# Reallocative Auctions and Core Selection*

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**Abstract**

When selling goods like wireless spectrum or electricity contracts, designers often opt for *core-selecting* mechanisms — i.e., those that induce outcomes in the core — in order to balance revenue and efficiency goals. But increasingly, auctions — such as the FCC's Incentive Auction and those explored for natural resources — seek to reallocate goods, not just sell them. We show that when bidders can both buy and sell, substitutability among goods is no longer sufficient or necessary for core selection. In particular, in these environments, core selection can fail even with a single good and positive revenue, and can succeed even when some or all bidders view goods as complements. Instead, we show that the key feature that determines core selection is heterogeneity among the bidders. With too much heterogeneity, reallocation mostly realizes *pre-existing* gains from trade among the bidders, and core selection fails. With limited heterogeneity, most gains from trade among the bidders are created by the quantity auctioned, and a core-selecting mechanism is possible.

**Keywords**: Core-selecting design; Reallocative auction; Market design for natural resources.
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# 1 Introduction

Increasingly, auctions — such as the U.S. FCC's Incentive Auction for wireless spectrum licenses — aim to serve bidders interested in buying and/or selling various goods by *reallocating* them, instead of just offering them for sale.[1] Such auctions can help realize surplus in settings where transaction costs restrain voluntary trade, e.g., by reallocating water rights (Milgrom (2022)), electric vehicle charging time (Morstyn et al. (2019)), fishing rights (Bichler et al. (2019)), carbon sequestration credits, groundwater abstraction permits, and other resources (see, e.g., the survey by Teytelboym (2019)).[2] They can also facilitate the reallocation of goods when an additional quantity provided by the auctioneer makes such reallocation efficient. These *reallocative* auctions are the focus of our paper.

The designers of these auctions often seek to balance multiple objectives. In particular, both revenue and allocative efficiency may be important when, e.g., the auction is designed on behalf of a government agency. Moreover, a design's ability to achieve these objectives may be challenged by the participants' ability to deviate in ways that the designer cannot fully control, such as by engaging in shill bidding. In response, designers often favor auctions that are *core-selecting* — that is, which select a core outcome. Such designs rule out incentive problems that compromise these objectives: By definition, core selection ensures that the auction allocates goods efficiently, and that its revenue is *competitive*, in the sense that no coalition of bidders could profitably persuade him to trade with them instead of conducting the auction. It also renders shill bidding unprofitable (Yokoo and Matsubara, 2004) and ensures that revenue is increasing in bidder participation (Ausubel and Milgrom, 2002).

The central question relevant for core-selecting auction design — and this paper — is when such a design is an option for the auctioneer.[3] In *one-sided* environments — those where each bidder is only interested in buying goods — it is well understood that core selection is possible when the goods being auctioned are *gross substitutes* (Ausubel and Milgrom, 2002). We show that in *reallocative* environments — those where bidders are interested in both buying and selling goods — this condition is neither necessary nor sufficient. Specifically, core selection may be impossible even when goods are substitutes for all bidders (Example 3), and in fact, *even when only one good is being auctioned* (Example 1). Conversely, core selection can be possible even when some bidders view goods as comple-

[1] Reallocating the spectrum controlled by television stations into uses of higher value was a major part of the FCC's objective in the Incentive Auction; see http://www.fcc.gov/incentiveauctions and Ausubel et al. (2017). Bidders in spectrum auctions may wish to switch to a different frequency or exit a market; reallocative auctions allow them to do so.

[2] See also Klemperer (2010) for a discussion, as well as a description of the Product-Mix Auction, a uniform-price auction that can be used in a way that permits reallocation.

[3] The question of *which* design has a simple answer, since when core selection is possible, any auction that achieves it must be outcome-equivalent to the Vickrey (1961) auction. See the discussion below (and Proposition B.1 in the Supplementary Appendix).

mentary (Example 2). In fact, our main results (Theorems 1 and 2) show that in reallocative auctions, the main obstacle to core selection is *heterogeneity* among bidders, rather than complementarity among goods.

Many applications feature bidders that hold non-zero quantities of some of the goods traded — such as spectrum licenses or natural resources — and could thus both buy and sell them. These environments typically involve multiple goods: For instance, spectrum licenses differ in their geographical coverage and technical characteristics such as interference with adjacent frequency bands (e.g., Milgrom (2019)); appropriative rights to surface water differ in location of use, purpose of use, season of use, and seniority (Ferguson and Milgrom, 2024); electricity contracts differ in duration and location characteristics; and emission permits are issued for different time periods and pollutants. Notably, bidders generally have complex preferences over the goods being auctioned, and may consider some goods to be complements and others to be substitutes; in particular, complementarities are often an essential feature of bidders' preferences.[4]

We integrate these features in an environment with multiple divisible goods. Because any auction that achieves core selection is outcome-equivalent to the Vickrey (1961) auction (in the one-sided case, Goeree and Lien (2016); in the reallocative case, Proposition B.1 in the Supplementary Appendix) we focus, without loss of generality, on whether the Vickrey outcome lies in the core. As we show in Section 3, the possibility of reallocation changes the meaning of core selection by making new blocking coalitions relevant, but the design implications of core selection remain unchanged (Proposition 3).

**Main results**

We present two sets of main results. Together, they show that when reallocation is possible, different characteristics of the environment determine the possibility of core-selecting design. In particular, it is heterogeneity among bidders that matters, not substitutability among goods.

Theorem 1 establishes a *joint* sufficient condition — on bidders' preferences and the efficient allocation — that allows for the design of a core-selecting auction: There is a set of *packages* (bundles of goods) that the bidders regard as substitutable, and it is efficient to allocate non-negative quantities of them to each bidder. Mathematically, these substitutable packages must form a basis for the set of goods; the additional condition on package allocations then ensures that the reallocative auction for goods functions like a non-reallocative (i.e., one-sided) auction for *packages* that are gross substitutes for all bidders. Intuitively, the basis change represents a pattern of behavior relevant for core selection, and the joint condition captures how the bidders' incentives to substitute are aligned with the trades

[4] See, e.g., the discussion of wireless spectrum in Milgrom (2019) and natural resources in Teytelboym (2019).

necessary to realize the surplus created by participants' pre-auction allocations and the quantity auctioned. While predefined packages are often explicitly used in auction design in practice (e.g., Xiao and Yuan (2022)), Theorem 1 allows them to be implicit.

We also provide sufficient conditions that apply directly to bidders' primitive valuations. Theorem 2 shows that one can find a set of substitutable packages that are always allocated to bidders in positive quantities (and hence, by Theorem 1, core selection is possible) whenever bidders' *substitution patterns* — their valuations' Hessian derivative matrices — are homogeneous (in the sense that they are simultaneously diagonalizable) and bidders' marginal utility at their pre-auction allocations is the same. Thus, we can design a reallocative auction that selects outcomes in the core whenever the bidders (and in particular, their substitution patterns and pre-auction marginal utilities) are not too heterogeneous.

Our second set of main results shows that limited heterogeneity is not just sufficient for core selection, but also necessary (in a maximal domain sense). In fact, with sufficient heterogeneity in marginal utility at the pre-auction allocation, Propositions 4 and 5 show that core selection fails *independently* of heterogeneity in bidders' substitution patterns. Likewise, with sufficient heterogeneity in substitution patterns, Proposition 6 shows that core selection fails even when the pre-auction allocation is efficient.

Thus, in practice, the possibility of core selection in a reallocative auction depends on the intended design objective: If the primary objective is to eliminate the inefficiencies in bidders' *pre*-auction allocation (as in, e.g., the FCC Incentive Auction), core selection is less likely. But if the majority of the gains from trade to be realized among the bidders are created by the *new* quantity of goods offered by the auctioneer, a reallocative auction can be core-selecting. Which of these dominates depends crucially on the quantity of goods that the auctioneer offers for sale.

Together, these results illustrate that, in contrast to one-sided environments, the Vickrey auction may perform poorly in reallocative environments (in the sense that it is not core-selecting) even when goods are substitutable.[5] Furthermore, its poor performance is due precisely to heterogeneity among bidders that creates gains from trade between them, and causes reallocation of goods to occur. This raises the question of whether, when reallocation is important, other auction designs, such as the *uniform-price auction (UPA)* used in the clock phases of the FCC Incentive Auction, may be more appropriate: Even though the uniform-price auction does not result in an efficient allocation, it guarantees nonnegative revenue. However, it is known that even with a single good, the uniform-price auction can sometimes yield lower revenue than a Vickrey auction would (Ausubel et al., 2014). In environments where bidders have i.i.d. quadratic valuations, we show in Theorem 3 that

[5]That is, its revenue may be uncompetitively low, shill bidding may be profitable, and the auctioneer may have an incentive to restrict participation, and not just because negative revenue is possible (as is well understood; see, e.g., Myerson and Satterthwaite (1983)).

Vickrey outperforms the UPA when the expected heterogeneity (i.e., the variance) of the bidders' marginal utilities for the packages they view as substitutes (the eigenvectors of their substitution patterns) is sufficiently small — precisely when it is also likely to be core selecting (Theorem 2).

Finally, we make two observations about implementation. As the FCC Incentive Auction demonstrated, when the prospective buyers and prospective sellers in a reallocative auction are separate groups, the auction's complexity can be reduced considerably by splitting it into two non-reallocative auctions: a *reverse auction* that buys goods, and a *forward auction* that sells them. These auctions are then linked through a *clearing rule* that equalizes price across them. When an auction uses a uniform-price design, such as the one used in the reverse and forward clock phases of the Incentive Auction, it can be split without affecting its outcome. However, splitting a Vickrey auction would generally not be an outcome-neutral design choice: bidding according to their marginal utility schedules is no longer a dominant strategy for bidders (Proposition 8). Thus, the equilibrium allocation in a split Vickrey auction will generally be inefficient.

On the other hand, in environments with limited heterogeneity among bidders' substitution patterns, we show that there is an alternative way to simplify reallocative auctions that works just as well with either a Vickrey or uniform-price design. Specifically, suppose that bidders have homogeneous substitution patterns, in the sense that their valuations' Hessian matrices are simultaneously diagonalizable (or equivalently, commute with one another). Then, a reallocative Vickrey or uniform-price auction that allows package bidding is equivalent to a series of independent Vickrey or uniform-price auctions for the packages formed by the eigenvectors of those matrices (Proposition 9).

Consequently, while incorporating the substitutable packages from Theorems 1 and 2 into the auction's design is unnecessary for those results to hold, doing so may be useful: If bidder substitution patterns are not too different, then instead of conducting a package auction for, e.g., complementary spectrum licenses for Chicago and New York, Proposition 9 shows that we can separately auction off a spectrum license for *both* Chicago and New York, and a license that gives the holder the right to swap some amount of spectrum in Chicago for some amount of spectrum in New York. This allows auction designers a way to sidestep the known complexity issues associated with package bidding (see, e.g., surveys by De Vries and Vohra (2003) or Vohra (2015)).

### Related literature

Our paper belongs to three strands of the literature.

**Core-selecting design.** The literature on core-selecting auction design has explored one-sided environments (where reallocation is not possible), and has also largely focused on environments with indivisible goods. There, the key role of gross substitutability for an incentive compatible design's ability to select a core outcome is well understood (e.g., Ausubel and Milgrom (2002) or Bikhchandani and Ostroy (2002)). This paper examines the possibility of core-selecting design in environments where goods can be reallocated among the bidders, which have more recently attracted attention in market design, and considers bidders with multi-unit demands for divisible goods. An important predecessor is Milgrom and Strulovici (2009), who show that in one-sided environments, divisibility does not fundamentally alter the features of the environment that allow core-selecting design: Gross substitutability is still sufficient, and in a maximal domain sense, necessary. Since it reduces to gross substitutability when it is applied to goods in a one-sided environment, the sufficient condition we give in Theorem 1 can be seen as unifying the features that make core selection possible across environments where trade is necessarily one-sided and those where participants and the auctioneer can both buy and sell.[6]

Some authors have investigated how an auction should function when the Vickrey outcome is not in the core, and so core selection conflicts with incentive compatibility. One approach is to relax the requirement of strategy-proofness, and seek to achieve revenue objectives using payment rules that select outcomes in the core as long as bidders truthfully reveal their valuations. Authors such as Ausubel et al. (2006), Day and Raghavan (2007), Day and Milgrom (2008), Erdil and Klemperer (2010), Day and Cramton (2012), and Ausubel and Baranov (2020) consider the design of such payment rules, as well as methods for implementing them in practice.

**Patterns of behavior and basis changes.** Theorem 1 is based on *package substitutability* — substitutability under a change of basis. The idea that analyzing transformations through basis changes can help to identify relevant patterns of behavior has been noted for other questions. Notably, Galeotti et al. (2020) introduce a principal component approach to analyze the optimal interventions in a network of interacting agents, aimed at changing their individual incentives to take action. Applying these tools in a market context, Galeotti et al. (2024) show how the use of a basis change to transform the representative consumer's Slutsky matrix, which captures a network structure among firms in the space of goods, allows a characterization of the optimal tax-and-subsidy scheme (and its effect on welfare) in terms of the price pass-throughs of *eigenbundles* of goods produced by firms.[7]

Substitutability under a basis change of the space of goods has also allowed new existence results in the literature on competitive equilibrium with imperfectly divisible goods.[8]

[6]Milgrom's (2007) Fisher-Schultz lecture emphasizes that the theory of package exchanges has few predictive results. Our results characterize the possibilities for and limitations of core-selecting design in reallocative environments with multiple heterogeneous divisible goods, including the design of exchanges (where the auctioneer's quantity vector is zero).

[7]Weinstein (2022) explores the role of basis changes in defining substitutes and complements.

[8]Examples include the gross substitutes and complements condition of Sun and Yang (2006); its close relative, the full substitutability condition of Ostrovsky (2008) and Hatfield et al. (2013); and, more generally, the basis changes employed in Baldwin and Klemperer (2019).

In our setting, nonemptiness of the core (or the set of competitive equilibria) does not require substitutability (Proposition C.1 in the Supplementary Appendix). On the other hand, substitutability under a basis change does not suffice on its own to establish core selection. Instead, Theorem 1 shows that an additional allocation condition — which is not relevant to the existence of the core or competitive equilibrium with either divisible or indivisible goods — is needed.[9]

**Applied market design.** Several recent papers explore and evaluate design features of the FCC Incentive Auction, which reallocated wireless spectrum from TV broadcasters to higher-value uses such as mobile broadband (Kwerel et al. (2017), Doraszelski et al. (2019), Loertscher and Marx (2020), Newman et al. (2020), Milgrom and Segal (2020), Ausubel and Baranov (2023)). Teytelboym (2019) reviews related designs — proposed or implemented — for natural resources. We contribute by establishing conditions when core-selecting design is possible in reallocative environments such as these. We also analyze the impact of splitting a reallocative auction into forward and reverse stages (as in the Incentive Auction) when coupled with a core-selecting payment rule.

### Structure of the paper

Section 2 introduces the setting. Section 3 discusses the relevant notion of core selection in reallocative auctions and its link to the surplus function. Section 4 provides our motivating examples and main results on core selection in reallocative auctions. Section 5 discusses the uniform-price auction and compares its performance to the Vickrey auction. Finally, Section 6 discusses implementation. Section 7 concludes. All proofs are contained in the Appendix.

## 2 Setting

There is an auctioneer, denoted $a$, and a set $I$ of potential bidders. A finite subset $X \subseteq I$ of these bidders participate in an auction conducted by the auctioneer, where $|X| \geq 2$. The auctioneer has a quantity vector $\bar{q} \in \mathbb{R}^K$ of $K$ perfectly divisible goods, but obtains no value from them: His payoff is the sum of the transfers he receives from the bidders.[10] Each bidder $i \in I$ has a *valuation* $u_i : \mathbb{R}^K \to \mathbb{R}$ for bundles of those goods, where $u_i(0) = 0$; when

[9] In fact, even after the transformation of auctions for goods into those of packages, the arguments available for standard one-sided environments (e.g., Milgrom and Strulovici (2009)) are not sufficient in reallocative environments. (See the discussion in the appendix before Lemma 4 and after the proof of Theorem 1.)

[10] We consider an environment with perfectly divisible goods in order to accommodate bidders who can buy or sell multiple units of the same good, while abstracting away from the questions of equilibrium existence associated with indivisible goods in the absence of gross substitutability (see, e.g., Gul and Stacchetti (1999) and Baldwin and Klemperer (2019)). The issues created by reallocation that we examine are relevant beyond divisible-good environments. We conjecture that results similar to ours can be found in indivisible good environments, with a restriction on the change of basis due to the integer choice set.

he receives the bundle $q_i \in \mathbb{R}^K$ and sends the transfer $x_i \in \mathbb{R}$ to the auctioneer, his payoff is given by $u_i(q_i) - x_i$.[11] When the $k$th entry of $q_i$ is negative, it represents a sale of good $k$ by agent $i$ to the auctioneer; when it is positive, it represents a purchase of good $k$ from the auctioneer. We assume that each bidder's valuation $u_i$ is twice continuously differentiable and strictly concave, and that the image of each agent's marginal utility function is the same: There is a convex set $M \subseteq \mathbb{R}^K$ such that for each $i \in I$, $\nabla u_i(\mathbb{R}^K) = M$.[12]

In contrast to the *one-sided* environments typically considered in the core-selection literature, our setting does not constrain bidders' allocations of goods $q_i$ to be positive. Instead, the only feasibility constraint on allocations $\{q_i\}_{i\in X}$ is that they must sum to the quantity vector $\bar{q}$. This allows us to accommodate new applications where bidders are able to both buy and sell goods; e.g., those in which bidders have non-zero quantities of some of the goods auctioned.[13]

### The Core

We are primarily interested in auctions that produce allocations in the *core* of the auction environment: the set of payoff profiles such that no coalition can be better off by abandoning the mechanism and trading on its own. To define the core, we first define the *surplus function*

$$v(Z, \bar{q}) := \max_{\{q_i\}_{i\in Z}} \sum_{i\in Z} u_i(q_i) \text{ s.t. } \sum_{i\in Z} q_i = \bar{q}. \tag{1}$$

$v(Z, \bar{q})$ represents the maximum value that can be obtained by allocating the bundle $\bar{q}$ among the bidders $Z \subseteq I$. Similarly, for a coalition $Z \subseteq I$, let $\{q_i^e(Z, \bar{q})\}_{i\in Z}$ denote the Pareto efficient allocation of the bundle $\bar{q}$ among the bidders in $Z$:[14]

$$\{q_i^e(Z, \bar{q})\}_{i\in Z} := \arg\max_{\{q_i\}_{i\in Z}} \sum_{i\in Z} u_i(q_i) \text{ s.t. } \sum_{i\in Z} q_i = \bar{q}. \tag{2}$$

[11] We can think of $u_i$ as a reduced form describing the bidders' pre-auction allocations $\{q_i^0\}_i$ and preferences over total quantities $\hat{u}_i$; then $u_i(q_i) \equiv \hat{u}_i(q_i^0 + q_i) - \hat{u}_i(q_i^0)$, and $u_i(0) = 0$. We work directly with the reduced form $u_i$, since when the bidders can buy and sell in arbitrary quantities, this is all that matters for the properties of the auction designs that we consider.

[12] For instance, with quadratic valuations (Section 4), $M = \mathbb{R}^K$, whereas if marginal utility is bounded, then $M$ is some strict subset of $\mathbb{R}^K$. These regularity assumptions ensure that there is a competitive equilibrium in finite quantities and that aggregate demand is everywhere well-defined. This is stronger than necessary for equilibrium existence, but ensures the invertibility of agents' demand functions.

[13] While not our main focus, the model accommodates environments in which the auctioneer seeks only to reallocate goods among the bidders (and so $\bar{q} = 0$) or procurement auctions in which the auctioneer seeks to purchase goods (and so $\bar{q} < 0$).

[14] Since the bidders' payoffs are quasilinear with strictly concave valuations, and the images of their marginal utility functions are identical, there is a unique efficient allocation (Lemma E.3 in the Supplementary Appendix).

