Online Appendix for "Matching with Strategic Consistency"*

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S.1 Aggregate IRC as a Necessary Condition

In general, preservation of the irrelevance of rejected contracts condition under aggregation — either among all agents, or among the agents on each of two implicit sides — is not a necessary condition for the generic existence of stable outcomes. However, there is a related condition that is necessary in a maximal domain sense. We say that C_J has *no 3-cycles* if there exist no distinct contracts $\{y(1), y(2), y(3)\}$ such that for each m, $y(m)$ is chosen by J when both $y(m)$ and $y(m-1)$ are available, and $y(m-1)$ is rejected; i.e., $y(m) = C_J(\{y(m), y(m-1 \mod 3)\})$.[1]

When C_J satisfies irrelevance of rejected contracts, it has no 3-cycles.[2] This relationship allows us to give a partial converse to Theorem 1 that is related to those of Pycia (2012) (coalition formation), Abeledo and Isaak (1991) (two sided matching), and Hatfield and Kominers (2012) (matching on networks). Each of these results from the literature demonstrates that the presence of a certain type of cycle rules out stable outcomes: for Pycia (2012), a 3-cycle of agents' preferences; for Abeledo and Isaak (1991), an odd cycle of acceptable partners; and

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[1] This property is similar in principle to the no 3-cycles condition used by Pycia (2012), but rules out a 3-element loop of contracts in a coalition's choice function rather than a triple of coalitions which cause the preferences of a triple of agents to form a loop.

[2] No 3-cycles is a weaker version of the *strong axiom of revealed preference (SARP)*, which requires that there is no loop of choice sets $\{Y(m)\}_{m=1}^{M}$ such that $C_J(Y(m)) \subseteq Y(m+1 \mod M)$. When contracts are substitutes, Alva (2018) shows that IRC and SARP are equivalent, and hence either implies the absence of 3-cycles.

for Hatfield and Kominers (2012), a directed cycle of contracts. Here, we show that a similar result holds in the absence of explicit restrictions on the market structure (Theorem S.1).

Specifically, suppose an environment has two sides that are implicit, in the sense that agents on the same side may be involved in or affected by the same contracts. Then Theorem S.1 shows that unless we can rule out cycles of three contracts in the aggregate choice of each implicit side, stable outcomes will not generically exist, at least within any class of choice functions that includes those satisfying substitutability.

Theorem S.1 (Stable Outcomes and 3-Cycles). *Suppose that each contract names an agent in K, i.e., $X_K = X$, and let $J = N \setminus K$. If the aggregate choice of group J, C_J, has a 3-cycle, then there exist choice functions C_k for the agents in K such that C_K satisfies IRC and substitutability but no stable outcome exists.*

Theorem S.1 tells us that if we want to guarantee the existence of stable outcomes, the requirement that IRC survives aggregation — either on two implicit sides, or among all agents — cannot be weakened in isolation (at least not by much). Instead, we must replace it with different conditions on the environment. For instance, we show in Rostek and Yoder (2020) that in settings where contracts are complementary instead of substitutable, stable outcomes exist even when IRC does not survive aggregation. Alternatively, Bando and Hirai (2021) show that certain conditions on the structure of the market ensure existence of a stable outcome, even in the absence of substitutability or complementarity.

S.1.1 Proofs for Appendix S.1

Proof of Theorem S.1 (Stable Outcomes and 3-Cycles) Let $\{y(1), y(2), y(3)\}$ denote the 3-cycle in C_J. For any $Z \subseteq X$, define $m(Z) = 2$ if $y(2) \in Z$, and $m(Z) = 3$ otherwise. For each $k \in K$, let $C_k(Z_k|Z_{-k}) = Z_k \cap \{y(1), y(m(Z))\}$. Then since $Z_K = Z$, $C_K(Z) = Z \cap \{y(1), y(m(Z))\}$.

Contracts are substitutes for K: For $Z \subset Z'$, we must have $m(Z) \geq m(Z')$. Now $R_K(Z) = Z \setminus \{y(1), y(m(Z))\} \subseteq Z' \setminus \{y(1), y(m(Z))\} = R_K(Z')$ if $m(Z) = m(Z')$. If $m(Z) > m(Z')$, then $m(Z) = 3$ and $m(Z') = 2$, so $y(2) \in Z' \setminus Z$ and $R_K(Z) = Z \setminus \{y(1), y(3)\} = Z \setminus \{y(1), y(2), y(3)\} \subseteq Z' \setminus \{y(1), y(2)\} = R_K(Z')$.

