PSTAT 194: Homework 4:
Chapter 6 Monte Carlo Methods in Inference

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Please refer to the **detailed homework policy document** on Course Page for information about homework formatting, submission, and grading.

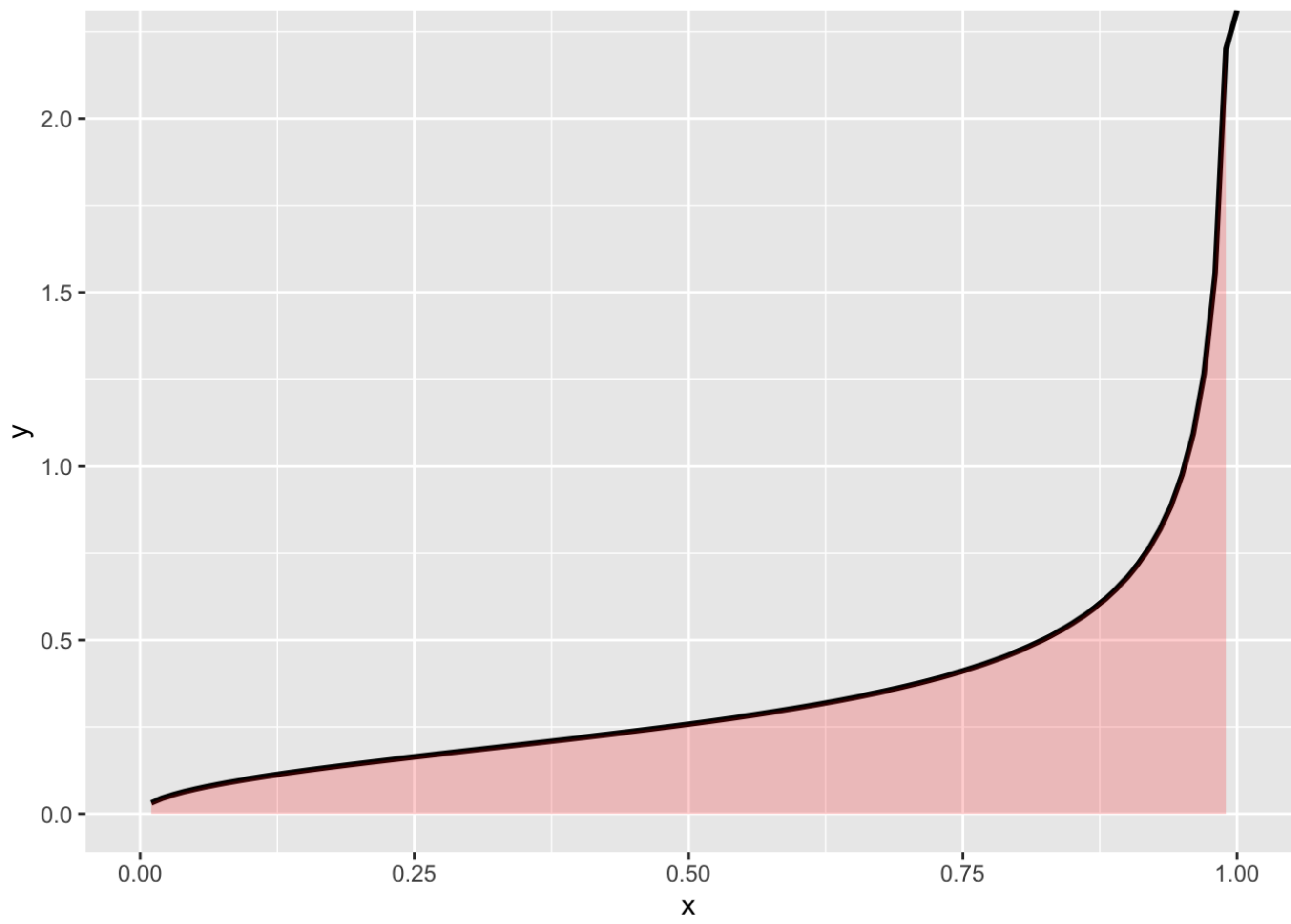
Exercise 1

An Exploration of Standard Error in Monte Carlo Estimation

Consider the following integral:

$$\int_0^1 \frac{\ln(x+1)}{\pi\sqrt{x(1-x)}}dx.$$

a. Estimate the integral using naive Monte Carlo. What is the standard error of this estimate?




```
## [1] 0.3411
```

```
## [1] 3.411e-06
```

```
## [1] 0.3413
```

b. Let's see if we can improve the standard error. Implement Monte Carlo with antithetic sampling to estimate this integral. What is the standard error of this estimate?

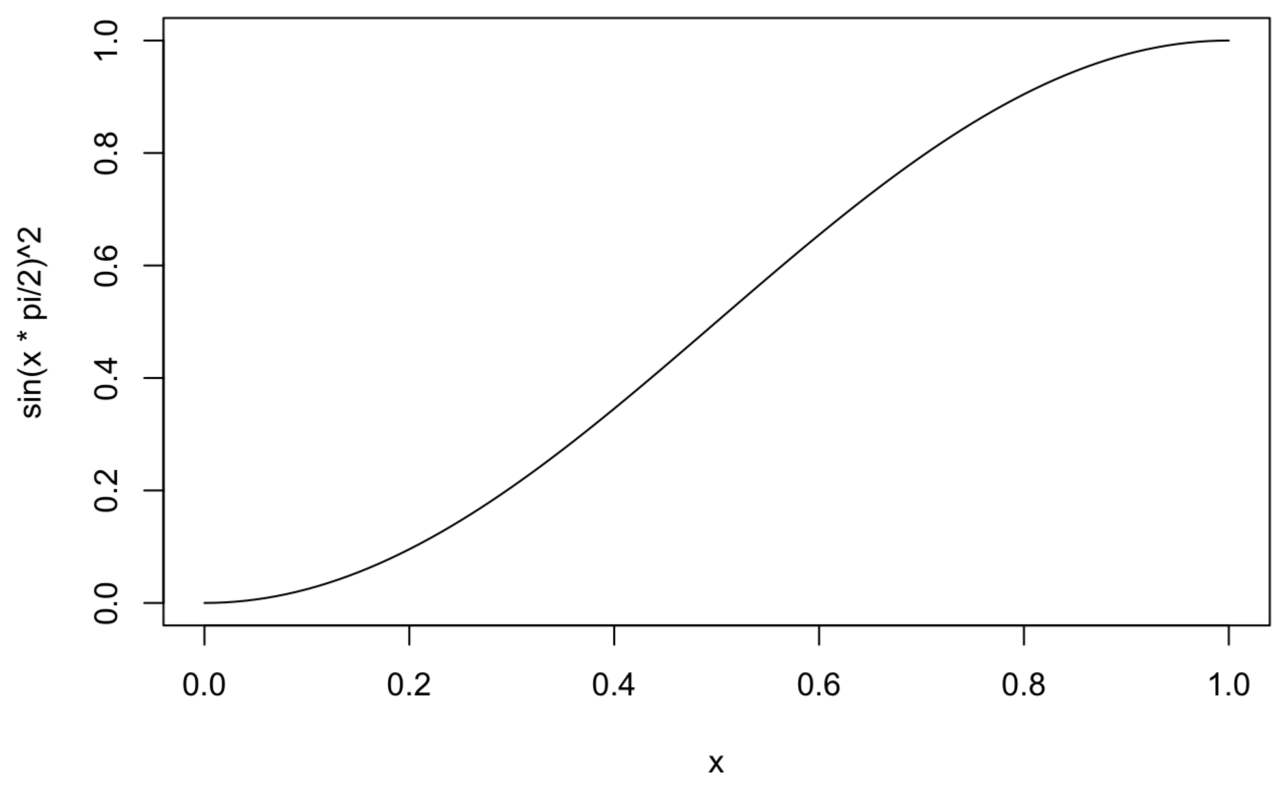
```
## [1] NA
```

c. Would stratified sampling seem to help here? Why or why not? (Whatever you decide, you do not need to implement it).