The *coalitional value function*

$$V(Z,\bar{q}) := \begin{cases} v(Z \setminus \{a\},\bar{q}), & a \in Z; \\ v(Z,0), & a \notin Z, \end{cases}$$

gives the maximum surplus that can be produced by reallocating goods among the agents $Z \subseteq I \cup \{a\}$ when the auctioneer's quantity vector is $\bar{q}$. Given the set of participating bidders $X$, the payoff profile $\pi \in \mathbb{R}^{X \cup \{a\}}$ is in the core if

$$\sum_{i \in X \cup \{a\}} \pi_i = V(X \cup \{a\},\bar{q}), \text{ and } \sum_{i \in Z} \pi_i \geq V(Z,\bar{q}) \text{ for all } Z \subseteq X \cup \{a\}.$$

If $\sum_{i \in Z} \pi_i < V(Z,\bar{q})$, we say that the coalition $Z$ *blocks* the payoff profile $\pi$. If $\pi$ is unblocked by any $Z \subseteq X$ except $Z = \{a\}$, we say that $\pi$ is in the *bidder-core*.[15,16]

Because we study an environment where goods can be reallocated among the bidders, our definitions of the coalitional value function and core are generalizations of those that appear in most of the core-selecting auction literature. In particular, coalitions that do not involve the auctioneer are relevant, since the bidders may be able to realize gains from trade among themselves without access to the auctioneer's quantity vector. In *one-sided* environments commonly studied in the literature, on the other hand, reallocation is not feasible, and so coalitions are only relevant to the core if they include the auctioneer. We discuss this difference, and its consequences for (bidder-)core selection, in Section 3.

**The Vickrey Auction**

Since we are interested in core selection, we mainly focus on the Vickrey (1961) auction. This focus is without loss of generality: if an auction mechanism is incentive compatible and (bidder-)core-selecting on some convex domain of bidder preferences, it must produce the Vickrey outcome there (Goeree and Lien, 2016).[17,18] Many ways of implementing

[15]We note that the set of bidder-core outcomes (a *superset* of the set of core outcomes) is different from the set of *bidder-optimal* core outcomes (the minimum-revenue *subset* of the set of core outcomes) often considered in the literature on core selection in one-sided auctions.

[16]There is a large literature on the existence of payoff profiles in the core (e.g., Bondareva (1963); Shapley (1967)). This is not our focus: In our reallocative auction environment, the (bidder-)core is never nonempty. (See Proposition C.1 in the Supplementary Appendix.) Hence, when our results show that core selection is not possible, it is not because the core does not contain *any* payoff profiles, but rather that it just does not contain the *Vickrey* payoff profile.

[17]Note that in our setting, the Vickrey auction is necessarily reallocative: Since it is possible for a design to reallocate goods among the bidders, efficiency requires that the auction is capable of doing so.

[18]Specifically, Goeree and Lien (2016) show that in a one-sided environment where bidder preferences are drawn from a convex set, if an auction mechanism is Bayesian incentive compatible and core-selecting, it must be outcome-equivalent to the Vickrey auction. In Section B of the Supplementary Appendix, we extend this result to reallocative environments (Proposition B.1). We note that while auctions that use different pricing schemes (e.g., the Vickrey-Nearest rule) may select outcomes in the core of an environment where bidders' valuations are replaced by their *reported* valuations, these outcomes are not necessarily in the core of

the Vickrey auction have appeared in the literature.[19] Throughout the paper, we use the version that (as in Vickrey (1961)) solicits bids in the form of inverse demand schedules $b_i : \mathbb{R}^K \to M$, awards each bidder the bundle $q_i^*(b)$ at which his bid is equal to the market-clearing price $p^*(b)$, and charges him a payment $t_i^V(b)$ equal to the area under his residual supply curve.[20] As we show in Section 5, this format facilitates a straightforward comparison with the commonly used *uniform-price* auction.

Proposition 1 gives the usual characterization of the Vickrey auction's dominant-strategy (truthful) equilibrium, which we assume the bidders play throughout the paper.

**Proposition 1 (Equilibrium in the Vickrey Auction).**

*i. For each participating bidder $i \in X$, submitting a bid function $b_i^V(q_i) = \nabla u_i(q_i)$ that coincides with his marginal utility function is a weakly dominant strategy in the Vickrey auction.*

*ii. The Vickrey auction implements the efficient allocation: $\{q_i^*(b^V)\}_{i\in X} = \{q_i^e(X, \bar{q})\}_{i\in X}$.*

*iii. The Vickrey auction gives each bidder $i$ a payoff $\pi_i^V(X)$ equal to their marginal contribution to the surplus produced by the grand coalition, and the remainder of that surplus to the auctioneer:*

$$\pi_i^V(X) := u_i(q_i^*(b^V)) - t_i^V(b^V) = v(X, \bar{q}) - v(X \setminus \{i\}, \bar{q}) \text{ for each } i \in X;$$
$$\pi_a^V(X) := \sum_{i\in X} t_i^V(b^V) = v(X, \bar{q}) - \sum_{i\in X} \pi_i^V(X).$$

When the Vickrey payoff profile $\pi^V(X) \equiv \{\pi_i^V(X)\}_{i\in X\cup\{a\}}$ is in the core, we say that the Vickrey auction is *core-selecting*; when it is in the bidder-core, we say that the Vickrey auction is *bidder-core-selecting*.

## Discussion

As Proposition 1 shows, equilibrium in the Vickrey auction is ex post. When we introduce the uniform-price auction in Section 5, we show that the same is true of its equilibrium in the quadratic environment where we consider it. This allows us to avoid assumptions about the information available to the bidders (other than that they know their own valuations) when analyzing the properties of the two auctions. Instead, the only uncertainty that we need to consider is the auctioneer's uncertainty about which bidders will participate in the auction. In our main results, this uncertainty is implicit: the auctioneer may

the underlying environment, because when the payment rule differs from Vickrey, truthful reporting is not always a dominant strategy.

[19]The Vickrey auction can equivalently be implemented dynamically (as in, e.g., the Combinatorial Clock Auction (Ausubel et al., 2006)) or by directly soliciting bidders' payoff functions $u_i$, allocating $\bar{q}$ efficiently given their reports, and charging a payment of $u_i(q_i^e(X, \bar{q})) - v(X, \bar{q}) + v(X \setminus \{i\}, \bar{q})$.

[20]See the Appendix for a formal description of the auction.

be concerned with whether the Vickrey auction is core-selecting *for all possible participants* $X \subseteq I$, instead of one particular set of participants $X$. When we compare the revenue performance of the Vickrey and uniform-price auctions in Section 5, we make the auctioneer's uncertainty explicit by giving him a prior over the participants $X$ drawn from $I$.

# 3 Reallocation and Core Selection

The practical implications of core selection in one-sided environments are well known. There, it is equivalent to the statement that the auctioneer's revenue is *competitive*, in the sense that no coalition of bidders could profitably collude with the auctioneer;[21] it ensures that shill bidding is unprofitable (Yokoo and Matsubara (2004), Day and Milgrom (2008)); and it rules out any incentive for the auctioneer to restrict entry to the auction, by ensuring that his revenue is monotone in the number of bidders (Ausubel and Milgrom, 2002).

Before we ask when core selection is possible in environments where bidders can both buy and sell, we should examine whether these practical implications are the same in such settings. The results of this section show that they are, but that core selection is slightly stronger than necessary: Instead, the relevant criterion is *bidder*-core selection.

### Reallocation and Blocks by Coalitions Without the Auctioneer

In a one-sided environment, the value of any coalition that does not include the auctioneer is zero, since no reallocation can occur. Thus, a payoff profile can only be blocked by such a coalition if it offers some bidder a negative payoff. This makes these coalitions irrelevant for core selection, since participation in the Vickrey auction is individually rational. Instead, core selection is strictly about the incentives of coalitions *that include the auctioneer* — and hence the competitiveness of the auction's revenue.

With reallocation, however, coalitions that only consist of bidders can realize gains from trade among themselves. Consequently, they can block a payoff profile even when each individual bidder receives a nonnegative payoff. As it turns out, however, they can never block the *Vickrey* payoff profile: Proposition 2 shows that participation in the reallocative Vickrey auction is not only individually rational, but *coalitionally* rational as well. Thus, core selection is still fundamentally a question about the auctioneer's revenue.

**Proposition 2 (Coalitional Rationality of Participation).** *Coalitions which do not include the auctioneer never block the Vickrey payoff profile $\pi^V(X)$.*

To understand Proposition 2, observe that the special case of the coalition of all participating bidders and $\bar{q} = 0$ is just the classical "no budget balance" result of Vickrey (1961):

[21] Formally, the Vickrey auction's revenue is competitive if its payoff profile is unblocked by any coalition that includes the auctioneer and at least one bidder.

When $\bar{q} = 0$, and thus all surplus is derived from reallocation among the bidders, the auctioneer must provide a subsidy in order to realize that surplus with a reallocative Vickrey auction. Proposition 2 generalizes this insight by showing that *every* coalition is so subsidized for *any* $\bar{q}$. Thus, any blocking coalition must include the auctioneer, and so core selection is still fundamentally a property of the auctioneer's revenue.

**Reallocation and Blocks by the Auctioneer Alone**

Vickrey's result also points at another important difference between auctions with and without reallocation that affects the interpretation of core selection: Unlike a Vickrey auction in a one-sided environment, a reallocative Vickrey auction can generate negative revenue. Hence, the Vickrey payoff profile may be blocked by the coalition consisting of the auctioneer alone.

While the possibility of negative revenue is an important design consideration, it is a separate issue from those that usually focus attention on core-selecting auctions: For instance, we might expect revenue to be negative in a procurement auction (where $\bar{q} < 0$) but still be interested in whether the payment made to the bidders is competitive. Thus, a designer may want to abstract away from the sign of revenue and focus instead on its competitiveness. This means considering whether the auction is *bidder*-core selecting, since (by Proposition 2) the coalitions that are relevant for competitiveness are exactly those that are relevant for the bidder-core.

**Corollary 1 (Revenue and (Bidder-)Core Selection).** *The reallocative Vickrey auction's revenue is competitive if and only if it is bidder-core selecting. Its revenue is competitive and nonnegative if and only if it is core-selecting.*

In fact, besides ensuring that the auctioneer's revenue is competitive, bidder-core selection suffices to ensure that two of the most well-known implications of core selection in one-sided environments — shill bidding is unprofitable and revenue is monotone in the set of participants — carry over to reallocative auctions. We thus focus mainly on bidder-core selection for the remainder of the paper.

**Proposition 3 (Implications of Bidder-Core Selection).** *If the Vickrey auction is bidder-core-selecting for each finite $X \subseteq I$, then*

*i. The auctioneer's revenue $\pi_a^V(X)$ is increasing in the set of participants $X$, and*

*ii. Shill bidding is never profitable for bidders, regardless of the set of participating bidders $X \subseteq I$: There is no bidder $i \in X$ and finite collection of shills $B \subset I$ such that when the set of bidders $B \cup X \setminus \{i\}$ participate in the auction and bid their dominant strategies $b_j^V(q_j) = \nabla u_j(q_j)$,*

$$u_i\left(\sum_{j \in B} q_j^*(b^V)\right) - \sum_{j \in B} t_j(b^V) > \pi_i^V(X).$$

# 4 Core Selection and Preferences

This section contains the main results of the paper. We begin with examples that illustrate how the relationship between bidder preferences and (bidder-)core selection is affected when bidders can both buy and sell.

## 4.1 Motivating Examples

The literature has established a close connection between core selection — or equivalently, submodularity of the surplus function in bidders — and substitutability between goods.[22]

**Definition (Demand and Substitutability).** The *demand function* of a bidder $i \in I$ is given by $d_i(p) := \arg\max_{q_i} u_i(q_i) - p \cdot q_i$.[23] We say that *goods are substitutable for bidder i* if, when the price of one good goes up, bidder $i$'s demand for each other good does not go down.

Because goods are divisible in our environment, substitutability is a property of the bidders' Slutsky matrices $Dd_i(p)$. Since the inverse of a bidder's Slutsky matrix is just the second derivative matrix $D^2 u_i(q_i)$ of their valuation[24] — what we call their *substitution pattern* — this amounts to a condition on the valuation itself: Goods are substitutes for bidder $i$ if his substitution pattern has an inverse with nonnegative off-diagonal entries.

In one-sided environments, existing results show that when goods are substitutes, the Vickrey auction is core-selecting (e.g., Ausubel and Milgrom (2002); Milgrom and Strulovici (2009)). The next three examples show that in reallocative environments, substitutability is neither sufficient (Examples 1 and 3) nor necessary (Example 2) for core- or bidder-core-selection. In each, we consider environments where bidders have *quadratic* valuations for goods.[25] This allows us to solve for the surplus function and efficient allocation in closed form (Lemma 3 in the Appendix).

**Definition (Quadratic Valuations).** We say that bidders have *quadratic valuations* if for each bidder $i$, $u_i(q_i) = \theta_i' q_i - \frac{1}{2} q_i' S_i q_i$ for some positive definite matrix $S_i \in \mathbb{R}^{K \times K}$ and *marginal utility parameter* $\theta_i \in \mathbb{R}^K$.

Example 1 shows that when the bidders and/or the auctioneer can buy and sell, the Vickrey outcome may not lie in the core *even when substitutability is trivially satisfied* (e.g., in environments with only one good).

[22] As an intermediate step in establishing our main results, we extend this equivalence to our setting where reallocation is feasible (Lemma 1) and show that the Vickrey auction's ability to select outcomes in the bidder-core is equivalent to a weaker submodularity condition (Lemma 2).

[23] Since $u_i$ is strictly concave, $d_i$ is a function, not a correspondence.

[24] Follows from the first-order condition $\nabla u_i(d_i(p)) = p$ and the implicit function theorem.

[25] Models with quadratic payoffs are commonly studied in the divisible good auctions and financial markets literatures. See, e.g., Ausubel et al. (2014); Malamud and Rostek (2017). Unlike in much of the literature, however, we allow for heterogeneity in bidders' substitution patterns $\{-S_i\}_{i=1}^N$.

**Example 1 (No Core Selection with a Single Good and Positive Revenue).** An auctioneer has $\bar{q} = 1$ units of a single divisible good for sale in a Vickrey auction. There are three participating bidders $X = \{1, 2, 3\}$, with valuations

$$u_1(q_1) = 4q_1 - \frac{1}{2}q_1^2, \qquad u_2(q_2) = 4q_2 - \frac{1}{2}q_2^2, \qquad u_3(q_3) = 2q_3 - \frac{1}{2}q_3^2.$$

The Vickrey auction efficiently allocates $q_1^e(\{1,2,3\}, \bar{q}) = q_2^e(\{1,2,3\}, \bar{q}) = 1$ unit of the good to each of bidders 1 and 2, by causing bidder 3 to sell a unit of the good: $q_3^e(\{1,2,3\}, \bar{q}) = -1$. Or, put differently, it allocates the auctioneer's quantity to bidder 1, and also *reallocates* a unit of the good from bidder 3 to bidder 2. Since the Vickrey payment rule ensures that the auctioneer pays more for goods than he receives by reselling them, this makes it attractive for him to cancel the auction and negotiate with bidder 1 instead: the coalition $\{a, 1\}$ blocks the Vickrey outcome, and so (bidder-)core selection fails.[26]

Intuitively, bidder 3's marginal utility is low enough that even after bidders 1 and 2 have obtained the auctioneer's unit of the good, they can realize further surplus by buying from bidder 3. Facilitating that reallocation requires a subsidy from the auctioneer, decreasing the revenue he obtains from the auction. If each bidder had instead had the *same* marginal utility at zero auction allocation, reallocation would be unnecessary, and (as we show in Theorem 2) the Vickrey outcome would be in the (bidder-)core.

Example 1 illustrates that reallocative environments present new challenges for core selection that are independent of complementarity or substitutability among goods. In particular, bidder-core selection can fail when bidders' marginal valuations are heterogeneous enough that the auctioneer must subsidize reallocation in order to realize surplus in the auction. In contrast, when bidders' marginal valuations are the same *at their initial allocations*, Example 2 shows that allowing them to both buy and sell need not cause the Vickrey payoff profile to lie outside the core. In fact, as we will show in Theorem 2, Example 2 illustrates the conditions under which bidder-core selection always holds in reallocative auctions.

**Example 2 (Core Selection with Substitutability and Complementarity).** An auctioneer wishes to sell a bundle $\bar{q} = [1\ 1]'$ of two goods using a Vickrey auction. There are three participating bidders $X = \{1, 2, 3\}$, with valuations

$$u_1(q_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}' q_1 - \frac{1}{2}q_1' \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} q_1; \quad u_i(q_i) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}' q_i - \frac{1}{2}q_i' \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} q_i, \quad i \in \{2, 3\}.$$

Hence, goods are complements for bidder 1, but substitutes for bidders 2 and 3. Nevertheless, as we show in the Supplementary Appendix, the Vickrey auction is bidder-core

[26] Intuitively, a net buyer is complementary to a net seller in the surplus function $v$, which can cause core selection to fail (Lemma 1 in the Appendix). Here, core selection fails because bidder 2 (a net buyer) and bidder 3 (a net seller) are complementary.

selecting (in fact, core-selecting, since the auctioneer's revenue is positive).

Intuitively, while bidders 2 and 3 view goods as substitutes, and bidder 1 views them as complements, all three bidders have substitution patterns $D^2 u_i(q_i)$ with the same eigenvectors, $[1\ 1]'$ and $[-1\ 1]'$. Hence, they view the *packages* described by these eigenvectors as *perfect* substitutes. Since the pre-auction allocation is efficient, the reallocation problems observed in Example 1 are avoided, and core selection follows.

As we will show in Theorems 1 and 2, in reallocative auctions, it is not the presence of complementarities *per se*, but instead a type of *heterogeneity* in substitution patterns, that can challenge (bidder-)core selection. In particular, core selection is possible in Example 2, even though some bidders view goods as complements, while others view them as substitutes, because substitution patterns $D^2 u_i(q_i)$ are symmetric across bidders in the sense that they *commute* with one another.[27] This allows the setting to be transformed from one where bidders heterogeneously view goods as complements or substitutes to one where they all view *packages* as substitutes. On the other hand, Example 3 shows that with too much heterogeneity in bidders' substitution patterns, a reallocative auction may not be bidder-core selecting, even when goods (or packages) are substitutable, and the pre-auction allocation is efficient.

**Example 3 (No Core Selection with Substitutes and Efficient Pre-Auction Allocation).** An auctioneer wishes to sell 1 unit of good 1 and 0 units of good 2 using a reallocative Vickrey auction; that is, he wishes to auction the vector $\bar{q} = [1, 0]$.[28] Three bidders with the following valuations participate:

$$u_1(q_1) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}' q_1 - \tfrac{1}{2} q_1' \begin{bmatrix} 0.25 & 0.19 \\ 0.19 & 1.49 \end{bmatrix} q_1, \ \ u_2(q_2) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}' q_2 - \tfrac{1}{2} q_2' \begin{bmatrix} 5.8 & 0 \\ 0 & 5.8 \end{bmatrix} q_2,$$
$$u_3(q_3) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}' q_3 - \tfrac{1}{2} q_3' \begin{bmatrix} 3.07 & 3.47 \\ 3.47 & 4.47 \end{bmatrix} q_3.$$

Here, no gains to trade exist among the bidders prior to the auction, i.e., all bidders have the same marginal utility at zero. Moreover, the quantity of each good auctioned is nonnegative, and goods are substitutes for all bidders. Thus, if efficiency did not require reallocation of the goods among the bidders, we know from the literature that the Vickrey auction would be core-selecting.

However, as we show in the Supplementary Appendix, the Vickrey payoff profile is blocked by $\{a, 1\}$, and thus outside the bidder-core. Because the bidders' substitution patterns are heterogeneous, the new supply of good 1 creates gains from trade among the

[27] Two matrices $A$ and $B$ commute if $AB = BA$. Diagonalizable matrices $A$ and $B$ commute if, and only if, they have the same eigenvectors.

[28] The logic of this example can be mimicked when the auctioneer wants to both buy and sell; e.g., when he wishes to exchange 0.5 units of good 1 for 0.9 units of good 2 using a reallocative Vickrey auction; that is, he wishes to auction the vector $\bar{q} = [0.5, -0.9]$.

bidders from reallocating good 2, even though (unlike in Example 1) no gains from trade existed prior to the auction. Since the auctioneer subsidizes this reallocation in a Vickrey auction, these gains from trade are sufficient for core selection to break down.

These examples demonstrate that in a reallocative environment, substitutability among goods is neither necessary (Example 2) nor sufficient (Examples 1 and 3) for a core-selecting design. Moreover, heterogeneity in bidders' substitution patterns $D^2u_i(q_i)$ (as in Example 3) or in their marginal utilities at the pre-auction allocation (Example 1) can independently challenge core selection, because each can lead to "too much" reallocation among the bidders that must be subsidized by the auctioneer. But conversely, when bidders' substitution patterns have the same eigenvectors and the pre-auction allocation is efficient (e.g., when an efficient auction has been conducted in the past), core selection is possible (Example 2). Our main results formalize these insights.

## 4.2 Sufficient Conditions for Core Selection

In one-sided environments, the literature has shown that substitutability among goods is sufficient — and, in a maximal domain sense, necessary — for core selection to be possible (Ausubel and Milgrom, 2002). This condition can be decomposed into two separate statements:

(a) When the price vector for goods changes in one of $K$ linearly independent directions, the quantity vectors that bidders demand move in directions opposite (or *dual* to) that direction.

