Online Appendix for "Reallocative Auctions and Core Selection"

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A Quantity Auctioned Comparative Statics

Theorem 1 shows that in a reallocative auction, core selection depends not just on the existence of packages that all bidders view as substitutes, but also on the efficiency of allocating positive quantities of each of those packages to each bidder. This allocation condition depends on the quantity auctioned: When $Q = 0$, for instance, it can only be satisfied if each bidder's pre-auction allocation is the same. And in Example 1 — which shows that Theorem 1's allocation condition, and thus core selection, can fail in a setting with a single good — if the auctioneer had 5 units of the good for sale instead of 1 unit, each bidder would always receive a positive quantity, regardless of which bidders participate.

This section gives conditions under which a change in quantity auctioned Q can restore core selection in the general multiple-good case. First, we describe the set of directions in which a change in Q increases the efficient quantity of packages allocated to a bidder.

Proposition A.1 (Package Allocations and Quantity Auctioned). *Let $\Phi \subset \mathbb{R}^K$ be a set of K linearly independent vectors. As a function of Q, the efficient allocation of packages $T_\Phi^{-1} q_i^e(X, Q)$ to bidder $i \in X$ is increasing in direction $x \in \mathbb{R}^K$ if and only if*

$$x \in C_i(\Phi, X, Q) \equiv \left\{ x \in \mathbb{R}^K \mid T_\Phi^{-1} (D^2 u_i(q_i^e(X,Q)))^{-1} \left(\sum_{j \in X} D^2 u_j(q_j^e(X,Q))^{-1} \right)^{-1} x \geq 0 \right\}.$$

Note that if bidders have quadratic valuations, $C_i(\Phi, X, Q) = \{x \mid T_\Phi^{-1} S_i^{-1} H(X) x \geq 0\}$ for each Q.

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Proposition A.1 yields two conditions on quantity auctioned that are each sufficient for Theorem 1's allocation condition to hold. The first is for settings without heterogeneity in marginal utility at the initial allocation:

Corollary A.1 (Theorem 1's Allocation Condition and Quantity Auctioned I). *Suppose that $\nabla u_i(0) = \nabla u_j(0)$ for all $i, j \in I$. If $Q \in \bigcap_{Z\subseteq I} \bigcap_{x\in\mathbb{R}^K} \bigcap_{i\in Z} C_i(\Phi, Z, x)$, then $T_\Phi^{-1} q_i^e(X, Q) \geq 0$ for each set of participating bidders $X \subseteq I$.*

The second allows heterogeneity in marginal utility at the initial allocation, while requiring that increases in package allocations are uniformly bounded away from zero.

Corollary A.2 (Theorem 1's Allocation Condition and Quantity Auctioned II). *For any $\epsilon > 0$, let*

$$\tilde{C}_i(\Phi, X, Q, \epsilon) \equiv \left\{ x \in \mathbb{R}^K \mid T_\Phi^{-1} (D^2 u_i(q_i^e(X, Q)))^{-1} \left(\sum_{j\in X} D^2 u_j(q_j^e(X, Q))^{-1} \right)^{-1} x \geq \epsilon \mathbb{1} \right\}.$$

For any quantity vector Q and any $\Delta Q \in \bigcap_{Z\subseteq I} \bigcap_{x\in\mathbb{R}^K} \bigcap_{i\in Z} \tilde{C}_i(\Phi, Z, x, \epsilon)$ for some $\epsilon > 0$, there exists a scalar $a > 0$ such that $T_\Phi^{-1} q_i^e(X, Q + a\Delta Q) \geq 0$ for each set of participating bidders $X \subseteq I$.

