Supplementary Appendix for "Reallocative Auctions and Core Selection"

Marzena Rostek* and Nathan Yoder†

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A Omitted Computations from Examples 1-3.

Example 1. The surplus function is

$$v(\{1,2,3\},\bar{q}) = \frac{1}{2}(16+16+4) - \frac{1}{6}(4+4+2-1)^2 = \frac{9}{2}; \quad v(\{1,3\},\bar{q}) = v(\{2,3\},\bar{q}) = \frac{15}{4};$$
$$v(\{1,2\},\bar{q}) = \frac{15}{4}; \qquad v(\{3\},\bar{q}) = \frac{3}{2}; \qquad v(\{1\},\bar{q}) = v(\{2\},\bar{q}) = \frac{7}{2}.$$

Then by Proposition 1(iii), the sum of the Vickrey payoffs for the coalition of the auctioneer and bidder 1 is

$$\pi_a^V(X) + \pi_1^V(X) = -V(\{0,1,2,3\},\bar{q}) + V(\{0,1,2\},\bar{q}) + V(\{0,1,3\},\bar{q}) = 3 < V(\{0,1\},\bar{q}).$$

Example 2. The efficient allocation among the participating bidders is

$$q_1^e(X,\bar{q}) = \begin{bmatrix} 0.4286 & 0.4286 \end{bmatrix}'; \qquad q_2^e(X,\bar{q}) = q_3^e(X,\bar{q}) = \begin{bmatrix} 0.2857 & 0.2857 \end{bmatrix}',$$

and the surplus function is

$$v(\{1\},\bar{q}) = \frac{3}{2}, \; v(\{2\},\bar{q}) = v(\{3\},\bar{q}) = \frac{5}{4},$$
$$v(\{2,3\},\bar{q}) = \frac{13}{8}, \; v(\{1,2\},\bar{q}) = v(\{1,3\},\bar{q}) = \frac{17}{10}, \; v(\{1,2,3\},\bar{q}) = \frac{25}{14}.$$

Then the Vickrey payoff profile $\pi^V(X)$ can be computed from Proposition 1(iii), and is not blocked by any coalition of the auctioneer and a single bidder: $V(\{a,1\},\bar{q}) = \frac{3}{2} <$

* University of Wisconsin-Madison, Department of Economics; E-mail: mrostek@ssc.wisc.edu.

† University of Georgia, Department of Economics; E-mail: nathan.yoder@uga.edu.

$\pi_a^V(X) + \pi_1^V(X) = \frac{113}{70}$, and for each $i \in \{2,3\}$, $V(\{a,i\},\bar{q}) = \frac{5}{4} < \pi_a^V(X) + \pi_i^V(X) = \frac{431}{280}$. Since it can never be blocked by a coalition of the auctioneer and all but one participating bidder (Proposition 1(iii))[1] or a coalition that does not include the auctioneer (Proposition 2), it follows that it is in the bidder-core. In fact, it is in the core, since the auctioneer's revenue is $\pi_a^V(X) = v(X,\bar{q}) - \sum_{i=1}^{3} \pi_i^V(X) = \frac{25}{14} - \frac{9}{56} - \frac{6}{35} = \frac{407}{280} > 0$.

Example 3. The efficient allocation among the participating bidders is

$$\begin{aligned} q_1^e(X,\bar{q}) &= [0.784374 \;\; 0.0673261]', \\ q_2^e(X,\bar{q}) &= [0.0360147 \;\; 0.0313829]', \\ q_3^e(X,\bar{q}) &= [0.179611 \;\; -0.098709]'. \end{aligned}$$

By Lemma 3, the surplus function has a simple form: $v(Z,\bar{q}) = \theta'\bar{q} - \frac{1}{2}\bar{q}'H(Z)\bar{q}$, where $H(Z) = \left(\sum_{i\in Z} S_i^{-1}\right)^{-1}$ is the harmonic mean of the substitution patterns of the bidders in Z. Then by Proposition 1(iii), the incentive for the auctioneer to cancel the auction and negotiate with bidder 1 is

$$\begin{aligned} v(\{1\},\bar{q}) - \pi_a^V(X) - \pi_1^V(X) &= v(\{1\},\bar{q}) + v(X,\bar{q}) - v(\{1,2\},\bar{q}) - v(\{1,3\},\bar{q}) \\ &= \bar{q}'\left(H(\{1,2\}) + H(\{1,3\}) - S_1 - H(X)\right)\bar{q} = \bar{q}'\begin{bmatrix} 0.0015 & -0.0146 \\ -0.0146 & -0.2398 \end{bmatrix}\bar{q} = 0.0015. \end{aligned}$$

Then the Vickrey payoff profile is not in the bidder-core, since it is blocked by the coalition $\{a,1\}$.

B Core-Selecting Auctions are Vickrey Auctions in Reallocative Environments

As is well known, if a mechanism is dominant-strategy incentive compatible and produces an efficient outcome on a smoothly connected domain of bidder preferences, it must be a Groves scheme (Holmström, 1979). In a one-sided auction environment, Goeree and Lien (2016) extend this result, showing that with independent private values, if an auction is *Bayesian* incentive compatible and *core-selecting*, it must be a Vickrey auction. (A similar argument shows that the same is true for dominant-strategy incentive compatiblity even without an independence assumption.) Here, we extend this result to the reallocative environments that we consider.

We begin with some definitions.