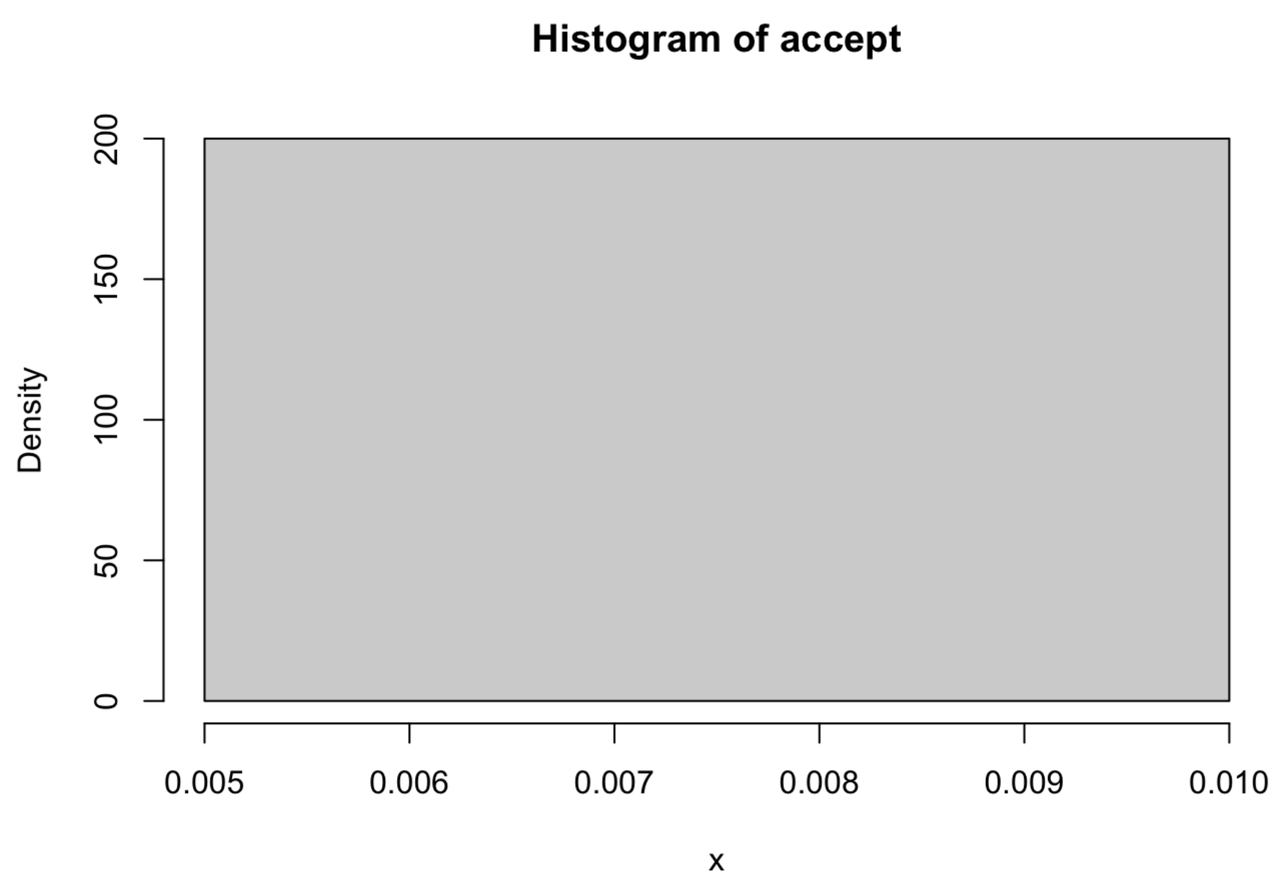
Stratified sampling would not seem to help here because it is used in a case where variance is high and is used to reduce variance by splitting the the interval into strata and then integrating based on each strata. Since the interval is 0 to 1 there really is no use for stratified sampling in this case.

d. $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ for $x \in (0, 1)$ is the probability density function for the Arcsine distribution (https://en.wikipedia.org/wiki/Arcsine_distribution). Using inverse transformation method, sample 1000 random values from the Arcsine distribution.

