieces, y_{12} δ_2-sized pieces, and z_1 $\frac{1}{2}$-sized pieces.
- x_2 people will get y_{21} δ_1-sized pieces, y_{22} $(1-\delta_2)$-sized pieces, and z_2 $\frac{1}{2}$-sized pieces.
- x_3 people will get y_{31} $(1-\delta_1)$-sized pieces, y_{32} δ_2-sized pieces, and z_3 $\frac{1}{2}$-sized pieces.
- x_4 people will get y_{41} $(1-\delta_1)$-sized pieces, y_{42} $(1-\delta_2)$-sized pieces, and z_4 $\frac{1}{2}$-sized pieces.

The following procedure show $f(m,s) \geq \delta$.

1. Divide $M_1, \ldots, M_{x_1y_1}$ into $(\delta, 1-\delta)$.

   (There are x_1y_1 δ-sized pieces, and x_2y_2 $(1-\delta)$-sized pieces.)

2. Divide $M_{x_1y_1+1}, \ldots, M_m$ into $(\frac{1}{2}, \frac{1}{2})$.

   (There are $2(m - x_1y_1) = z_1x_1 + z_2x_2$ $\frac{1}{2}$-pieces.)

3. $S_1, \ldots, S_{x_1}$ each get y_{11} δ_1-sized pieces, y_{12} δ_2-sized pieces, z_1 $\frac{1}{2}$-sized pieces. (Each get $y_{11}\delta_1 + y_{12}\delta_2 + \frac{z_1}{2} = \frac{m}{s}$.)

4. $S_{x_1+1}, \ldots, S_{x_1+x_2}$ each get y_{21} δ_1-sized pieces and y_{22} $(1-\delta_2)$-sized z_2 $\frac{1}{2}$-sized pieces. (Each get $y_{21}\delta_1 + y_{22}(1 = \delta_2) + \frac{z_2}{2} = \frac{m}{s}$.)

5. $S_{x_1+x_2+1}, \ldots, S_{x_1+x_2+x_3}$ each get y_{31} $(1 - \delta_1)$-sized pieces and y_{32} δ_2–sized pieces, and z_3 $\frac{1}{2}$-sized pieces. (Each get $y_{31}(1 - \delta_1) + y_{32}(\delta_2) + \frac{z_2}{2} = \frac{m}{s}$.)

6. $S_{x_1+x_2+x_3+1}, \ldots, S_{x_1+x_2+x_3+x_4}$ each get y_{41} $(1 - \delta_1)$-sized pieces and y_{42} $(1 - \delta_2)$–sized pieces, and z_4 $\frac{1}{2}$-sized pieces. (Each get $y_{41}(1 - \delta_1) + y_{42}(1 - \delta_2) + \frac{z_2}{2} = \frac{m}{s}$.)

$\blacksquare$

ALEX- I WANT YOU TO WRITE A PROGRAM THAT DOES THE FOLLOWING.

ON INPUT (m,s)

IF $m = 1$ OR $m = 2$ THEN USE THEOREMS THAT ALREADY GIVE $f(m, s)$

IF s divides m THEN OUTPUT $f(m, s) = 1$.

FIND THE UPPER BOUND ON $f(m, s)$ USING THEOREM 2.2 AND THEOREM 2.4. IF THE SMALLER UPPER BOUND IS a/b (KEEP IT AS (a, b)) THEN (NOT SURE HOW THIS WORKS FOR SEEING IF ITS ¡1/3 OR NOT.)

a) FIRST USE THEOREM 2.8 TO TRY TO FIND THE ALGORITHM WITH MATCHING LOWER BOUND. OUTPUT ALL PARAMETERS THAT WORK- I AM CURIOUS IF ITS JUST ONE SET. IF YOU GET BOUNDS MATCH, DONE. IF YOU GET BOUNDS THAT DO NOT MATCH REPORT THAT BUT GO TO NEXT STEP IF YOU DO NOT GET ANYTHING THEN GOTO NEXT STEP.

b) USE THEOREM 2.9 1WITH $\delta 1 = a/b$ AND $\delta 2$ VARIES FROM $(a+1)/b$, $(a+2)/b$ ETC SO LONG AS ITS $< 1/2$.