(b) Those directions represent a change in the price of a single good.

We show that for the purposes of core selection, (a) is the key part of the substitutes condition. The second part (b) is only important insofar as it ensures that the directions in question represent changes in the prices of bundles that the auction allocates to each bidder in non-negative quantities. Condition (b) always performs this function in a one-sided environment, but it cannot do so in environments where reallocation is possible.

Hence, we dispense with (b) and show that the relevant substitutability condition for core selection is among *packages* rather than goods. To formalize this, for any set of linearly independent vectors $\Phi = \{\phi_k\}_{k=1}^K$, let $T_\Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_K \end{bmatrix}$. $T_\Phi^{-1}$ maps quantity vectors to their representation in terms of the packages $\Phi$. $T'_\Phi$, on the other hand, maps price vectors to their representation in terms of the *prices* of the packages $\Phi$.

**Definition (Package Substitutability).** If $\Phi \subset \mathbb{R}^K$ is a set of $K$ linearly independent vectors, we say that *packages* $\Phi$ *are substitutes* if whenever the price $\phi_k \cdot p$ of a package $\phi_k \in \Phi$ increases, each bidder $i$'s quantity demanded $[T_\Phi^{-1} d_i(p)]_{-k}$ of the other packages (weakly) increases.

Package substitutability ensures that when prices change in certain directions (the row vectors of $T_\Phi^{-1}$), the quantities demanded by all bidders move in a direction opposite that of the price change (linear combinations of the column vectors of $T_\Phi$ with all but one coefficient positive). In other words, package substitutability is just the substitutes property under a basis change of the space of goods and a compatible basis change of the space of prices.[29] This basis change can arise naturally from bidders' valuations: For instance, in Theorem 2, we provide a condition on bidders' substitution patterns ensuring substitutability among the packages formed by their eigenvectors.

As Example 1 shows, reallocation can rule out core selection, even when goods are substitutes. The same is true in the more general case where packages are substitutes. Instead, Theorem 1 establishes a *joint* condition on the substitutability and non-negative allocation of packages that is sufficient for bidder-core selection to be possible in a reallocative auction.

**Theorem 1 (Core Selection: Package Substitutability and Allocations).** *Suppose that (i) packages $\Phi$ are substitutes and (ii) the efficient allocation always gives each bidder a non-negative quantity of them: $T_\Phi^{-1} q_i^e(Z, \bar{q}) \geq 0$ for each $Z \subseteq I$ and all $i \in Z$. Then the Vickrey auction is bidder-core selecting for each set of participating bidders $X \subseteq I$. If, in addition, for each $Z \subseteq I$, $T'_\Phi \nabla_Q v(Z, \bar{q}) \geq 0$, then the Vickrey auction is core selecting for each set of participating bidders $X \subseteq I$.*

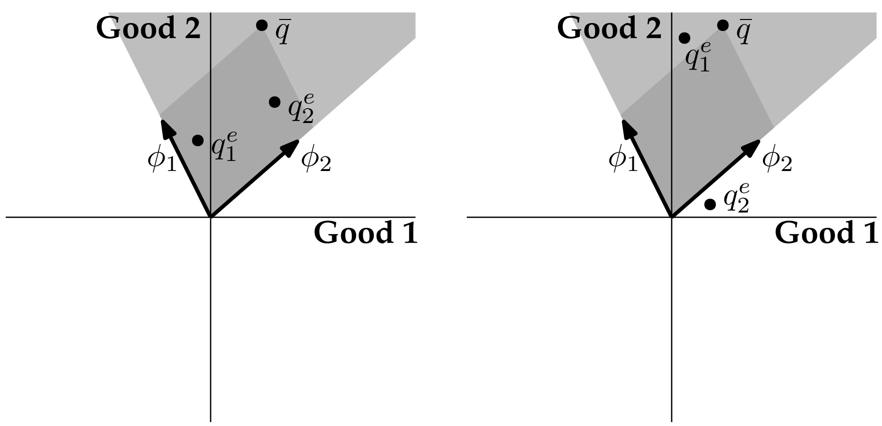


**Figure 1: Theorem 1's allocation condition.** Both panels illustrate Theorem 1's allocation condition with two bidders and two goods. In the left panel, both bidders receive positive allocations of both $\phi_1$ and $\phi_2$, and the condition is satisfied for the packages $\Phi = \{\phi_1, \phi_2\}$. In the right panel, this is no longer true.

Theorem 1 provides sufficient conditions for core selection in environments where bidders can both buy goods and sell them. In particular, it shows that bidder-core selection is

[29] In the appendix, Lemma 4 shows that package substitutability can also be formulated as a condition directly on the bidders' valuations — and in particular, their substitution patterns. In fact, Theorem 1 only requires this condition only needs to hold locally on the polyhedron $0 \leq T_\Phi^{-1} q_i \leq T_\Phi^{-1} \bar{q}$, where the auction allocation is guaranteed by assumption to lie (Lemma 5).

possible even when bidders do not receive positive quantities of each good, as long as they receive positive quantities of each of a set of substitutable packages; i.e., bidders are all net package buyers.

Relative to results in the literature on core selection in one-sided environments, Theorem 1 generalizes substitutability to packages and includes a new condition on allocations. This additional condition is necessary because it ensures that *the reallocative auction for goods functions like a non-reallocative (i.e., one-sided) auction for packages*; i.e., we can restrict attention to the positive orthant in *package* space. This clarifies the role that one-sidedness plays in core selection results in the literature.[30] In particular, when the packages are goods, Theorem 1's allocation condition is always satisfied in a one-sided environment — by definition, a one-sided auction must allocate goods to the bidders in non-negative quantities — and so substitutability among *goods* is sufficient for (bidder-)core selection.[31] However, when it is efficient for some bidders to sell some goods, goods substitutability generally will not ensure that the Vickrey payoff profile is in the bidder-core. Nevertheless, if efficient reallocation always gives each bidder a positive quantity of certain *packages*, then package substitutability ensures bidder-core selection.

Our second main result, Theorem 2, gives conditions that guarantee this. As foreshadowed by our motivating examples, these conditions must limit heterogeneity among the bidders, both in their substitution patterns (the problem in Example 3) and in their pre-auction marginal utilities (the problem in Example 1).

**Theorem 2 (Core Selection: Package Substitutability and Heterogeneity).** *Suppose that*

*(a) Bidders' pre-auction allocations are efficient: $\nabla u_i(0) = \nabla u_j(0)$ for all $i, j \in I$, and*

*(b) Bidders' substitution patterns $\{D^2 u_i(q_i)\}_{i \in I, q_i \in \mathbb{R}^K}$ are commuting matrices.*

*Then the bidders' substitution patterns have a common orthonormal eigenbasis $\Phi$ such that*

*i. The packages $\Phi$ are substitutes; and*

*ii. The efficient allocation always gives each bidder a positive quantity of each package in $\Phi$, i.e., $T_\Phi^{-1} q_i^e(Z, \bar{q}) \geq 0$ for each $Z \subseteq I$ and all $i \in Z$.*

*Hence, the Vickrey auction is core-selecting for each set of participating bidders $X \subseteq I$.*

[30] Even though transforming a reallocative auction for goods into an auction for packages unifies Theorem 1's sufficient conditions with those for one-sided auctions, we cannot establish Theorem 1 simply by applying standard results on one-sided auctions (e.g., Theorem 31 in Milgrom and Strulovici (2009)) under a change of basis. Instead, as we discuss in the appendix before Lemma 4 and after the proof of Theorem 1, new arguments are needed because of the interaction between the change of basis and the reallocative nature of the auction.

[31] Theorem 1's derivative condition, on the other hand, ensures that adding packages to the quantity vector $\bar{q}$ increases the surplus available to any potential group of participating bidders. As we show, this ensures that the auctioneer will receive non-negative payments for packages from all bidders, just as in a one-sided auction.

Intuitively, commutativity (b) (or equivalently, simultaneous diagonalizability) is a homogeneity property of substitution patterns. It ensures that there is some set of *implicit* packages — the common eigenbasis $\Phi$ — in which bidders' valuations are separable (i). When we orient these implicit packages so that the quantity of each for sale in the auction is non-negative — i.e., so that $T_\Phi^{-1}\bar{q} \geq 0$ — the homogeneity of bidders' marginal utilities at zero (a) makes it efficient to allocate a positive quantity of each package to each bidder, no matter which bidders participate in the auction (ii). As we show in Section 6, these implicit packages can also be useful in implementation.

We emphasize that heterogeneity among bidders does not necessarily mean that the Vickrey outcome lies outside the bidder-core.[32] In particular, a continuity argument demonstrates that when this heterogeneity is *limited*, Theorem 1's allocation and package substitutability conditions — and thus bidder-core selection — continue to hold: that is, "near commutativity" and "near pre-auction efficiency" imply bidder-core selection.

Revisiting our motivating examples from Section 4.1 clarifies the scope of Theorems 1 and 2. (Figure 2 illustrates.) In Example 1, core selection fails with a single good: Bidders' marginal utilities differ at zero, so Theorem 2 does not apply. And since the only possible packages are (multiples of) $\pm 1$, Theorem 1's allocation condition would require all bidders to be net buyers or net sellers, which is not efficient. In Example 2, on the other hand, Theorem 2 ensures that the auction is core-selecting: While some bidders view the goods as substitutes and others view them as complements, the commutativity of their substitution patterns and efficiency of the pre-auction allocation jointly guarantee the existence of a set of substitutable packages (here, $[1\ 1]'$ and $[-1\ 1]'$) that are allocated in non-negative quantities. Finally, in Example 3, core selection fails with substitutable goods and an efficient pre-auction allocation: The bidders' substitution patterns do not commute, so Theorem 2 is not applicable. In fact, they are different enough that even though the auctioneer has a positive quantity of each good, efficiency still requires him to reallocate them among the bidders, violating Theorem 1's allocation condition.

#### Discussion

Generalizing substitutes from goods to packages by changing how we think of 'more' is natural in many applications. For instance, suppose that the goods are licenses for wireless spectrum in Los Angeles and Chicago. A unit of package $\phi_1 = [1\ 1]'$ is thus equivalent to a license for the a unit of wireless spectrum in both cities, while a unit of package $\phi_2 = [1\ -1]'$ is equivalent to a contract to swap a unit of wireless spectrum in Chicago for the same amount in Los Angeles.[33] The bidders may differ in whether they consider spectrum in

[32] In fact, the condition that Proposition 6 shows is necessary in a maximal domain sense is strictly weaker than Theorem 2's commutativity condition.

[33] Alternatively, if the goods sold in the auction were contracts for electricity generation in the two markets, $\phi_1$ would represent a contract to deliver the same amount of electricity to both cities, while $\phi_2$ could represent

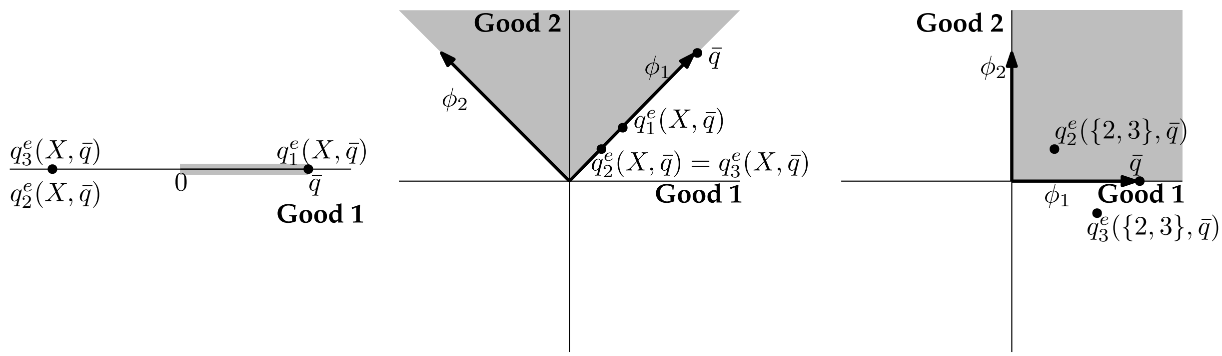


**Figure 2: Allocations and core-selection conditions in examples.** These three panels illustrate Theorem 1 in Examples 1-3. First panel, Example 1: There are no packages that are efficiently allocated in positive quantities to all three bidders. Second panel, Example 2: Each bidder's efficient allocation lies in the cone generated by the packages $\phi_1 = [1\ 1]'$ and $\phi_2 = [-1\ 1]'$ that form an eigenbasis for the bidders' substitution patterns $D^2u_i(q_i)$. Third panel, Example 3: When only bidders 2 and 3 participate, bidder 3's efficient allocation $q_3^e(\{2,3\},\bar{q})$ is outside the positive orthant of $\mathbb{R}^2$ where Theorem 1's allocation condition holds with substitutable goods.

the two markets to be complements (e.g., if they want to expand their coverage nationally) or substitutes (e.g., if they want to focus on achieving a high level of coverage in as many markets as they can). Nevertheless, core selection can still obtain, as it does in Example 2, if all bidders view contracts to *swap* spectrum between the two markets and contracts to acquire *more* spectrum as substitutes.

This taxonomy of *switching packages* and *buying packages* applies more generally than the spectrum context. If $\Phi \subset \mathbb{R}^K$ is a set of packages, then each of its elements can be classified as a switching package (if it has both positive and negative entries), a buying package (if its entries are all nonnegative), or a selling package (if its entries are all nonpositive). For instance, in Example 2, $\phi_1$ is a buying package, and $\phi_2$ is a switching package.

When there are packages that are seen as substitutes by all bidders, the success or failure of Theorem 1's allocation condition depends on the auctioneer's quantity vector, not just the bidders' valuations. In Section D of the Supplementary Appendix, we show that for any set of packages, there is a set of directions (which may be empty) such that if $\bar{q}$ is large enough in one of those directions, each bidder receives a non-negative allocation of each package. Thus, a quantity vector that is sufficiently large in that direction can mitigate the impact of heterogeneity among bidders on the possibility of bidder-core selection.

### 4.3 Maximal Domain Results for Core Selection

Theorem 2 shows that in reallocative auctions, the combination of homogeneity in bidders' substitution patterns and homogeneity in their marginal utility at the pre-auction allocation is sufficient for bidder-core selection to be possible. On the other hand, Examples 1

a contract to transmit electricity from Chicago to Los Angeles.

and 3 demonstrate that with heterogeneity in either bidders' substitution patterns or their pre-auction marginal utilities, the Vickrey auction need not be bidder-core-selecting. Our next results establish that both types of homogeneity are *necessary* for bidder-core selection in a maximal domain sense.

First, Proposition 4 demonstrates that in fact, for *any* profile of bidders' valuations over goods, there exist pre-auction allocations such that the Vickrey outcome is not in the core for some profile of participants.

**Proposition 4 (No Core Selection: Pre-Auction Allocations and Heterogeneity in Pre-Auction Marginal Utility).** *There is a pre-auction allocation $\{t_i\}_{i\in I}$ such that if each bidder $i$ is endowed with the valuation $\tilde{u}_i(q_i) = u_i(q_i + t_i)$, the Vickrey auction is not bidder-core-selecting for some set of participants $X \subseteq I$.*

Similarly, Proposition 5 shows that if the pre-auction allocation is not efficient, there is a quantity vector $\bar{q}$ for the auctioneer such that the Vickrey auction is not bidder-core-selecting.

**Proposition 5 (No Core Selection: Quantity Auctioned and Heterogeneity in Pre-Auction Marginal Utility).** *If $\nabla u_\ell(0) \neq \nabla u_j(0)$ for some $j, \ell \in I$, there is a quantity vector $\bar{q}$ such that the Vickrey auction is not bidder-core-selecting for some set of participants $X \subseteq I$.*

Together, Propositions 4 and 5 make precise the sense in which an efficient pre-auction allocation is necessary for bidder-core selection: The necessary condition is placed *jointly* on the bidders' pre-auction quantities and the quantity supplied by the auctioneer, and is thus weaker than a condition on the former alone. Whenever the pre-auction allocation is inefficient, there is a quantity vector for the auctioneer for which bidder-core selection fails (Proposition 5); conversely, for any quantity vector for the auctioneer, there is an inefficient pre-auction allocation that makes bidder-core selection fail (Proposition 4).[34]

However, as Example 3 shows, an efficient pre-auction allocation is not sufficient for core selection. Instead, Theorem 2 also requires bidders' substitution patterns to be homogeneous, in the sense that their eigenvectors are the same. Proposition 6 clarifies the sense in which the latter homogeneity condition is necessary, at least in the case of quadratic valuations: Suppose that substitution patterns $-S_i$ are heterogeneous enough that even though $S_\ell^{-1}$, the harmonic mean $\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}$, and $S_j^{-1}$ are all positive definite, their product has a negative eigenvalue. Then there is some quantity vector $\bar{q}$ for the auctioneer such that core selection is impossible, even when each bidder has the same marginal utility at the pre-auction allocation.[35]

[34] We emphasize that the challenges that bidder heterogeneity presents for core selection in reallocative auctions are not due to the divisibility of goods in the setting we study. In particular, in the Supplementary Appendix, we contribute a converse similar to Proposition 4 (Proposition F.1) that applies in environments with indivisible goods and multi-unit demands (i.e., where allocations are constrained to some subset of $\mathbb{Z}^K$).

[35] Equivalently, we can fix $\bar{q}$ and find a *rotation* of the substitution patterns — that is, right-multiply each $S_i$ by some orthonormal $A$ and left-multiply it by $A'$ — for which bidder-core selection fails.

**Proposition 6 (No Core Selection: Heterogeneity in Substitution Patterns).** *Suppose that bidders have quadratic valuations with substitution patterns $\{-S_i\}_{i\in I}$ and the same marginal utility parameter $\theta_i = \theta$. If the matrix $S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1}$ has a negative eigenvalue for some coalition $Z \subseteq I$ and agents $\ell, j \in Z$, then there is a quantity vector $\bar{q}$ such that the Vickrey auction is not bidder-core-selecting for some set of participants $X \subseteq I$.*

### 4.4 Discussion

As has been frequently noted (e.g., Milgrom (2007, 2019); Milgrom and Segal (2020)), the Vickrey auction has drawbacks. It is not group strategy-proof; it can be computationally demanding to implement (Day and Raghavan, 2007); and perhaps most importantly, revenue may be uncompetitively low. However, (bidder-)core selection rules out the last of these; consequently, auction designs that coincide with the Vickrey auction in such environments (e.g., Vickrey-nearest core pricing (Day and Raghavan, 2007; Day and Cramton, 2012)) are widely used.

Our main results show that in reallocative auctions, the conditions under which these revenue drawbacks are severe differ from those in one-sided auctions:[36] Substitutability among goods is no longer necessary or sufficient for (bidder-)core selection (Examples 1-3). Instead, a design's revenue competitiveness depends crucially on the degree of *heterogeneity* among the bidders, both in their substitution patterns and pre-auction marginal utilities (Theorem 2, Propositions 4 and 6), rather than the presence of complementarities.

One takeaway from our analysis is that in practice, the possibility of (bidder-)core selection in reallocative environments depends on the intended design objective: A reallocative Vickrey auction has poor revenue performance (i.e., isn't bidder-core selecting) when bidders have very different marginal utilities at their pre-auction allocations, and the auction is meant to rectify the inefficiency in these allocations. But when it is meant to allow bidders to realize gains from trade *that appear when the auctioneer offers a new quantity of goods for sale*, a reallocative Vickrey auction can have good revenue performance (both in the sense of bidder-core selection, and relative to other designs — see Section 5) whenever bidders substitute between goods in similar ways. In particular, the underlying $K$ goods need not be substitutable — only a set of $K$ bundles, or *packages*.

## 5 Revenue in the Vickrey and Uniform-Price Auctions

In practice, concerns about the Vickrey auction's revenue often lead designers to employ alternative auction mechanisms. (See, e.g., Milgrom (2007, 2019) for a discussion).

[36] Moreover, our results describe the environments where Vickrey-nearest core pricing coincides with Vickrey, and thus inherits its strategyproofness.

These concerns are especially prominent in the environments we consider: When efficiency requires reallocating goods among the bidders, the Vickrey auction subsidizes that reallocation (Proposition 2). This motivates the consideration of *uniform-price* designs, such as the one used in the clock phases of the FCC Incentive Auction: Unlike the Vickrey auction, a uniform-price auction never subsidizes reallocation.