In order to use the inverse transformation method we need to find the inverse of the CDF.




```
## [1] 0.006156
```




```
## [1] 1.368e-06
```

e. Use importance sampling and the code you wrote in part d to estimate this integral. What is the standard error?

f. Are all methods equally effective? Which method is the most efficient?

The method that seems to be the most effective is stratified sampling.

Exercise 2

Comparing MSE of estimators using MC.

Let $f(x|\theta) = \mathrm{t}(\nu, \mu)$, the non-central t-distribution (https://en.wikipedia.org/wiki/Noncentral_t-distribution), where μ is a location parameter and ν is the degrees of freedom.

Estimate the MSE of the level k trimmed means for random samples of size 20 generated from a a non-central t-distribution with degrees of freedom 3 and mean 4 (with $\nu = 2$ and $\mu = 3$). Summarize the estimates of MSE in a table for k = 1, 2, ... , 9.

```
##    k t-mean      se
## 1  0  31.22 0.02780
## 2  1  27.51 0.02131
## 3  0   0.00 0.00000
## 4  1   0.00 0.00000
## 5  0   0.00 0.00000
## 6  1   0.00 0.00000
## 7  0   0.00 0.00000
## 8  1   0.00 0.00000
## 9  0   0.00 0.00000
## 10 1   0.00 0.00000
## 11 0   0.00 0.00000
## 12 1   0.00 0.00000
```

Exercise 3

Bayesian Statistics Suppose $X_1, \dots, X_n$ are n independent and identical distributed random variables from $Exp(\theta)$, where θ is the unknown parameter. So,

$$f(x|\theta) = \theta\ e^{-\theta x}, \quad x \geq 0.$$

We assume the prior distribution on θ is the Gamma distribution ($Gamma(3, 2)$).

$$g(\theta) = 4\theta^2 e^{-2\theta}, \quad x \geq 0.$$

1. Write down the posterior distribution of θ, $g(\theta|X)$.

For some reason my file will not knit due to my latex code so for the pdf I am putting the LaTex into comments.

The likelihood function is

$f(x|\theta) = f(X1 = x1, \dots, Xn = xn|\theta) = \pi\theta e^-\theta xi = \theta^n e^-\theta$

Then by Bayes' Theroem

$g(\theta|x) \sim f(x|\theta)g(\theta)$

$ =$ $ ^n e^-_{i=1}^{n}xi 4^2 e^-2$

$\theta^n + 2e^- \theta(2 + \sum_{i=1}^n xi)$

This is proportional to a Gamma Distribution, thus $\theta|x\ Gamma(n + 3, 2 + \sum_{i=1}^n xi)$, and