Proof of Proposition A.1 (Bidder-Submodularity and Quantity Auctioned): Setting $Z = X$, the Kuhn-Tucker conditions for a maximum in (2) are

$$\sum_{i\in X} q_i = Q, \qquad \nabla u_i(q_i) = p \text{ for each } i \in X \text{ for some } p \in \mathbb{R}^K.$$

By definition, these are satisfied by setting $p = p(X, Q)$ and $q_i = d_i(p)$ for each $i \in X$, where $\sum_{i\in X} d_i(p(X, Q)) = Q$. By the implicit function theorem,

$$Dd_i(p) = (D^2 u_i(d_i(p)))^{-1};$$
$$Dp(X, Q) = \left(\sum_{i\in X} Dd_i(p(X, Q)) \right)^{-1} = \left(\sum_{i\in X} (D^2 u_i(d_i(p(X, Q))))^{-1} \right)^{-1} = \left(\sum_{i\in X} (D^2 u_i(q_i^e(X, Q)))^{-1} \right)^{-1}.$$

Then by the chain rule,

$$Dq_i^e(X, Q) = (D^2 u_i(q_i^e(X, Q)))^{-1} \left(\sum_{i\in X} (D^2 u_i(q_i^e(X, Q)))^{-1} \right)^{-1}.$$

The statement follows. □

Proof of Corollary A.2 (Theorem 1's Allocation Condition and Quantity Auctioned II):

From the proof of Proposition A.1 and the fundamental theorem of calculus, we have

$$T_\Phi^{-1} q_i^e(X, Q + a\Delta Q) =$$
$$T_\Phi^{-1} q_i^e(X,Q) + a \int_0^1 T_\Phi^{-1}(D^2 u_i(q_i^e(X, Q + ar\Delta Q)))^{-1} \left(\sum_{j\in X}(D^2 u_j(q_j^e(X, Q + ar\Delta Q)))^{-1}\right)^{-1} \Delta Q dr.$$

Then let

$$a = \frac{\max_{k\in K, i\in Z, Z\subseteq I} -\{T_\Phi^{-1} q_i^e(Z,Q)\}_k}{\epsilon} \geq \frac{\max_{k\in K, i\in Z, Z\subseteq I} -\{T_\Phi^{-1} q_i^e(Z,Q)\}_k}{\inf_{k\in K, \{x_j\}_{j\in Z}\in\mathbb{R}^{|Z|K}, i\in Z, Z\subseteq I}\{T_\Phi^{-1}(D^2 u_i(x_i))^{-1}\left(\sum_{j\in Z}(D^2 u_j(x_j))^{-1}\right)^{-1}\Delta Q\}_k}.$$

It follows that $Tq_i^*(J, Q + a\Delta Q) \geq 0$ for all J and all $i \in J$. $\square$

B Omitted Proofs

To begin, for any $f : \mathbb{R}^K \to \mathbb{R} \cup \{\infty\}$, let $f^* : \mathbb{R}^K \to \mathbb{R} \cup \{\infty\}$ be the convex conjugate of f, i.e., $f^*(p) \equiv \sup_{q\in\mathbb{R}^K}\{p \cdot q - f(q)\}$.

Lemma B.1. *Let $\{f_i : \mathbb{R}^K \to \mathbb{R}\}_{i=1}^N$ be continuously differentiable, strictly convex, and have $\nabla f_i(\mathbb{R}^K) = -M$. Then for each $Q \in \mathbb{R}^K$, $\underline{f}_N(Q) \equiv \inf_{\{q_i\}_{i=1}^N}\{\sum_{i=1}^N f_i(q_i)$ s.t. $\sum_{i=1}^N q_i = Q\}$ has a solution, $\underline{f}_N$ is convex, and $\underline{f}_N = (\sum_{i=1}^N f_i^*)^*$.*

Proof. First note that for any convex, upper semicontinuous $f : \mathbb{R}^K \to \mathbb{R} \cup \{\infty\}$, $f = f^{**}$ (e.g., Hiriart-Urruty and Lemaréchal (2004) Corollary E.1.3.1) and f^* is upper semicontinuous and convex (e.g., Hiriart-Urruty and Lemaréchal (2004) Theorem E.1.1.2).

Moreover, since each f_i is continuously differentiable and strictly convex, for each $p \in \nabla f_i(\mathbb{R}^K)$, there exists a unique q^* such that $\nabla f(q^*) = p$, and so $f_i^*(p) = p \cdot q^* - f(q^*)$; hence, $\nabla f_i(\mathbb{R}^K) \subseteq \operatorname{dom} f_i^*$ for each i.