AS A TEST CASE: $f(13,5)$ YOUR OTHER PROGRAM COULD NOT FIND THE ANSWER. BY HAND I FOUND IT. δ FROM THEOREM 2.2 was $13/30$. I USED $(13/30, 17/30)$ AND $(7/15, 8/15)$ WHICH IS $(14/30, 16/30)$. THE PROGRAM YOU WRITE SHOULD BE ABLE TO FIND THIS.

I SUSPECT THAT WHEN THIS IS DONE ALMOST ALL OF THE $1 \le m \le 30$, $s = 5$ CASES WILL BE KNOWN AND MANY OF THE PATTERNS WILL BE KNOWN. I WONDER IF THE MOD WILL END UP BEING LESS THAN 30. OR MORE THAN 30.

CLEARLY ONE CAN WRITE A THEOREM WITH $\delta_1, \delta_2, \delta_3$ BUT WE WILL HOLD OFF ON THAT AND SEE WHAT WE NEED.

Theorems 2.8 and 2.9 are useful when $f(m,s)$ is large (in particular larger than $1/3$). The following theorem is useful when $f(m,s)$ is small.

ALEX- THE THEOREM BELOW IS JUST FOR $s = 3$. I WILL LATER TRY TO GENERALIZE IT.

Theorem 2.10 *Let $m, s \in \mathsf{N}$. For all $1 \le x, y \le s$ let $A(x,y)$ be the least A such that $(m-y)$ divides $A(s-xy)$. Let $x, y \in \mathsf{N}$ such that $xy \le s$ and $m-y$ divides xy. Then there is a procedure that shows*

$$f(x,s) \ge \min\left\{\frac{1}{x}, \frac{mx-s}{sx}, \frac{m}{A(x,y)s}\right\}$$

Hence $f(x,s)$ is the max over all such x, y of this quantity.

Proof: Let x, y be as in the premise. Let $A = A(x,y)$.

Consider the following procedure.

1. Divide $M_1, \ldots, M_y$ into $(\frac{1}{x}, \ldots, \frac{1}{x})$

   (There are xy pieces of size $\frac{1}{x}$.)

2. Divide each of $M_{y+1}, \ldots, M_m$ into $\frac{xy}{m-y}$ pieces of size $\frac{mx-s}{sx}$ and $\frac{A(s-xy)}{m-y}$ pieces of size $\frac{m}{As}$. (There are $(m-y)\frac{xy}{m-y} = xy$ pieces of size $\frac{mx-s}{sx}$ and $(m-y)\frac{A(s-xy)}{m-y} = A(s-xy)$ pieces of size $\frac{m}{As}$.)

3. $S_1, \ldots, S_{xy}$ each get one $\frac{1}{x}$-sized piece and one $\frac{mx-s}{sx}$-sized piece.

4. $S_{xy+1}, \ldots, S_s$ each get A $\frac{m}{As}$-sized piece.

Clearly

$$f(m,s) \ge \min\left\{\frac{1}{x}, \frac{mx-s}{sx}, \frac{m}{As}\right\}.$$

▮

ALEX- THIS SHOULD BE EASY TO CODE- LOOK AT ALL $(x, y, A(x,y))$ THAT SATISFY THE CRITERIA AND TAKE THE MAX OF

$$\min\left\{\frac{1}{x}, \frac{mx-s}{sx}, \frac{m}{As}\right\}.$$

3 Five Muffins, Three Students

In the introduction we showed that $f(5,3) \geq \frac{1}{3}$. We show that $f(5,3) = \frac{5}{12}$.

Theorem 3.1 $f(5,3) = \frac{5}{12}$.

Proof:

The following procedure shows $f(5,3) \geq \frac{5}{12}$.