In this section, we compare the revenue performance of the Vickrey and uniform-price auctions, and show that the same feature of the environment that makes the Vickrey auction select outcomes in the bidder-core — namely, limited heterogeneity among the bidders — also leads it to outperform the uniform-price auction on revenue (Theorem 3). For simplicity, we work in an environment where bidders have quadratic valuations with a common substitution pattern $-S$.[37]

**Definition (Uniform-Price Auction).** The *uniform-price auction* (UPA) is conducted the same way as the Vickrey auction, except that the payment rule requires bidders to pay the market-clearing price for each unit of the good they are awarded: $t_i^U(b) = p^*(b) \cdot q_i^*(b)$.

Unlike in the Vickrey auction, bidders do not have weakly dominant strategies in the uniform-price auction. However, we can still remain agnostic about the bidders' information when computing the auction's equilibrium: The UPA has a unique linear *ex post* equilibrium such that each bidder's bid is optimal, regardless of the identities of the other bidders.[38] Proposition 7 offers the standard characterization of this equilibrium.

**Proposition 7 (Equilibrium in the Uniform-Price Auction).** *Suppose that bidders have quadratic valuations with a common substitution pattern $-S$, and that $|X| > 2$.*

*i. The profile of bids $b^U = \{b_i^U\}_{i \in I}$, where $b_i^U(q_i) \equiv \theta_i - \frac{|X|-1}{|X|-2} S q_i$, is an ex post equilibrium of the uniform-price auction.*

*ii. In the ex post equilibrium $b^U$, unless all participating bidders have identical valuations, the UPA implements an inefficient allocation: $\{q_i^*(\{b_j^U\}_{j \in X})\}_{i \in X} \neq \{q_i^e(X, \bar{q})\}_{i \in X}$.*

*iii. In the ex post equilibrium $b^U$, the UPA produces revenue $\pi_a^U(X) \equiv \frac{1}{|X|} \left( \left( \sum_{i \in X} \theta_i' \bar{q} \right) - \frac{|X|-1}{|X|-2} \bar{q}' S Q \right)$.*

In a uniform-price auction, bidders *shade their bids* by submitting a bid function $b_i^U$ that

[37] While we do not provide results for nonquadratic utilities, we anticipate that the results we establish — and in particular, the role of heterogeneity they highlight — will continue to hold qualitatively. Nonquadratic utilities will change the conditions quantitatively by introducing effects due to the third derivative of the utility function.

[38] Formally, we say that a profile of bids $\{b_i\}_{i \in I}$ for *all* agents is an *ex post equilibrium* of the uniform-price auction if for each set of participating bidders $X' \subseteq I$ with $|X'| = |X|$, $\{b_i\}_{i \in X'}$ is a Nash equilibrium of a uniform-price auction with participants $X'$.

is steeper than their marginal utility schedule $\nabla u_i(q_i) = \theta_i - Sq_i$.[39] As a consequence, unlike the Vickrey auction, the uniform-price auction does not allocate goods efficiently among the bidders. Its revenue properties, on the other hand, may be more favorable than those of the Vickrey auction: Corollary 2 shows that revenue in the UPA can never fall below zero, as long as each bidder's marginal utility for goods is high enough that the auctioneer could sell his entire quantity vector $\bar{q}$ to her for a positive price. In contrast, revenue in a reallocative Vickrey auction can be negative — and when $\bar{q} = 0$, it will never be positive (Vickrey, 1961).

**Corollary 2 (Nonnegative Revenue in the Uniform-Price Auction).** *Suppose that bidders have quadratic valuations with a common substitution pattern $-S$, and that $|X| > 2$. If the quantity auctioned $\bar{q}$ and the bidders' marginal utilities at $\bar{q}$, $\{\theta_i - SQ\}_{i \in I}$, are each non-negative, then the auctioneer's revenue $\pi_a^U(X)$ from a uniform-price auction is non-negative.*

However, the UPA's revenue performance is not always better than that of the Vickrey auction. Even though the auctioneer may receive a larger *share* of the surplus realized in a uniform-price auction than the share he would receive in a Vickrey auction, the *amount* of surplus realized in a uniform-price auction is smaller, since it does not produce an efficient allocation. The revenue ranking of the two auctions thus depends on which of the two effects dominates.

For clarity, we compare the revenue produced by the Vickrey and uniform-price auctions from an *ex ante* perspective, before the auctioneer knows the identities of the participating bidders. This requires informational assumptions that we have thus far avoided. In particular, we assume that the auctioneer knows the *number* of participants, and that the bidders' marginal utility parameters are i.i.d.

**Definition (I.I.D. Quadratic Valuations).** We say bidders have *i.i.d. quadratic valuations* if

(a) The set of potential bidders is $I = \coprod_{i=1}^{N} \Theta$ for some set $\Theta \subset \mathbb{R}^K$ and $N > 2$;[40]

(b) Bidders have quadratic valuations with a common substitution pattern $-S$: For each bidder $\theta \in I$, $u_\theta(q) = \theta' q - \frac{1}{2} q' S q$; and

(c) The auctioneer has a prior $\mu$ over sets of participants $X$ that assigns positive probability only to sets of participants $\{\theta_i\}_{i=1}^{N}$ with cardinality $N$, and does so such that each $\theta_i$ is i.i.d. with distribution $F$.[41]

[39] This is because bidders in a uniform-price auction have *price impact*: In either a Vickrey or uniform-price auction, the only way for a bidder to increase the quantity of each good sold to her in the auction is by placing a higher bid that causes market-clearing prices to rise. With a Vickrey payment rule, this price increase only affects the bidder's payment for the *marginal* units of the goods that she obtains by submitting a higher bid. But in a uniform-price auction, the price increase also affects the bidder's payment for the *inframarginal* units of the goods — the bundle that she would have received if she had placed a lower bid instead.

[40] That is, $I$ is the coproduct, or disjoint union, of $N$ copies of $\Theta$.

[41] Formally, $\mu$ is a product distribution on $\prod_{i=1}^{N} \Theta$ with identical marginals $F$.

When bidders have quadratic valuations that are perfectly correlated, rather than i.i.d., Ausubel et al. (2014) show that with a single good, the Vickrey auction produces more revenue than the uniform-price auction. Theorem 3 shows that when we allow for heterogeneity by assuming i.i.d. quadratic valuations instead, this revenue ranking can reverse. In particular, the UPA outperforms the Vickrey auction when expected heterogeneity in bidders' marginal utility schedules is large, and so (by Propositions 4 and 5) bidder-core selection is likely to fail. Conversely, the Vickrey auction outperforms the UPA when expected heterogeneity in bidders' marginal utility parameters is small, and so it is likely to be core selecting (Theorem 2).[42]

**Theorem 3 (Expected Revenue in the Uniform-Price and Vickrey Auctions).** *Suppose that bidders have i.i.d. quadratic valuations, let $\Phi = \{\phi_k\}_{k=1}^K$ be the orthonormal eigenbasis of the common substitution pattern $-S$, and let $\{\lambda_k\}_{k\in K}$ be the corresponding eigenvalues of $S$.*

*i. In the Vickrey auction, expected revenue is $E_\mu[\pi_a^V(X)] = \bar{q}' E_F(\theta) - \frac{1}{2}\sum_{k=1}^K \frac{1}{\lambda_k} \mathrm{Var}_F(\phi_k \cdot \theta) - \frac{2N-1}{2N(N-1)}\bar{q}' SQ$.*

*ii. In the uniform-price auction, expected revenue is $E_\mu[\pi_a^U(X)] = \bar{q}' E_F(\theta) - \frac{N-1}{N(N-2)}\bar{q}' S\bar{q}$.*

*iii. Expected revenue is higher in the Vickrey auction than in the uniform-price auction if and only if*

$$\sum_{k=1}^K \frac{1}{\lambda_k} \mathrm{Var}_F(\phi_k \cdot \theta) < \frac{1}{(N-1)(N-2)}\bar{q}' SQ.$$

Theorem 3 (i) shows that in expectation, Vickrey revenue is increasing in the number of participating bidders $N$ and in their expected marginal utility in the direction of the quantity auctioned $\bar{q}$, but decreasing in expected *heterogeneity* in bidders' marginal utility for the packages $\Phi$. Intuitively, since the bidders have identical substitution patterns, we can think of the Vickrey auction as having two stages: first, the auctioneer efficiently reallocates goods among the bidders; second, the auctioneer sells his quantity vector to the bidders, who now have identical marginal utility schedules. In the first stage, the auctioneer loses revenue by subsidizing reallocation among the bidders; the more heterogeneous the bidders are, the larger the revenue loss. In the second stage, the auctioneer makes money by selling his quantity vector; the higher the bidders' marginal utility in the direction of the quantity vector, and the more of them there are, the more money he makes.[43]

[42]In a working paper version of Ausubel et al. (2014), the authors compare Vickrey and UPA revenue in the independent private values case with one good. By providing a comparison for multiple-good environments, Theorem 3 highlights the role of heterogeneity in bidder valuations, thus providing an intuitive link between the environments where UPA is outperformed by the Vickrey auction and those in which core selection is likely to hold.

[43]Theorem 3 (i) also highlights another important difference between core selection in environments where

In the uniform-price auction, on the other hand, no such subsidization is necessary. Thus, Theorem 3 (ii) shows that the auctioneer's expected revenue only depends on the bidders' expected marginal utility parameter, the number of participating bidders, and the quantity auctioned. Together, parts (i) and (ii) of Theorem 3 show that (iii) the uniform-price auction outperforms the Vickrey auction when expected heterogeneity is large enough, and this threshold is decreasing in the number of participating bidders and the quantity auctioned.

# 6 Implementation

The auctions that we consider present bidders with a complex task. Each allows *package bidding*: bidders are asked to submit bids for *combinations* of goods, rather than *amounts* of each good. And because these auctions are reallocative, bidders must consider both the prices they are willing to pay to buy bundles of goods *and* the prices they are willing to take to sell them — and moreover, the prices at which they are willing to buy some goods and sell others.

In this section, we consider the extent to which both types of complexity can be ameliorated with an alternative implementation of the Vickrey or uniform-price auctions. First, we consider the effects of splitting a reallocative auction into independent *forward* (selling) and *reverse* (buying) components linked by a *clearing rule*, as in the FCC Incentive Auction. We show that, while splitting an auction does not change the revenue or allocation that it produces when (as in the FCC Incentive Auction) pricing is uniform, it can change both in a Vickrey auction (Proposition 8): In particular, truthful bidding is no longer a dominant strategy in the split Vickrey auction, and so the allocation it produces need not be efficient. Instead, the VCG principle requires the payment rules in the forward and reverse auctions, not just their market-clearing prices and quantities, to each depend on the bids submitted in the other auction, as in Andreyanov et al. (2023).

Then, in Proposition 9, we show that when bidders' substitution patterns are homogeneous (in the sense of Theorem 2), a Vickrey or uniform-price auction where bidders bid on *all* bundles is equivalent to $K$ independent auctions where they bid on specific packages *individually*. This allows a designer to bypass the complexity problem that bidders face in an auction where they can bid on each possible bundle, without leading to the well-known

reallocation is feasible and those where it is not. In one-sided Vickrey auctions, the auctioneer captures all surplus as the number of participants becomes large, and so such large auctions are approximately core selecting. In contrast, the auctioneer does not capture all surplus from a reallocative Vickrey auction as it becomes large, even in expectation: $\lim_{N\to\infty} E_\mu[\pi_a^V(X)] = \bar{q}' E_F(\theta) - \frac{1}{2}\sum_{k=1}^K \frac{1}{\lambda_k}\mathrm{Var}_F(\phi_k \cdot \theta) < \bar{q}' E_F(\theta) = \lim_{N\to\infty} E_\mu[v(X,\bar{q})]$. Intuitively, when reallocation is feasible, efficiency generically requires goods to be reallocated between bidders, even as the number of those bidders becomes large. Since a Vickrey auction must subsidize that reallocation, the fraction of total surplus obtained by the auctioneer is bounded away from 1.

*exposure* problems present in auctions where *goods* are sold independently.[44]

## 6.1 Split Auctions

Throughout, we have focused on *reallocative* auctions that allow bidders to both buy and sell goods. These auctions facilitate the realization of gains from trade among the bidders, rather than just between the bidders and the auctioneer. This is a central objective of many auctions, including the FCC Incentive Auction. But when some agents are known to be buyers, and others are known to be sellers, we could also seek to achieve a designer's reallocation objectives by using a pair of independent *non-reallocative* auctions whose market-clearing prices and quantities are linked, as the FCC Incentive Auction did. We call such a design choice a *split auction*.

**Definition.** A *split auction* consists of three parts:

- a *forward auction* mechanism that auctions a quantity $\bar{q}_+$ to participating bidders $X_+$;
- a *reverse auction* mechanism that auctions a quantity $\bar{q}_-$ to participating bidders $X_- = X \setminus X_+$; and
- a *clearing rule* specifying that given the profiles of bids $b_{X_+} = \{b_i\}_{i \in X_+}$ and $b_{X_-} = \{b_i\}_{i \in X_-}$ submitted in the forward and reverse auctions, the quantities $\bar{q}_-$ and $\bar{q}_+$ are chosen so that (i) they sum to the quantity auctioned (i.e., $\bar{q}_- + \bar{q}_+ = \bar{q}$) and (ii) the market-clearing prices $p^*(b_{X_+})$ and $p^*(b_{X_-})$ in the forward and reverse auctions are the same.

Because of the clearing rule, if the profiles of bids submitted in the forward and reverse auctions are $b_{X_+}$ and $b_{X_-}$, the market-clearing prices $p^*(b_{X_+})$ and $p^*(b_{X_-})$ must each be equal to the market-clearing price $p^*(b)$ that would obtain if the same bids were submitted in a single reallocative auction. Consequently, the split auction must award bidders the same bundles that a single reallocative auction would. Since payments in a uniform-price auction only depend on the market-clearing price and the bundles obtained by the bidders, it follows that a split UPA and a reallocative UPA are equivalent. Hence, with a uniform-price design like the one used in the clock phases of the FCC Incentive Auction, splitting the auction into buyers and sellers reduces the level of complexity facing the bidders without affecting the revenue or allocation produced by the auction.

Splitting a Vickrey auction in this way would be less innocuous: Truthful bidding would no longer be a dominant strategy. This should not be surprising: given the same bids $b$, payments in a split Vickrey auction are different from those in a single reallocative

[44]For a more detailed description of the exposure problem, see, e.g., De Vries and Vohra (2003) or Milgrom (2007). For empirical evidence on its impact in FCC spectrum auctions, see Xiao and Yuan (2022).

Vickrey auction, but the Vickrey payment rule uniquely implements the efficient allocation (Holmström, 1979).

**Proposition 8 (No Truthful Bidding in a Split Vickrey Auction).** *There exist valuations $\{u_i\}_{i \in X}$ for the set of participating agents such that for any bidder $i \in X$, submitting the bid function $b_i^V(q_i) = \nabla u_i(q_i)$ is not a weakly dominant strategy in a split Vickrey auction.*

Observe that in a reallocative auction, a bidder's residual supply is the price that clears the market, given the quantity that he receives, but in the forward part of a split auction, residual supply is the price that clears the market *holding the quantity awarded to bidders in $X_-$ constant.* Consequently, splitting the auction causes each bidder's residual supply curve to pivot around the market-clearing price and quantity. Since the Vickrey payment is just the area under the residual supply curve, this shift changes the payment rule and gives bidders incentives to submit bids that differ from their marginal utility schedules.

## 6.2 Independent Package Auctions

When an auction solicits bids for all possible bundles, it places a computational burden on both the bidders (who must compute their valuations for each bundle) and on the auctioneer (who faces a difficult problem when mapping bids for many different bundles to an allocation).[45] This burden can be alleviated substantially by using a dynamically implemented auction (as in, e.g., Ausubel and Milgrom (2002)) or by constructing predefined *packages* and requiring bidders to place independent bids on them. Both approaches are common in practice (e.g., Milgrom (2019)); for instance, the U.S. FCC employs both concurrently in many of its auctions for wireless spectrum (Xiao and Yuan, 2022).

But unlike implementing an auction dynamically, restricting bidders to independent bids on specific packages can affect the outcome of the auction and the revenue it produces. This is because it introduces an *exposure problem*: The amount a bidder is willing to pay for a package may depend on the quantities of other packages it obtains in the auction, but its bids for that package cannot be contingent on those quantities. Consequently, the design of these packages is important for both the revenue an auction yields and the surplus it creates: For instance, Xiao and Yuan (2022) use bids from FCC Auction 73 to show that different package designs could change the revenue and surplus generated by the auction by billions of dollars.

Proposition 9 shows how to design these packages when heterogeneity among bidders' substitution patterns is limited. In particular, it shows that when bidders' substitution patterns commute with one another, the exposure problem can be eliminated by choosing packages that coincide with the substitution patterns' common eigenvectors. Intuitively,

[45] See Parkes and Ungar (2000) and De Vries and Vohra (2003) for a discussion of the computational problem facing the bidders and the auctioneer, respectively.

designing the packages in this way leverages the homogeneity of bidders' substitution patterns to reduce the auction's complexity without affecting bidders' incentives.

**Proposition 9 (Equivalent Implementation with Eigenvector Packages).** *Suppose that bidders' substitution patterns $\{D^2 u_i(q_i)\}_{i \in I, q_i \in \mathbb{R}^K}$ are commuting matrices. Then they have a common orthonormal eigenbasis $\Phi$, and conducting independent Vickrey auctions for each package in $\Phi$ yields the same allocation and payoff profile as a single Vickrey auction in which goods are sold jointly. Moreover, if bidders have quadratic valuations with identical substitution patterns, and there are at least three participating bidders, then conducting independent uniform-price auctions for each package in $\Phi$ yields the same allocation and payoff profile as a single uniform-price auction in which goods are sold jointly.*

For intuition, recall from the discussion following Theorem 2 that when bidders' substitution patterns commute, their valuations are additively separable in those substitution patterns' common eigenbasis $\Phi$. As a consequence, their equilibrium bids in either the Vickrey or uniform-price auction are also separable in $\Phi$. It follows that the equilibrium of either auction can be decentralized among $K$ separate auctions for the packages in $\Phi$.

We emphasize that when the eigenbases of bidders' substitution patterns are very different, they cannot be used to design packages that eliminate (or nearly eliminate) the exposure problem for all bidders simultaneously. But if the data suggests that the bidders' substitution patterns have approximately the same eigenvectors, Proposition 9 indicates how packages could be defined in practice. Moreover, as noted in Section 4.2, each of these packages can be described to bidders very simply as a switching package (a package that swaps some goods for others — what Ferguson and Milgrom (2024) call a "swap bid") or a buying package (a more traditional package that contains non-negative amounts of each good). Employing both buying and switching packages allows the auction to be decentralized into independent auctions for packages, even when bidders do not regard any buying packages as independent.

# 7 Conclusion

Our results provide a more nuanced view of core-selection as well as the usefulness of the Vickrey auction (and equivalent designs). As we show, it is not the presence of complementarities per se, but rather heterogeneity in bidder valuations — both in pre-auction marginal utility, and (independently) in substitution patterns — that can challenge core selection.

Our results also qualify both the conditions under which the drawbacks of the Vickrey auction noted by the literature are or are not severe, thus adding to the discussion of the relative merits of the Vickrey auction vs. alternative designs (see, e.g., Milgrom (2019), Ausubel and Baranov (2023)). In particular, when bidders' substitution patterns are sufficiently similar, and the gains from trade in the auction come primarily from the new quantity vector offered by the designer, the reallocative Vickrey auction performs well. When bidder heterogeneity is significant — such as when some are TV broadcasters and others are providers of wireless broadband — uniform-price rules like those used in the clock phase of the FCC's Incentive auction may perform better (Theorem 3).

More generally, our analysis demonstrates that environments where bidders are interested in both buying and selling (e.g., markets for natural resources; Teytelboym (2019)) offer new opportunities for market design research beyond classical market design.

# References


ANDREYANOV, P., J. PARK, AND T. SADZIK (2023): "Optimal Robust Double Auctions," Working paper.

AUSUBEL, L. M., C. APERJIS, AND O. BARANOV (2017): "Market Design and the FCC Incentive Auction," Working paper.

AUSUBEL, L. M. AND O. BARANOV (2020): "Core-Selecting Auctions with Incomplete Information," *International Journal of Game Theory*, 49, 251–273.

——— (2023): "The VCG Mechanism, the Core, and Assignment Stages in Auctions," Working paper.

AUSUBEL, L. M., P. CRAMTON, AND P. MILGROM (2006): "The Clock-Proxy Auction: A Practical Combinatorial Auction Design," *Handbook of Spectrum Auction Design*, 120–140.

AUSUBEL, L. M., P. CRAMTON, M. PYCIA, M. ROSTEK, AND M. WERETKA (2014): "Demand Reduction and Inefficiency in Multi-Unit Auctions," *Review of Economic Studies*, 81, 1366–1400.