[1]Since each bidder's Vickrey payoff is equal to their marginal product to the coalition of participating bidders X, the other bidders and the auctioneer cannot benefit from excluding them.

Definition 1 (Convex and Rich Preference Domains, Independent Preferences, Direct Mechanisms). We say that *the domain of bidder preferences is convex* if

(a) The set of potential bidders is $I = \coprod_{n=1}^{N} \Theta_n$ for some convex sets $\Theta_n \subset \mathbb{R}^K$ and $N > 0$; and[2]

(b) For each $n \in \{1, \ldots, N\}$, there is a function $U_n : \mathbb{R}^K \times \Theta_n$, convex in Θ_n, such that for each $\theta_n \in \Theta_n$, $u_{\theta_n}(\cdot) = U_n(\cdot, \theta_n)$.

When the domain of bidder preferences is convex:

- We say bidders' preferences are *independent* if there is a common prior μ over sets of participants X that assigns positive probability only to sets of participants $\{\theta_n\}_{n=1}^{N}$ such that $\theta_n \in \Theta_n$ for each n, and does so such that the θ_n are independent.

- We say that *the domain of bidder preferences is rich* if for every $n \in \{1, \ldots, N\}$, there exists θ such that $\nabla u_{\theta_n}(0) = \nabla u_{\theta_\ell}(q^e_{\theta_\ell}(\{\theta_j\}_{j \neq n}, \bar{q}))$ for each $\ell \neq n$. We say it is *uniformly rich* if for every $n \in \{1, \ldots, N\}$ and every vector $p \in M$, there is a $\theta_n \in \Theta_n$ such that $\nabla u_{\theta_n}(0) = p$.

  Richness means that for every n, there is a type profile for which it is efficient to assign the nth bidder a quantity of zero. Uniform richness means that we can always find such a type profile *fixing* θ_{-n}.

- A *direct mechanism* is a pair of functions $(\chi : \prod_{n=1}^{N} \Theta_n \to \mathbb{R}^{K \times N}, t : \prod_{n=1}^{N} \Theta_n \to \mathbb{R}^{K \times N})$ such that for each $\theta \in \prod_{n=1}^{N} \Theta_i$, $\sum_{n=1}^{N} \chi_n(\theta) = \bar{q}$.

- (χ, t) is *dominant-strategy incentive compatible* if for all $n \in \{1, \ldots, N\}$, all $\theta_{-n} = \{\theta_\ell\}_{\ell \neq n} \in \prod_{\ell \neq n} \Theta_\ell$, and all $\theta_n, \theta'_n \in \Theta_n$,

$$U_n(\chi_n(\theta_n, \theta_{-n}), \theta_n) - t_n(\theta_n, \theta_{-n}) \geq U_n(\chi_n(\theta'_n, \theta_{-n}), \theta_n) - t_n(\theta'_n, \theta_{-n}),$$

  and is *Bayesian incentive compatible* if for all $n \in \{1, \ldots, N\}$ and all $\theta_n, \theta'_n \in \Theta_n$,

$$E_\mu[U_n(\chi_n(\theta_n, \theta_{-n}), \theta_n) - t_n(\theta_n, \theta_{-n}) | \theta_n] \geq E_\mu[U_n(\chi_n(\theta'_n, \theta_{-n}), \theta_n) - t_n(\theta'_n, \theta_{-n}) | \theta_n].$$

- (χ, t) is *(bidder-)core-selecting* if, for any profile $\theta \in \prod_{n=1}^{N} \Theta_n$, the payoff profile $\pi^{(\chi,t)}(\theta) = \{\{U_n(\chi_n(\theta), \theta_n) - t_n(\theta)\}_{\theta_n \in X}, \sum_{n=1}^{N} t_n(\theta)\}$ is in the (bidder-)core given the set of participating bidders $\{\theta_n\}_{n=1}^{N}$.

Proposition B.1 (Vickrey is Uniquely Bidder-Core-Selecting). *Suppose that the domain of bidder preferences is convex, and that the direct mechanism (χ, t) is bidder-core-selecting. If either*

[2] That is, I is the coproduct, or disjoint union, of N copies of Θ.

i. (χ, t) is dominant-strategy incentive compatible and the domain of bidder preferences is uniformly rich, or

ii. bidders' preferences are independent, (χ, t) is Bayesian incentive compatible, and the domain of bidder preferences is rich,

then (χ, t) produces the same allocation and payoff profile as the Vickrey auction: For each profile $\theta \in \prod_{n=1}^N \Theta_n$, we have $\chi_n(\theta) = q^e_{\theta_n}(\{\theta_n\}_{n=1}^N, \bar{q})$ for each $n \in \{1, \ldots, N\}$ and $\pi^{(\chi,t)}(\theta) = \pi^V(\{\theta_n\}_{n=1}^N)$.