1. Divide M_1, M_2, M_3, M_4 into $(\frac{5}{12}, \frac{7}{12})$.
2. Divide M_5 into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1 and S_2 each get two of the $\frac{7}{12}$-sized pieces and one of the $\frac{1}{2}$-sized pieces.
4. S_3 gets four $\frac{5}{12}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{5}{12} > \frac{1}{3}$ so it applies,

$$f(5,3) \leq \min\left\{\frac{5}{3 \times \lceil 10/3 \rceil}, 1 - \frac{5}{3 \times \lfloor 10/3 \rfloor}\right\} = \min\left\{\frac{5}{12}, 1 - \frac{5}{9}\right\} = \frac{5}{12}$$

∎

Note 3.2 We could also have obtained Theorem 3.1 by applying Theorem 2.8 with $m = 5$, $s = 3$, $\delta = \frac{5}{12}$, $x_1 = 1$, $y_1 = 4$, $x_2 = 2$, $y_2 = 2$.

4 m Muffins, Three Students

Theorem 4.1

0) *If* $m \equiv 0 \pmod 3$ *then* $f(m,3) = 1$.

1) $f(1,3) = \frac{1}{3}$. *If* $m \equiv 1 \pmod 3$ *and* $m = 3k+1$, *with* $k \geq 1$, *then* $f(m,3) = \frac{3k-1}{6k}$.

2) If $m \equiv 2 \pmod 3$ and $m = 3k+2$, with $k \geq 0$, then $f(m,3) = \frac{3k+2}{6k+6}$.

Proof:

For parts 2 and 3 we use Theorem 2.2.2 to obtain an upper bound on $f(m,s)$. To apply this we need that $f(m,s) > \frac{1}{3}$. This is the case for every $f(m,s)$ except in part 2 with $k=0$. In this case we have $f(m,s) \geq \frac{1}{3}$; therefore we could structure the proof as a proof by contradiction.

0) This follows from Theorem 2.1.1.

1a) $f(1,3) = \frac{1}{3}$ by Theorem 2.1.4

1b) $m = 3k+1$ with $k \geq 1$. The following procedure shows $f(m,3) \geq \frac{3k-1}{6k}$.

1. Divide $M_1, \ldots, M_{2k}$ into $(\frac{3k-1}{6k}, \frac{3k+1}{6k})$.
2. Divide $M_{2k+1}, \ldots, M_{3k+1}$ into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1 gets $2k$ of the $\frac{3k+1}{6k}$-sized pieces.
4. S_2 and S_3 each get k of the $\frac{3k-1}{6k}$-sized pieces and $k+1$ of the $\frac{1}{2}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{3} = \frac{6k+2}{3} = 2k + \frac{2}{3}$.

$$f(3k+1,3) \leq \min\left\{\frac{3k+1}{3(2k+1)}, 1 - \frac{3k+1}{3(2k)}\right\} = \frac{3k-1}{6k}.$$

2) $m = 3k+2$. The following procedure shows $f(m,3) \geq \frac{3k+2}{6k+6}$.

1. Divide $M_1, \ldots, M_{2k+2}$ into $(\frac{3k+2}{6k+6}, \frac{3k+4}{6k+6})$.
2. Divide $M_{2k+3}, \ldots, M_{3k+2}$ into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1 gets $2k+2$ of the $\frac{3k+2}{6k+6}$-sized pieces.
4. S_2 and S_3 each get $k+1$ of the $\frac{3k+4}{6k+6}$-sized pieces and k $\frac{1}{2}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{3} = \frac{6k+4}{3} = 2k + \frac{4}{3}$,

$$f(3k+2,3) \leq \min\left\{\frac{3k+2}{3(2k+2)}, 1 - \frac{3k+2}{3(2k+1)}\right\} = \frac{3k+2}{6k+6}$$

▮

5 m Muffins, Four Students

Theorem 5.1

0) If $m \equiv 0 \pmod 4$ *then* $f(m,4) = 1$.

1) $f(1,4) = \frac{1}{4}$. *If* $m \equiv 1 \pmod 4$ *and* $m = 4k+1$, *with* $k \geq 1$, *then* $f(m,4) = \frac{4k-1}{8k}$.

2) If $m \equiv 2 \pmod 4$ *then* $f(m,4) = \frac{1}{2}$.