If $X^* \nsubseteq Y \equiv \{y(1), y(2), y(3)\}$, then $C_K(X^*) \subseteq X^* \cap Y \neq X^*$. Thus, X^* cannot be individually rational (and hence is not stable).

If $X^* = Y$ then $C_K(X^*) = \{y(1), y(3)\} \neq X^*$. Thus, X^* cannot be individually rational (and hence is not stable).

If $X^* = y(m)$, then $y(m+1 \mod 3)$ blocks X^*, since $y(m+1 \mod 3) \in C_K(\{y(m), y(m+1 \mod 3)\})$ and $y(m+1 \mod 3) = C_J(\{y(m), y(m+1 \mod 3)\})$; hence, X^* is not stable.

If $X^* = \{y(m), y(m-1 \mod 3)\}$, then X^* is not individually rational since $C_J(X^*) = y(m)$; hence, X^* is not stable.

Since the above list of possibilities for X^* is exhaustive, we conclude that no stable outcome exists. □

S.2 Other Stability Concepts

While the matching literature generally focuses on the solution concept — stability — that we adopt in this paper, many papers also consider other related matching-theoretic solution concepts. Two of the most common are *setwise stability* (Sotomayor, 1999) and *weak setwise stability* (Klaus and Walzl, 2009).

These solution concepts make the same predictions in two-sided one-to-one matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009). But in two-sided many-to-many matching markets (Echenique and Oviedo, 2006; Klaus and Walzl, 2009) or environments with multilateral contracts (Bando and Hirai, 2021), there is, in general, a gap between them. Because it requires that agents' beliefs must be correct, strategic consistency closes this gap between the predictions of stability and weak setwise stability: Whenever a block (in the sense used in stability) is successful, each agent must agree on the set of contracts that will be chosen (Lemma 4). Since the difference between stability and weak setwise stability is that the latter only considers blocks with this property, the two solution concepts must coincide.[3]

S.3 Strategic Consistency and Nash Equilibrium

Consider the normal form game $G_J(Y)$ defined as follows:

Players The agents $i \in J$.

Actions Sets of contracts $S_i \in 2^{Y_i}$.

Payoffs $\pi_i(\{S_i\}_{i \in J}) = u_i\left(\cap_{i \in J}(S_i \cup Y_{-i})\right)$.

$G_J(Y)$ extends the link-announcement game discussed in, e.g., Myerson (1991) and Jackson (2010) to our matching with contracts setting, limiting the available contracts to Y. Proposition S.1 shows that a tuple of choice functions for the group J is part of a strategically consistent assessment if it maps each set of contracts Y to the contracts signed in a Nash equilibrium of $G_J(Y)$. Conversely, the choice functions in a strategically consistent assessment map each $Y \subseteq X$ to a Nash equilibrium of $G_J(Y)$.

[3] In the literature, weak setwise stability is defined for environments without externalities, but it can be extended to accommodate externalities in the same way as we extended stability. Formally, a set of contracts Y is weakly setwise stable if it is individually rational and there is no $Z \subseteq X \setminus Y$ and $Y' \subseteq Z \cup Y$ such that $Z_i \subseteq Y'_i = C_i(Z_i \cup Y_i | Z_{-i} \cup Y_{-i})$.

Proposition S.1 (Strategic Consistency and Nash Equilibrium). *If $\{C_i, \mu_i\}_{i\in J}$ is a strategically consistent assessment, then for each $Y \subseteq X$, $\{C_i(Y_i|Y_{-i})\}_{i\in J}$ is a Nash equilibrium of $G_J(Y)$.*

Conversely, if for each $Y \subseteq X$, there is a Nash equilibrium $\{S_i(Y)\}_{i\in J}$ of $G_J(Y)$ such that $C_i(Y_i|Y_{-i}) = S_i(Y) \cap \bigcap_{j\in J\setminus\{i\}}(S_j(Y) \cup Y_{-j})$ for each $i \in I$, then $\{C_i, \mu_i\}_{i\in J}$ is a strategically consistent assessment for some $\{\mu_i\}_{i\in J}$.