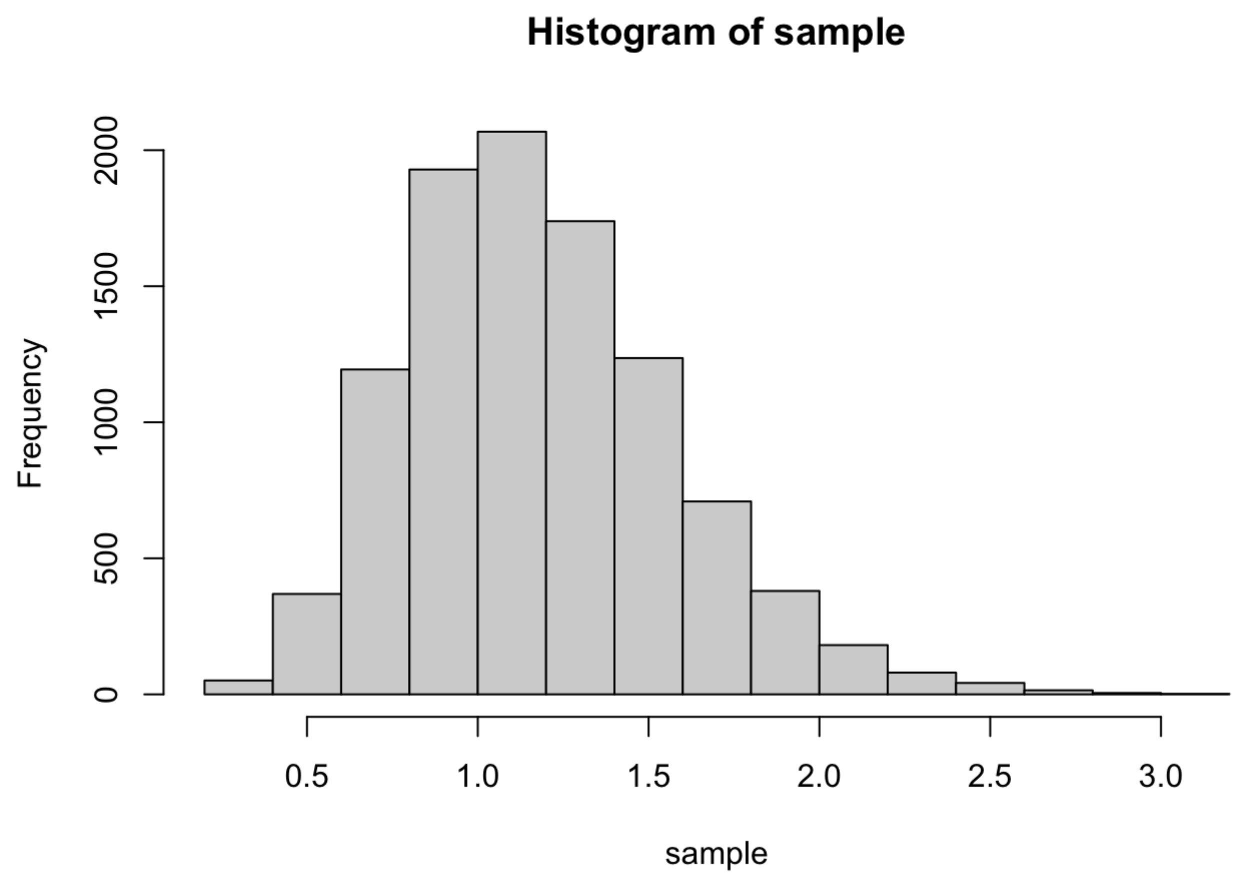
$g(\theta|x) = [(2 + \sum_{i=1}^n xi)^n + 3/\Gamma(n + 3)]\theta^n + 2e^- \theta(2 + \sum_{i=1}^n xi)$

2. Suppose $n = 6$ and we observe that $x_1 = 0.4, x_2 = 1.1, x_3 = 0.2, x_4 = 1.6, x_5 = 1.4, x_6 = 0.9$. Estimate the posterior mean of θ based on 1000 simulated θ from its prior distribution.

```
## [1] 1.184
```

```
## [1] 1.188
```

3. Suppose $n = 6$ and we observe that $x_1 = 0.4, x_2 = 1.1, x_3 = 0.2, x_4 = 1.6, x_5 = 1.4, x_6 = 0.9$.
   a. Design an acceptance-rejection sampling algorithm to generate 1000 (accepted) samples of θ from the posterior distribution of θ. Write down your algorithm with your instrumental distribution/proposal distribution $g(\theta)$. (Hint: for the acceptance-rejection sampling method, the normalizing constant in the posterior distribution can be ignored.)



   b. Implement your acceptance-rejection sampling algorithm with R code. Plot the histogram of your generated sample and compare your sample mean with your estimated posterior mean obtained in Ex.3.2.

```
## [1] 1.182
```