Now we induct on N. For the initial step, suppose $N = 2$. Since $-M = \nabla f_i(\mathbb{R}^K) \subseteq \operatorname{dom} f_i^*$ for each $i \in I$, $\operatorname{dom} f_1^* \cap \operatorname{dom} f_2^* \neq \emptyset$. Then by Hiriart-Urruty and Lemaréchal (2004) Theorem E.2.3.2,[1] for each $Q \in \mathbb{R}^K$, $\underline{f}_2(Q) = \inf_{q_1,q_2}\{f_1(q_1) + f_1(q_2)$ s.t. $q_1 + q_2 = Q\}$ has a solution $(q_1^*(Q), q_2^*(Q))$, and $\underline{f}_2(Q) = (f_1^* + f_2^*)^*(Q)$ Then since it is the conjugate of the convex, upper semicontinuous function $f_1^* + f_2^*$, $\underline{f}_2$ is upper semicontinuous and convex.

Now suppose that the claim holds for $N-1$: for each $Q \in \mathbb{R}^K$, $\underline{f}_{N-1}(Q) \equiv \inf_{\{q_i\}_{i=1}^{N-1}}\{\sum_{i=1}^{N-1} f_i(q_i)$ s.t. $\sum$ $Q\}$ has a solution $\{q_i^*(Q)\}_{i=1}^{N-1}$, $\underline{f}_{N-1}$ is convex, and $\underline{f}_{N-1} = (\sum_{i=1}^{N-1} f_i^*)^*$. Since $\underline{f}_{N-1} = (\sum_{i=1}^{N-1} f_i^*)^*$, $\operatorname{dom} \underline{f}_{N-1}^* = \bigcap_{i=1}^{N-1} \operatorname{dom} f_i^* \supseteq -M$. Moreover, observe that we can write $\underline{f}_N(Q) = \inf_{q_{-N},q_N}\{f_{N-1}(q_{-N}) + f_N(q_N)$ s.t. $q_{-N} + q_N = Q\}$. Then by Hiriart-Urruty and Lemaréchal (2004) Theorem E.2.3.2, for each $Q \in \mathbb{R}^K$, $\inf_{q_{-N},q_N}\{\underline{f}_{N-1}(q_{-N}) + f_N(q_N)$ s.t. $q_{-N} +$

[1] Take $g_i = f_i^*$.

$q_N = Q\}$ has a solution $(q^*_{-N}(Q), q^*_N(Q))$, and $\underline{f}_N(Q) = (\underline{f}^*_{N-1} + f^*_N)^*(Q)$; since $\underline{f}_{N-1} = (\sum_{i=1}^{N-1} f_i^*)^*$, we have $\underline{f}_N(Q) = (\sum_{i=1}^{N-1} f_i^* + f_N^*)^*(Q)$. Then since it is the conjugate of the convex, upper semicontinuous function $\sum_{i=1}^N f_i^*$, $\underline{f}_N$ is upper semicontinuous and convex. Moreover, $\underline{f}_N(Q) \equiv \inf_{\{q_i\}_{i=1}^N} \{\sum_{i=1}^N f_i(q_i) \text{ s.t. } \sum_{i=1}^N q_i = Q\}$ has a solution given by $\{\{q_i^*(q^*_{-N}(Q))\}_{i=1}^{N-1}, q_N^*(Q)\}$.

The claim follows by induction. □

Lemma B.2 (Existence of Market-Clearing Prices)**.** *For each $i \in X$, let $b_i : \mathbb{R}^K \to M$ be continuously differentiable, surjective, and have a negative definite Jacobian derivative matrix. Then*

i. Each b_i is bijective, and its inverse b_i^{-1} is continuously differentiable with a negative definite Jacobian.

ii. $d(p) \equiv \sum_{i \in X} b_i^{-1}(p)$ is a bijective map from M to $\mathbb{R}^K$, and so there is a unique $p^*(b)$ such that $d(p^*(b)) = Q$.