Strategic consistency among *all* agents, on the other hand, implies that choice functions map each set of contracts to equilibria of a single game $G \equiv G_I(X)$ where the set of contracts is unrestricted.

Proposition S.2 (Strategic Consistency Among All Agents and Nash Equilibrium). *If $\{C_i, \mu_i\}_{i\in I}$ is a strategically consistent assessment, then for each $Y \subseteq X$, $\{C_i(Y_i|Y_{-i})\}_{i\in I}$ is a Nash equilibrium of G.*

Conversely, if for each $Y \subseteq X$, there is a Nash equilibrium $\{S_i(Y)\}_{i\in I}$ of G such that $S_i(Y) \subseteq Y_i$ and $C_i(Y_i|Y_{-i}) = S_i(Y) \cap \bigcap_{j\neq i}(S_j(Y) \cup Y_{-j})$ for each $i \in I$, then $\{C_i, \mu_i\}_{i\in I}$ is a strategically consistent assessment for some $\{\mu_i\}_{i\in I}$.

S.3.1 Proofs for Appendix S.3

Proof of Proposition S.1 (Strategic Consistency and Nash Equilibrium) (Only if) Suppose $\{C_i, \mu_i\}_{i\in J}$ is a strategically consistent assessment, and fix $Y \subseteq X$. Since $\{\mu_i\}_{i\in J}$ are correct given $\{C_i\}_{i\in J}$, for each $i \in J$, $\mu_i(Y) = C_{J\setminus\{i\}}(Y) = \bigcap_{j\in J\setminus\{i\}}(C_j(Y_j|Y_{-j}) \cup Y_{-j})$. Then since $\{C_i\}_{i\in J}$ are optimal given $\{\mu_i\}_{i\in J}$, $C_i(Y_i|Y_{-i}) \subseteq \mu_i(Y) = \bigcap_{j\in J\setminus\{i\}}(C_j(Y_j|Y_{-j}) \cup Y_{-j})$ and

$$
\begin{aligned}
C_i(Y_i|Y_{-i}) \in \arg\max_{S\subseteq\mu_i(Y)_i} u_i(S \cup \mu_i(Y)_{-i}) &\subseteq \arg\max_{S_i\in 2^{Y_i}} u_i((S_i \cup Y_{-i}) \cap \mu_i(Y)) \\
&= \arg\max_{S_i\in 2^{Y_i}} u_i\left((S_i \cup Y_{-i}) \cap \bigcap_{\substack{j\in J\\ j\neq i}}(C_j(Y_j|Y_{-j}) \cup Y_{-j})\right),
\end{aligned}
$$

for each $i \in J$. It follows that $\{C_i(Y_i|Y_{-i})\}_{i\in J}$ is a Nash equilibrium of $G_J(Y)$.

(If) Suppose that for each $Y \subseteq X$, $\{S_i(Y)\}_{i\in J}$ is a Nash equilibrium of $G_J(Y)$. Then for each $Y \subseteq X$ and $i \in J$, let $C_i(Y_i|Y_{-i}) = S_i(Y) \cap \bigcap_{j\in J\setminus\{i\}}(S_j(Y) \cup Y_{-j})$ and $\mu_i(Y) =$

$(Y_i \setminus X_{J\setminus\{i\}}) \cup \bigcap_{j\in J}(S_j(Y) \cup Y_{-j})$. Then for each $i \in J$,

$$\begin{aligned}
C_{J\setminus\{i\}}(Y) &= \bigcap_{j\in J\setminus\{i\}} \left(S_j(Y) \cap \bigcap_{k\in J\setminus\{j\}} (S_k(Y) \cup Y_{-k}) \right) \cup Y_{-j} \\
&= \bigcap_{j\in J\setminus\{i\}} \left(\bigcap_{k\in J} (S_k(Y) \cup Y_{-k}) \right) \cup Y_{-j} \\
&= \left(\bigcap_{j\in J\setminus\{i\}} Y \setminus X_j \right) \cup \left(\bigcap_{k\in J} (S_k(Y) \cup Y_{-k}) \right) \\
&= (Y \setminus X_{J\setminus\{i\}}) \cup \bigcap_{k\in J} (S_k(Y) \cup Y_{-k}) = (Y_i \setminus X_{J\setminus\{i\}}) \cup \bigcap_{k\in J} (S_k(Y) \cup Y_{-k}) = \mu_i(Y),
\end{aligned}$$