3) If $m \equiv 3 \pmod 4$ *and* $m = 4k+3$ *then* $f(m,4) = \frac{4k+1}{8k+4}$.

Proof:

For parts 2 and 3 we use Theorem 2.2.2 to obtain an upper bound on $f(m,s)$. To apply this we need that $f(m,s) > \frac{1}{3}$. There is one case where it does not apply. We mention that when it happens.

0) This follows from Theorem 2.1.1.

0) This follows from Theorem 2.1.1.

1a) $f(1,4) = \frac{1}{4}$ by Theorem 2.1.4.

1b) 1) $m = 4k+1$. The following procedure shows $f(m,4) \geq \frac{4k-1}{8k}$.

1. Divide $M_1, \ldots, M_{4k}$ into $(\frac{4k-1}{8k}, \frac{4k+1}{8k})$.
2. Divide M_{4k+1} into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1 and S_2 each get $2k$ of the $\frac{4k+1}{8k}$-sized pieces.

4. S_3 and S_4 each get $2k$ of the $\frac{4k-1}{8k}$-sized pieces and one of the $\frac{1}{2}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{4} = \frac{8k+2}{4} = 2k + \frac{1}{2}$,

$$f(4k+1,4) \le \min\left\{\frac{4k+1}{4(2k+1)}, 1 - \frac{4k+1}{4(2k)}\right\} = \frac{4k-1}{8k}$$

2) This follows from Theorem 2.1.2.

3) $m = 4k+3$. The following procedure shows $f(m,4) \ge \frac{4k+1}{8k+4}$.

1. Divide $M_1, \ldots, M_{4k+2}$ into $(\frac{4k+1}{8k+4}, \frac{4k+3}{8k+4})$.
2. Divide M_{4k+3} into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1 and S_2 each get $2k+1$ of the $\frac{4k+3}{8k+4}$-sized pieces.
4. S_3 and S_4 each get $2k+1$ of the $\frac{4k+1}{8k+4}$-sized pieces and one of the $\frac{1}{2}$-sized pieces.

If $k \ge 1$ then $f(4k+3) > \frac{1}{3}$ then, by Theorem 2.2.2, noting that $\frac{2m}{4} = \frac{8k+6}{4} = 2k+1+\frac{1}{2}$,

$$f(4k+3,4) \le \min\left\{\frac{4k+3}{4(2k+2)}, 1 - \frac{4k+3}{4(2k+1)}\right\} = \frac{4k+1}{8k+4}$$

If $k = 0$ then by Theorem 2.4

$$f(3,4) \le \max\left\{1 - \frac{3}{4}, \frac{3}{3*4}, \frac{3}{4} - \frac{1}{2}\right\} = \frac{1}{4}$$

∎

6 m Muffins, Five Students

We first prove a general theorem that covers many but not all cases.

Theorem 6.1 *The following table gives the value of $f(m,5)$ depending on what m is mod 30.*

1. *If 5 divides m then the table says 1 and nothing is put in the other columns.*
2. *If Theorem 2.8 applies then we give the upper bound and the parameters $x_1, x_2, y_1, y_2, y_3, z_1, z_2$. In all of these cases the upper bound equals the lower bound.*
3. *The table is mod 30. Some of the results need a finer subdivision. In these cases we note this and give the finer subdivision.*
4. *If 5 does not divide s and Theorem 2.8 does not apply we just note that its hard.*

BILL- CHECK EDGE CASES, $L = 0$ OR $L = 1$.

