

Online Appendix for “Matching with Strategic Consistency”^{*}

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S.1 Strategic Consistency and Bargaining Power: Example 3 Revisited

Suppose we want use Theorem 6 to construct the Pareto-optimal profiles for which the outcomes found in Example 3 are stable. To do so, we must first assign cardinal values to the agents’ payoffs. Suppose that each agent receives payoff e^2 from their most preferred outcome, e from their second most preferred outcome, and e^{-10} from outcomes that they regard as worse than autarky.¹ We consider two social welfare functions and contrast the profiles they produce. First, consider the Nash product $\phi(x) = \prod_{i \in I} x_i^{\alpha_i}$ with asymmetric weights $(\alpha_a, \alpha_b, \alpha_1, \alpha_2) = (0.3, 0.2, 0.35, 0.15)$. Then in the profile $\{C_i^\phi, \mu_i^\phi\}_{i \in I}$, (5) tells us that the agents choose (and believe others will choose) the contracts in the nonstrategically individually rational outcome that yields the highest value of ϕ among all such outcomes that are subsets of the set of available contracts: i.e., the algorithm (3) can construct $\{C_i^\phi, \mu_i^\phi\}_{i \in I}$

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¹Formally, we let $u_i(\emptyset) = 1$ for each $i \in I$, and

$$\begin{array}{lll}
 u_a(\{x_{a1}\}) = e^2; & u_a(\{x_{a2}\}) = e; & u_a(\{x_{a1}, x_{a2}\}) = e^{-10}; \\
 u_b(\{x_{b2}\}) = e^2; & u_b(\{x_{b1}\}) = e; & u_b(\{x_{b1}, x_{b2}\}) = e^{-10}; \\
 u_1(\{x_{a1}\}) = e^{-10}; & u_1(\{x_{b1}\}) = e; & u_1(\{x_{a1}, x_{b1}\}) = e^2; \\
 u_2(\{x_{a2}\}) = e^2; & u_2(\{x_{b2}\}) = e; & u_2(\{x_{a2}, x_{b2}\}) = e^{-10}.
 \end{array}$$

using the order \succ^ϕ given by

$$\{x_{a1}, x_{b1}\} \succ^\phi \{x_{a2}, x_{b1}\} \succ^\phi \{x_{b2}\} \succ^\phi \{x_{a2}\} \succ^\phi \{x_{b1}\} \succ^\phi \emptyset,$$

and so the stable outcome for $\{C_i^\phi, \mu_i^\phi\}_{i \in I}$ is given by $\mu^\phi(X) = \{x_{a1}, x_{b1}\}$.²

Now suppose that we change the weights to be more favorable to firm 2: Let $(\alpha'_a, \alpha'_b, \alpha'_1, \alpha'_2) = (0.3, 0.2, 0.15, 0.35)$, and $\phi'(x) = \prod_{i \in I} x_i^{\alpha'_i}$. Then, the values taken by the social welfare function become higher at outcomes where firm 2 hires Bob relative to those where it hires no one, and at outcomes where firm 2 hires Alice relative to those where it hires Bob: the ordering $\succ^{\phi'}$ is

$$\{x_{a2}, x_{b1}\} \succ^{\phi'} \{x_{a1}, x_{b1}\} \succ^{\phi'} \{x_{a2}\} \succ^{\phi'} \{x_{b2}\} \succ^{\phi'} \{x_{b1}\} \succ^{\phi'} \emptyset,$$

and the stable outcome for $\{C_i^{\phi'}, \mu_i^{\phi'}\}_{i \in I}$ is $\mu^{\phi'}(X) = \{x_{a2}, x_{b1}\}$. ■

S.2 Other Stability Concepts

While the matching literature generally focuses on the solution concept — stability — that we adopt in this paper, many papers also consider other related matching-theoretic solution concepts. Two of the most common are *setwise stability* (Sotomayor, 1999) and *weak setwise stability* (Klaus and Walzl, 2009).

These solution concepts make the same predictions in two-sided one-to-one matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009). But in two-sided many-to-many matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009) or environments with multilateral contracts (Bando and Hirai, 2021), there is, in general, a gap between them. Because it requires that agents' beliefs must be correct, strategic consistency closes this gap between the predictions of stability and weak setwise stability: Whenever a block (in the sense used in stability) is successful, each agent must agree on the set of contracts that will be chosen (Lemma 1). Since the difference between stability and weak setwise stability is that the latter only considers blocks with this property, the two solution concepts must coincide.³ Hence,