and $\{\mu_i\}_{i\in J}$ are correct given $\{C_i\}_{i\in J}$. Then for each $Y \subseteq X$, since $\{S_i(Y)\}_{i\in J}$ is a Nash equilibrium of $G_J(Y)$,

$$\begin{aligned}
& S_i(Y) \in \arg\max_{S_i\in 2^{Y_i}} u_i \left((S_i \cup Y_{-i}) \cap \bigcap_{j\in J\setminus\{i\}} (S_j(Y) \cup Y_{-j}) \right) \\
\Rightarrow C_i(Y_i|Y_{-i}) = S_i(Y) \cap \bigcap_{j\in J\setminus\{i\}} (S_j(Y) \cup Y_{-j}) &\in \arg\max_{S_i\in 2^{Y_i}} u_i \left((S_i \cup Y_{-i}) \cap \bigcap_{j\in J\setminus\{i\}} (S_j(Y) \cup Y_{-j}) \right),
\end{aligned}$$

and since $C_i(Y_i|Y_{-i}) = S_i(Y) \cap \bigcap_{j\in J\setminus\{i\}}(S_j(Y) \cup Y_{-j}) \subseteq \bigcap_{j\in J}(S_j(Y) \cup Y_{-j}) \subseteq \mu_i(Y)$,

$$\begin{aligned}
C_i(Y_i|Y_{-i}) &\in \arg\max_{S\subseteq\mu_i(Y)_i} u_i \left((S \cup Y_{-i}) \cap \bigcap_{j\in J\setminus\{i\}} (S_j(Y) \cup Y_{-j}) \right) \\
&= \arg\max_{S\subseteq\mu_i(Y)_i} u_i \left(\left(S \cap \bigcap_{j\in J\setminus\{i\}} (S_j(Y) \cup Y_{-j}) \right) \cup \left(Y_{-i} \cap \bigcap_{j\in J\setminus\{i\}} (S_j(Y) \cup Y_{-j}) \right) \right) \\
&= \arg\max_{S\subseteq\mu_i(Y)_i} u_i \left(\left(S \cap \bigcap_{j\in J\setminus\{i\}} (S_j(Y) \cup Y_{-j}) \right) \cup \mu_i(Y)_{-i} \right) \\
&= \arg\max_{S\subseteq\mu_i(Y)_i} u_i(S \cup \mu_i(Y)_{-i}),
\end{aligned}$$

since $\mu_i(Y) \subseteq (Y_i \setminus X_{J\setminus\{i\}}) \cup \bigcap_{k\in J\setminus\{i\}}(S_k(Y) \cup Y_{-k}) = \bigcap_{k\in J\setminus\{i\}}(S_k(Y) \cup Y_{-k})$. It follows that $\{C_i\}_{i\in J}$ are optimal given $\{\mu_i\}_{i\in J}$, and so $\{C_i, \mu_i\}_{i\in J}$ is a strategically consistent assessment.

Proof of Proposition S.2 (Strategic Consistency Among All Agents and Nash Equilibrium) By Proposition S.1, it suffices to show that $\{S_i\}_{i\in I}$ is a pure strategy Nash equilibrium of $G_I(Y)$ if and only if it is a pure strategy Nash equilibrium of G and $S_i \subseteq Y_i$

for each $i \in I$.

First suppose $S_i \subseteq Y_i$ for each $i \in I$. Then for each $\ell \in I$, since $|N(x)| \geq 2$ for each $x \in X$, if $x \in \bigcap_{j \neq \ell}(S_j \cup X_{-j})$, then $x \in S_j$ for some $j \neq \ell$. Then since $S_j \subseteq Y_j$ for each $j \in I$, $\bigcap_{j \neq \ell}(S_j \cup X_{-j}) \subseteq \bigcup_{j \neq \ell} S_j \subseteq Y$, and consequently, $\bigcap_{j \neq \ell}(S_j \cup X_{-j}) = Y \cap \bigcap_{j \neq \ell}(S_j \cup X_{-j}) = \bigcap_{j \neq \ell}(S_j \cup Y_{-j})$.