Lemma B.1 (Ausubel and Milgrom (2002) Theorem 5). *Suppose that given the set of participating bidders X, the payoff profile π is in the bidder-core. Then for each $i \in X$, $0 \leq \pi_i \leq \pi_i^V(X)$.*

Proof. Since π is in the bidder-core, for each $i \in X$, $\pi_i \geq V(\{i\}, \bar{q}) = v(\{i\}, 0) = 0$. By Proposition 1(iii), $\pi_i^V(X) = v(X, \bar{q}) - v(X \setminus \{i\}, \bar{q})$. Suppose toward a contradiction that $\pi_i > \pi_i^V(X)$ for some $i \in X$. Then $\sum_{j \in \{a\} \cup X \setminus \{i\}} \pi_j = v(X, \bar{q}) - \pi_i < v(X \setminus \{i\}, \bar{q}) = V(\{a\} \cup X \setminus \{i\}, \bar{q})$, and $\{a\} \cup X \setminus \{i\}$ blocks π, a contradiction. □

Proof of Proposition B.1 (Vickrey is Uniquely Bidder-Core-Selecting) Since (χ, t) is bidder-core-selecting, we have $\sum_{n=1}^N u_{\theta_n}(\chi_n(\theta)) = \sum_{i \in \{\theta_n\}_{n=1}^N \cup \{a\}} \pi_i^{(\chi,t)}(\theta) = v(\{\theta_n\}_{n=1}^N, \bar{q})$. It follows from (2) that χ is efficient: $\chi_n(\theta) = q^e_{\theta_n}(\{\theta_n\}_{n=1}^N, \bar{q})$ for each $n \in \{1, \ldots, N\}$.

Consequently, by Theorem 1 in Holmström (1979) (when (i) holds) or Proposition 1 in Krishna and Maenner (2001) (when (ii) holds), there are functions $\{h_n : \prod_{\ell \neq n} \theta_\ell \to \mathbb{R}\}$ such that for each $\theta \in \prod_{n=1}^N \Theta_n$ and each $n \in \{1, \ldots, N\}$, $\pi_{\theta_n}^{(\chi,t)}(\theta) = \pi_{\theta_n}^V(\{\theta_n\}_{n=1}^N) + h_n(\theta_{-n})$; when (ii) holds, these functions are constant. Then by Lemma B.1, $h_n(\theta_{-n}) \in [-\pi_{\theta_n}^V(\{\theta_n\}_{n=1}^N), 0]$.

Recall that for each $Z \subseteq I$, the Kuhn-Tucker conditions for a maximum in (2) are

$$\sum_{i \in Z} q_i = \bar{q}, \qquad \nabla u_i(q_i) = p \text{ for each } i \in Z \text{ for some } p \in \mathbb{R}^K.$$

Choose $n \in \{1, \ldots, N\}$. Suppose first that (i) holds, choose $\theta_{-n} \in \prod_{\ell \neq n} \Theta_n$, and let $\hat{p} \in \mathbb{R}^K$ be the p that satisfies these conditions, along with $\{q_i^e(Z, \bar{q})\}_{i \in Z}$, for $Z = \{\theta_\ell\}_{\ell \neq n}$. Since the preference domain is uniformly rich, there exists $\hat{\theta}_n \in \Theta_n$ such that $\nabla u_{\hat{\theta}_n}(0) = \hat{p}$. Then $\{\hat{p}, \{q^e_{\theta_\ell}(\{\theta_\ell\}_{\ell \neq n}, \bar{q})\}_{\ell \neq n}, 0\}$ satisfies these conditions for $Z = \{\theta_\ell\}_{\ell \neq n} \cup \{\hat{\theta}_n\}$. Hence, $v(\{\theta_\ell\}_{\ell \neq n} \cup \{\hat{\theta}_n\}, \bar{q}) = v(\{\theta_\ell\}_{\ell \neq n}, \bar{q})$ and $\pi_{\hat{\theta}_n}^V(\{\theta_\ell\}_{\ell \neq n} \cup \{\hat{\theta}_n\}) = 0$. Then $h_n(\theta_{-n}) = 0$. Since θ_{-n} was arbitrary, it follows that for all $\theta \in \prod_{n=1}^N \Theta_n$, we have $\pi_{\theta_n}^{(\chi,t)}(\theta) = \pi_{\theta_n}^V(\{\theta_n\}_{n=1}^N)$.

Alternatively, suppose that (ii) holds. Then there exists $\theta' \in \prod_{n=1}^N \Theta_n$ such that $\nabla u_{\theta'_n}(0) = \nabla u_{\theta'_\ell}(q^e_{\theta'_\ell}(\{\theta'_j\}_{j \neq n}, \bar{q})) = \hat{p}$ for each $\ell \neq n$. Then $\{\hat{p}, \{q_i^e(Z, \bar{q})\}_{i \in Z}\}$ satisfies the Kuhn-Tucker conditions for $Z = \{\theta'_\ell\}_{\ell \neq n}$, and $\{\hat{p}, \{q^e_{\theta'_\ell}(\{\theta_{\ell'}\}_{\ell \neq n}, \bar{q})\}_{\ell \neq n}, 0\}$ satisfies these conditions for $Z = \{\theta'_\ell\}_{n=1}^N$. Hence, $v(\{\theta'_\ell\}_{\ell \neq n} \cup \{\hat{\theta}'_n\}, \bar{q}) = v(\{\theta'_\ell\}_{\ell \neq n}, \bar{q})$ and $\pi_{\hat{\theta}'_n}^V(\{\theta'_\ell\}_{\ell \neq n} \cup \{\hat{\theta}'_n\}) = 0$.

Then $h_n(\theta'_{-n}) = 0$. Since (ii) holds, h_n is constant; it follows that for all $\theta \in \prod_{n=1}^N \Theta_n$, we have $\pi_{\theta_n}^{(\chi,t)}(\theta) = \pi_{\theta_n}^V(\{\theta_n\}_{n=1}^N)$.