²The log values taken by the two social welfare functions ϕ and ϕ' at the nonstrategically individually rational outcomes are given by

$$\begin{array}{lll} \log \phi((u_i(\emptyset))_{i \in I}) = 0; & \log \phi((u_i(\{x_{a2}\}))_{i \in I}) = 0.6; & \log \phi((u_i(\{x_{a2}, x_{b1}\}))_{i \in I}) = 1.15; \\ \log \phi((u_i(\{x_{b1}\}))_{i \in I}) = 0.55; & \log \phi((u_i(\{x_{b2}\}))_{i \in I}) = 0.7; & \log \phi((u_i(\{x_{a1}, x_{b1}\}))_{i \in I}) = 1.5, \\ \log \phi'((u_i(\emptyset))_{i \in I}) = 0; & \log \phi'((u_i(\{x_{a2}\}))_{i \in I}) = 1; & \log \phi'((u_i(\{x_{a2}, x_{b1}\}))_{i \in I}) = 1.35; \\ \log \phi'((u_i(\{x_{b1}\}))_{i \in I}) = 0.35; & \log \phi'((u_i(\{x_{b2}\}))_{i \in I}) = 0.75; & \log \phi'((u_i(\{x_{a1}, x_{b1}\}))_{i \in I}) = 1.1. \end{array}$$

³In the literature, weak setwise stability is defined for environments without externalities, but it can be extended to accommodate externalities in the same way as we extended stability. Formally, a set of contracts Y is weakly setwise stable if it is individually rational and there is no $Z \subseteq X$ such that $Z_i = C_i(Z_i \cup Y_i | Z_{-i} \cup Y_{-i})$ for each $i \in N(Z \setminus Y)$.

strategic consistency allows stability to capture outcomes that, with nonstrategic choice, are only captured by weak setwise stability. Example S.1 illustrates.

Example S.1 (Strategic Consistency vs. Nonstrategic Choice). Consider a market with two agents $I = \{1, 2\}$ and two contracts $X = \{x, z\}$ they can sign with each other, and suppose that they have preferences

$$u_1(\{x, z\}) > u_1(\{z\}) > u_1(\{x\}) > u_1(\emptyset); \quad u_2(\{x\}) > u_2(\{z\}) > u_2(\emptyset) > u_2(\{x, z\}).$$

This is a two-sided market where contracts are substitutable, so with nonstrategic choice, there is a (unique) stable outcome, $\{x\}$. But another outcome is also compelling. Observe that $\{z\}$ is blocked by $\{x\}$ when choice is nonstrategic, since $\hat{C}_1(\{x, z\}) = \{x, z\}$ and $\hat{C}_2(\{x, z\}) = \{x\}$. But this block only occurs because agent 1 incorrectly assumes that agent 2 will choose z as well as x . With strategic consistency, however, the agents must agree: either that the block will result in z being dropped (in profiles where $\{x\}$ is stable) or that it will lead both agents to choose z alone (in profiles where $\{z\}$ is stable). This allows strategic consistency to identify both outcomes as consistent with stability, whereas nonstrategic choice identifies only one. ■

S.3 Multilateral vs. Bilateral Contracts

We emphasize that interactions between pairs of agents specified by a multilateral contract cannot be represented as independent bilateral contracts in a matching on networks model. We illustrate this fact using an example from professional sports.

Example S.2 (Multilateral vs. Bilateral Agreements). On July 31, 2014, Major League Baseball's Detroit Tigers, Tampa Bay Rays, and Seattle Mariners traded a total of five players as part of a single multilateral agreement. Detroit sent two players to Tampa Bay; Tampa Bay sent one player to Detroit; Detroit sent one player to Seattle; and Seattle sent one player to Tampa Bay.⁴

If we tried to model this transaction as three independent bilateral contracts, instead of a single multilateral contract, we would generally fail to predict that it would take place. Indeed, a contract representing the bilateral interaction between the Seattle Mariners and Detroit Tigers would not be individually rational for the Tigers, as they would send Seattle a valuable player without receiving anything in return. Likewise, a contract representing the bilateral interaction between the Mariners and Rays would not be individually rational for the Mariners. These interactions are only possible because they were conducted as part of a single multilateral agreement.

⁴Source: <http://mlb.mlb.com/mlb/transactions/index.jsp#month=7&year=2014>.

S.4 Strategic Consistency and Nash Equilibrium

Consider the normal form game G defined as follows:

Players The agents $i \in I$.

Actions Sets of contracts $S_i \in 2^{X_i}$.

Payoffs $\pi_i(\{S_i\}_{i \in I}) = u_i(\cap_{i \in I}(S_i \cup X_{-i}))$.

G extends the link-announcement game discussed in, e.g., Myerson (1991) and Jackson (2010) to our matching with contracts setting. Proposition S.1 shows that an outcome is nonstrategically individually rational if and only if it is the set of contracts signed in a Nash equilibrium of G . The nonstrategically individually rational outcomes are precisely those that can be pinned down by the choice functions and beliefs in a strategically consistent profile (Lemmas 1 and 2), so a profile of choice functions and beliefs is strategically consistent if and only if it is cross-set consistent and maps each set of contracts Y to the contracts signed in a pure strategy Nash equilibrium of G .