Then if $S_i \subseteq Y_i$ for each $i \in I$, we have

$$
\begin{aligned}
& S_i \in \arg\max_{S_i \in 2^{X_i}} u_i \left((S_i \cup X_{-i}) \cap \bigcap_{j \neq i} (S_j \cup X_{-j}) \right) \\
\Leftrightarrow\ & S_i \in \arg\max_{S_i \in 2^{X_i}} u_i \left((S_i \cup X_{-i}) \cap \bigcap_{j \neq i} (S_j \cup Y_{-j}) \right) \\
\Leftrightarrow\ & S_i \in \arg\max_{S_i \in 2^{Y_i}} u_i \left((S_i \cup Y_{-i}) \cap \bigcap_{j \neq i} (S_j \cup Y_{-j}) \right) \qquad \left(\text{since } \bigcap_{j \neq i} (S_j \cup Y_{-j}) \subseteq Y \right).
\end{aligned}
$$

The if part follows directly; the only if part follows from the additional observation that for any action profile $\{S_i\}_{i \in I}$ in $G_I(Y)$, $S_i \subseteq Y_i$ for each $i \in I$.

S.4 An Alternative Forward Induction Criterion

As mentioned in Section 5.1, weak forward induction requires agents to believe that others will go along with credible blocking proposals — those for which no agent has an incentive to refuse to go along with, either directly (by increasing their payoff) or indirectly (by proposing further blocks) — when those blocks add contracts. Here, we introduce a stronger criterion that also applies to blocks that *change* the set of contracts by adding some contracts and deleting others.

Definition (Strong Forward Induction). Given a strategically consistent assessment $\{C_i, \mu_i\}_{i \in I}$, we say that Z is a *credible blocking proposal* for $Y \not\supseteq Z$ if

(a) each agent would like to choose all of the contracts in Z that name him when every other agent does the same: for each $i \in I$, $\hat{C}_i(Z_i | Z_{-i}) = Z_i$;

(b) when the other agents agree to the contracts in Z, each agent would prefer to drop their contracts in $Y \setminus Z$ for those in Z, even if the other agents leave their existing contracts intact: for each $i \in I$, $\hat{C}_i(Y_i \cup Z_i | Y_{-i} \cup Z_{-i}) = Z_i$; and

(c) adopting the new set of contracts cannot lead to further blocking proposals, except those that only involve existing contracts: if $Z' \subseteq C(Z' \cup Z)$ for some $Z' \nsubseteq Z$ then $Z' \subseteq Y$

If for each Y, and each Z that is a credible blocking proposal for Y, we have $\mu_i(Y \cup Z) \neq Y$ for each $i \in I$, then we say $\{C_i, \mu_i\}_{i \in I}$ satisfies *strong forward induction (SFI)*.

Intuitively, when Z is a credible blocking proposal for Y, a group of agents has no reason to propose changing the set of contracts from Y to Z unless they intend to follow through with that proposal: they cannot benefit from further blocks, except those that lead to an outcome that was already available before the blocking proposal (c), and they each prefer to go through with the block, either when they behave strategically and expect everyone else to do so as well (a) or when they behave myopically and expect other agents to leave their existing contracts intact (b). Strong forward induction requires each agent to believe that the others will never ignore such a credible proposal by continuing to choose only their existing contracts.

This notion of credibility used in SFI applies to a larger class of blocking proposals than the notion of credibility implicit in weak forward induction: A credible blocking proposal may add some contracts and delete others, whereas WFI only requires agents to evaluate the credibility of proposals that add new contracts while leaving the existing set of contracts intact. However, its third prong, which ensures that the blocking proposal was not made in order to prepare the ground for a *subsequent* blocking proposal, is slightly different: Instead of merely requiring that any successful block of the proposed set of contracts would also block the existing set of contracts, SFI's notion of credibility requires that any successful block of the proposed set of contracts is *contained in* the existing set of contracts.