All of the bounds below hold when $L \geq 1$.

m	$f(m,s)$	x_1	x_2	y_1	y_2	y_3	z_1	z_2	*Comment*
$30L$	1								$m \equiv 0 \pmod{s}$
$30L+1$									*Hard*
$30L+2$		2	3	0	$8L$	$12L$	0	$12L+4$	
$30L+3$									*Hard*
$30L+4$									*Hard*
$30L+5$	1								$m \equiv 0 \pmod{s}$
$30L+6$									*3 cases*
$90L+6$		3	2	$6L$	$36L+3$	$36L+2$	0	0	
$90L+36$		3	2	$6L+2$	$36L+15$	$30L+12$	0	0	
$90L+66$		2	3	$6L+4$	$16L+12$	$30L+22$	0	$60L+44$	
$30L+7$									*Hard*
$30L+8$									*2 cases*
$60L+8$		4	1	$9L+1$	$24L+4$	$15L+2$	0	0	
$60L+38$		3	2	$9L+5$	$9L+6$	$15L+9$	3	$30L+19$	
$30L+9$		2	3	0	$8L+2$	$12L+3$	0	$12L+6$	
$30L+10$	1								$m \equiv 0 \pmod{s}$
$30L+11$									*Hard*
$30L+12$									*4 cases*
$120L+12$		4	1	$18L+2$	$48L+4$	$30L+3$	0	0	
$120L+42$		2	3	$15L+5$	$12L+4$	$33L+11$	1	$108L+39$	
$120L+72$		4	1	$18L+11$	$48L+28$	$30L+18$	0	0	
$120L+102$		2	3	$15L+12$	$12L+10$	$33L+27$	3	$108L+93$	

m	$f(m,s)$	x_1	x_2	y_1	y_2	y_3	z_1	z_2	*Comment*
$30L+13$		3	2	$2L$	$12L+6$	$10L+4$	2	0	
$30L+14$									*Hard*
$30L+15$	1								$m \equiv 0 \pmod s$
$30L+16$									*Hard*
$30L+17$		2	3	0	$8L+4$	$12L+6$	0	$12L+10$	
$30L+18$									*2 cases*
$60L+18$		3	2	$9L+2$	$9L+3$	$15L+4$	3	$30L+9$	
$60L+48$		4	1	$9L+7$	$24L+20$	$15L+12$	0	0	
$30L+19$									*Hard*
$30L+20$	1								$m \equiv 0 \pmod s$
$30L+21$									*3 cases*
$90L+21$		2	3	$6L+1$	$16L+4$	$30L+7$	0	$60L+14$	
$90L+51$		3	2	$6L+3$	$36L+21$	$30L+17$	0	0	*Check*
$90L+81$		3	2	$6L+5$	$36L+33$	$30L+27$	0	6	*Check*
$30L+22$									
$120L+22$									
$120L+52$									
$120L+82$									
$120L+112$									
$30L+23$									*need data*
$30L+24$		2	3	0	$8L+6$	$12L+9$	0	$12L+12$	
$30L+25$	1								$m \equiv 0 \pmod s$
$30L+26$									*Hard*
$30L+27$									*Hard*
$30L+28$									*need data*
$30L+29$									*Hard*

Theorem 6.2

1. *If* $m \equiv 0 \pmod 5$ *then* $f(m,s) = 1$.
2. $f(1,5) = \frac{1}{5}$.
3. $f(2,5) = \frac{1}{5}$
4. $f(3,5) = \frac{1}{4}$
5. $f(4,5) = \frac{3}{10}$
6. $f(6,5) = \frac{2}{5}$
7. $f(7,5) = \frac{1}{3}$
8. $f(8,5) = \frac{2}{5}$
9. $f(9,5) = \frac{2}{5}$
10. $\frac{2}{5} \le f(11,5) \le \frac{11}{25}$.
11. $f(12,5) = \frac{2}{5}$
12. $f(13,5) = \frac{13}{30}$
13. $f(14,5) = \frac{11}{25}$
14. $f(16,5) = \frac{16}{35}$
15. $f(17,5) = \frac{13}{30}$
16. $f(18,5) = \frac{9}{20}$
17. $f(19,5) = \frac{16}{35}$
18. $f(21,5) = \frac{7}{15}$

19. $f(22,5) = \frac{9}{20}$

Proof:

1) This follows from Theorem 2.1.1.

2) This follows from Theorem 2.1.5.

3) The following procedure shows $f(3,5) \geq \frac{1}{4}$.