Proposition S.1 (Nonstrategic Individual Rationality and Nash Equilibrium). *If Y is nonstrategically individually rational, then $\{Y_i\}_{i \in I}$ is a Nash equilibrium of G . Conversely, if $\{S_i(Y)\}_{i \in I}$ is a Nash equilibrium of G , then $S(Y) \equiv \cap_{i \in I}(S_i(Y) \cup X_{-i})$ is nonstrategically individually rational.*

Proposition S.2 (Strategic Consistency and Nash Equilibrium). *If $\{C_i, \mu_i\}_{i \in I}$ is a strategically consistent profile of choice functions and beliefs, then for each $Y \subseteq X$, $\{C_i(Y_i|Y_{-i})\}_{i \in I}$ is a Nash equilibrium of G .*

Conversely, if for each $Y \subseteq X$, there is a Nash equilibrium $\{S_i(Y)\}_{i \in I}$ of G such that for each $i \in I$, $\mu_i(Y) = S(Y) \equiv \cap_{i \in I}(S_i(Y) \cup X_{-i})$ and $C_i(Y_i|Y_{-i}) = S(Y) \cap X_i$, then the choices $\{C_i\}_{i \in I}$ are optimal given $\{\mu_i\}_{i \in I}$, and the beliefs $\{\mu_i\}_{i \in I}$ are correct given $\{C_i\}_{i \in I}$. If, in addition, we have $S(Y) = S(Z)$ whenever $S(Y) \subseteq Z \subseteq Y$, then $\{C_i, \mu_i\}_{i \in I}$ is a strategically consistent profile.

S.4.1 Proofs for Appendix S.4

Proof of Proposition S.1 (Nonstrategic Individual Rationality and Nash Equilibrium) (Nash Equilibrium \Rightarrow Nonstrategic IR) Suppose that for each $Y \subseteq X$, $\{S_i(Y)\}_{i \in I}$ is a Nash equilibrium of G . By definition and since $S_i(Y) \subseteq X_i$, we have

$$S_i(Y) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) = (S_i(Y) \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_i = S(Y)_i, \text{ and}$$

$$\left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} = (S_i(Y) \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} = S(Y)_{-i}.$$

Then since $\{S_i(Y)\}_{i \in I}$ is a Nash equilibrium of G , for each $Y \subseteq X$ and $i \in I$,

$$\begin{aligned}
& S_i(Y) \in \arg \max_{S_i \subseteq X_i} u_i((S_i \cup X_{-i}) \bigcap_{j \neq i} (S_j(Y) \cup X_{-j})); \\
\Leftrightarrow S(Y)_i &= S_i(Y) \cap \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \in \arg \max_{S_i \subseteq (\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}))_i} u_i \left(S_i \cup \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} \right); \\
\Rightarrow S(Y)_i &\in \arg \max_{S_i \subseteq S(Y)_i} u_i \left(S_i \cup \left(\bigcap_{j \neq i} (S_j(Y) \cup X_{-j}) \right) \cap X_{-i} \right) \\
&= \arg \max_{S_i \subseteq S(Y)_i} u_i(S_i \cup S(Y)_{-i}); \\
\Leftrightarrow S(Y)_i &= \hat{C}_i(S(Y)_i | S(Y)_{-i}),
\end{aligned}$$

and so $S(Y)$ is nonstrategically individually rational.

(If) Suppose that Y is nonstrategically individually rational. Since $N(x) \geq 2$ for each $x \in X$, $\bigcap_{j \neq i} (Y_j \cup X_{-j}) = Y$. Then for each $i \in I$,

$$\begin{aligned}
Y_i &\in \arg \max_{S_i \subseteq Y_i} u_i(S_i \cup Y_{-i}) = \arg \max_{S_i \in 2^{Y_i}} u_i \left((S_i \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (Y_j \cup X_{-j}) \right) \right) \\
&\subseteq \arg \max_{S_i \in 2^{X_i}} u_i \left((S_i \cup X_{-i}) \cap \left(\bigcap_{j \neq i} (Y_j \cup X_{-j}) \right) \right).
\end{aligned}$$

It follows that $\{Y_i\}_{i \in I}$ is a Nash equilibrium of G . \square

Proof of Proposition S.2 (Strategic Consistency and Nash Equilibrium) (Strategic Consistency \Rightarrow Nash Equilibrium) Suppose that $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent. By Lemmas 1 and 2, at each set $Y \subseteq X$, $\mu_i(Y)$ is nonstrategically individually rational, and $C_i(Y_i | Y_{-i}) = \mu_i(Y) \cap X_i$, for each $i \in I$. Then from Proposition S.1, $\{C_i(Y_i | Y_{-i})\}_{i \in I}$ is a Nash equilibrium of G .

(Nash Equilibrium \Rightarrow Strategic Consistency) By Proposition S.1, $S(Y)$ is nonstrategically individually rational for each $Y \subseteq X$. Then by Lemma 2, $\{C_i\}_{i \in I}$ are optimal given $\{\mu_i\}_{i \in I}$ and $\{\mu_i\}_{i \in I}$ are correct given $\{C_i\}_{i \in I}$. If, in addition, $S(Y) = S(Z)$ whenever $S(Y) \subseteq Z \subseteq Y$, then for each $i \in I$, $\mu_i(Y) \subseteq Z \subseteq Y$ implies $\mu_i(Y) = \mu_i(Z)$; cross-set (and hence strategic) consistency of $\{C_i, \mu_i\}_{i \in I}$ then follows from Lemma 4. \square

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