Then in either case, for all $\theta \in \prod_{n=1}^N \Theta_n$, we have $\pi_i^{(\chi,t)}(\theta) = \pi_i^V(\{\theta_n\}_{n=1}^N)$ for each $i \in \{\theta_n\}_{n=1}^N$; since (χ,t) is bidder-core selecting, $\sum_{i\in\{\theta_n\}_{n=1}^N \cup \{a\}} \pi_i^{(\chi,t)}(\theta) = v(\{\theta_n\}_{n=1}^N, \bar{q}) = \sum_{i\in\{\theta_n\}_{n=1}^N \cup \{a\}} \pi_i^V(\{\theta_n\}_{n=1}^N)$. Hence, we must have $\pi_a^{(\chi,t)}(\theta) = \pi_a^V(\{\theta_n\}_{n=1}^N)$ as well. $\square$

C Nonemptiness of the Core

Here, we show that when (bidder-)core selection is not possible, it is never because the (bidder-)core is empty: there is always some payoff profile in the core. Rather, (bidder-)core selection fails because none of those payoff profiles coincide with the one induced by the Vickrey auction.

Proposition C.1. *The core is nonempty. Hence, so is the bidder-core.*

Proof. We show that the cooperative game $\langle X \cup \{a\}, V(\cdot,\bar{q})\rangle$ is *balanced* in the sense of the Bondareva-Shapley theorem. Let $\lambda : 2^{X\cup\{a\}} \setminus \{\emptyset\} \to [0,1]$ be a function such that for each $i \in X \cup \{a\}$, $\sum_{Z\subseteq X\cup\{a\}:Z\ni i} \lambda(Z) = 1$. Then

$$\begin{aligned}
\sum_{Z\subseteq X\cup\{a\}} \lambda(Z)V(Z,\bar{q}) &= \sum_{Z\subseteq X} \lambda(Z)v(Z,0) + \sum_{Z\subseteq X:Z\ni a} \lambda(Z)v(Z,\bar{q}) \\
&= \sum_{Z\subseteq X\cup\{a\}} \lambda(Z) \max_{\{q_i^Z\}_{i\in Z\setminus\{a\}}} \left\{ \sum_{i\in Z\setminus\{a\}} u_i(q_i^Z) \text{ s.t. } \sum_{i\in Z\setminus\{a\}} q_i^Z = \begin{matrix} \bar{q}, & a\in Z; \\ 0, & a\notin Z \end{matrix} \right\} \\
&= \max_{\{q_i^Z\}_{Z\subseteq X\cup\{a\}:Z\ni i}^{i\in X}} \left\{ \sum_{i\in X} \sum_{Z\subseteq X\cup\{a\}:Z\ni i} \lambda(Z)u_i(q_i^Z) \text{ s.t. } \sum_{i\in Z\setminus\{a\}} q_i^Z = \begin{matrix} \bar{q}, & a\in Z; \\ 0, & a\notin Z \end{matrix} \right\} \\
&\leq \max_{\{q_i^Z\}_{Z\subseteq X\cup\{a\}:Z\ni i}^{i\in X}} \left\{ \sum_{i\in X} u_i\left(\sum_{Z\subseteq X\cup\{a\}:Z\ni i} \lambda(Z)q_i^Z \right) \text{ s.t. } \sum_{i\in Z\setminus\{a\}} q_i^Z = \begin{matrix} \bar{q}, & a\in Z; \\ 0, & a\notin Z \end{matrix} \right\},
\end{aligned}$$

since the u_i are concave. Now observe that if $\{q_i^Z\}_{Z\subseteq X\cup\{a\}:Z\ni i}^{i\in X}$ satisfies the constraints $\sum_{i\in Z\setminus\{a\}} q_i^Z = \begin{cases} \bar{q}, & a\in Z; \\ 0, & a\notin Z, \end{cases}$ then we must have

$$\begin{aligned}
&\sum_{Z\subseteq X\cup\{a\}:Z\ni a} \lambda(Z) \sum_{i\in Z} q_i^Z = \sum_{Z\subseteq X\cup\{a\}:Z\ni a} \lambda(Z)\bar{q} = \bar{q}; \\
&\sum_{Z\subseteq X\cup\{a\}:Z\not\ni a} \lambda(Z) \sum_{i\in Z} q_i^Z = 0; \\
\Rightarrow &\sum_{i\in X} \sum_{Z\subseteq X\cup\{a\}:Z\ni i} \lambda(Z)q_i^Z = \bar{q}.
\end{aligned}$$

Hence, we have

$$\sum_{Z\subseteq X\cup\{a\}} \lambda(Z)V(Z,\bar{q}) \leq \max_{\{q_i^Z\}^{i\in X}_{Z\subseteq X\cup\{a\}:Z\ni i}} \left\{ \sum_{i\in X} u_i \left(\sum_{Z\subseteq X\cup\{a\}:Z\ni i} \lambda(Z) q_i^Z \right) \text{ s.t. } \sum_{i\in X} \sum_{Z\subseteq X\cup\{a\}:Z\ni i} \lambda(Z) q_i^Z = \bar{q} \right\}$$
$$= \max_{\{q_i\}_{i\in X}} \left\{ \sum_{i\in X} u_i(q_i) \text{ s.t. } \sum_{i\in X} q_i = \bar{q} \right\} = v(X,\bar{q}) = V(X\cup\{a\},\bar{q}).$$

Then $\langle X\cup\{a\}, V(\cdot,\bar{q})\rangle$ is balanced. It follows from the Bondareva-Shapley theorem that the core is nonempty. □

D Quantity Auctioned Comparative Statics

Theorem 1 shows that in a reallocative auction, core selection depends not just on the existence of packages that all bidders view as substitutes, but also on the efficiency of allocating positive quantities of each of those packages to each bidder. This allocation condition depends on the quantity auctioned: When $\bar{q} = 0$, for instance, it can only be satisfied if each bidder's pre-auction allocation is the same. And in Example 1 — which shows that Theorem 1's allocation condition, and thus core selection, can fail in a setting with a single good — if the auctioneer had 5 units of the good for sale instead of 1 unit, each bidder would always receive a positive quantity, regardless of which bidders participate.