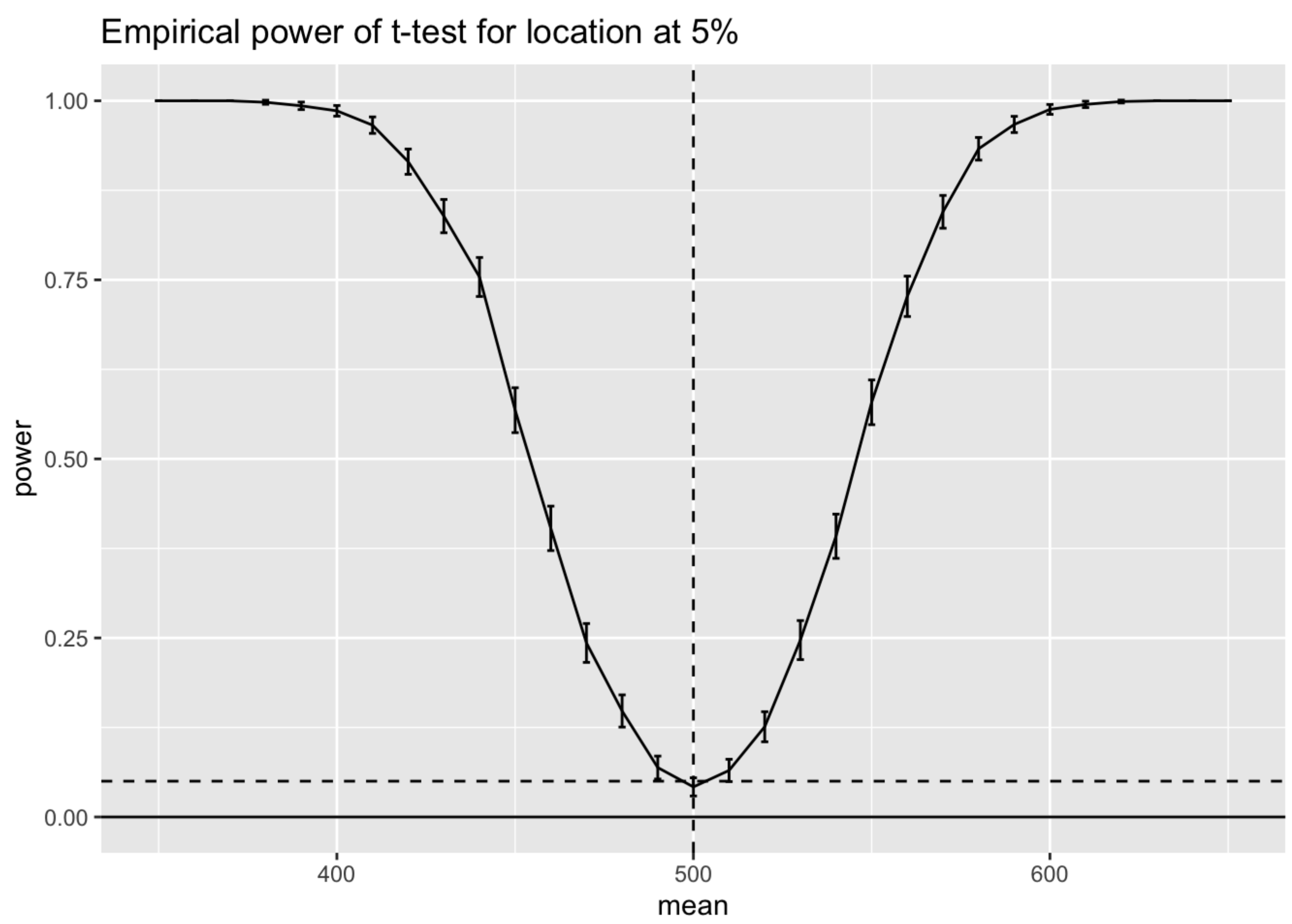
Exercise 4

Do 6.1 in the book, except with $n = 25$ and $k = 1, 2, \cdots, 10$.

```
##    k  t-mean      se
## 1  0 564.404 0.75116
## 2  1   9.202 0.09581
## 3  0   0.000 0.00000
## 4  1   0.000 0.00000
## 5  0   0.000 0.00000
## 6  1   0.000 0.00000
## 7  0   0.000 0.00000
## 8  1   0.000 0.00000
## 9  0   0.000 0.00000
## 10 1   0.000 0.00000
## 11 0   0.000 0.00000
## 12 1   0.000 0.00000
```