AUSUBEL, L. M. AND P. MILGROM (2002): "Ascending Auctions with Package Bidding," *Frontiers of Theoretical Economics*, 1, 1–42.

BALDWIN, E. AND P. KLEMPERER (2019): "Understanding Preferences: "Demand Types", and the Existence of Equilibrium with Indivisibilities," *Econometrica*, 87, 867–932.

BICHLER, M., V. FUX, AND J. K. GOEREE (2019): "Designing combinatorial exchanges for the reallocation of resource rights," *Proceedings of the National Academy of Sciences*, 116, 786–791.

BIKHCHANDANI, S. AND J. M. OSTROY (2002): "The Package Assignment Model," *Journal of Economic Theory*, 107, 377–406.

BONDAREVA, O. N. (1963): "Some Applications of Linear Programming Methods to the Theory of Cooperative Games," *Problemy Kibernetiki*, 10, 119.

DAY, R. AND P. MILGROM (2008): "Core-Selecting Package Auctions," *International Journal of Game Theory*, 36, 393–407.

DAY, R. W. AND P. CRAMTON (2012): "Quadratic Core-selecting Payment Rules for Combinatorial Auctions," *Operations Research*, 60, 588–603.

DAY, R. W. AND S. RAGHAVAN (2007): "Fair Payments for Efficient Allocations in Public Sector Combinatorial Auctions," *Management Science*, 53, 1389–1406.

De Vries, S. and R. V. Vohra (2003): "Combinatorial Auctions: A Survey," *INFORMS Journal on Computing*, 15, 284–309.

Doraszelski, U., K. Seim, M. Sinkinson, and P. Wang (2019): "Ownership Concentration and Strategic Supply Reduction," National Bureau of Economic Research Working Paper 23034.

Erdil, A. and P. Klemperer (2010): "A New Payment Rule for Core-selecting Package Auctions," *Journal of the European Economic Association*, 8, 537–547.

Ferguson, B. A. and P. Milgrom (2024): "Market Design for Surface Water," Tech. rep., National Bureau of Economic Research, nBER Working Paper 32010.

Galeotti, A., B. Golub, and S. Goyal (2020): "Targeting Interventions in Networks," *Econometrica*, 88, 2445–2471.

Galeotti, A., B. Golub, S. Goyal, E. Talamàs, and O. Tamuz (2024): "Robust Market Interventions," *arXiv preprint arXiv:2112.08153*.

Goeree, J. K. and Y. Lien (2016): "On the Impossibility of Core-Selecting Auctions," *Theoretical Economics*, 11, 41–52.

Gul, F. and E. Stacchetti (1999): "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory*, 87, 95–124.

Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2013): "Stability and Competitive Equilibrium in Trading Networks," *Journal of Political Economy*, 121, 966–1005.

Holmström, B. (1979): "Groves' Scheme on Restricted Domains," *Econometrica*, 1137–1144.

Johnson, C. R. (1982): "Inverse M-matrices," *Linear Algebra and its Applications*, 47, 195–216.

Klemperer, P. (2010): "The Product-Mix Auction: A New Auction Design for Differentiated Goods," *Journal of the European Economic Association*, 8, 526–536.

Kwerel, E., P. Sanyal, K. Seim, M. Stancill, and P. Sun (2017): "Economics at the FCC, 2016–2017: Auction Designs for Spectrum Repurposing and Universal Service Subsidies," *Review of Industrial Organization*, 51, 451–486.

Loertscher, S. and L. M. Marx (2020): "Asymptotically Optimal Prior-Free Clock Auctions," *Journal of Economic Theory*, 187, Article 105030.

Malamud, S. and M. Rostek (2017): "Decentralized Exchange," *American Economic Review*, 107, 3320–3362.

Milgrom, P. (2007): "Package Auctions and Exchanges," *Econometrica*, 75, 935–965.

——— (2019): "Auction Market Design: Recent Innovations," *Annual Review of Economics*, 11, 383–405.

——— (2022): "Markets for Water in California," https://mediatheque.lindau-nobel.org/recordings/40259/paul-milgrom, lecture, Lindau Nobel Laureate Meetings.

Milgrom, P. and I. Segal (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583–601.

——— (2020): "Clock Auctions and Radio Spectrum Reallocation," *Journal of Political Economy*, 128, 1–31.

Milgrom, P. and B. Strulovici (2009): "Substitute Goods, Auctions, and Equilibrium," *Journal of Economic Theory*, 144, 212–247.

MORSTYN, T., A. TEYTELBOYM, AND M. D. MCCULLOCH (2019): "Designing Decentralized Markets for Distribution System Flexibility," *IEEE Transactions on Power Systems*, 34, 2128–2139.

MYERSON, R. B. AND M. A. SATTERTHWAITE (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265–281.

NEWMAN, N., K. LEYTON-BROWN, P. MILGROM, AND I. SEGAL (2020): "Incentive Auction Design Alternatives: A Simulation Study," in *Proceedings of the 21st ACM Conference on Economics and Computation*, 603–604.

OSTROVSKY, M. (2008): "Stability in Supply Chain Networks," *American Economic Review*, 98, 897–923.

PARKES, D. C. AND L. H. UNGAR (2000): "Iterative Combinatorial Auctions: Theory and Practice," in *Proceedings of the Seventeenth National Conference on Artificial Intelligence and Twelfth Conference on Innovative Applications of Artificial Intelligence*, 74–81.

SHAPLEY, L. S. (1967): "On Balanced Sets and Cores," *Naval Research Logistics Quarterly*, 14, 453–460.

SUN, N. AND Z. YANG (2006): "Equilibria and Indivisibilities: Gross Substitutes and Complements," *Econometrica*, 74, 1385–1402.

TEYTELBOYM, A. (2019): "Natural capital market design," *Oxford Review of Economic Policy*, 35, 138–161.

TOPKIS, D. M. (1998): *Supermodularity and Complementarity*, Princeton University Press.

VICKREY, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

VOHRA, R. V. (2015): "Chapter 8 - Combinatorial Auctions," Elsevier, vol. 4 of *Handbook of Game Theory with Economic Applications*, 455–476.

WEINSTEIN, J. (2022): "Direct Complementarity," Working Paper.

XIAO, M. AND Z. YUAN (2022): "License Complementarity and Package Bidding: US Spectrum Auctions," *American Economic Journal: Microeconomics*, 14, 420–464.

YOKOO, M., Y. S. AND S. MATSUBARA (2004): "The Effect of False-name Bids in Combinatorial Auctions: New Fraud in Internet Auctions," *Games and Economic Behavior*, 46, 174–188.

# Appendix

**Vickrey Auction: Formal Description.** Here, we formally describe the version of the Vickrey auction that solicits bids in the form of inverse demand schedules. Each participating bidder $i \in X$ submits a *bid function* (i.e., a (net) inverse demand schedule) $b_i : \mathbb{R}^K \to M$ that is continuously differentiable, surjective, and downward sloping (in the sense that it has a negative definite Jacobian derivative matrix), and hence has a continuously differentiable inverse $b_i^{-1}$ with a negative definite Jacobian. This ensures that for each profile of bids $b = \{b_i\}_{i \in X}$, there is a unique market-clearing price $p^*(b) \in M$ that solves $\bar{q} = \sum_{i \in X} b_i^{-1}(p)$. (See Lemma E.2 in the Supplementary Appendix.) Each bidder $i$ is then awarded the bundle $q_i^*(b) := b_i^{-1}(p^*(b))$ at which his bid is equal to the market-clearing price. The amount $t_i^V(b)$ he pays for this bundle is given by the area under the (inverse) *residual supply curve* $r_i$ that he faces: $t_i^V(b) = \int_0^{q_i^*(b)} r_i(q, b_{-i}) \cdot dq$, where $r_i(q_i, b_{-i})$ is the unique value of $p$ such that $q_i + \sum_{j \in X \setminus \{i\}} b_j^{-1}(p) = \bar{q}$.

**Proof of Proposition 1 (Equilibrium in the Vickrey Auction)** (i) First, observe that if

$$q_i^*(b_i^V, b_{-i}) \in \arg\max_{q_i} u_i(q_i) - \int_0^{q_i} r_i(q, b_{-i}) \cdot dq, \tag{3}$$

then given $b_{-i}$, a bid of $b_i^V$ maximizes $u_i(q_i^*(b)) - t_i^V(b)$. Further, observe that $\int_0^{q_i} r_i(q, b_{-i})) \cdot dq$ is strictly convex: its Hessian matrix is $Dr_i(q_i, b_{-i})$, which is positive definite by the implicit function theorem, since $Db_j^{-1}(p)$ is negative definite for each $j \neq i$. It follows that the objective in (3) is strictly concave, and so is maximized at $q_i^*(b_i^V, b_{-i})$ if and only if $\nabla u_i(q_i^*(b_i^V, b_{-i})) = r_i(q_i^*(b_i^V, b_{-i}), b_{-i})$. By definition, we have $\nabla u_i(q_i^*(b_i^V, b_{-i})) = b_i^V(q_i^*(b_i^V, b_{-i})) = p^*(b_i^V, b_{-i}) = r_i(q_i^*(b_i^V, b_{-i}), b_{-i})$. It follows that $b_i^V$ is weakly dominant.

(ii) Setting $Z = X$, the Kuhn-Tucker conditions for a maximum in (2) are

$$\sum_{i \in X} q_i = \bar{q}, \qquad \nabla u_i(q_i) = p \text{ for each } i \in X \text{ for some } p \in \mathbb{R}^K.$$

By definition, these are satisfied by setting $q_i = q_i^*(b^V)$ for each $i \in X$ and $p = p^*(b^V)$; the statement follows.

(iii) Setting $Z = X \setminus \{i\}$ and replacing $\bar{q}$ with $\bar{q} - q_i$, the Kuhn-Tucker conditions for a maximum in (2) are

$$\sum_{j \in X \setminus \{i\}} q_j = \bar{q} - q_i, \qquad \nabla u_j(q_j) = p \text{ for each } j \in X \setminus \{i\} \text{ for some } p \in \mathbb{R}^K.$$

By definition, these are satisfied by setting $p = r_i(q_i, b_{-i}^V)$ and $q_j = (b_j^V)^{-1}(p)$. By the envelope theorem, then, $\nabla_{q_i} v(X \setminus \{i\}, \bar{q} - q_i) = -r_i(q_i, b_{-i}^V)$. Then by the fundamental theorem of calculus, for each $i \in X$,

$$\begin{aligned}
\pi_i^V(X) &= u_i(q_i^*(b^V)) + \int_0^{q_i^*(b^V)} \nabla_{q_i} v(X \setminus \{i\}, \bar{q} - q_i) \cdot dq_i \\
&= u_i(q_i^*(b^V)) + v(X \setminus \{i\}, \bar{q} - q_i^*(b^V)) - v(X \setminus \{i\}, \bar{q}) \\
&= u_i(q_i^e(X, \bar{q})) + v(X \setminus \{i\}, \bar{q} - q_i^e(X, \bar{q})) - v(X \setminus \{i\}, \bar{q}) \text{ (by (ii))} \\
&= v(X, \bar{q}) - v(X \setminus \{i\}, \bar{q}).
\end{aligned}$$

The expression for $\pi_a^V(X)$ then follows from (ii). □

**Proof of Proposition 2 (Coalitional Rationality of Participation)** Suppose $Z \subseteq X$ blocks $\pi_i^V(X)$. Then by Proposition 1 (iii),

$$\begin{aligned}
&|Z| v(X, \bar{q}) - \sum_{i \in Z} v(X \setminus \{i\}, \bar{q}) < v(Z, 0) \\
\Leftrightarrow\ &|Z| \max_{\{q_i\}_{i \in X}} \left\{ \sum_{i \in X} u_i(q_i) \text{ s.t. } \sum_{i \in X} q_i = \bar{q} \right\} < \max_{\{q_i\}_{i \in Z}} \left\{ \sum_{i \in Z} u_i(q_i) \text{ s.t. } \sum_{i \in Z} q_i = 0 \right\} \\
&\qquad + \sum_{i \in Z} \max_{\{q_j\}_{j \in X \setminus \{i\}}} \left\{ \sum_{j \in X \setminus \{i\}} u_j(q_j) \text{ s.t. } \sum_{j \in X \setminus \{i\}} q_j = \bar{q} \right\},
\end{aligned}$$

$$\Leftrightarrow \max_{\{q_{i,j}\}_{\substack{i\in Z\\ j\in X}}} \left\{ \sum_{\substack{i\in Z\\ j\in X}} u_i(q_{i,j}) \text{ s.t. } \sum_{\substack{i\in Z\\ j\in X}} q_{i,j} = |Z|\bar{q} \right\} < \max_{\{q_{i,i}\}_{i\in Z}} \left\{ \sum_{i\in Z} u_i(q_{i,i}) \text{ s.t. } \sum_{i\in Z} q_{i,i} = 0 \right\}$$

$$+ \max_{\{q_{i,j}\}_{\substack{i\in Z\\ j\in X\setminus\{i\}}}} \left\{ \sum_{\substack{i\in Z\\ j\in X\setminus\{i\}}} u_j(q_{i,j}) \text{ s.t. } \sum_{j\in X\setminus\{i\}} q_{i,j} = \bar{q}\ \forall i\in Z \right\},$$

$$\Leftrightarrow \max_{\{q_{i,j}\}_{\substack{i\in Z\\ j\in X}}} \left\{ \sum_{\substack{i\in Z\\ j\in X}} u_i(q_{i,j}) \text{ s.t. } \sum_{\substack{i\in Z\\ j\in X}} q_{i,j} = |Z|\bar{q} \right\} < \max_{\{q_{i,j}\}_{\substack{i\in Z\\ j\in X}}} \left\{ \sum_{\substack{i\in Z\\ j\in X}} u_i(q_{i,j}) \text{ s.t. } \begin{array}{c} \sum_{\substack{i\in Z\\ j\in X}} q_{i,j} = |Z|\bar{q}, \\ \sum_{j\in X\setminus\{i\}} q_{i,j} = \bar{q}\forall i\in Z \end{array} \right\},$$

a contradiction, since the problem on the right hand side of the inequality has $|Z|$ additional constraints. $\square$

The intuition for Proposition 2 is as follows. Observe that if $Z \subseteq X$ opts out of the auction, they can achieve the surplus $v(Z,0)$; unlike in a one-sided setting, this surplus may be strictly positive. Hence, by Proposition 1, $Z \subseteq X$ blocks $\pi^V(X)$ if the surplus $v(Z,0)$ they can create through reallocation is greater than the sum of their marginal contributions to the surplus created in the auction:

$$v(Z,0) + \sum_{i\in Z} v(X\setminus\{i\},\bar{q}) > |Z|v(X,\bar{q}). \tag{4}$$

The term on the right-hand side of (4) is equal to the total surplus in an auction where the participants are replicated $|Z|$ times, and the quantity auctioned is $|Z|$ times larger. This must be at least the surplus that could be created in that auction if we were constrained to allocate $\bar{q}$ to each of the coalitions $\{X\setminus\{i\}\}_{i\in Z}$, and nothing to the remaining participants $Z$ — which is precisely the left-hand side of (4).

Proposition 2 allows us to restrict attention to coalitions that include the auctioneer for the purposes of core selection. Lemma 1 shows that the condition on the surplus function $v$ that ensures that these coalitions do not block the Vickrey payoff profile is identical to the one given by Ausubel and Milgrom (2002) in the one-sided case. In particular, the Vickrey payoff profile is in the core precisely when the surplus function is *bidder-submodular*, i.e., when $v(Z,\bar{q})$ is submodular in $Z$ under the usual set order $\subseteq$.[46]

**Lemma 1 (Core Selection and Bidder-Submodularity).** *The Vickrey auction is core selecting for each set of participating bidders $X \subseteq I$ if, and only if, $v(Z,\bar{q})$ is submodular in $Z$.*

Given Proposition 2, the argument for Lemma 1 is familiar from Ausubel and Milgrom (2002): if each bidder in a group $Z$ is more valuable to smaller coalitions than larger ones, then the sum of their individual contributions to the surplus generated by allocating $\bar{q}$ among the participants $X$ — their payoffs in the Vickrey auction — must be less than the marginal contribution of the group as a whole. Consequently, removing $Z$ from the auction decreases the total surplus available by a larger

[46]That is, for all coalitions $Z \subseteq I$ that include bidder $j$, $v(Z,\bar{q}) - v(Z\setminus\{j\},\bar{q})$ does not increase when more bidders are added to $Z$. Intuitively, bidder-submodularity requires that adding a bidder increases the surplus produced in the auction by more when the set of participants is smaller.

amount than the share of the surplus that the Vickrey auction allocates to $Z$ — and so the coalition $(X \setminus Z) \cup \{a\}$ cannot block the Vickrey payoff profile $\pi^V(X)$.

**Proof of Lemma 1 (Core Selection and Bidder-Submodularity)** (If) Fix a set of participating bidders $X$ and label each bidder in $X$ as an integer $i \in \mathbb{Z}$. By Proposition 1, $\pi^V(X)$ is unblocked by $Z \cup \{a\}$ if, and only if,

$$
\begin{aligned}
& v(X,\bar{q}) - \sum_{i \in X \setminus Z} \pi_i^V(X) \geq v(Z,\bar{q}) \\
\Leftrightarrow & v(X,\bar{q}) - v(Z,\bar{q}) \geq \sum_{i \in X \setminus Z} (v(X,\bar{q}) - v(X \setminus \{i\},\bar{q})) \qquad (5) \\
\Leftrightarrow & \sum_{i \in X \setminus Z} \left( \begin{array}{c} v(Z \cup \{j \in X \setminus Z | j \leq i\},\bar{q}) - v(Z \cup \{j \in X \setminus Z | j < i\},\bar{q}) \\ -v(X,\bar{q}) + v(X \setminus \{i\},\bar{q}) \end{array} \right) \geq 0 \qquad (6)
\end{aligned}
$$

Since each term in (6) is nonnegative when $v(Y,\bar{q})$ is submodular in $Y$, the "if" part follows by Proposition 2.

(Only if) Suppose that $v(Y,\bar{q})$ is not submodular in $Y$. Then there exists some finite $Y, Z \subset I$ such that $v(Y \cup Z,\bar{q}) - v(Y,\bar{q}) > v(Z,\bar{q}) - v(Y \cap Z,\bar{q})$. Since $2^{Y \cup Z}$ is a finite product of chains, submodularity on $2^{Y \cup Z}$ is equivalent to decreasing differences, so it follows that for some $X \subseteq Y \cup Z$ and some agents $i, j \in X$,

$$
\begin{aligned}
v(X,\bar{q}) - v(X \setminus \{i\},\bar{q}) &> v(X \setminus \{j\},\bar{q}) - v(X \setminus \{i,j\},\bar{q}) \\
\Leftrightarrow \pi_i^V(X) + \pi_j^V(X) &> v(X,\bar{q}) - v(X \setminus \{i,j\},\bar{q}).
\end{aligned}
$$

It follows that $\pi^V(X)$ is blocked by $\{a\} \cup X \setminus \{i,j\}$. □

Lemma 2 shows that *bidder*-core selection is equivalent to a slightly weaker version of the bidder-submodularity condition from Lemma 1.

**Lemma 2 (Bidder-Core Selection and Bidder-Submodularity).** *The Vickrey auction is bidder-core-selecting for each set of participating bidders $X \subseteq I$ if, and only if, for each $i \in I$, $v(Z,\bar{q})$ is submodular in $Z$ on the sublattice $\{Y \cup \{i\} | Y$ is a finite subset of $I\}$.*[47]

For intuition for Lemma 2, observe that Lemma 2's submodularity condition is equivalent to saying that $v(Z,\bar{q}) + v(Y,\bar{q}) \geq v(Z \cup Y,\bar{q}) + v(Z \cap Y,\bar{q})$ for all $Y, Z'$ such that $Y \cap Z \neq \emptyset$. Hence, bidders are more valuable to smaller *nonempty* coalitions than larger ones, so the sum of the individual contributions of a *strict subset of the participants* $Z \subset X$ to the grand coalition must be less than the marginal contribution of $Z$ as a whole. Consequently, the intuition for Lemma 1 still applies, so long as the coalition being removed from the auction is not the grand coalition $X$.

**Proof of Lemma 2 (Bidder-Core Selection and Bidder-Submodularity)** For the "if" part, suppose that $v(Z,\bar{q})$ is submodular in $Z$ on the sublattice $\{Y \cup \{\ell\} | Y$ is a finite subset of $I\}$. Then if $\ell \in Z$, each term in (6) is nonnegative. It follows that if $\ell \in Z$, $Z \cup \{a\}$ does not block $\pi^V(X)$. Thus, when $v(Z,\bar{q})$ is submodular in $Z$ on the sublattice $\{Y \cup \{\ell\} | Y$ is a finite subset of $I\}$ for each $\ell \in I$, it follows from Proposition 2 that no $Z \neq \{a\}$ blocks $\pi^V(X)$.