Proof. (i): Suppose toward a contradiction that b_i is not injective. Then there exist distinct $q, q' \in \mathbb{R}^K$ such that $b_i(q) = b_i(q')$. Since b_i is C^1, by the fundamental theorem of calculus, $b_i(q') = b_i(q) + \int_0^1 Db_i(rq' + (1-r)q)(q' - q)dr$. Then $0 = \int_0^1 Db_i(rq' + (1-r)q)(q' - q)dr$ and hence $0 = \int_0^1 (q' - q)' Db_i(rq' + (1-r)q)(q' - q)dr$, a contradiction because the integrand is strictly negative for each r.

Then b_i is bijective, and so has an inverse b_i^{-1}; existence, continuity, and negative definiteness of the Jacobian Db_i^{-1} follows from the inverse function theorem.

(ii): First suppose toward a contradiction that d is not injective. Then there exist distinct $p, p' \in M$ such that $d(p) = d(p')$. By part (i), d is C^1 and has a negative definite Jacobian matrix. Then by the fundamental theorem of calculus, $d(p') = d(p) + \int_0^1 Dd(rp' + (1-r)p)(p' - p)dr$. Then $0 = \int_0^1 Dd(rp' + (1-r)p)(p' - p)dr$ and hence $0 = \int_0^1 (p' - p)' Dd(rp' + (1-r)p)(p' - p)dr$, a contradiction because the integrand is strictly negative for each r.

We now show that d is surjective. For each $i \in X$, let $f_i(q_i) = -\int_0^1 b_i(rq_i) \cdot q_i dr$; then f_i is twice continuously differentiable with positive definite Hessian derivative matrix $D^2 f_i(q_i) = -Db_i(q_i)$. Then f_i is strictly convex, and $\nabla f_i(\mathbb{R}^K) = -b_i(\mathbb{R}^K) = -M$.

Then by Lemma B.1, for each $q \in \mathbb{R}^K$, $\inf_{\{q_i\}_{i \in X}} \{\sum_{i \in X} f_i(q_i) \text{ s.t. } \sum_{i \in X} q_i = q\}$ has a solution $\{q_i^*(q)\}_{i \in X}$. Since the f_i are strictly convex, this solution must satisfy a necessary Kuhn-Tucker condition: there exists $p(q) \in \mathbb{R}^K$ such that $-b_i(q_i^*(q)) = \nabla f_i(q_i^*(q)) = p(q)$ for each $i \in X$. It follows that for each $q \in \mathbb{R}^K$, there exists $p = -p(q)$ such that $q = d(p)$, and so d is surjective. □

Lemma B.3 (Existence and Uniqueness of Efficient Allocations)**.** *The social planner's problem* (1) *has a unique solution $\{q_i^e(Z, Q)\}_{i \in Z}$ for each $Z \subseteq I$.*

Proof. Existence of a solution follows immediately from Lemma B.1 by letting $f_i = -u_i$. Uniqueness follows from strict convexity of the u_i. $\square$

Proof of Lemma 7 (Eigenvalues and Harmonic Means) We have

$$\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1} = \sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1} + S_\ell^{-1}$$

$$\Rightarrow \frac{1}{2}I = \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right) + \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}$$

Left-multiplying by $S_j^{-1} = \sum_{i\in Z\setminus\{\ell\}} S_i^{-1} - \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}$ yields

$$\begin{aligned}
\frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} - \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right) =& \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\\
&- \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\\
&+ \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\\
&- \frac{1}{2}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1} + \sum_{i\in Z} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\\
&+ \frac{1}{2}S_j^{-1}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1} S_\ell^{-1}
\end{aligned}$$

Since the $\{S_i^{-1}\}_{i\in I}$ are symmetric, adding the transpose of both sides yields

$$\begin{aligned}
\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} - \sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1} =& \left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1}\right)\\
&- \left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1} + \sum_{i\in Z} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell,j\}} S_i^{-1}\right)\\
&+ \frac{1}{2}S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1} + \frac{1}{2}S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1},
\end{aligned}$$

Then from the harmonic mean identity $(A^{-1}+B^{-1})^{-1} = A - A(A+B)^{-1}A$, we have

$$(H(Z\setminus \ell)+H(Z\setminus j))^{-1} = (H(Z)+H(Z\setminus\{\ell,j\}))^{-1} + \frac{1}{2}S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1}$$
$$+ \frac{1}{2}S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}.$$