This section gives conditions under which a change in quantity auctioned $\bar{q}$ can restore core selection in the general multiple-good case. First, we describe the set of directions in which a change in $\bar{q}$ increases the efficient quantity of packages allocated to a bidder.

Proposition D.1 (Package Allocations and Quantity Auctioned). *Let $\Phi \subset \mathbb{R}^K$ be a set of K linearly independent vectors. As a function of $\bar{q}$, the efficient allocation of packages $T_\Phi^{-1} q_i^e(X,\bar{q})$ to bidder $i \in X$ is increasing in direction $x \in \mathbb{R}^K$ if and only if*

$$x \in C_i(\Phi, X, \bar{q}) \equiv \left\{ x \in \mathbb{R}^K \mid T_\Phi^{-1}(D^2 u_i(q_i^e(X,\bar{q})))^{-1} \left(\sum_{j\in X} D^2 u_j(q_j^e(X,\bar{q}))^{-1} \right)^{-1} x \geq 0 \right\}.$$

Note that if bidders have quadratic valuations, $C_i(\Phi, X, \bar{q}) = \{x \mid T_\Phi^{-1} S_i^{-1} H(X) x \geq 0\}$ for each $\bar{q}$.

Proposition D.1 yields two conditions on quantity auctioned that are each sufficient for Theorem 1's allocation condition to hold. The first is for settings without heterogeneity in marginal utility at the initial allocation:

Corollary D.1 (Theorem 1's Allocation Condition and Quantity Auctioned I). *Suppose that $\nabla u_i(0) = \nabla u_j(0)$ for all $i, j \in I$. If $\bar{q} \in \bigcap_{Z\subseteq I} \bigcap_{x\in\mathbb{R}^K} \bigcap_{i\in Z} C_i(\Phi, Z, x)$, then $T_\Phi^{-1} q_i^e(X,\bar{q}) \geq 0$ for each set of participating bidders $X \subseteq I$.*

The second allows heterogeneity in marginal utility at the initial allocation, while requiring that increases in package allocations are uniformly bounded away from zero.

Corollary D.2 (Theorem 1's Allocation Condition and Quantity Auctioned II). *For any $\epsilon > 0$, let*

$$\tilde{C}_i(\Phi, X, \bar{q}, \epsilon) \equiv \left\{ x \in \mathbb{R}^K \mid T_\Phi^{-1}(D^2 u_i(q_i^e(X, \bar{q})))^{-1} \left(\sum_{j \in X} D^2 u_j(q_j^e(X, \bar{q}))^{-1} \right)^{-1} x \geq \epsilon \mathbb{1} \right\}.$$

For any quantity vector $\bar{q}$ and any $\Delta\bar{q} \in \bigcap_{Z \subseteq I} \bigcap_{x \in \mathbb{R}^K} \bigcap_{i \in Z} \tilde{C}_i(\Phi, Z, x, \epsilon)$ for some $\epsilon > 0$, there exists a scalar $a > 0$ such that $T_\Phi^{-1} q_i^e(X, \bar{q} + a\Delta\bar{q}) \geq 0$ for each set of participating bidders $X \subseteq I$.

Proof of Proposition D.1 (Bidder-Submodularity and Quantity Auctioned): Setting $Z = X$, the Kuhn-Tucker conditions for a maximum in (2) are

$$\sum_{i \in X} q_i = \bar{q}, \qquad \nabla u_i(q_i) = p \text{ for each } i \in X \text{ for some } p \in \mathbb{R}^K.$$

By definition, these are satisfied by setting $p = p(X, \bar{q})$ and $q_i = d_i(p)$ for each $i \in X$, where $\sum_{i \in X} d_i(p(X, \bar{q})) = \bar{q}$. By the implicit function theorem,

$$Dd_i(p) = (D^2 u_i(d_i(p)))^{-1};$$
$$Dp(X, \bar{q}) = \left(\sum_{i \in X} Dd_i(p(X, \bar{q})) \right)^{-1} = \left(\sum_{i \in X} (D^2 u_i(d_i(p(X, \bar{q}))))^{-1} \right)^{-1} = \left(\sum_{i \in X} (D^2 u_i(q_i^e(X, \bar{q})))^{-1} \right)^{-1}.$$

Then by the chain rule,

$$Dq_i^e(X, \bar{q}) = (D^2 u_i(q_i^e(X, \bar{q})))^{-1} \left(\sum_{i \in X} (D^2 u_i(q_i^e(X, \bar{q})))^{-1} \right)^{-1}.$$