Despite the fact that SFI's notion of credibility is more demanding on blocking proposals than WFI's — and hence its requirement that agents take credible proposals seriously is weaker — it offers at least as much predictive power as WFI: Theorem S.2 (i) shows that any outcome consistent with stability among strategically sophisticated agents with reasonable beliefs and strong forward induction is among those outcomes predicted by Theorem 5. But conversely, despite the fact that SFI requires agents to evaluate the credibility of a larger set of blocking proposals, its predictive power is not much greater than WFI's: Theorem S.2 (ii) shows that applying SFI instead of WFI only eliminates outcomes characterized in Theorem 5 when they fail to satisfy a weak nondomination criterion.

To characterize the set of outcomes that are consistent with stability and strong forward induction, we need to introduce new objects. Let $\mathcal{M} = \{Z \subseteq X | \hat{C}_i(Z_i|Z_{-i}) = Z_i \text{ for each } i \in I\} \subseteq 2^X$ denote the collection of myopically individually rational outcomes. Let $\overline{\mathcal{M}} = \{Z \in \mathcal{M} | \nexists Z' \in \mathcal{M}, Z' \supset Z\}$ denote the collection of maximally myopically individually rational outcomes. Then define the *myopically Blair-undominated* binary relation $\trianglerighteq$ on $\mathcal{M}$ as follows: $Y \trianglerighteq Z \Leftrightarrow Z \not\succ_{\hat{C}_i} Y$ for each $i \in I$. Since each $\not\succ_{\hat{C}_i}$ is irreflexive, $\trianglerighteq$ is complete: $Y \trianglerighteq Z$ or $Y \trianglelefteq Z$ for each $Y, Z \in \mathcal{M}$.

We say that $Y \in \overline{\mathcal{M}}$ is *comparably myopically undominated (CMU)* if there exists $Z^*(Y) \in$

$\overline{\mathcal{M}}$ such that $Z^*(Y) \trianglelefteq Y$ and for every $Z \in \mathcal{M}$ with $Z \triangleright Y$, $Y \nsubseteq Z^*(Y) \cup Z$. In words, an outcome Y is comparably myopically undominated if there is a maximal myopically IR $Z^*(Y)$ that does not myopically Blair-dominate it, and that can be compared to any myopically IR outcome that myopically Blair-dominates Y without making Y available. This is not a very restrictive condition: An outcome Y satisfies CMU, for instance, whenever there is a disjoint maximal myopically IR outcome Z' that does not myopically Blair-dominate it.

Theorem S.2 (Stability with Strong Forward Induction).

i. If Y^* is stable for some strategically consistent assessment $\{C_i, \mu_i\}_{i \in I}$ satisfying SFI with beliefs $\{\mu_i\}$ satisfying IRC, it is maximally myopically individually rational: $Y^* \in \overline{\mathcal{M}}$.

ii. If $Y^* \in \overline{\mathcal{M}}$ is comparably myopically undominated, it is stable for some strategically consistent assessment $\{C_i, \mu_i\}_{i \in I}$ satisfying SFI with beliefs $\{\mu_i\}$ satisfying IRC.

S.4.1 Proofs for Appendix S.4

Lemma S.1. *Suppose that $\{C_i, \mu_i\}_{i \in I}$ is strategically consistent. If $Y \subseteq X$ is individually rational, it is myopically individually rational: $Y \in \mathcal{M}$. The converse holds if $\{\hat{C}_i\}_{i \in I}$ satisfy WFI.*

Proof. Since Y is individually rational, we have $C_i(Y_i|Y_{-i}) = Y_i \Leftrightarrow C_i(Y_i|Y_{-i}) \cup Y_{-i} = Y \Leftrightarrow C(Y) = Y$. Then by Lemma 10 (ii), $\mu_i(Y) = C(Y) = Y$ for each $i \in I$. Then by Lemma 5 we have $\hat{C}_i(Y_i|Y_{-i}) = \hat{C}_i(\mu_i(Y)_i|\mu_i(Y)_{-i}) = C_i(Y_i|Y_{-i}) = Y_i$ for each $i \in I$, as desired.