[47] That is, for any finite sets $Z, Z' \subset I$ containing $i$, $v(Z \cup Z',\bar{q}) + v(Z \cap Z',\bar{q}) \leq v(Z,\bar{q}) + v(Z',\bar{q})$.

For the "only if" part, suppose that $v(Z,\bar{q})$ is not submodular in $Z$ on the sublattice $\{Y \cup \{\ell\} | Y \text{ is a finite subset of } I\}$ for some $\ell \in I$. Then there exist some finite $Y, Z \subset I$ such that $\ell \in Y \cap Z$ and $v(Y \cup Z, \bar{q}) - v(Y, \bar{q}) > v(Z, \bar{q}) - v(Y \cap Z, \bar{q})$ Since $2^{Y \cup Z}$ is a product of chains, submodularity on $2^{Y \cup Z}$ is equivalent to decreasing differences, so it follows that for some $X \ni \ell$ and some agents $i, j \in X$ with $i, j \neq \ell$,

$$\begin{aligned} & v(X,\bar{q}) - v(X \setminus \{i\}, \bar{q}) > v(X \setminus \{j\}, \bar{q}) - v(X \setminus \{i,j\}) \\ & \Leftrightarrow \pi_i^V(X) + \pi_j^V(X) > v(X,\bar{q}) - v(X \setminus \{i,j\}). \end{aligned}$$

It follows that $\pi^V(X)$ is blocked by $\{a\} \cup X \setminus \{i,j\} \supset \{a, \ell\}$.

**Proof of Proposition 3 (Implications of Bidder-Core Selection)** First note that since the Vickrey auction is bidder-core-selecting for all $X \subseteq I$, then by Lemma 2, for each $\ell \in I$, $v(Z,\bar{q})$ is submodular in $Z$ on the sublattice $\{Y \cup \{\ell\} | Y \text{ is a finite subset of } I\}$. (i): By Proposition 1, $\pi_a^V(X) = v(X,\bar{q}) - \sum_{i \in X}(v(X,\bar{q}) - v(X \setminus \{i\}, \bar{q}))$. Suppose that $Y \subseteq Z \subseteq I$, and that $|Y| \geq 2$, so that $Y$ and $Z$ are valid sets of auction participants. Then (labeling each $i \in Z$ with an integer) we have

$$\begin{aligned} \pi_a^V(Z) - \pi_a^V(Y) = & \begin{array}{l} v(Z,\bar{q}) - v(Y,\bar{q}) - \sum_{i \in Z}(v(X,\bar{q}) - v(X \setminus \{i\}, \bar{q})) \\ + \sum_{i \in Y}(v(X,\bar{q}) - v(X \setminus \{i\}, \bar{q})) \end{array} \\ & = v(Z,\bar{q}) - v(Y,\bar{q}) - \textstyle\sum_{i \in Z \setminus Y}(v(X,\bar{q}) - v(X \setminus \{i\}, \bar{q})) \\ & \geq v(Z,\bar{q}) - v(Y,\bar{q}) - \textstyle\sum_{i \in Z \setminus Y}(v(Y \cup \{j \in Z | j \leq i\}, \bar{q}) - v(Y \cup \{j \in Z | j < i\}, \bar{q})) \\ & = 0, \end{aligned}$$

as desired.

(ii): Given a finite $B \subset I \setminus (X \setminus \{i\})$, let $b_B(q) = \nabla_q v(B,q)$. Suppose that, given the set of participants $X$, bidder $i$ plays $b_B$ and each bidder $j \in X \setminus \{i\}$ plays $b_j^V$. Then for each $j \in X \setminus \{i\}$, $\nabla_q v(B, q_i^*(b_B, b_{-i}^V)) = p^*(b_B, b_{-i}^V) = \nabla u_j(q_j^*(b_B, b_{-i}^V))$. Then since $u_j$ is strictly concave for each $j \in I$ — and hence $v(B,q)$ is strictly concave in $q$ — we have

$$\begin{aligned} q_i^*(b_B, b_{-i}^*) &= \arg\max_q \left\{ v(B,q) + \sum_{j \in X \setminus \{i\}} u_j(q_j) \text{ s.t. } q + \sum_{j \in X \setminus \{i\}} q_j = \bar{q} \right\} \\ &= \arg\max_q \left\{ \max_{\{q_j\}_{j \in B}} \left\{ \sum_{j \in B} u_j(q_j) \text{ s.t. } \sum_{j \in B} q_j = q \right\} + \sum_{j \in X \setminus \{i\}} u_j(q_j) \text{ s.t. } q + \sum_{j \in X \setminus \{i\}} q_j = \bar{q} \right\} \\ &= \sum_{j \in B} q_j^e(B \cup (X \setminus \{i\}), \bar{q}). \end{aligned} \tag{7}$$

Moreover, since by the envelope theorem, $r_i(q_i, b_{-i}^V) = -\nabla_{q_i} v(X \setminus \{i\}, \bar{q} - q_i)$, we have

$$\begin{aligned} t_i^V(b_B, b_{-i}^V) &= -\int_0^{q_i^*(b_B, b_{-i}^V)} \nabla_{q_i} v(X \setminus \{i\}, \bar{q} - q_i) dq_i \\ &= v(X \setminus \{i\}, \bar{q}) - v\left(X \setminus \{i\}, \textstyle\sum_{j \in X \setminus \{i\}} q_j^e(B \cup (X \setminus \{i\}), \bar{q})\right) \\ &= v(X \setminus \{i\}, \bar{q}) - \textstyle\sum_{j \in X \setminus \{i\}} u_j(q_j^e(B \cup (X \setminus \{i\}), \bar{q})). \end{aligned}$$

Now suppose that the set of bidders $B \cup X \setminus \{i\}$ participate in the auction and bid their dominant strategies $b_j^V$. Then by Proposition 1, for each $j \in B$,

$$
\begin{aligned}
q_j^*(\{b_\ell^V\}_{\ell \in B \cup (X \setminus \{i\})}) &= q_j^e(B \cup (X \setminus \{i\}), \bar{q}), \\
t_j^V(\{b_\ell^V\}_{\ell \in B \cup (X \setminus \{i\})}) &= u_j(q_j^e(B \cup (X \setminus \{i\}), \bar{q})) - \pi_j^V(B \cup (X \setminus \{i\})) \\
&= u_j(q_j^e(B \cup (X \setminus \{i\}), \bar{q})) - v(B \cup (X \setminus \{i\}), \bar{q}) + v((B \setminus \{j\}) \cup (X \setminus \{i\}), \bar{q}).
\end{aligned}
$$

Since $|X| \geq 2$, there is some $\ell \in X \setminus \{i\}$. Since $v(Z, \bar{q})$ is submodular in $Z$ on the sublattice $\{Y \cup \{\ell\} | Y \subseteq I\}$, it follows that (assigning each $j \in B$ an integer label)

$$
\begin{aligned}
t_i^V(b_B, b_{-i}^V) - \sum_{j \in B} t_j^V(\{b_\ell^V\}_{\ell \in B \cup (X \setminus \{i\})}) &= \begin{array}{l} v(X \setminus \{i\}, \bar{q}) - v(B \cup (X \setminus \{i\}), \bar{q}) \\ + \sum_{j \in B} (v(B \cup (X \setminus \{i\}), \bar{q}) - v((B \setminus \{j\}) \cup (X \setminus \{i\}), \bar{q})) \end{array} \\
&\leq \begin{array}{l} v(X \setminus \{i\}, \bar{q}) - v(B \cup (X \setminus \{i\}), \bar{q}) \\ + \sum_{j \in B} \begin{pmatrix} v(\{\ell \in B | \ell \leq j\} \cup (X \setminus \{i\}), \bar{q}) \\ -v(\{\ell \in B | \ell < j\} \cup (X \setminus \{i\}), \bar{q}) \end{pmatrix} \end{array} = 0.
\end{aligned}
$$

Then by Proposition 1 (i),

$$
\begin{aligned}
\pi_i^V(X) &\geq u_i(q_i^*(b_B, b_{-i}^V)) - t_i^V(b_B, b_{-i}^V) \\
&= u_i\left(\sum_{j \in B} q_j^e(B \cup (X \setminus \{i\}), \bar{q})\right) - t_i^V(b_B, b_{-i}^V) \text{ (by (7))} \\
&= u_i\left(\sum_{j \in B} q_j^*(\{b_\ell^V\}_{\ell \in B \cup (X \setminus \{i\})})\right) - t_i^V(b_B, b_{-i}^V) \\
&\geq u_i\left(\sum_{j \in B} q_j^*(\{b_\ell^V\}_{\ell \in B \cup (X \setminus \{i\})})\right) - \sum_{j \in B} t_j^V(\{b_\ell^V\}_{\ell \in B \cup (X \setminus \{i\})}),
\end{aligned}
$$

as desired. □

**Lemma 3 (Coalitional Value Function: Quadratic Valuations).** *With quadratic valuations, the surplus function and efficient allocation are given by*

$$
\begin{aligned}
v(Z, \bar{q}) &= \frac{1}{2} \sum_{i \in Z} \theta_i' S_i^{-1} \theta_i - \frac{1}{2} \left(\sum_{i \in Z} S_i^{-1} \theta_i - \bar{q}\right)' H(Z) \left(\sum_{i \in Z} S_i^{-1} \theta_i - \bar{q}\right), \\
q_i^e(Z, \bar{q}) &= S_i^{-1} \theta_i - S_i^{-1} H(Z) \left(\sum_{j \in Z} S_j^{-1} \theta_j - \bar{q}\right),
\end{aligned}
$$

*where $H(Z) \equiv \left(\sum_{j \in Z} S_j^{-1}\right)^{-1}$ is the harmonic mean of the matrices $\{S_i\}_{i \in Z}$. In particular, note that when $\theta_i = \theta$ for each $i \in I$, $v(Z, \bar{q}) = \theta' \bar{q} - \frac{1}{2} \bar{q}' H(Z) \bar{q}$.*

*Proof.* The Lagrangian for (1) is

$$
\mathcal{L}(\{q_i\}_{i \in Z}, \mu) = \sum_{i \in Z} \left(\theta_i' q_i - \tfrac{1}{2} q_i' S_i q_i\right) - \mu' \left(\sum_{i \in Z} q_i - \bar{q}\right).
$$

The first-order condition (which is sufficient, by strict concavity of the objective) is $\theta_i - S_i q_i =$

$\mu$, $\forall i \in J$. Substituting $q_i = S_i^{-1}(\theta_i - \mu)$ in the feasibility constraint yields

$$\bar{q} = \left(\sum_{i\in Z} S_i^{-1}\theta_i\right) - \left(\sum_{i\in Z} S_i^{-1}\right)\mu \qquad\qquad \Leftrightarrow \mu = H(Z)\left(\sum_{i\in Z} S_i^{-1}\theta_i - \bar{q}\right).$$

Substituting back into the FOC:

$$\theta_i - S_i q_i = H(Z)\left(\sum_{j\in Z} S_j^{-1}\theta_j - \bar{q}\right) \qquad \Leftrightarrow q_i = S_i^{-1}\theta_i - S_i^{-1}H(Z)\left(\sum_{j\in Z} S_j^{-1}\theta_j - \bar{q}\right),\ \forall i \in Z.$$

Now substituting our optimal choice of quantity back into the coalitional payoff:

$$\begin{aligned} v(Z,\bar{q}) &= \begin{array}{l} \sum_{i\in Z}\theta_i' S_i^{-1}\theta_i - \left(\sum_{i\in Z}\theta_i' S_i^{-1}\right) H(Z)\left(\sum_{i\in Z} S_i^{-1}\theta_i - \bar{q}\right) \\ -\frac{1}{2}\sum_{i\in Z}\theta_i' S_i^{-1}\theta_i + \left(\sum_{i\in Z}\theta_i' S_i^{-1}\right) H(Z)\left(\sum_{i\in Z} S_i^{-1}\theta_i - \bar{q}\right) \\ -\frac{1}{2}\left(\sum_{i\in Z} S_i^{-1}\theta_i - \bar{q}\right)' H(Z)\left(\sum_{i\in Z} S_i^{-1}\theta_i - \bar{q}\right) \end{array} \\ &= \frac{1}{2}\sum_{i\in Z}\theta_i' S_i^{-1}\theta_i - \frac{1}{2}\left(\sum_{i\in Z} S_i^{-1}\theta_i - \bar{q}\right)' H(Z)\left(\sum_{i\in Z} S_i^{-1}\theta_i - \bar{q}\right). \end{aligned}$$

□

As an intermediate step in establishing Theorem 1, Lemmas 4 and 5 show that under the theorem's hypotheses, the reallocative auction for goods functions like a one-sided auction for packages $\Phi$, but one in which bidders' valuations may be decreasing in the quantity of a package that they consume, and package prices may be negative. We must thus provide a new proof of the surplus function's bidder-submodularity, rather than applying off-the-shelf results to the one-sided package auction.

**Lemma 4 (Package Substitutability as a Condition on Primitives).** *The packages $\Phi$ are substitutes if, and only if, for each bidder $i$ and each $q_i$, $-(T_\Phi' D^2 u_i(T_\Phi \tilde{d}_i^\Phi(p_\Phi))T_\Phi)^{-1}$ is an M-matrix.*[48]

*Proof.* Agent $i$'s first-order condition (which is necessary and sufficient for a maximum, since $u_i$ is strictly concave) is $\nabla u_i(d_i(p)) - p = 0$. Then by the implicit function theorem, $d_i$ is continuously differentiable, and we have $Dd_i(p) = (D^2 u_i(d_i(p)))^{-1}$. Then packages $\Phi$ are substitutes if and only if $D_{p_\Phi} T_\Phi^{-1} d_i(T_\Phi^{-1} p_\Phi)$ has nonnegative off-diagonal entries for each $p_\Phi$. By the chain rule,

$$\begin{aligned} D_{p_\Phi} T_\Phi^{-1} d_i(T_\Phi^{-1\prime} p_\Phi) &= T_\Phi^{-1}(D^2 u_i(d_i(T_\Phi^{-1\prime} p_\Phi)))^{-1} T_\Phi^{-1\prime} \\ &= (T_\Phi' D^2 u_i(d_i(T_\Phi^{-1\prime} p_\Phi))T_\Phi)^{-1}. \end{aligned}$$

Since $u_i$ is strictly concave, $-(T_\Phi' D^2 u_i(d_i(T_\Phi^{-1\prime} p_\Phi))T_\Phi)^{-1}$ is positive definite; then it has nonpositive off-diagonal entries if and only if it is an M-matrix.

Now observe that for each $q_i$, $d_i(T_\Phi^{-1} p_\Phi) = q_i$ for $p_\Phi = T_\Phi \nabla u_i(q_i)$. It follows that $D_{p_\Phi} T_\Phi^{-1} d_i(T_\Phi^{-1} p_\Phi)$ has nonnegative off-diagonal entries for each $p_\Phi$ if and only if $-(T_\Phi' D^2 u_i(q_i)T_\Phi)^{-1}$ is an M-matrix for each $q_i$. The claim follows. □

**Lemma 5 (Constrained Submodularity and Monotonicity in Package Prices).** *If the packages $\Phi$ are substitutes, then*

$$\tilde{\Pi}_i^\Phi(p_\Phi) \equiv \max_{q_i}\{u_i(q_i) - p_\Phi \cdot T_\Phi^{-1} q_i \text{ s.t. } 0 \le T_\Phi^{-1} q_i\}$$

[48] Recall that a positive definite matrix is an M-matrix iff its off-diagonal entries are nonpositive.

*is submodular and nonincreasing.*

*Proof.* First note that $\tilde{\Pi}_i^{\Phi}(p_{\Phi}) = \max_x\{u_i(T_{\Phi}x) - p_{\Phi} \cdot x \text{ s.t. } 0 \leq x\}$. Since $u_i$ is strictly concave, $\tilde{d}_i^{\Phi}(p_{\Phi}) \equiv \arg\max_x\{u_i(T_{\Phi}x) - p_{\Phi} \cdot x \text{ s.t. } 0 \leq x\}$ is single-valued. Then by the envelope theorem (e.g., Milgrom and Segal (2002)), $\nabla\tilde{\Pi}_i^{\Phi}(p_{\Phi}) = -\tilde{d}_i^{\Phi}(p_{\Phi}) \leq 0$. Hence, $\tilde{\Pi}_i^{\Phi}$ is nonincreasing.

By the maximum theorem, $\tilde{d}_i^{\Phi}$ is continuous. Then for each $k \in \{1, \ldots, K\}$, $\hat{P}_k = \{p_{\Phi} : [\tilde{d}_i^{\Phi}(p_{\Phi})]_k > 0\}$ is open. For each $L \subseteq \{1, \ldots, K\}$, let $\hat{P}_L = \bigcap_{k \in L} \hat{P}_k$. Then since $u_i$ is strictly concave, for any $p_{\Phi} \in \hat{P}_L$, we have

$$[T'_{\Phi}\nabla u_i(T_{\Phi}\tilde{d}_i^{\Phi}(p_{\Phi})) - p_{\Phi}]_k = 0, k \in L; \qquad [\tilde{d}_i^{\Phi}(p_{\Phi})]_k = 0, k \notin L.$$

Then by the implicit function theorem, for each $p_{\Phi} \in \hat{P}_L$,

$$D\tilde{d}_i^{\Phi}(p_{\Phi}) = \begin{bmatrix} \left[\left[T'_{\Phi}D^2u_i(T_{\Phi}\tilde{d}_i^{\Phi}(p_{\Phi}))T_{\Phi}\right]_{L\times L}\right]^{-1} & 0_{L\times(K\setminus L)} \\ 0_{(K\setminus L)\times L} & 0_{(K\setminus L)\times(K\setminus L)} \end{bmatrix}$$

By Lemma 4, $-(T'_{\Phi}D^2u_i(T_{\Phi}\tilde{d}_i^{\Phi}(p_{\Phi}))T_{\Phi})^{-1}$ is an M-matrix. Then by, e.g., Johnson (1982) Corollary 3, $-\left[\left[T'_{\Phi}D^2u_i(T_{\Phi}\tilde{d}_i^{\Phi}(p_{\Phi}))T_{\Phi}\right]_{L\times L}\right]^{-1}$ is an M-matrix.

Then for each $L \subseteq \{1, \ldots, K\}$ and each $p_{\Phi} \in \hat{P}_L$, $D^2\tilde{\Pi}_i^{\Phi}(p_{\Phi}) = -D\tilde{d}_i^{\Phi}(p_{\Phi})$ has nonpositive off-diagonal entries. Since $\bigcup_{L\subseteq\{1,\ldots,K\}} \hat{P}_L = \mathbb{R}^K$, it follows from the fundamental theorem of calculus for line integrals that $\tilde{\Pi}_i^{\Phi}$ has decreasing differences, and so is submodular. □

**Proof of Theorem 1 (Core Selection: Package Substitutability and Allocations)** For each nonempty $Z \subseteq I$, since $T_{\Phi}^{-1}q_i^e(Z, \bar{q}) \geq 0$, we have

$$\begin{aligned}
v(Z, \bar{q}) &= \max_{\{q_i\}_{i\in Z}} \left\{\sum_{i\in Z} u_i(q_i) \text{ s.t. } 0 \leq T_{\Phi}^{-1}q_i \forall i \in Z \text{ and } \sum_{i\in Z} q_i = \bar{q}\right\} \\
&= \max_{\{q_i\}_{i\in Z}} \left\{\textstyle\sum_{i\in Z} u_i(q_i) \text{ s.t. } 0 \leq T_{\Phi}^{-1}q_i \forall i \in Z \text{ and } \sum_{i\in Z} T_{\Phi}^{-1}q_i = T_{\Phi}^{-1}\bar{q}\right\} &(8)\\
&= \min_{p_{\Phi}} \max_{\{q_i\}_{i\in Z}} \left\{\textstyle\sum_{i\in Z} u_i(q_i) - p_{\Phi} \cdot \left(\sum_{i\in Z} T_{\Phi}^{-1}q_i - T_{\Phi}^{-1}\bar{q}\right) \text{ s.t. } 0 \leq T_{\Phi}^{-1}q_i \forall i \in Z\right\} &(9)\\
&= \min_{p_{\Phi}} \left\{\textstyle\sum_{i\in Z} \max_{q_i}\{u_i(q_i) - p_{\Phi} \cdot T_{\Phi}^{-1}q_i \text{ s.t. } 0 \leq T_{\Phi}^{-1}q_i\} + p_{\Phi} \cdot T_{\Phi}^{-1}\bar{q}\right\} \\
&= \min_{p_{\Phi}} \left\{\textstyle\sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(p_{\Phi}) + p_{\Phi} \cdot T_{\Phi}^{-1}\bar{q}\right\}, &(10)
\end{aligned}$$

where the primal (8) and dual (9) optimization problems are equivalent because the primal problem is maximization of a concave function over a convex region under a feasible linear constraint, and hence Slater's condition ensures that strong duality holds.

