BHOMANI PROBLEMS

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Abstract. Below are compiled solutions to two olympiad math problems sent by Aariz Bhamani.

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Problem 1. *Determine all positive integers $n \geq 3$ for which;*

$$\frac{(n-1)^{n-1} - n^2 + 291(n-1)}{(n-2)^2}$$

is an integer.

Solution. Note: Here, $a \mid b$ means that a divides b, or b is a multiple of a. Also, $a \equiv b \pmod{c}$ means a and b have the same remainder when divided by c. First, we let $k = n-2$. Since $n \geq 3$, we must have $k = n-2 \geq 1$, we get that;

$$\frac{(n-1)^{n-1} - n^2 + 291(n-1)}{(n-2)^2} = \frac{(k+1)^{k+1} - (k+2)^2 + 291(k+1)}{k^2}.$$

We can rearrange and simplify this as;

$$\begin{aligned}
\frac{(k+1)^{k+1} - (k+2)^2 + 291(k+1)}{k^2} &= \frac{(k+1)^{k+1} + 291(k+1) - (k+2)^2}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - (k+2)^2}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - k^2 - 4k - 4}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - k^2 - 4(k+1)}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - 4(k+1)}{k^2} - 1 \\
&= \frac{(k+1)((k+1)^k + 291 - 4)}{k^2} - 1 \\
&= \frac{(k+1)((k+1)^k + 287)}{k^2} - 1.
\end{aligned}$$

This expression is an integer. So, clearly, k^2 divides $(k+1)((k+1)^k + 287)$. Since k^2 and $k+1$ are coprime, we have k^2 divides $(k+1)^k + 287$. But;

$$(k+1)^k = \binom{k}{0}k^k + \binom{k}{1}k^{k-1} + \cdots + \binom{k}{k-2}k^2 + \binom{k}{k-1}k + 1 \equiv 1 \pmod{k^2}.$$

So clearly, $287 \equiv -(k+1)^k \pmod{k^2} \equiv -1 \pmod{k^2}$, so $k^2 \mid 288$.

But, the divisors of 288 are;

$$1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144, 288.$$

So $k = 1, 2, 3, 4,$ or 6. $\square$

Problem 2. *A sequence is defined by $t_1 = 1$ and $t_2 = 2$ and $t_n = \frac{kt_{n-1}+1}{k^2 t_{n-2}}$ for $n \geq 3$, where k is a positive integer. Determine t_{2024} (possibly in terms of k).*

Solution. We manually compute the first few terms of the sequence in terms of k;

$$\begin{aligned}
t_1 &= 1 \\
t_2 &= 2 \\
t_3 &= \frac{2k+1}{k^2} \\
t_4 &= \frac{k\left(\frac{2k+1}{k^2}\right)+1}{2k^2} \\
&= \frac{3k+1}{2k^3} \\
t_5 &= \frac{k\left(\frac{3k+1}{2k^3}\right)+1}{k^2 \times \frac{2k+1}{k^2}} \\
&= \frac{2k^2+3k+1}{2k^2(2k+1)} \\
&= \frac{k+1}{2k^2} \\
t_6 &= \frac{k\left(\frac{k+1}{2k^2}\right)+1}{k^2 \times \frac{3k+1}{2k^3}} \\
&= \frac{2k+k+1}{2k\left(\frac{3k+1}{2k}\right)} \\
&= \frac{3k+1}{3k+1} \\
&= 1 \\
t_7 &= \frac{k+1}{k^2 \times \frac{k+1}{2k^2}} \\
&= \frac{1}{2}
\end{aligned}
\qquad
\begin{aligned}
t_8 &= \frac{\frac{k}{2}+1}{k^2} \\
&= \frac{k+2}{2k^2} \\
t_9 &= \frac{k\left(\frac{k+2}{2k^2}\right)+1}{k^2\left(\frac{1}{2}\right)} \\
&= \frac{3k+2}{k^3} \\
t_{10} &= \frac{k\left(\frac{3k+2}{k^3}\right)+1}{k^2\left(\frac{k+2}{2k^2}\right)} \\
&= \frac{2(k^2+3k+2)}{k^2(k+2)} \\
&= \frac{2k+2}{k^2} \\
t_{11} &= \frac{k\left(\frac{2k+2}{k^2}\right)+1}{k^2\left(\frac{3k+2}{k^3}\right)} \\
&= \frac{2(k^2+3k+2)}{k^2(k+2)} \\
&= \frac{3k+2}{3k+2} \\
&= 1 \\
t_{12} &= \frac{k+1}{k^2\left(\frac{2k+2}{k^2}\right)} \\
&= \frac{k+1}{2(k+1)} \\
&= \frac{1}{2}.
\end{aligned}$$

Since $t_6 = t_{11}$ and $t_7 = t_{12}$, we can see that the sequence cycles after t_6, and takes the values;

$$\begin{cases} \frac{2k+2}{k^2} & x \equiv 0 \pmod 5 \\ 1 & x \equiv 1 \pmod 5 \\ \frac{1}{2} & x \equiv 2 \pmod 5 \\ \frac{k+2}{2k^2} & x \equiv 3 \pmod 5 \\ \frac{3k+2}{k^3} & x \equiv 4 \pmod 5. \end{cases}$$

Since $2024 \equiv 4 \pmod 5$, we get;

$$t_{2024} = \frac{3k+2}{k^3}.$$

□

Problem 3. *The set S contains 18 distinct (but unknown) four-digit positive integers. Pavak calculates the sum of the elements in each non-empty subset of S. Prove that it is impossible for all of these sums to be distinct.*

Proof. The number of subsets of S is; 2^{18} including the empty set. So the number of non-empty subsets of S is $2^{18} - 1 = 262143$. So there are total 262143 sums that Pavak calculates.

Each of these sums is an integer from

$$1000 + 1001 + \cdots + 1017 = 18153$$

$$\text{to } 9982 + 9982 + \cdots + 9999 = 179829.$$

So each of the sums is one of the numbers; $\{18153, 18154, \ldots, 179829\}$. So there are total $179829 - 18153 + 1 = 161677$ possible values of each sum.

Now, there are 262143 sums from a range of 161677 possible values, so by the Pigeonhole Principle, there must be two sums that are equal, completing the proof. □