The statement follows. $\square$

Proof of Corollary D.2 (Theorem 1's Allocation Condition and Quantity Auctioned II): From the proof of Proposition D.1 and the fundamental theorem of calculus, we have

$$T_\Phi^{-1} q_i^e(X, \bar{q} + a\Delta\bar{q}) =$$
$$T_\Phi^{-1} q_i^e(X, \bar{q}) + a \int_0^1 T_\Phi^{-1} (D^2 u_i(q_i^e(X, \bar{q} + ar\Delta\bar{q})))^{-1} \left(\sum_{j \in X} (D^2 u_j(q_j^e(X, \bar{q} + ar\Delta\bar{q})))^{-1} \right)^{-1} \Delta Q dr.$$

Then let

$$a = \frac{\max_{k \in K, i \in Z, Z \subseteq I} -\{T_\Phi^{-1} q_i^e(Z, \bar{q})\}_k}{\epsilon} \geq \frac{\max_{k \in K, i \in Z, Z \subseteq I} -\{T_\Phi^{-1} q_i^e(Z, \bar{q})\}_k}{\inf_{k \in K, \{x_j\}_{j \in Z} \in \mathbb{R}^{|Z|K}, i \in Z, Z \subseteq I} \{T_\Phi^{-1}(D^2 u_i(x_i))^{-1} \left(\sum_{j \in Z} (D^2 u_j(x_j))^{-1} \right)^{-1} \Delta\bar{q}\}_k}.$$

It follows that $Tq_i^*(J, \bar{q} + a\Delta\bar{q}) \geq 0$ for all J and all $i \in J$. $\square$

E Omitted Proofs

To begin, for any $f : \mathbb{R}^K \to \mathbb{R} \cup \{\infty\}$, let $f^* : \mathbb{R}^K \to \mathbb{R} \cup \{\infty\}$ be the convex conjugate of f, i.e., $f^*(p) \equiv \sup_{q \in \mathbb{R}^K}\{p \cdot q - f(q)\}$.

Lemma E.1. *Let $\{f_i : \mathbb{R}^K \to \mathbb{R}\}_{i=1}^N$ be continuously differentiable, strictly convex, and have $\nabla f_i(\mathbb{R}^K) = -M$. Then for each $\bar{q} \in \mathbb{R}^K$, $\underline{f}_N(\bar{q}) \equiv \inf_{\{q_i\}_{i=1}^N}\{\sum_{i=1}^N f_i(q_i)$ s.t. $\sum_{i=1}^N q_i = \bar{q}\}$ has a solution, $\underline{f}_N$ is convex, and $\underline{f}_N = (\sum_{i=1}^N f_i^*)^*$.*

Proof. First note that for any convex, upper semicontinuous $f : \mathbb{R}^K \to \mathbb{R} \cup \{\infty\}$, $f = f^{**}$ (e.g., Hiriart-Urruty and Lemaréchal (2004) Corollary E.1.3.1) and f^* is upper semicontinuous and convex (e.g., Hiriart-Urruty and Lemaréchal (2004) Theorem E.1.1.2).

Moreover, since each f_i is continuously differentiable and strictly convex, for each $p \in \nabla f_i(\mathbb{R}^K)$, there exists a unique q^* such that $\nabla f(q^*) = p$, and so $f_i^*(p) = p \cdot q^* - f(q^*)$; hence, $\nabla f_i(\mathbb{R}^K) \subseteq \operatorname{dom} f_i^*$ for each i.

Now we induct on N. For the initial step, suppose $N = 2$. Since $-M = \nabla f_i(\mathbb{R}^K) \subseteq \operatorname{dom} f_i^*$ for each $i \in I$, $\operatorname{dom} f_1^* \cap \operatorname{dom} f_2^* \neq \emptyset$. Then by Hiriart-Urruty and Lemaréchal (2004) Theorem E.2.3.2,[3] for each $\bar{q} \in \mathbb{R}^K$, $\underline{f}_2(\bar{q}) = \inf_{q_1,q_2}\{f_1(q_1) + f_1(q_2)$ s.t. $q_1 + q_2 = \bar{q}\}$ has a solution $(q_1^*(\bar{q}), q_2^*(\bar{q}))$, and $\underline{f}_2(\bar{q}) = (f_1^* + f_2^*)^*(\bar{q})$ Then since it is the conjugate of the convex, upper semicontinuous function $f_1^* + f_2^*$, $\underline{f}_2$ is upper semicontinuous and convex.