From here we follow the proofs of Theorems 2.6.2 and 2.7.6 in Topkis (1998). Suppose that packages $\Phi$ are substitutes. If $p'_{\Phi} \geq p_{\Phi}$, then by Lemma 5, $\tilde{\Pi}_i^{\Phi}(p'_{\Phi}) - \tilde{\Pi}_i^{\Phi}(p_{\Phi}) \leq 0$ for each $i \in I$, and hence $\sum_{i\in Z}(\tilde{\Pi}_i^{\Phi}(p'_{\Phi}) - \tilde{\Pi}_i^{\Phi}(p_{\Phi}))$ is decreasing in $Z$. Then $\sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y)$ has decreasing differences in $(y, Z)$. Then for all $y, y' \in \mathbb{R}^K$ and all $Y, Z \subseteq I$, we have

$$\sum_{i\in Y\cup Z} \tilde{\Pi}_i^{\Phi}(y' \vee y) - \sum_{i\in Y} \tilde{\Pi}_i^{\Phi}(y) = \sum_{i\in Y\cup Z} \tilde{\Pi}_i^{\Phi}(y' \vee y) - \sum_{i\in Y\cup Z} \tilde{\Pi}_i^{\Phi}(y) + \sum_{i\in Y\cup Z} \tilde{\Pi}_i^{\Phi}(y) - \sum_{i\in Y} \tilde{\Pi}_i^{\Phi}(y)$$

$$
\begin{aligned}
&\leq \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y' \vee y) - \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y) && \text{(decreasing differences in } (y,Z))\\
&\quad + \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y) - \sum_{i\in Y\cap Z} \tilde{\Pi}_i^{\Phi}(y) && ((Y\cup Z)\setminus Y = Z\setminus (Y\cap Z))\\
&\leq \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y') - \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y\wedge y') && \text{(Lemma 5)}\\
&\quad + \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y\wedge y') - \sum_{i\in Y\cap Z} \tilde{\Pi}_i^{\Phi}(y\wedge y') && \text{(decreasing differences in } (y,Z))\\
&= \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y') - \sum_{i\in Y\cap Z} \tilde{\Pi}_i^{\Phi}(y'\wedge y).
\end{aligned}
$$

Then since $(y' \vee y) - y = y' - (y' \wedge y)$,

$$
\begin{aligned}
\begin{matrix} \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y') + y'\cdot T_{\Phi}^{-1}\bar{q} \\ + \sum_{i\in Y} \tilde{\Pi}_i^{\Phi}(y) + y\cdot T_{\Phi}^{-1}\bar{q} \end{matrix}
&\geq \begin{matrix} \sum_{i\in Y\cup Z} \tilde{\Pi}_i^{\Phi}(y'\vee y) + (y'\vee y)\cdot T_{\Phi}^{-1}\bar{q} \\ + \sum_{i\in Y\cap Z} \tilde{\Pi}_i^{\Phi}(y'\wedge y) + (y'\wedge y)\cdot T_{\Phi}^{-1}\bar{q} \end{matrix} \\
&\geq v(Y\cup Z,\bar{q}) + v(Y\cap Z,\bar{q}) \text{ whenever } Y\cap Z \neq \emptyset.
\end{aligned}
\tag{11}
$$

By (10), choosing $y$ and $y'$ to minimize the left-hand side yields

$$
v(Z,\bar{q}) + v(Y,\bar{q}) \geq v(Y\cup Z,\bar{q}) + v(Y\cap Z,\bar{q}) \text{ whenever } Y\cap Z \neq \emptyset, \tag{12}
$$

so for each $i \in I$, $v(Z,\bar{q})$ is submodular in $Z$ on the sublattice $\{Y\cup\{i\} \mid Y \subseteq I\}$. It follows from Lemma 2 that the Vickrey auction is bidder-core-selecting for each set of participating bidders $X \subseteq I$.

Now suppose that in addition, $T'_{\Phi}\nabla v(Z,\bar{q}) \geq 0$ for each $Z \subseteq I$. By the envelope theorem and (10), for each $Z \subseteq I$,

$$
\begin{aligned}
\nabla v(Z,\bar{q}) &= T_{\Phi}^{-1\prime} \arg\min_{p_{\Phi}} \left\{ \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(p_{\Phi}) + p_{\Phi}\cdot T_{\Phi}^{-1}\bar{q} \right\};\\
&\Rightarrow 0 \leq \arg\min_{p_{\Phi}} \left\{ \sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(p_{\Phi}) + p_{\Phi}\cdot T_{\Phi}^{-1}\bar{q} \right\}.
\end{aligned}
\tag{13}
$$

Since $T_{\Phi}^{-1}q_i^e(Z,\bar{q}) \geq 0$ for each $Z \subseteq I$ and $i \in Z$, we must have $T_{\Phi}^{-1}\bar{q} \geq 0$. Then for any $Y, Z \subseteq I$ such that $Y \cap Z = \emptyset$, for each $y, y' \geq 0$, (11) yields

$$
\begin{aligned}
\sum_{i\in Z} \tilde{\Pi}_i^{\Phi}(y') + y'\cdot T_{\Phi}^{-1}\bar{q} + \sum_{i\in Y} \tilde{\Pi}_i^{\Phi}(y) + y\cdot T_{\Phi}^{-1}\bar{q} &\geq v(Y\cup Z,\bar{q}) + (y'\wedge y)\cdot T_{\Phi}^{-1}\bar{q}\\
&\geq v(Y\cup Z,\bar{q}) + 0 = v(Y\cup Z,\bar{q}) + v(Y\cap Z,\bar{q}).
\end{aligned}
$$

By (13) and (10), choosing $y, y' \geq 0$ to minimize the left-hand side yields $v(Z,\bar{q}) + v(Y,\bar{q}) \geq v(Y \cup Z,\bar{q}) + v(Y\cap Z,\bar{q})$. It follows from (12) that $v(Z,\bar{q})$ is submodular in $Z$; hence, by Lemma 1, the Vickrey auction is core-selecting for each set of participating bidders $X \subseteq I$. □

In the proof of Theorem 1, substitutable packages form a change of basis that allows us to aggregate bidders' indirect utility functions into a submodular objective function for the social planner's dual minimization problem, just as goods substitutability does in a one-sided auction (Lemma 5 in the Appendix). This argument is similar to the one used in core selection result for one-sided divisible good auctions by Milgrom and Strulovici (2009, Theorem 31), with an important difference: submodularity of the social planner's dual objective function in package prices and bidders only follows from submodularity of the bidders' indirect utility functions when their allocations are positive.[49]

**Lemma 6 (Commutativity and Separability).** *Suppose that bidders' substitution patterns $\{D^2 u_i(q_i)\}_{i \in I, q_i \in \mathbb{R}^K}$ are commuting matrices. Then $\{D^2 u_i(q_i)\}_{i \in I, q_i \in \mathbb{R}^K}$ have a common orthonormal eigenbasis $\Phi$ such that*

*i. The packages $\Phi$ are substitutes and $\phi \cdot \bar{q} \geq 0$ for each $\phi \in \Phi$; and*

*ii. Each $u_i$ is separable in $\Phi$: There exist $\{\hat{u}_i^k\}_{k=1}^K$ such that $u_i(T_\Phi x) = \sum_{k=1}^K \hat{u}_i^k(x_k)$ for all $x$.*

*Proof.* Since $u_i$ is strictly concave for each bidder $i$, each $D^2 u_i(q_i)$ is negative definite and thus has an orthonormal eigenbasis $\Phi(i, q_i)$; choose the signs of $\Phi(i, q_i)$ such that for each $\phi \in \Phi(i, q_i)$, $\phi \cdot \bar{q} \geq 0$. Then since the $D^2 u_i(q_i)$ commute, they must be simultaneously diagonalizable: $\Phi(i, q_i) = \Phi$ for each $i$ and each $q_i$. Then for each $i$ and each $q_i$, $D^2 u_i(q_i) = T_\Phi M_i(q_i) T_\Phi'$, where $M_i(q_i)$ is a diagonal matrix. Then since $\Phi$ are orthonormal, $-(T_\Phi' D^2 u_i(q_i) T_\Phi)^{-1} = -M_i(q_i)^{-1}$ is diagonal. (i) follows by Lemma 4.

Moreover, for each $i$, since $D^2 u_i(q_i) = T_\Phi M_i(q_i) T_\Phi'$ for each $q_i$, $D^2_{T_\Phi' q_i} u_i(T_\Phi x) = T_\Phi' D^2 u_i(x) T_\Phi = M_i(x)$ is diagonal. Then $u_i(T_\Phi x)$ is modular. Then since $\mathbb{R}$ is a chain, (ii) follows by Topkis (1998) Theorem 2.6.4. □

**Proof of Theorem 2 (Core Selection: Package Substitutability and Heterogeneity)** By Lemma 6 (i), $\{D^2 u_i(q_i)\}_{i \in I}$ have a common orthonormal eigenbasis $\Phi$ such that the packages $\Phi$ are substitutes and $T_\Phi' \bar{q} \geq 0$; (i) follows.

By Lemma 6 (ii), there exist $\{\hat{u}_i^k\}_{k=1}^K$ such that $u_i(T_\Phi x) = \sum_{k=1}^K \hat{u}_i^k(x_k)$ for all $x$. Since $u_i$ is strictly concave, each $\hat{u}_i^k$ must be as well. Further, by the chain rule, for each $k \in \{1, \ldots, K\}$, $\hat{u}_i^{k\prime}(x_k) = \nabla u_i(T_\Phi x) \cdot \phi_k$; equivalently, $\hat{u}_i^{k\prime}(q_i \cdot \phi_k) = \nabla u_i(q_i) \cdot \phi_k$ for each $q_i \in \mathbb{R}^K$ and each $k \in \{1, \ldots, K\}$.

For each $Z \subseteq I$, the Kuhn-Tucker conditions for a maximum in (2) are

$$\sum_{i \in Z} q_i = \bar{q}, \qquad \nabla u_i(q_i) = p \text{ for each } i \in Z \text{ for some } p \in \mathbb{R}^K.$$

Since the bidders' pre-auction allocation is efficient, for each $k \in \{1, \ldots, K\}$, either $\phi_k \cdot p > \phi_k \cdot \nabla u_i(0) = \hat{u}_i^{k\prime}(0)$ for each $i \in Z$, or $\phi_k \cdot p \leq \hat{u}_i^{k\prime}(0)$ for each $i \in Z$. If the former is true, since $u_i$ is separable and $\hat{u}_i^k$ is strictly concave, we must have $\phi_k \cdot q_i^e(Z, \bar{q}) < 0$ for each $i \in Z$, a contradiction since $\phi_k \cdot \bar{q} \geq 0$. Then the latter must hold, and so (since $u_i$ is separable and $\hat{u}_i^k$ is strictly concave) we have $\phi_k \cdot q_i^e(Z, \bar{q}) \geq 0$ for each $i \in Z$, as desired. □

[49] It is worth pointing out the specific point where the possibility of reallocation breaks standard arguments for core selection with substitutable goods — and thus requires us to prove a new result rather than simply appeal to standard results under a change of basis. In particular, in the study of one-sided auctions by Milgrom and Strulovici (2009), one step in Theorem 31 involves showing that the surplus function is bidder-submodular when $v(Z, \bar{q}) - v(Z, \bar{q}')$, $\bar{q}' \geq \bar{q}$, is nondecreasing in $Z$. This implication follows from the agents in $Z$ receiving fewer goods in aggregate when an additional agent $j$ participates, which is true only when that additional agent's efficient allocation $q_j^*(Z \cup \{j\}, \bar{q})$ is positive. Indeed, in Example 1, if the agents' valuations were such that they each received a positive allocation of the good — that is, if the auction behaved like a one-sided one — then the payoff profile would be in the core. Instead, bidder 3 ends up selling some of his pre-auction allocation in the Vickrey auction.

**Proof of Proposition 4 (No Core Selection: Pre-Auction Allocations and Heterogeneity in Pre-Auction Marginal Utility)** Choose $X$ with $|X| = 3$; label $X = \{1, 2, 3\}$. Since each $u_i$ is strictly concave and twice continuously differentiable, by the inverse function theorem, $\nabla u_2 : \mathbb{R}^K \to M$ has an inverse, $(\nabla u_2)^{-1} : M \to \mathbb{R}^K$. By assumption, $\nabla u_1(\bar{q}) \in M$. Then let

$$t_1 = 0; \qquad t_2 = (\nabla u_2)^{-1}(\nabla u_1(\bar{q})); \qquad t_3 = \begin{cases} 0, & \nabla u_1(\bar{q}) \neq \nabla u_3(0), \\ \mathbf{1}, & \nabla u_1(\bar{q}) = \nabla u_3(0). \end{cases}$$

Then we have $\nabla \tilde{u}_1(\bar{q}) = \nabla u_1(\bar{q}) = \nabla \tilde{u}_2(0)$.

Now suppose valuations for bundles are given by $\{\tilde{u}_i\}_{i \in I}$. From the first-order conditions for a maximum in (2), we have $q_1^e(\{1,2\}, \bar{q}) = q_1^e(\{1\}, \bar{q}) = \bar{q}$ and $q_2^e(\{1,2\}, \bar{q}) = 0$. Then $v(\{1,2\}, \bar{q}) = \tilde{u}_1(\bar{q}) + \tilde{u}_2(0) = u_1(\bar{q}) + \tilde{u}_2(0) = v(\{1\}, \bar{q}) + \tilde{u}_2(0)$.

By definition, $v(X, \bar{q}) \geq \tilde{u}_2(0) + v(\{1,3\}, \bar{q})$. Suppose this inequality binds. Then we must have $q_2^e(X, \bar{q}) = 0$. Then from the first-order conditions for a maximum in (2), $\nabla u_1(\bar{q}) = \nabla \tilde{u}_2(0) = \nabla \tilde{u}_2(q_2^e(X, \bar{q}))$, and so we must have $q_1^e(X, \bar{q}) = \bar{q}$, and hence $q_3^e(X, \bar{q}) = 0$. But then $\nabla \tilde{u}_3(q_3^e(X, \bar{q})) = \nabla \tilde{u}_3(0) \neq \nabla u_1(\bar{q})$, a contradiction. So we must have $v(X, \bar{q}) > \tilde{u}_2(0) + v(\{1,3\}, \bar{q})$. It follows that

$$\begin{aligned} \pi_a^V(X) + \pi_1^V(X) &= v(X, \bar{q}) - (v(X, \bar{q}) - v(\{1,2\}, \bar{q})) - (v(X, \bar{q}) - v(\{1,3\}, \bar{q})) \\ &= v(\{1\}, \bar{q}) + \tilde{u}_2(0) - v(X, \bar{q}) + v(\{1,3\}, \bar{q}) \\ &< v(\{1\}, \bar{q}) = V(\{a, 1\}, \bar{q}), \end{aligned}$$

and the coalition $\{0, 1\}$ blocks the Vickrey payoff profile $\pi^V(X)$. $\square$

**Proof of Proposition 5 (No Core Selection: Quantity Auctioned and Heterogeneity in Pre-Auction Marginal Utility)** Since each $u_i$ is strictly concave and twice continuously differentiable, by the inverse function theorem, $\nabla u_1 : \mathbb{R}^K \to M$ has an inverse, $(\nabla u_1)^{-1} : M \to \mathbb{R}^K$. By assumption, $\nabla u_j(0) \in M$. Then let $\bar{q} = (\nabla u_1)^{-1}(\nabla u_j(0))$, and suppose $X = \{1, j, \ell\}$.

We have $\nabla u_1(\bar{q}) = \nabla u_j(0)$, so from the first-order conditions for a maximum in (2), we have $q_1^e(\{1,j\}, \bar{q}) = q_1^e(\{1\}, \bar{q}) = \bar{q}$ and $q_j^e(\{1,j\}, \bar{q}) = 0$. Then $v(\{1,j\}, \bar{q}) = u_1(\bar{q}) + u_j(0) = v(\{1\}, \bar{q}) + u_j(0)$.

By definition, $v(X, \bar{q}) \geq \tilde{u}_j(0) + v(\{1,\ell\}, \bar{q})$. Suppose this inequality binds. Then we must have $q_j^e(X, \bar{q}) = 0$. Then from the first-order conditions for a maximum in (2), $\nabla u_1(\bar{q}) = \nabla u_j(0) = \nabla u_j(q_j^e(X, \bar{q}))$, and so we must have $q_1^e(X, \bar{q}) = \bar{q}$, and hence $q_\ell^e(X, \bar{q}) = 0$. But then $\nabla u_\ell(q_\ell^e(X, \bar{q})) = \nabla u_\ell(0) \neq \nabla u_j(0)$, a contradiction. So we must have $v(X, \bar{q}) > u_j(0) + v(\{1,\ell\}, \bar{q})$. It follows that

$$\begin{aligned} \pi_a^V(X) + \pi_1^V(X) &= v(X, \bar{q}) - (v(X, \bar{q}) - v(\{1,j\}, \bar{q})) - (v(X, \bar{q}) - v(\{1,\ell\}, \bar{q})) \\ &= v(\{1\}, \bar{q}) + u_j(0) - v(X, \bar{q}) + v(\{1,\ell\}, \bar{q}) \\ &< v(\{1\}, \bar{q}) = V(\{a, 1\}, \bar{q}), \end{aligned}$$

and the coalition $\{0, 1\}$ blocks the Vickrey payoff profile $\pi^V(X)$. $\square$

The proof of Proposition 6 relies on the following lemma, proven in the Supplementary Appendix:

**Lemma 7 (Eigenvalues and Harmonic Means).** *Suppose that $\{S_i\}_{i\in I}$ are positive definite $K \times K$ matrices, and that for some $Z \subseteq I$ and $\ell, j \in Z$, $S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1}$ has a negative eigenvalue. Then $H(Z) + H(Z \setminus \{\ell, j\}) - H(Z \setminus \{\ell\}) - H(Z \setminus \{j\})$ is not positive definite.*

To understand Proposition 6's eigenvalue condition, observe that with quadratic payoffs, Lemma 3 shows that the surplus function's difference-in-differences $v(Y \cup Y', \bar{q}) + v(Y \cap Y', \bar{q}) - v(Y, \bar{q}) - v(Y', \bar{q})$ is just $\frac{1}{2}\bar{q}'(H(Y) + H(Y') - H(Y \cap Y') - H(Y \cup Y'))\bar{q}$. When we let $Y = Z \setminus \{\ell\}$ and $Y' = Z \setminus \{j\}$, this difference-in-differences represents the incentive of the coalition $\{a\} \cup Z \setminus \{\ell, j\}$ to cancel the auction and trade among themselves. In this case, the central matrix in this expression — the harmonic mean's difference-in-differences — is just the sum of Proposition 6's triple product and its transpose, multiplied by $-\frac{1}{2}$. Hence, if this triple product has a negative eigenvalue, and the quantity vector $\bar{q}$ lies in the same direction as the associated eigenvector, the coalition $\{a\} \cup Z \setminus \{\ell, j\}$ must block the Vickrey payoff profile.

**Proof of Proposition 6 (No Core Selection: Heterogeneity in Substitution Patterns)**: We must have $Z \neq \{\ell, j\}$: Suppose not, and $Z = \{\ell, j\}$. Then

$$S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1} = S_\ell^{-1}(S_j^{-1} + S_\ell^{-1})^{-1} S_j^{-1} = (S_\ell + S_j)^{-1},$$

which is positive definite, a contradiction since by assumption it has a negative eigenvalue.