1. Divide M_1 into $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.
2. Divide M_2, M_3 into $(\frac{3}{10}, \frac{7}{20}, \frac{7}{20})$.
3. S_1, S_2, S_3, S_4 each get one of the $\frac{1}{4}$-sized pieces and one of the $\frac{7}{20}$-sized pieces.
4. S_5 gets two of the $\frac{3}{10}$-sized pieces.

$f(3,5) \leq \frac{1}{4}$ by Theorem 2.4.

4) The following procedure shows $f(4,5) \geq \frac{3}{10}$.

1. Divide M_1, M_2 into $(\frac{3}{10}, \frac{3}{10}, \frac{2}{5})$.
2. Divide M_3, M_4 into $(\frac{1}{2}, \frac{1}{2})$.
3. S_1, S_2, S_3, S_4 each get a $\frac{3}{10}$-sized piece and a $\frac{1}{2}$-sized piece.
4. S_5 gets two of the $\frac{2}{5}$-sized pieces.

By Theorem 2.4 $f(4,5) \leq \frac{3}{10}$.

5) This follows from Theorem 2.8 with $m = 6$, $s = 5$, $\delta = \frac{2}{5}$, $x_1 = 2$, $y_1 = 3$, $x_2 = 3$, $y_2 = 2$.

6) The following procedure shows $f(7,5) \geq \frac{1}{3}$.

1. Divide $M_1, \ldots, M_6$ into $(\frac{7}{15}, \frac{8}{15})$.

2. Divide M_7 into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

3. S_1, S_2 each get three $\frac{7}{15}$-sized pieces.

4. $S_3, \ldots, S_7$ each get two $\frac{1}{3}$-sized pieces and two $\frac{8}{15}$-sized pieces.

To show $f(7,5) \le \frac{1}{3}$ there are two cases.

Case 1: An optimal division cuts some muffin into ≥ 3 pieces. One of those pieces will be of size $\le \frac{1}{3}$, hence $f(7,5) \le \frac{1}{3}$.

Case 2: An optimal division cuts all muffins into ≤ 2 pieces. We can assume that every muffin is in exactly 2 pieces since if a muffin is not cut we can still cut it $(\frac{1}{2}, \frac{1}{2})$. Hence there are 10 pieces. By Theorem 2.2.2, noting that $\frac{2m}{5} = \frac{14}{5} = 2 + \frac{4}{5}$,

$$f(7,5) \le \min\left\{\frac{7}{5 \times 3}, 1 - \frac{7}{5 \times 2}\right\} = \frac{3}{10}$$

Since we have a division where the smallest piece is $\frac{1}{3}$ this case does not happen.

7) The following procedure shows $f(8,5) \ge \frac{2}{5}$.

ALEX- I CAN GET A GENERAL THEOREM FROM WHICH THIS FALLS OUT.

1. Divide $M_1, \ldots, M_8$ into $(\frac{2}{5}, \frac{3}{5})$.

2. S_1, S_2, S_3, S_4 each get one $\frac{2}{5}$-sized pieces and two $\frac{3}{5}$-sized piece.

3. S_5 gets four $\frac{2}{5}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{5} = \frac{16}{5} = 3 + \frac{1}{5}$,

$$f(8,5) \le \min\left\{\frac{8}{5 \times 4}, 1 - \frac{8}{5 \times 3}\right\} = \frac{2}{5}.$$

8) The following procedure shows $f(9,5) \geq \frac{2}{5}$.

ALEX- I WILL HAVE A GEN THEOREM FROM WHICH THIS FALLS OUT.

1. Divide $M_1, \ldots, M_6$ into $(\frac{2}{5}, \frac{3}{5})$.

2. Divide M_7, M_8, M_9 into $(\frac{1}{2}, \frac{1}{2})$.

3. S_1, S_2 each get three $\frac{3}{5}$-sized piece.

4. $S_3, S_4, S_5, S_6, S_7, S_8, S_9$ each get two $\frac{1}{2}$-sized pieces and two $\frac{2}{5}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{5} = \frac{18}{5} = 3 + \frac{3}{5}$,

$$f(9,5) \leq \min\left\{\frac{9}{5 \times 4}, 1 - \frac{9}{5 \times 3}\right\} = \frac{2}{5}.$$

9) By Theorem 2.1.6 $f(11,5) \geq \frac{2}{5}$. By Theorem 2.2.2 $f(11,5) \leq \frac{11}{25}$.