Now suppose that the claim holds for $N-1$: for each $\bar{q} \in \mathbb{R}^K$, $\underline{f}_{N-1}(\bar{q}) \equiv \inf_{\{q_i\}_{i=1}^{N-1}}\{\sum_{i=1}^{N-1} f_i(q_i)$ s.t. $\sum$ $\bar{q}\}$ has a solution $\{q_i^*(\bar{q})\}_{i=1}^{N-1}$, $\underline{f}_{N-1}$ is convex, and $\underline{f}_{N-1} = (\sum_{i=1}^{N-1} f_i^*)^*$. Since $\underline{f}_{N-1} = (\sum_{i=1}^{N-1} f_i^*)^*$, $\operatorname{dom} \underline{f}_{N-1}^* = \bigcap_{i=1}^{N-1} \operatorname{dom} f_i^* \supseteq -M$. Moreover, observe that we can write $\underline{f}_N(\bar{q}) = \inf_{q_{-N},q_N}\{f_{N-1}(q_{-N}) + f_N(q_N)$ s.t. $q_{-N} + q_N = \bar{q}\}$. Then by Hiriart-Urruty and Lemaréchal (2004) Theorem E.2.3.2, for each $\bar{q} \in \mathbb{R}^K$, $\inf_{q_{-N},q_N}\{\underline{f}_{N-1}(q_{-N}) + f_N(q_N)$ s.t. $q_{-N} + q_N = \bar{q}\}$ has a solution $(q_{-N}^*(\bar{q}), q_N^*(\bar{q}))$, and $\underline{f}_N(\bar{q}) = (\underline{f}_{N-1}^* + f_N^*)^*(\bar{q})$; since $\underline{f}_{N-1} = (\sum_{i=1}^{N-1} f_i^*)^*$, we have $\underline{f}_N(\bar{q}) = (\sum_{i=1}^{N-1} f_i^* + f_N^*)^*(\bar{q})$. Then since it is the conjugate of the convex, upper semicontinuous function $\sum_{i=1}^N f_i^*$, $\underline{f}_N$ is upper semicontinuous and convex. Moreover, $\underline{f}_N(\bar{q}) \equiv \inf_{\{q_i\}_{i=1}^N}\{\sum_{i=1}^N f_i(q_i)$ s.t. $\sum_{i=1}^N q_i = \bar{q}\}$ has a solution given by $\{\{q_i^*(q_{-N}^*(\bar{q}))\}_{i=1}^{N-1}, q_N^*(\bar{q})\}$.

The claim follows by induction. □

Lemma E.2 (Existence of Market-Clearing Prices)**.** *For each $i \in X$, let $b_i : \mathbb{R}^K \to M$ be continuously differentiable, surjective, and have a negative definite Jacobian derivative matrix. Then*

i. Each b_i is bijective, and its inverse b_i^{-1} is continuously differentiable with a negative definite Jacobian.

ii. $d(p) \equiv \sum_{i \in X} b_i^{-1}(p)$ is a bijective map from M to $\mathbb{R}^K$, and so there is a unique $p^*(b)$ such that $d(p^*(b)) = \bar{q}$.

[3] Take $g_i = f_i^*$.

Proof. (i): Suppose toward a contradiction that b_i is not injective. Then there exist distinct $q, q' \in \mathbb{R}^K$ such that $b_i(q) = b_i(q')$. Since b_i is C^1, by the fundamental theorem of calculus, $b_i(q') = b_i(q) + \int_0^1 Db_i(rq' + (1-r)q)(q'-q)dr$. Then $0 = \int_0^1 Db_i(rq' + (1-r)q)(q' - q)dr$ and hence $0 = \int_0^1 (q'-q)' Db_i(rq' + (1-r)q)(q'-q)dr$, a contradiction because the integrand is strictly negative for each r.

Then b_i is bijective, and so has an inverse b_i^{-1}; existence, continuity, and negative definiteness of the Jacobian Db_i^{-1} follows from the inverse function theorem.

(ii): First suppose toward a contradiction that d is not injective. Then there exist distinct $p, p' \in M$ such that $d(p) = d(p')$. By part (i), d is C^1 and has a negative definite Jacobian matrix. Then by the fundamental theorem of calculus, $d(p') = d(p) + \int_0^1 Dd(rp' + (1-r)p)(p'-p)dr$. Then $0 = \int_0^1 Dd(rp' + (1-r)p)(p'-p)dr$ and hence $0 = \int_0^1 (p'-p)' Dd(rp' + (1-r)p)(p'-p)dr$, a contradiction because the integrand is strictly negative for each r.

We now show that d is surjective. For each $i \in X$, let $f_i(q_i) = -\int_0^1 b_i(rq_i) \cdot q_i dr$; then f_i is twice continuously differentiable with positive definite Hessian derivative matrix $D^2 f_i(q_i) = -Db_i(q_i)$. Then f_i is strictly convex, and $\nabla f_i(\mathbb{R}^K) = -b_i(\mathbb{R}^K) = -M$.

Then by Lemma E.1, for each $q \in \mathbb{R}^K$, $\inf_{\{q_i\}_{i\in X}} \{\sum_{i\in X} f_i(q_i) \text{ s.t. } \sum_{i \in X} q_i = q\}$ has a solution $\{q_i^*(q)\}_{i \in X}$. Since the f_i are strictly convex, this solution must satisfy a necessary Kuhn-Tucker condition: there exists $p(q) \in \mathbb{R}^K$ such that $-b_i(q_i^*(q)) = \nabla f_i(q_i^*(q)) = p(q)$ for each $i \in X$. It follows that for each $q \in \mathbb{R}^K$, there exists $p = -p(q)$ such that $q = d(p)$, and so d is surjective. □

Lemma E.3 (Existence and Uniqueness of Efficient Allocations)**.** *The social planner's problem* (1) *has a unique solution* $\{q_i^e(Z, \bar{q})\}_{i \in Z}$ *for each finite* $Z \subseteq I$.