Since $A \succeq B$ iff $B^{-1} \succeq A^{-1}$,[2] it follows that $H(Z)+H(Z\setminus\{\ell,j\}) \succeq H(Z\setminus\{\ell\})+H(Z\setminus\{j\})$ iff $\frac{1}{2}S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1} + \frac{1}{2}S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}$ is positive semidefinite. But since $S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1}$ has a negative eigenvalue, this cannot hold: if x is the eigenvector associated with the negative eigenvalue, $x'\left(S_\ell^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_j^{-1} + S_j^{-1}\left(\sum_{i\in Z\setminus\{\ell\}} S_i^{-1} + \sum_{i\in Z\setminus\{j\}} S_i^{-1}\right)^{-1} S_\ell^{-1}\right)x < 0$.

C Imperfect Divisibility

For settings with imperectly divisible goods, the following result gives a converse analogous to Proposition 1 to show that the effects due to two-sidedness – particularly those illustrated in Example 1 – are not unique to divisible goods. In particular, if bidders are endowed with the same preferences, there is some pre-auction allocation for which core selection fails.

Proposition C.1 (No Core Selection with Indivisible Goods). *Suppose allocations are constrained to be integer vectors. If $|I| \geq 4$, then for any strictly concave valuation u which has a unimodular demand type (in the sense of Baldwin and Klemperer (2019)) when restricted to $q_i \in \mathbb{Z}^K$, there exists a pre-auction allocation $\{t_i\}_{i\in I} \subseteq \mathbb{Z}^K$ such that if each bidder $i \in X$ is endowed with with the valuation $\tilde{u}_i(q_i) = u(q_i + t_i)$, there exists a set of participating bidders $X \subseteq I$ such that the Vickrey auction is not core-selecting.*

Proof of Proposition C.1 (No Core Selection with Indivisible Goods): By Baldwin and Klemperer (2019)'s Unimodularity Theorem, a competitive equilibrium exists in the exchange economy consisting of any set of participating bidders and any quantity Q. Consequently, if p is a market-clearing price for the exchange economy consisting of bidders Z and quantity Q, we have $q_i^e(Z,Q) \in d_i(p)$ for each $i \in Z$ for some efficient allocation $\{q_i^e(Z,Q)\}_{i\in Z}$.

Choose some integer vector $z \leq Q - 4\cdot\mathbf{1}$. Now choose $u_2(x) = u_3(x) = u_1(x+2z-Q) - u_1(2z-Q)$ and $u_4(x) = u_1(x+z) - u_1(z)$.

[2] $\succeq$ denotes the positive definite order.

For coalitions $\{1,2\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}$ any price in the subdifferential of $u_1(z)$ clears the market. It follows that the allocations $\{z, Q-z\}, \{z, Q-z\}, \{z, Q-z, 0\}, \{z, Q-z, 0\}$ are efficient when the set of participating agents is, respectively, $\{1,2\}, \{1,3\}, \{1,2,4\}$, or $\{1,3,4\}$. Consequently, $v(Z, Q)$ is the same for each $Z \in \{\{1,2\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}\}$.

When the set of participating agents is $\{1,2,3\}$, since $z + 2(Q-z) > Q + 3 \cdot \mathbf{1}$, some agent must receive a quantity of some good at least 2 units smaller than he did in any of the two-person coalitions he was a member of. It follows from strict concavity that the new market-clearing price p' is such that $z \notin q_1(p') = q_2(p') + (2z - Q) = q_3(p') + (2z - Q) = q_4(p') + z$. Since $0 \notin q_4(p')$, it follows that in the social planner's problem (1) for $Z = \{1,2,3,4\}$, allocating 0 to agent 4 is strictly suboptimal, and hence that $v(\{1,2,3,4\}, Q) > v(\{1,2,3\}, Q)$. Then $v(\{1,2\}, Q) > v(\{1,2,3\}, Q) + v(\{1,2,4\}, Q) - v(\{1,2,3,4\}, Q) = \pi_0 + \pi_1 + \pi_2$ and so $\{0,1,2\}$ blocks the Vickrey payoff profile. $\square$

References

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