For the converse, suppose $Y \in \mathcal{M}$. Then since $\{\hat{C}_i\}_{i \in I}$ satisfy WFI, $\mu_i(Y) = Y$ for each $i \in I$. Then by Lemma 5 we have $C_i(Y_i|Y_{-i}) = \hat{C}_i(\mu_i(Y)_i|\mu_i(Y)_{-i}) = \hat{C}_i(Y_i|Y_{-i}) = Y_i$ for each $i \in I$, and Y is individually rational. $\square$

Proof of Theorem S.2 (Stability with Strong Forward Induction) (i) By definition of stability, Y^* is individually rational. Then by Lemma S.1, $Y^* \in \mathcal{M}$. Suppose $Y^* \notin \overline{\mathcal{M}}$. Then there exists $Z \in \mathcal{M}$ such that $Z \supset Y^*$. Then Z is a credible blocking proposal for Y: $Z \in \mathcal{M}$, satisfying (a). Since $Z \in \mathcal{M}$ and $Z \supset Y^*$, for each $i \in I$, $\hat{C}_i(Y_i^* \cup Z_i|Y_{-i}^* \cup Z_{-i}) = \hat{C}_i(Z_i|Z_{-i}) = Z_i$, satisfying (b). Since $\{\mu_i\}_{i \in I}$ satisfy IRC, C satisfies IRC by Theorem 3. Then by Theorem 2, $C(X) = Y^*$. Then since $Z \supset Y^*$, and C satisfies IRC, we have $C(Z \cup Z') = Y^*$ for any $Z' \subseteq X$. Then for any $Z' \nsubseteq Z$ such that $Z' \subseteq C(Z' \cup Z)$, $Z' \subseteq Y^*$, satisfying (c).

Then since $\{C_i, \mu_i\}_{i \in I}$ satisfies SFI, we must have $\mu_i(Y^* \cup Z) \neq Y$ for each $i \in I$, and so by Lemma 10 (ii), $C(Y^* \cup Z) = \mu_i(Y^* \cup Z) \neq Y^*$. But since C satisfies IRC, $Y^* = C(X) = C(Y^* \cup Z)$ by Theorem 2, a contradiction.

(ii) Label the elements of $\mathcal{M}$ according to the sequence $\{Y^n\}_{n=1}^{|\mathcal{M}|}$, constructed inductively as follows. For the initial element, choose $Y^1 = Y^*$. Since Y^* is CMU, there exists $Z^*(Y^*) \in \overline{\mathcal{M}}$

such that $Z^*(Y^*) \trianglelefteq Y^*$, and for every $Z \in \mathcal{M}$ with $Z \triangleright Y^*$, $Y^* \nsubseteq Z^*(Y^*) \cup Z$; choose $Y^2 = Z^*(Y^*)$.

Then, given elements $\{Y^n\}_{n=1}^m$, choose $Y^{m+1} \in \mathcal{M} \setminus \{Y^n\}_{n=1}^m$ such that there is no $Y' \in \mathcal{M} \setminus \{Y^n\}_{n=1}^m$ with $Y' \supset Y^{m+1}$. Then by construction, $Y^n \nsubseteq Y^m$ for all $n < m$.

For each $Z \subseteq X$, let $n^*(Z) \equiv \max\{n | Y^n \in 2^Z\}$. $n^*(Z)$ is well-defined for each Z: Since $\{\hat{C}_i\}_{i\in I}$ are myopically consistent, we have $\hat{C}_i(\emptyset|\emptyset) = \emptyset$ for each $i \in I$, and so $\emptyset \in \mathcal{M}$; consequently, for each $Z \subseteq X$, 2^Z contains at least one element of $\mathcal{M}$. Then define $\{C_i, \mu_i\}_{i\in I}$ as follows:

$$\mu_i(Z) = Y^{n^*(Z)}, \qquad\qquad C_i(Z_i|Z_{-i}) = Y_i^{n^*(Z)}.$$

$\{\mu_i\}_{i\in I}$ **are correct given** $\{C_i\}_{i\in I}$**:** For each $Z \subseteq X$ and $i \in I$, we have $C_{I\setminus\{i\}}(Z) = \bigcap_{j\neq i}(Y_j^{n^*(Z)} \cup Z_{-j}) = Y^{n^*(Z)} \cup \left(\bigcap_{j\neq i}(Z \setminus Y^{n^*(Z)})_j\right)$. By Lemma 11, $\bigcap_{j\neq i}(Z \setminus Y^{n^*(Z)})_j = \emptyset$; hence $C_{I\setminus\{i\}}(Z) = Y^{n^*(Z)} = \mu_i(Z)$, as desired.