10) The following procedure shows $f(12,5) \geq \frac{2}{5}$.

1. Divide $M_1, \ldots, M_{12}$ into $(\frac{2}{5}, \frac{3}{5})$.

2. S_1 and S_2 each get six of the $\frac{2}{5}$-sized pieces.

3. S_3, S_4, S_5 each get four of the $\frac{3}{5}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{24}{5} = 4 + \frac{4}{5}$,

$$f(12,5) \leq 1 - \frac{12}{5 \times 4} = \frac{2}{5}$$

11) The following procedure shows $f(13,5) \geq \frac{13}{30}$.

1. Divide $M_1, \ldots, M_6$ into $(\frac{13}{30}, \frac{17}{30})$.

2. Divide M_7, M_8, M_9 into $(\frac{7}{15}, \frac{8}{15})$.

3. Do not divide $M_{10}, M_{11}, M_{12}, M_{13}$.

4. S_1 gets six of the $\frac{13}{30}$-sized pieces.

5. S_2, S_3, S_4 each get one of the $\frac{7}{15}$-sized pieces, two of the $\frac{17}{30}$-sized pieces, and one of the 1-sized pieces,

6. S_5 gets three of the $\frac{8}{15}$-sized pieces and one of the 1-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{26}{5} = 5 + \frac{1}{5}$,

$$f(13,5) \leq \frac{13}{5 \times 6} = \frac{13}{30}$$

12) The following procedure shows $f(14,5) \geq \frac{11}{25}$.

1. Divide $M_1, \ldots, M_{10}$ into $(\frac{11}{25}, \frac{14}{25})$.

2. Divide $M_{11}, \ldots, M_{14}$ into $(\frac{12}{25}, \frac{13}{25})$.

3. S_1, S_2 each get five of the $\frac{14}{25}$-sized pieces.

4. S_3, S_4 each get four of the $\frac{11}{25}$-sized pieces, and two of the $\frac{11}{25}$-sized pieces.

5. S_5 gets two of the $\frac{11}{25}$-sized pieces and four of the $\frac{12}{25}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{28}{5} = 5 + \frac{3}{5}$,

$$f(14,5) \leq 1 - \frac{14}{5 \times 5} = \frac{11}{25}$$

13) The following procedure shows $f(16,5) \geq \frac{16}{35}$.

1. Divide $M_1, \ldots, M_{14}$ into $(\frac{16}{35}, \frac{19}{35})$

2. Divide M_{15}, M_{16} into $(\frac{17}{35}, \frac{18}{35})$

3. S_1, S_2 each get seven of the $\frac{16}{35}$-sized pieces.

4. S_3, S_4 each get five of the $\frac{19}{35}$-sized pieces and one of the $\frac{17}{35}$-sized pieces.

5. S_5 gets two of the $\frac{18}{35}$-sized pieces and four of the $\frac{19}{35}$-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{32}{5} = 6 + \frac{2}{5}$,

$$f(16,5) \leq \frac{16}{5 \times 7} = \frac{16}{35}$$

14) The following procedure shows $f(17,5) \geq \frac{13}{30}$.

1. Divide $M_1, \ldots, M_6$ into $(\frac{13}{30}, \frac{17}{30})$

2. Divide M_7, M_8, M_9 into $(\frac{9}{15}, \frac{8}{15})$

3. $M_{10}, \ldots, M_{17}$ are uncut.

4. S_1 gets six of the $\frac{17}{30}$-sized pieces.

5. S_2, S_3, S_4 each get two of the $\frac{13}{30}$-sized pieces, one of the $\frac{16}{30}$-sized pieces, and two 1-sized pieces.

6. S_5 gets three of the $\frac{14}{30}$-sized pieces and two 1-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{34}{5} = 6 + \frac{4}{5}$,

$$f(17,5) \leq 1 - \frac{17}{5 \times 6} = \frac{13}{30}$$

15) The following procedure shows $f(18,5) \geq \frac{9}{20}$.

