BHOMANI PROBLEMS

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Abstract. Below are compiled solutions to two olympiad math problems sent by Aariz Bhamani.

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Problem 1. *Determine all positive integers $n \geq 3$ for which;*

$$\frac{(n-1)^{n-1} - n^2 + 291(n-1)}{(n-2)^2}$$

is an integer.

Solution. Note: Here, $a \mid b$ means that a divides b, or b is a multiple of a. Also, $a \equiv b \pmod{c}$ means a and b have the same remainder when divided by c. First, we let $k = n-2$. Since $n \geq 3$, we must have $k = n-2 \geq 1$, we get that;

$$\frac{(n-1)^{n-1} - n^2 + 291(n-1)}{(n-2)^2} = \frac{(k+1)^{k+1} - (k+2)^2 + 291(k+1)}{k^2}.$$

We can rearrange and simplify this as;

$$\begin{aligned}
\frac{(k+1)^{k+1} - (k+2)^2 + 291(k+1)}{k^2} &= \frac{(k+1)^{k+1} + 291(k+1) - (k+2)^2}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - (k+2)^2}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - k^2 - 4k - 4}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - k^2 - 4(k+1)}{k^2} \\
&= \frac{(k+1)((k+1)^k + 291) - 4(k+1)}{k^2} - 1 \\
&= \frac{(k+1)((k+1)^k + 291 - 4)}{k^2} - 1 \\
&= \frac{(k+1)((k+1)^k + 287)}{k^2} - 1.
\end{aligned}$$

This expression is an integer. So, clearly, k^2 divides $(k+1)((k+1)^k + 287)$. Since k^2 and $k+1$ are coprime, we have k^2 divides $(k+1)^k + 287$. But;

$$(k+1)^k = \binom{k}{0}k^k + \binom{k}{1}k^{k-1} + \cdots + \binom{k}{k-2}k^2 + \binom{k}{k-1}k + 1 \equiv 1 \pmod{k^2}.$$

So clearly, $287 \equiv -(k+1)^k \pmod{k^2} \equiv -1 \pmod{k^2}$, so $k^2 \mid 288$.

But, the divisors of 288 are;

$$1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144, 288.$$

So $k = 1, 2, 3, 4,$ or 6. $\square$

Problem 2. *A sequence is defined by $t_1 = 1$ and $t_2 = 2$ and $t_n = \frac{kt_{n-1}+1}{k^2 t_{n-2}}$ for $n \geq 3$, where k is a positive integer. Determine t_{2024} (possibly in terms of k).*

Solution. $\square$