

BHOMANI PROBLEMS

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ABSTRACT. Below are compiled solutions to two olympiad math problems sent by Aariz Bhamani.

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Problem 1. Determine all positive integers $n \geq 3$ for which;

$$\frac{(n-1)^{n-1} - n^2 + 291(n-1)}{(n-2)^2}$$

is an integer.

Solution. Note: Here, $a \mid b$ means that a divides b , or b is a multiple of a . Also, $a \equiv b \pmod{c}$ means a and b have the same remainder when divided by c . First, we let $k = n - 2$. Since $n \geq 3$, we must have $k = n - 2 \geq 1$, we get that;

$$\frac{(n-1)^{n-1} - n^2 + 291(n-1)}{(n-2)^2} = \frac{(k+1)^{k+1} - (k+2)^2 + 291(k+1)}{k^2}.$$

We can rearrange and simplify this as;

$$\begin{aligned} \frac{(k+1)^{k+1} - (k+2)^2 + 291(k+1)}{k^2} &= \frac{(k+1)^{k+1} + 291(k+1) - (k+2)^2}{k^2} \\ &= \frac{(k+1)((k+1)^k + 291) - (k+2)^2}{k^2} \\ &= \frac{(k+1)((k+1)^k + 291) - k^2 - 4k - 4}{k^2} \\ &= \frac{(k+1)((k+1)^k + 291) - k^2 - 4(k+1)}{k^2} \\ &= \frac{(k+1)((k+1)^k + 291) - 4(k+1)}{k^2} - 1 \\ &= \frac{(k+1)((k+1)^k + 291 - 4)}{k^2} - 1 \\ &= \frac{(k+1)((k+1)^k + 287)}{k^2} - 1. \end{aligned}$$

This expression is an integer. So, clearly, k^2 divides $(k+1)((k+1)^k + 287)$. Since k^2 and $k+1$ are coprime, we have k^2 divides $(k+1)^k + 287$. But;

$$(k+1)^k = \binom{k}{0}k^k + \binom{k}{1}k^{k-1} + \cdots + \binom{k}{k-2}k^2 + \binom{k}{k-1}k + 1 \equiv 1 \pmod{k^2}.$$

So clearly, $287 \equiv -(k+1)^k \pmod{k^2} \equiv -1 \pmod{k^2}$, so $k^2 \mid 288$.
But, the divisors of 288 are;

1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144, 288.

So $k = 1, 2, 3, 4,$ or 6 .

□

Problem 2. A sequence is defined by $t_1 = 1$ and $t_2 = 2$ and $t_n = \frac{kt_{n-1}+1}{k^2t_{n-2}}$ for $n \geq 3$, where k is a positive integer. Determine t_{2024} (possibly in terms of k).

Solution. We manually compute the first few terms of the sequence in terms of k ;

$$\begin{aligned}
t_1 &= 1 & t_8 &= \frac{\frac{k}{2} + 1}{k^2} \\
t_2 &= 2 & &= \frac{k + 2}{2k^2} \\
t_3 &= \frac{2k + 1}{k^2} & t_9 &= \frac{k \left(\frac{k+2}{2k^2} \right) + 1}{k^2 \left(\frac{1}{2} \right)} \\
t_4 &= \frac{k \left(\frac{2k+1}{k^2} \right) + 1}{2k^2} & &= \frac{3k + 2}{k^3} \\
&= \frac{3k + 1}{2k^3} & t_{10} &= \frac{k \left(\frac{3k+2}{k^3} \right) + 1}{k^2 \left(\frac{k+2}{2k^2} \right)} \\
t_5 &= \frac{k \left(\frac{3k+1}{2k^3} \right) + 1}{k^2 \times \frac{2k+1}{k^2}} & &= \frac{2(k^2 + 3k + 2)}{k^2(k + 2)} \\
&= \frac{2k^2 + 3k + 1}{2k^2(2k + 1)} & &= \frac{2k + 2}{k^2} \\
&= \frac{k + 1}{2k^2} & t_{11} &= \frac{k \left(\frac{2k+2}{k^2} \right) + 1}{k^2 \left(\frac{3k+2}{k^3} \right)} \\
t_6 &= \frac{k \left(\frac{k+1}{2k^2} \right) + 1}{k^2 \times \frac{3k+1}{2k^3}} & &= \frac{2(k^2 + 3k + 2)}{k^2(k + 2)} \\
&= \frac{2k + k + 1}{2k \left(\frac{3k+1}{2k} \right)} & &= \frac{3k + 2}{3k + 2} \\
&= \frac{3k + 1}{3k + 1} & &= 1 \\
&= 1 & t_{12} &= \frac{k + 1}{k^2 \left(\frac{2k+2}{k^2} \right)} \\
t_7 &= \frac{k + 1}{k^2 \times \frac{k+1}{2k^2}} & &= \frac{k + 1}{2(k + 1)} \\
&= \frac{1}{2} & &= \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
t_8 &= \frac{\frac{k}{2} + 1}{k^2} \\
&= \frac{k + 2}{2k^2} \\
t_9 &= \frac{k \left(\frac{k+2}{2k^2} \right) + 1}{k^2 \left(\frac{1}{2} \right)} \\
&= \frac{3k + 2}{k^3} \\
t_{10} &= \frac{k \left(\frac{3k+2}{k^3} \right) + 1}{k^2 \left(\frac{k+2}{2k^2} \right)} \\
&= \frac{2(k^2 + 3k + 2)}{k^2(k + 2)} \\
&= \frac{2k + 2}{k^2} \\
t_{11} &= \frac{k \left(\frac{2k+2}{k^2} \right) + 1}{k^2 \left(\frac{3k+2}{k^3} \right)} \\
&= \frac{2(k^2 + 3k + 2)}{k^2(k + 2)} \\
&= \frac{3k + 2}{3k + 2} \\
&= 1 \\
t_{12} &= \frac{k + 1}{k^2 \left(\frac{2k+2}{k^2} \right)} \\
&= \frac{k + 1}{2(k + 1)} \\
&= \frac{1}{2}.
\end{aligned}$$

Since $t_6 = t_{11}$ and $t_7 = t_{12}$, we can see that the sequence cycles after t_6 , and takes the values;

$$\begin{cases} \frac{2k+2}{k^2} & x \equiv 0 \pmod{5} \\ 1 & x \equiv 1 \pmod{5} \\ \frac{1}{2} & x \equiv 2 \pmod{5} \\ \frac{k+2}{2k^2} & x \equiv 3 \pmod{5} \\ \frac{3k+2}{k^3} & x \equiv 4 \pmod{5}. \end{cases}$$

Since $2024 \equiv 4 \pmod{5}$, we get;

$$t_{2024} = \frac{3k + 2}{k^3}.$$

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