1. Divide $M_1, \ldots, M_8$ into $(\frac{9}{20}, \frac{11}{20})$

2. Divide $M_9, M_{10}, M_{11}, M_{12}$ into $(\frac{1}{2}, \frac{1}{2})$

3. $M_{13}, \dots, M_{18}$ are uncut.

4. S_1 gets eight of the $\frac{9}{20}$-sized pieces.

5. $S_2, \dots, S_5$ each get one $\frac{1}{2}$-sized piece, two $\frac{11}{20}$-sized pieces, and two 1-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{36}{5} = 7 + \frac{1}{5}$,

$$f(18,5) \leq \frac{18}{5 \times 8} = \frac{9}{20}$$

16) The following procedure shows $f(19,5) \geq \frac{16}{35}$.

1. Divide $M_1, \dots, M_{14}$ into $(\frac{16}{35}, \frac{19}{35})$

2. Divide M_{15}, M_{16}, M_{17} into $(\frac{17}{35}, \frac{18}{35})$

3. Do not divide M_{18}, M_{19}.

4. S_1, S_2 each get seven $\frac{19}{35}$-sized pieces.

5. S_3 gets five $\frac{16}{35}$-sized piece, one $\frac{17}{35}$ piece and two $\frac{18}{35}$-sized pieces.

6. S_4 gets five $\frac{16}{35}$-sized pieces, one $\frac{18}{35}$-sized piece, and one 1-piece.

7. S_5 gets four $\frac{16}{35}$-sized pieces, two $\frac{17}{35}$-sized pieces, and one 1-piece.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{38}{5} = 7 + \frac{3}{5}$,
$f(19,5) \leq 1 - \frac{19}{5*7} = \frac{16}{35}$

17) The following procedure shows $f(21,5) \geq \frac{7}{15}$.

1. Divide $M_1, \dots, M_{18}$ into $(\frac{7}{15}, \frac{8}{15})$

2. M_{19}, M_{20}, M_{21} are uncut.

3. S_1, S_2 each get nine $\frac{7}{15}$-sized pieces.

4. S_3, S_4, S_5 gets six $\frac{8}{15}$-sized piece and one 1-sized piece.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{42}{5} = 8 + \frac{2}{5}$,

$f(21, 5) \leq \frac{21}{5*9} = \frac{7}{15}$

18) The following procedure shows $f(22, 5) \geq \frac{9}{20}$.

1. Divide $M_1, \ldots, M_{18}$ into $(\frac{9}{20}, \frac{11}{20})$

2. M_{19}, M_{20}, M_{21} are uncut.

3. S_1 gets eight $\frac{11}{20}$-sized pieces.

4. S_2, S_3, S_4, S_5 each get two $\frac{9}{20}$-sized piece, one $\frac{1}{2}$-sized piece, and three 1-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{44}{5} = 8 + \frac{4}{5}$,

$f(22, 5) \leq \frac{21}{5*9} = \frac{9}{20}$

19) $f(23, 5)$

20) The following procedure shows $f(24, 5) \geq \frac{7}{15}$.

STILL WORKING ON IT

$24/5 = 216/45$

1. Divide $M_1, \ldots, M_{18}$ into $(\frac{7}{15}, \frac{8}{15})$

2. $M_{19}, \ldots, M_{24}$ are uncut.

3. S_1, S_2 each get nine $\frac{7}{15}$-sized pieces.

4. S_2, S_3, S_4, S_5 each get two $\frac{9}{20}$-sized piece, one $\frac{1}{2}$-sized piece, and three 1-sized pieces.

By Theorem 2.2.2, noting that $\frac{2m}{s} = \frac{48}{5} = 9 + \frac{3}{5}$,

$f(24, 5) \leq 1 - \frac{24}{5*9} = \frac{21}{45}$

■

References

[1] A. Frank. The muffin problem, 2013. Described to Jeremy Copeland and in the New York Times Numberplay Online Blog `wordplay.blogs.nytimes.com/2013/08/19/cake`.