Proof. Existence of a solution follows immediately from Lemma E.1 by letting $f_i = -u_i$. Uniqueness follows from strict convexity of the u_i. □

Proof of Lemma 7 (Eigenvalues and Harmonic Means) We have

$$\sum_{i \in Z\setminus \ell} S_i^{-1} + \sum_{i \in Z \setminus j} S_i^{-1} = \sum_{i \in Z \setminus \ell} S_i^{-1} + \sum_{i \in Z \setminus \{\ell \cup j\}} S_i^{-1} + S_\ell^{-1}$$

$$\Rightarrow \frac{1}{2} I = \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i \in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right) + \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}$$

Left-multiplying by $S_j^{-1} = \sum_{i\in Z\setminus\{\ell\}} S_i^{-1} - \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}$ yields

$$\begin{aligned}
\frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} - \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right) =& \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right) \\
&- \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right) \\
&+ \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right) \\
&- \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1} + \sum_{i\in Z} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right) \\
&+ \frac{1}{2} S_j^{-1}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1} S_\ell^{-1}
\end{aligned}$$

Since the $\{S_i^{-1}\}_{i\in I}$ are symmetric, adding the transpose of both sides yields

$$\begin{aligned}
\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} - \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1} =& \left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right) \\
&- \left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1} + \sum_{i\in Z} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right) \\
&+ \frac{1}{2} S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1} + \frac{1}{2} S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1},
\end{aligned}$$

Then from the harmonic mean identity $(A^{-1}+B^{-1})^{-1} = A - A(A+B)^{-1}A$, we have

$$\begin{aligned}
(H(Z\setminus\ell) + H(Z\setminus j))^{-1} = (H(Z) + H(Z\setminus\{\ell,j\}))^{-1} &+ \frac{1}{2} S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1} \\
&+ \frac{1}{2} S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}.
\end{aligned}$$

Since $A \succeq B$ iff $B^{-1} \succeq A^{-1}$,[4] it follows that $H(Z) + H(Z\setminus\{\ell,j\}) \succeq H(Z\setminus\{\ell\}) + H(Z\setminus\{j\})$ iff $\frac{1}{2} S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1} + \frac{1}{2} S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}$ is positive semidefinite. But since $S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1}$ has a negative eigenvalue, this cannot hold: if x is the eigenvector associated with the negative eigenvalue, $x'\left(S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1} + S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}\right) x < 0$.

[4] $\succeq$ denotes the positive definite order.

F Imperfect Divisibility

For settings with imperectly divisible goods, the following result gives a converse analogous to Proposition 1 to show that the effects due to two-sidedness – particularly those illustrated in Example 1 – are not unique to divisible goods. In particular, if bidders are endowed with the same preferences, there is some pre-auction allocation for which core selection fails.

Proposition F.1 (**No Core Selection with Indivisible Goods**). *Suppose allocations are constrained to be integer vectors. If $|I| \geq 4$, then for any strictly concave valuation u which has a unimodular demand type (in the sense of Baldwin and Klemperer (2019)) when restricted to $q_i \in \mathbb{Z}^K$, there exists a pre-auction allocation $\{t_i\}_{i \in I} \subseteq \mathbb{Z}^K$ such that if each bidder $i \in X$ is endowed with with the valuation $\tilde{u}_i(q_i) = u(q_i + t_i)$, there exists a set of participating bidders $X \subseteq I$ such that the Vickrey auction is not core-selecting.*

Proof of Proposition F.1 (**No Core Selection with Indivisible Goods**): By Baldwin and Klemperer (2019)'s Unimodularity Theorem, a competitive equilibrium exists in the exchange economy consisting of any set of participating bidders and any quantity $\bar{q}$. Consequently, if p is a market-clearing price for the exchange economy consisting of bidders Z and quantity $\bar{q}$, we have $q_i^e(Z, \bar{q}) \in d_i(p)$ for each $i \in Z$ for some efficient allocation $\{q_i^e(Z, \bar{q})\}_{i \in Z}$.

Choose some integer vector $z \leq \bar{q} - 4 \cdot \mathbf{1}$. Now choose $u_2(x) = u_3(x) = u_1(x + 2z - \bar{q}) - u_1(2z - \bar{q})$ and $u_4(x) = u_1(x + z) - u_1(z)$.

For coalitions $\{1,2\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}$ any price in the subdifferential of $u_1(z)$ clears the market. It follows that the allocations $\{z, \bar{q} - z\}, \{z, \bar{q} - z\}, \{z, \bar{q} - z, 0\}, \{z, \bar{q} - z, 0\}$ are efficient when the set of participating agents is, respectively, $\{1,2\}, \{1,3\}, \{1,2,4\}$, or $\{1,3,4\}$. Consequently, $v(Z, \bar{q})$ is the same for each $Z \in \{\{1,2\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}\}$.

When the set of participating agents is $\{1,2,3\}$, since $z + 2(\bar{q} - z) > \bar{q} + 3 \cdot \mathbf{1}$, some agent must receive a quantity of some good at least 2 units smaller than he did in any of the two-person coalitions he was a member of. It follows from strict concavity that the new market-clearing price p' is such that $z \notin q_1(p') = q_2(p') + (2z - \bar{q}) = q_3(p') + (2z - \bar{q}) = q_4(p') + z$. Since $0 \notin q_4(p')$, it follows that in the social planner's problem (1) for $Z = \{1,2,3,4\}$, allocating 0 to agent 4 is strictly suboptimal, and hence that $v(\{1,2,3,4\}, \bar{q}) > v(\{1,2,3\}, \bar{q})$. Then $v(\{1,2\}, \bar{q}) > v(\{1,2,3\}, \bar{q}) + v(\{1,2,4\}, \bar{q}) - v(\{1,2,3,4\}, \bar{q}) = \pi_0 + \pi_1 + \pi_2$ and so $\{0,1,2\}$ blocks the Vickrey payoff profile. $\square$

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