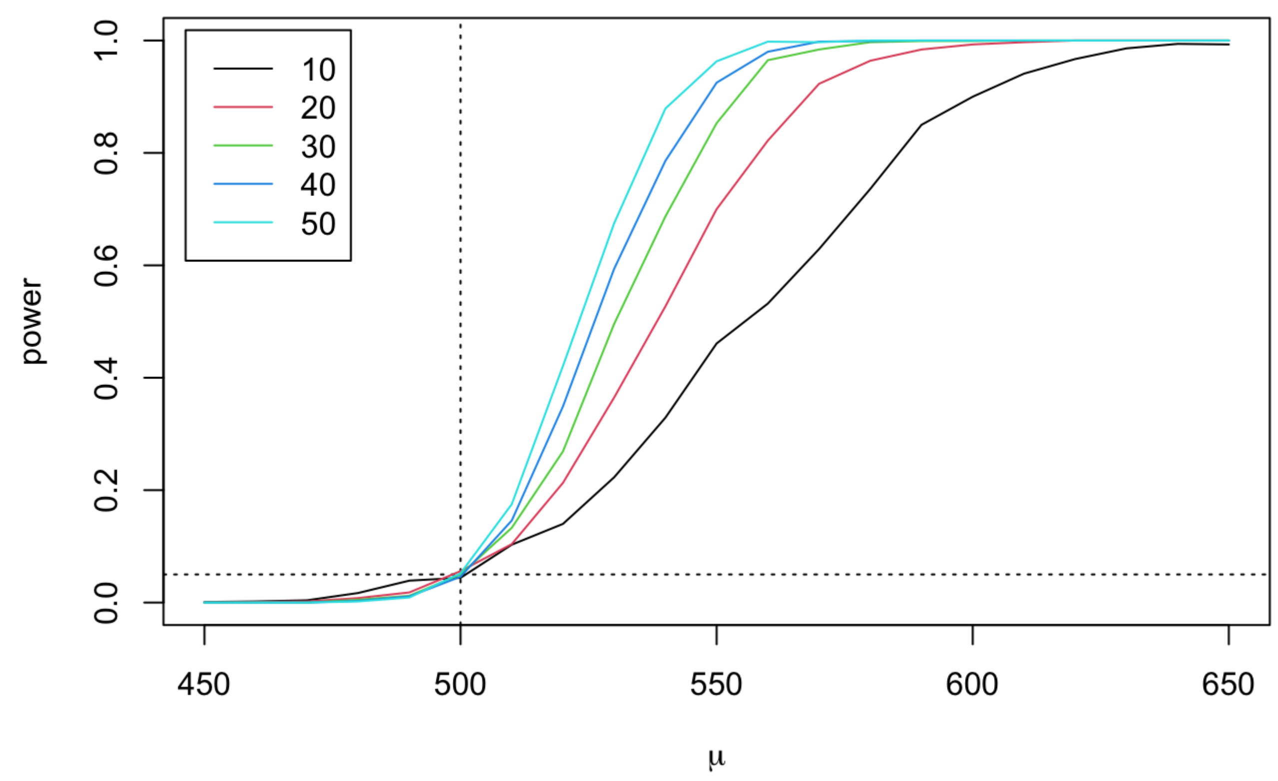
Exercise 5

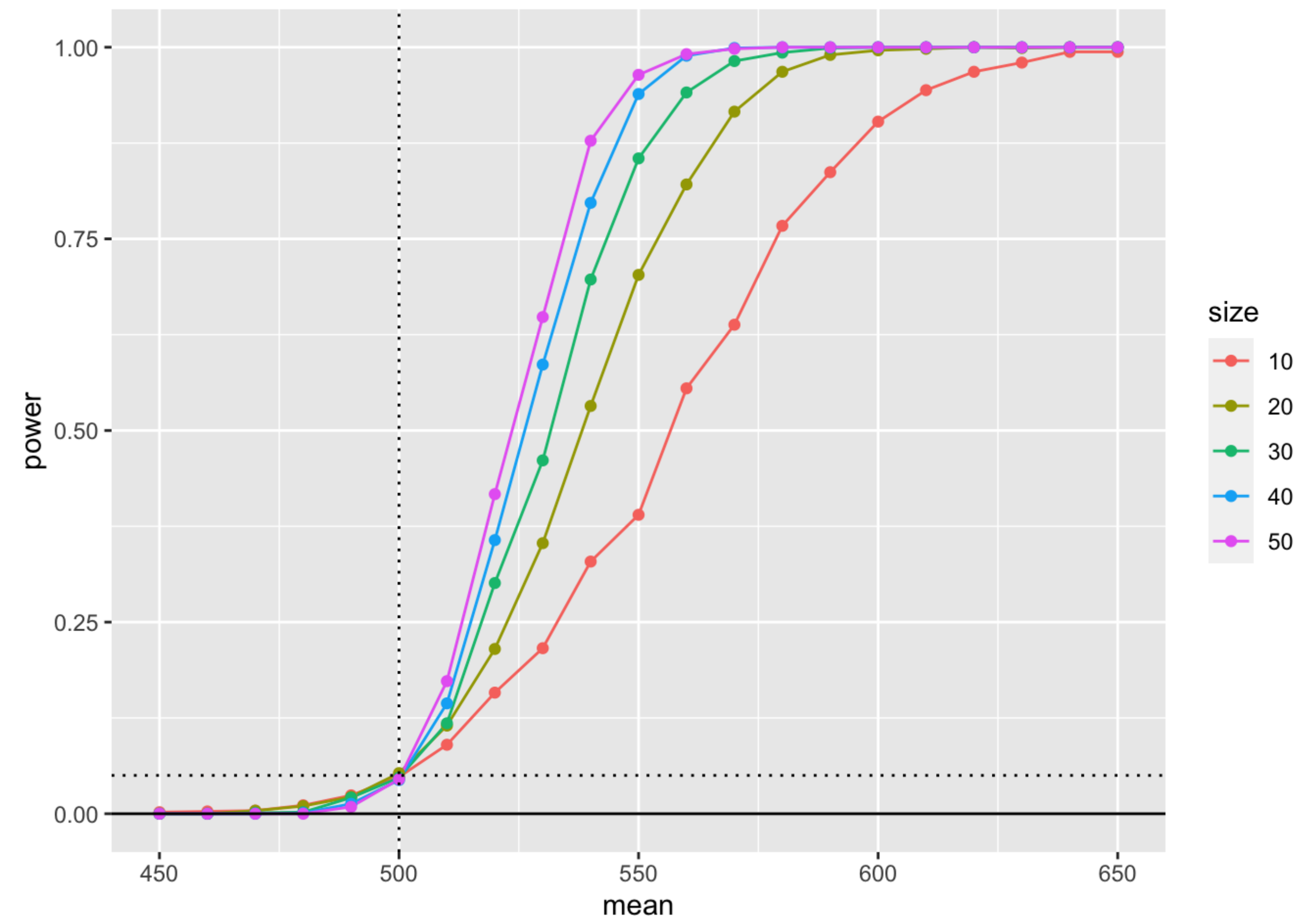
Do exercise 6.2 from the book.



Exercise 6

Do exercise 6.3 from the book.





Exercise 7

Do exercise 6.5 from the book.

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##     7.0     9.0    10.0    10.3    11.0    24.0
```

```
## [1] 1.885
```

```
## [1] 0.65
```

```
## [1] 0.388
```

Exercise 8

Do exercise 6.8 from the book. Use 15 as small sample size, 50 as medium sample size, and 250 as large sample size.

```
##  [1] 0.1488 0.0564 0.0220 0.0116 0.0040 0.0036 0.0004 0.0000 0.0004 0.0004
```