$\{C_i, \mu_i\}_{i\in I}$ **is strategically consistent:** For each $Z \subseteq X$ and $i \in I$, $\mu_i(Z) = Y^{n^*(Z)} \in \mathcal{M}$, and so $\hat{C}_i(\mu_i(Z)_i|\mu_i(Z)_{-i}) = Y_i^{n^*(Z)} = C_i(Z_i|Z_{-i})$. Since $\{\mu_i\}_{i\in I}$ are correct given $\{C_i\}_{i\in I}$, strategic consistency follows from Lemma 5.

$\{\mu_i\}_{i\in I}$ **satisfy IRC:** Suppose $\mu_i(Z) \subseteq Z' \subseteq Z$ for some $i \in I$ and $Z', Z \subseteq X$. Since $Z' \subseteq Z$, we have $2^{Z'} \subseteq 2^Z$, so $\{n|Y^n \in 2^{Z'}\} \subseteq \{n|Y^n \in 2^Z\}$, and hence $n^*(Z) \leq n^*(Z')$. Since $\mu_i(Z) = Y^{n^*(Z)} \subseteq Z'$, we have $n^*(Z) \in \{n|Y^n \in 2^{Z'}\}$, and hence $n^*(Z') \leq n^*(Z)$. Then $\mu_i(Z') = Y^{n^*(Z')} = Y^{n^*(Z)} = \mu_i(Z)$, as desired.

$\{C_i, \mu_i\}_{i\in I}$ **satisfies SFI:** Suppose Z is a credible blocking proposal for Y, but $\mu_i(Y \cup Z) = Y$ for each $i \in I$. Then $Z \in \mathcal{M}$, $Z \triangleright Y$, $Z' \subseteq Y$ for each $Z' = C(Z' \cup Z) \nsubseteq Z$, and by Lemma 10 (ii), $C(Y \cup Z) = Y$. Then by construction of $\{C_i, \mu_i\}_{i\in I}$, $Z \neq Y^*$, and since $Y^* \in \overline{\mathcal{M}}$ and $Z \in \mathcal{M}$, $Z \not\supseteq Y^*$.

Since $\{\mu_i\}_{i\in I}$ satisfy IRC, by Theorem 3, C satisfies IRC. Then $C(Y) = Y$, and by Lemma S.1, $Y \in \mathcal{M}$. Further, if $Z' = C(Z' \cup Z) \nsubseteq Z$, we have $Y \cup Z' = Y$, and so $C(Y \cup Z') = C(Y) = Y$.

By construction of $\{C_i, \mu_i\}_{i\in I}$, $Y^* = C(Y^* \cup Z)$; then since $Z \not\supseteq Y^*$, $C(Y \cup Y^*) = Y$. But by construction, $C(Y^* \cup Y) = Y^*$, so we must have $Y = Y^*$. Moreover, since $Z \triangleright Y = Y^*$, and by assumption $Y^* \in \mathcal{M}$, $Z \not\subset Y^*$.

By construction of $\{C_i, \mu_i\}_{i\in I}$, since $Z \in \mathcal{M}$ and $Z \neq Y = Y^*$, we have $Z = Y^k$ for some $k > 1$; since $Z \triangleright Y^*$, $Z \neq Z^*(Y^*) = Y^1$, and so $k > 2$. Then by construction of $Z^*(Y^*)$, $Y^* \nsubseteq Z^*(Y^*) \cup Z$, and so by construction of $\{C_i, \mu_i\}_{i\in I}$, $C(Z \cup Z^*(Y^*)) = Z^*(Y^*)$, and since $Y^*, Z^*(Y^*) \in \overline{\mathcal{M}}$, $Z^*(Y^*) \nsubseteq Y^*$, a contradiction since Z is a credible blocking proposal for $Y = Y^*$. The claim follows. $\square$

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