By Lemma 7, $H(Z) + H(Z \setminus \{\ell, j\}) - H(Z \setminus \{\ell\}) - H(Z \setminus \{j\})$ is not positive definite. Then since it is symmetric, there exists $\bar{q} \in \mathbb{R}^K$ such that $\bar{q}'(H(Z) + H(Z \setminus \{\ell, j\}) - H(Z \setminus \{\ell\}) - H(Z \setminus \{j\}))\bar{q} < 0$. Then by Lemma 3, $v(Z, \bar{q}) + v(Z \setminus \{\ell, j\}, \bar{q}) - v(Z \setminus \{\ell\}, \bar{q}) - v(Z \setminus \{j\}, \bar{q}) > 0$. The statement follows by Lemma 2. $\square$

**Proof of Proposition 7 (Equilibrium in the Uniform-Price Auction)** (i): Suppose that the set of participants in a uniform-price auction is $X' \subseteq I$ with $|X'| = |X|$. Choose $i \in X'$, and suppose that bidders $X' \setminus \{i\}$ submit bids $b^U_{X'\setminus\{i\}} = \{b^U_j\}_{j\in X'\setminus\{i\}}$. Then if bidder $i$ submits a bid $b_i$ such that he obtains $q_i^*(\{b_i, b^U_{X'\setminus\{i\}}\}) = q_i$, the market-clearing price must be $r_i(q_i, b^U_{X'\setminus\{i\}})$. Then his payoff from such a bid is

$$\theta_i' q_i - \frac{1}{2} q_i' S q_i - r_i(q_i, b^U_{X'\setminus\{i\}})' q_i. \tag{14}$$

Now since $b^U_j(q_j) = \theta_j - \frac{|X|-1}{|X|-2} S q_j$, we have $b^{U\ -1}_j(p) = \frac{|X|-2}{|X|-1} S^{-1}(\theta_j - p)$. It follows from the definition of residual supply that

$$\bar{q} = q_i + \tfrac{|X|-2}{|X|-1} S^{-1}\left(\textstyle\sum_{j\in X'\setminus\{i\}} \theta_j\right) - (|X|-2) S^{-1} r_i(q_i, b^U_{X'\setminus\{i\}})$$
$$\Rightarrow r_i(q_i, b^U_{X'\setminus\{i\}}) = \tfrac{1}{|X|-2} S(q_i - \bar{q}) + \tfrac{1}{|X|-1}\left(\textstyle\sum_{j\in X'\setminus\{i\}} \theta_j\right).$$

Then $Dr_i(q_i, b^U_{X'\setminus\{i\}}) = \frac{1}{|X|-2} S$, and so bidder $i$'s payoff (14) from a bid that obtains $q_i$ is concave in $q_i$. It follows that $b_i$ is a best response to $b^U_{X'\setminus\{i\}}$ for bidder $i$ if and only if the quantity $q_i =$

$q_i^*(\{b_i, b^U_{X'\setminus\{i\}}\})$ that bidder $i$ obtains by bidding $b_i$ satisfies

$$\begin{aligned} &\theta_i - Sq_i - r_i(q_i, b^U_{X'\setminus\{i\}}) - Dr_i(q_i, b^U_{X'\setminus\{i\}})' q_i = 0 \\ \Leftrightarrow\ &\theta_i - \frac{|X|-1}{|X|-2} Sq_i - r_i(q_i, b^U_{X'\setminus\{i\}}) = 0 \end{aligned}$$

Since $q_i = q_i^*(\{b_i, b^U_{X'\setminus\{i\}}\})$ whenever $b_i(q_i) = p^*(\{b_i, b^U_{X'\setminus\{i\}}\}) = r_i(q_i, b^U_{X'\setminus\{i\}})$, it follows that $b_i^U(q_i) = \theta_i - \frac{|X|-1}{|X|-2} Sq_i$ is a best response to $b^U_{X'\setminus\{i\}}$. Since this is true for each $i \in X'$, $\{b_i^U\}_{i\in X'}$ is a Nash equilibrium of the UPA with participants $X'$. The claim follows.

(ii): Since $b_\ell^{U\ -1}(p) = \frac{|X|-2}{|X|-1} S^{-1}(\theta_\ell - p)$, we have

$$\begin{aligned} &\bar q = \textstyle\sum_{\ell\in X} b_\ell^{U\ -1}(p^*(\{b_j^U\}_{j\in X})) \ = \frac{|X|-2}{|X|-1} S^{-1}\left( \left(\sum_{\ell\in X}\theta_\ell\right) - |X| p^*(\{b_j^U\}_{j\in X}) \right) \\ \Rightarrow\ & p^*(\{b_j^U\}_{j\in X}) = \tfrac{1}{|X|}\left(\textstyle\sum_{\ell\in X}\theta_\ell\right) - \tfrac{|X|-1}{(|X|-2)|X|} SQ \qquad (15) \\ \Rightarrow\ & q_i^*(\{b_j^U\}_{j\in X}) = b_i^{U\ -1}(p^*(\{b_j^U\}_{j\in X})) = \tfrac{|X|-2}{|X|-1} S^{-1}\left(\theta_i - \tfrac{1}{|X|}\left(\textstyle\sum_{\ell\in X}\theta_\ell\right) + \tfrac{|X|-1}{(|X|-2)|X|} SQ\right) \\ &= \tfrac{1}{|X|}\left(\bar q - \tfrac{|X|-2}{|X|-1} S^{-1} \textstyle\sum_{j\in X}(\theta_j - \theta_i)\right). \end{aligned}$$

From Lemma 3, $q_i^e(X, \bar q) = \frac{1}{|X|}\left(\bar q - S^{-1}\sum_{j\in X}(\theta_j - \theta_i)\right)$. Since $S^{-1}$ is positive definite, this coincides with $q_i^*(\{b_j^U\}_{j\in X})$ for each $i \in X$ iff $\theta_i = \frac{1}{|X|}\sum_{j\in X}\theta_j = \theta_\ell$ for each $i, \ell \in X$.

(iii): Revenue in a uniform-price auction is given by $\bar q \cdot p^*(\{b_j^U\}_{j\in X})$; the expression follows from (15). □

**Proof of Corollary 2 (Nonnegative Revenue in the Uniform-Price Auction)** For each $i \in X$, since $\bar q \geq 0$ and $\theta_i - SQ \geq 0$, we have $\theta_i' \bar q - \bar q' SQ \geq 0$. Then since $S$ is positive definite and $|X| > 2$, $\frac{|X|-1}{(|X|-2)|X|}\bar q' SQ \leq \frac{1}{|X|-2}\bar q' SQ \leq \bar q' SQ$. Then for each $i \in X$, $\theta_i'\bar q - \frac{|X|-1}{(|X|-2)|X|}\bar q' SQ \geq \theta_i'\bar q - \bar q' SQ \geq 0$. Then $\left(\sum_{i\in X}\theta_i'\bar q\right) - \frac{|X|-1}{|X|-2}\bar q' SQ \geq 0$. The statement follows from Proposition 7 (iii). □

**Proof of Theorem 3 (Expected Revenue in the Uniform-Price and Vickrey Auctions)**

(i): By Lemma 3, for any $Z \subseteq I$, we have

$$\begin{aligned} v(Z, \bar q) &= \tfrac12\left(\textstyle\sum_{\theta_i\in Z}\theta_i' S^{-1}\theta_i\right) - \tfrac{1}{2|Z|}\left(\left(\textstyle\sum_{\theta_i\in Z}\theta_i\right)' S^{-1}\left(\textstyle\sum_{\theta_i\in Z}\theta_i\right) - 2Q'\left(\textstyle\sum_{\theta_i\in Z}\theta_i\right) + \bar q' SQ\right) \\ &= \tfrac12\left(\textstyle\sum_{k=1}^K \frac{1}{\lambda_k}\sum_{\theta_i\in Z}(\theta_i'\phi_k)^2\right) - \tfrac{1}{2|Z|}\left(\left(\textstyle\sum_{k=1}^K \frac{1}{\lambda_k}\left(\sum_{\theta_i\in Z}\theta_i'\phi_k\right)^2\right) - 2Q'\left(\textstyle\sum_{\theta_i\in Z}\theta_i\right) + \bar q' SQ\right) \end{aligned}$$

Then by Proposition 1 (iii), for any $X = \{\theta_i\}_{i=1}^N \in \operatorname{supp}\mu$ and any $\theta_i \in X$,

$$\pi^V_{\theta_i}(X) = \begin{array}{l} \frac12\sum_{k=1}^K \frac{1}{\lambda_k}(\theta_i'\phi_k)^2 - \frac{1}{2N}\left(\left(\sum_{k=1}^K \frac{1}{\lambda_k}\left(\sum_{\theta_j\in X}\theta_j'\phi_k\right)^2\right) - 2Q'\left(\sum_{j\in X}\theta_j\right) + \bar q' SQ\right) \\ +\frac{1}{2(N-1)}\left(\left(\sum_{k=1}^K \frac{1}{\lambda_k}\left(\sum_{j\in X\setminus\{\theta_i\}}\theta_j'\phi_k\right)^2\right) - 2Q'\left(\sum_{j\in X\setminus\{\theta_i\}}\theta_j\right) + \bar q' SQ\right) \end{array}$$

Then by Proposition 1 (iii), since $\{\theta_i\}_{i=1}^N$ are independent, we have

$$
\begin{aligned}
E_\mu[\pi_a^V(X)] &= E_\mu[v(X,\bar{q})] - NE_\mu[\pi_{\theta_i}^V(X)|\theta_i \in X] \\
&= \begin{array}{l} \frac{N}{2}\left(\sum_{k=1}^K \frac{1}{\lambda_k}(\mathrm{Var}_F(\theta'\phi_k) + E_F(\theta'\phi_k)^2)\right) - \frac{1}{2N}\left(\sum_{k=1}^K \frac{1}{\lambda_k}(N\,\mathrm{Var}_F(\theta'\phi_k) + N^2 E_F(\theta'\phi_k)^2)\right) \\ +\bar{q}'E_F[\theta] - \frac{1}{2N}\bar{q}'SQ - \frac{N}{2}\left(\sum_{k=1}^K \frac{1}{\lambda_k}(\mathrm{Var}_F(\theta'\phi_k) + E_F(\theta'\phi_k)^2)\right) \\ +\frac{1}{2}\left(\sum_{k=1}^K \frac{1}{\lambda_k}(N\,\mathrm{Var}_F(\theta'\phi_k) + N^2 E_F(\theta'\phi_k)^2)\right) - NQ'E_F[\theta] + \frac{1}{2}\bar{q}'SQ \\ -\frac{N}{2(N-1)}\left(\sum_{k=1}^K \frac{1}{\lambda_k}((N-1)\,\mathrm{Var}_F(\theta'\phi_k) + (N-1)^2 E_F(\theta'\phi_k)^2)\right) + NQ'E_F[\theta] - \frac{N}{2(N-1)}\bar{q}'SQ \end{array} \\
&= \begin{array}{l} \bar{q}'E_F[\theta] + \frac{-(N-1)+N(N-1)-N^2}{2N(N-1)}\bar{q}'SQ + \frac{N^2-N-N(N-1)}{2}\left(\sum_{k=1}^K \frac{1}{\lambda_k}E_F(\theta'\phi_k)^2\right) \\ +\frac{-1+N-N}{2}\sum_{k=1}^K \frac{1}{\lambda_k}\mathrm{Var}_F(\theta'\phi_k) \end{array} \\
&= \bar{q}'E_F[\theta] - \frac{2N-1}{2N(N-1)}\bar{q}'SQ - \frac{1}{2}\sum_{k=1}^K \frac{1}{\lambda_k}\mathrm{Var}_F(\theta'\phi_k),
\end{aligned}
$$

as desired.

(ii): Follows immediately by Proposition 7 (iii) and independence of $\{\theta_i\}_{i=1}^N$.

(iii): From (i) and (ii), expected revenue is higher in the Vickrey auction if

$$
\begin{aligned}
&\bar{q}'E_F[\theta] - \frac{2N-1}{2N(N-1)}\bar{q}'SQ - \frac{1}{2}\sum_{k=1}^K \frac{1}{\lambda_k}\mathrm{Var}_F(\theta'\phi_k) > \bar{q}'E_F[\theta] - \frac{N-1}{N(N-2)}\bar{q}'SQ \\
\Leftrightarrow &\left(\frac{2N-2}{N(N-2)} - \frac{2N-1}{N(N-1)}\right)\bar{q}'SQ, = \frac{1}{(N-1)(N-2)}\bar{q}'SQ > \sum_{k=1}^K \frac{1}{\lambda_k}\mathrm{Var}_F(\theta'\phi_k),
\end{aligned}
$$

as desired. □

**Proof of Proposition 8 (No Truthful Bidding in a Split Vickrey Auction)** Observe that in a split auction, by the clearing rule, we must have

$$
\begin{aligned}
&\bar{q} = \bar{q}_- + \bar{q}_+ = \sum_{i\in X_-} b_i^{-1}(p^*(b_{X_-})) + \sum_{i\in X_+} b_i^{-1}(p^*(b_{X_+})) = \sum_{i\in X} b_i^{-1}(p^*(b_{X_-})) \\
\Rightarrow\; &p^*(b) = p^*(b_{X_+}) = p^*(b_{X_-}), \qquad q_i^*(b_{X_-}) = q_i^*(b_{X_+}) = b_i^{-1}(p^*(b)) = q_i^*(b).
\end{aligned}
$$

It follows from Proposition 1 (ii) that $q_i^*(b_{X_-}^V) = q_i^e(X,\bar{q})$ for each $i \in X_-$ and $q_i^*(b_{X_+}^V) = q_i^e(X,\bar{q})$ for each $i \in X_+$. Then since the Vickrey payment rule uniquely implements the efficient allocation in dominant strategies (Holmström, 1979), bidding $b_i^V = \nabla u_i$ is a dominant strategy for bidder $i$ if and only if his payment when the bids $b$ are submitted to the split auction is equal to $t_i^V(b)$.

To see that it is not, suppose $i \in X_+$. Let $r_i(q_i, b_{X_+\setminus\{i\}}|\bar{q}_+)$ be bidder $i$'s residual supply in a forward auction for quantity $\bar{q}_+$, and let $\bar{q}_+(b) \equiv \sum_{j\in X_+} q_j^*(b)$ be the quantity available in the forward auction when the profile of bids is $b$.

If $b$ is such that $q_i^*(b) = q_i$, then

$$
\bar{q}_+(b) = q_i + \sum_{j\in X_+\setminus\{i\}} b_j^{-1}(p^*(b_{X_+})) = q_i + \sum_{j\in X_+\setminus\{i\}} b_j^{-1}(r_i(q_i, b_{-i})).
$$

Hence, $\bar{q}_+(q_i, b_{-i}) \equiv q_i + \sum_{j\in X_+\setminus\{i\}} q_j^*(r_i(q_i, b_{-i}))$ is the total quantity sold in the forward auction when bidder $i$ places a bid that obtains $q_i$, given bids $b_{-i}$ by the other participants in the split auction. Now observe that given any $\bar{q}_+$, $r_i(q_i, b_{X_+\setminus\{i\}}|\bar{q}_+)$ is defined by

$$q_i + \sum_{j\in X_+\setminus\{i\}} b_j^{-1}(r_i(q_i, b_{X_+\setminus\{i\}}|\bar{q}_+)) = \bar{q}_+.$$

It follows that $r_i(q_i, b_{X_+\setminus\{i\}}|\bar{q}_+(q_i, b_{-i})) = r_i(q_i, b_{-i})$. Moreover, by the implicit function theorem,

$$D_{\bar{q}_+} r_i(q_i, b_{X_+\setminus\{i\}}|\bar{q}_+) = \left(\sum_{j\in X_+\setminus\{i\}} Db_j^{-1}(r_i(q_i, b_{X_+\setminus\{i\}}|\bar{q}_+))\right)^{-1}$$

And by definition,

$$\begin{aligned}
DQ_+(q_i, b_{-i}) &= I + \textstyle\sum_{j\in X_+\setminus\{i\}} Db_j^{-1}(p) Dr_i(q_i, b_{-i})|_{p=r_i(q_i,b_{-i})} \\
&= I - \left(\textstyle\sum_{j\in X_+\setminus\{i\}} Db_j^{-1}(r_i(q_i, b_{-i}))\right)\left(\textstyle\sum_{j\in X\setminus\{i\}} Db_j^{-1}(r_i(q_i, b_{-i}))\right)^{-1} \\
&= \left(\textstyle\sum_{j\in X_-\cup\{i\}} Db_j^{-1}(r_i(q_i, b_{-i}))\right)\left(\textstyle\sum_{j\in X\setminus\{i\}} Db_j^{-1}(r_i(q_i, b_{-i}))\right)^{-1}
\end{aligned}$$

Since each $b_i$ must have a negative definite Jacobian matrix, these derivative matrices have nonzero determinant. Then by the inverse function theorem, $\bar{q}_+(q_i, b_{-i})$ is a one-to-one function of $q_i$, and $r_i(q_i, b_{X_+\setminus\{i\}}|\bar{q}_+)$ is a one-to-one function of $\bar{q}_+$.

It follows that for any $b_{-i}$, $r_i(q, b_{X_+\setminus\{i\}}|\bar{q}_+(q_i, b_{-i})) \neq r_i(q, b_{X_+\setminus\{i\}}|\bar{q}_+(q, b_{-i})) = r_i(q, b_{-i})$ whenever $q \neq q_i$. Then for any $b_{-i}$, there exists $\tilde{q}_i$ such that

$$\int_0^{\tilde{q}_i} r_i(q, b_{X_+\setminus\{i\}}|\bar{q}_+(q_i, b_{-i})) \cdot dq \neq \int_0^{\tilde{q}_i} r_i(\mathbf{r}, b_{-i}) dq.$$

Let $\tilde{b}_i(q_i) = r_i(\tilde{q}_i, b_{-i})$. Then $q_i^*(\{\tilde{b}_i, b_{-i}\}) = \tilde{q}_i$. Then bidder $i$'s payment when the profile of bids $\{\tilde{b}_i, b_{-i}\}$ is submitted to the split auction is $\int_0^{\tilde{q}_i} r_i(q, b_{X_+\setminus\{i\}}|\bar{q}_+(q_i, b_{-i})) \cdot dq \neq \int_0^{\tilde{q}_i} r_i(\mathbf{r}, b_{-i}) dq = t_i^V(\{\tilde{b}_i, b_{-i}\})$, as desired. □

**Proof of Proposition 9 (Equivalent Implementation with Eigenvector Packages)** Existence of $\Phi$ and separability of valuations in $\Phi$ follows from Lemma 6. Then the social planner's problem (2) is separable in $\Phi$. Since bidder payoffs are quasilinear in transfers, it follows from applying Proposition 1 in the $K = 1$ case for each $\phi \in \Phi$ that conducting independent Vickrey auctions for each $\phi \in \Phi$ yields the Vickrey allocation and payoff profile.

For the UPA result, label $\Phi = \{\phi_k\}_{k=1}^K$ and observe that

$$u(T_\Phi x) = \sum_{k=1}^K \hat{u}_i^k(x_k), \qquad \hat{u}_i^k(x_k) = \theta_i'\phi_k x_k - \lambda_k x_k^2,$$

where $\lambda_k$ is the eigenvalue of $S$ associated with $\phi_k$. Then by Proposition 7 (ii), the ex post equilibrium of a uniform-price auction for package $\phi_k$ results in the allocation

$$q_i^{\phi_k}(X) = \frac{1}{|X|}\left(\phi_k'\bar{q} - \frac{|X|-2}{|X|-1}\frac{1}{\lambda_k}\sum_{j\in X}(\theta_j - \theta_i)'\phi_k\right) = \phi_k' q_i^*(\{b_j^U\}_{j\in X}),$$

Summing over the packages then yields $\sum_{k=1}^K \phi_k q_i^{\phi_k}(X) = T_\Phi T_\Phi q_i^*(\{b_j^U\}_{j\in X}) = q_i^*(\{b_j^U\}_{j\in X})$. Similarly, from (15), equilibrium price in the ex post equilibrium of a uniform-price auction for package

$\phi$ is given by $p^{\phi_k} = \frac{1}{|X|}(\sum_{\ell \in X} \phi_k' \theta_\ell) - \frac{|X|-1}{(|X|-2)|X|} \lambda_k \phi_k' \bar{q}$. It follows that each bidder's payment is

$$\begin{aligned}
\sum_{k=1}^{K} p^{\phi_k} q_i^{\phi_k}(X) &= \sum_{k=1}^{K} \frac{1}{|X|} q_i^*(\{b_j^U\}_{j\in X})' \phi_k \left( \left( \sum_{\ell \in X} \phi_k' \theta_\ell \right) - \frac{|X|-1}{|X|-2} \lambda_k \phi_k' \bar{q} \right) \\
&= q_i^*(\{b_j^U\}_{j\in X})' \left( \frac{1}{|X|} T_\Phi \left( \sum_{\ell \in X} T_\Phi' \theta_\ell \right) - \frac{|X|-1}{|X|-2} T_\Phi \Lambda T_\Phi' \bar{q} \right),
\end{aligned}$$

where $\Lambda$ is the diagonal matrix of eigenvalues associated with $\Phi$. From (15), this expression is just $q_i^*(\{b_j^U\}_{j\in X})' p^*(\{b_j^U\}_{j\in X})$. Then since each bidder's total transfer and allocation from the independent uniform-price package auctions is the same as their UPA transfer and allocation, their payoff must be the same as well. □