# Practice Set 3: Solutions

**Problem 1:**

```
Bubble-down(A,i)
    if i ≤ ⌊A.heapsize/2⌋                #make sure this is not a leaf.
        if 2i + 1 ≤ A.heapsize           #make sure that there are two children
            if A[2i] > A[2i+1] and A[2i] > A[i]    #the left child larger than parent and right child.
                Swap(A[i], A[2i])
                Bubble-down(A,2i)
            else if A[2i+1]>A[i]         #right child larger than parent
                Swap(A[i], A[2i+1])
                Bubble-down(A, 2i+1)
        else                                  # there is only one child
            if A[2i] > A[i]
                Swap(A[i],A[2i])
Bubble-up(A, i)
    if i > 1
        if A[i] > A[⌊i/2⌋]
            Swap(A[i],A[⌊i/2⌋]
            Bubble-up(A, ⌊i/2⌋)
```

**Problem 2**

Inserting into a heap is similar to the step carried out in the iterative heap building method. We simply add the new element to the array at the next available index, increase the heapsize, and perform a bubble-up. Since the bubble-up operation takes time $O(\log n)$, the insertion runs in time $O(\log n)$. Suppose the new element to be inserted is $k$:

```
A.heapsize ++
A[A.heapsize] = k
Bubble-up(A,A.heapsize)
```

**Problem 3**

Below is the pseudocode for BiggerThan(A,x,i). The parameter $i$ is necessary in order to efficiently implement a recursive algorithm. The solution works by examining the value at $A[i]$. If it is larger than $x$, we print out the value. Then we make a recursive call to the two children of the node at position $i$. When a node is discovered whose value is *smaller* than $x$, no recursive calls are made, because we know that all values in the subtree are out of range.

```
BiggerThan(A,x,i)
    if i ≤ A.heapsize          #only continue if index i is in the heap A
        if A[i] > x.              #if key is smaller than x, no recursive calls are necessary
            print A[i]
            BiggerThan(A,x,2i)          #repeat for left child
            BiggerThan(A,x,2i+1)          #repeat for right child
```

In the worst-case, the algorithm above searches through both subtrees at every node. The subtrees are not *exactly* the same size, but we will approximate them here as about $n/2$ nodes in each subtree. In this case, the recurrence for the runtime is $T(n) = 2T(n/2) + c$. The solution to this recurrence can be determined using Master method, where $k = \log_2 2 = 1$, $n^k = n$, and $f(n) = c$. Therefore $T(n)$ is $\Theta(n)$.

In the best-case, the algorithm may return in constant time. If the root node is smaller than $x$, then no recursive calls are made. Therefore $T(n) = O(1)$ in the best case.

**Problem 4:**

```
HeapDelete(A,i)
    if i = A.heapsize          #the element to delete is the last element
        A.heapsize = A.heapsize -1
    else if i = 1          #the element to delete is the root
        DeleteMax(A)
    else
        Swap A[i] and A[A.heapsize].          # put the item to be deleted into the last position
        A.heapsize = A.heapsize -1
        if A[i] > A[⌊i/2⌋]          #if key is smaller than x, no recursive calls are necessary
            BubbleUp(A,i)
        else
```

BubbleDown(A,i)

**Problem 5**

This question demonstrates that although max-heaps are excellent for finding the max and updating values in the tree, they are not very efficient when it comes to finding the *minimum*. A description of the algorithm is given below. The algorithm takes as input the heap $A$ rooted at index $i$. It finds the minimum of the heaps roots at each child (recursively), and compares those two results with the value at index $i$.

```
Find-min(A,i)
    if i > A.heapsize
        return infinity
    m1 = Find-min(A,2i)
    m2 = Find-min(A,2i+1)
    return min(m1,m2, A[i])
```

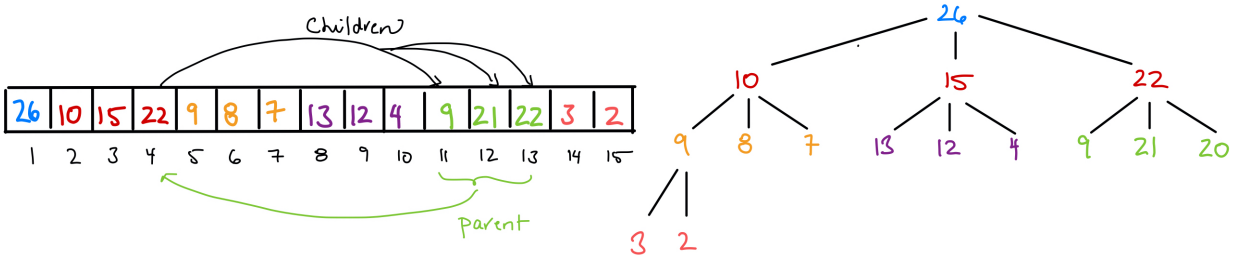
The runtime of this algorithm is $T(n) = 2T(n/2) + c$, in any case, which has solution $\Theta(n)$. This can be verified by the master method, where $n^k = n$ and $f(n) = c$, in which case the dominant function is $n$, and therefore the runtime is $\Theta(n)$. Notice that this is exactly the same runtime if we had just run a brute-force algorithm and looped through all the elements in the array looking for the minimum.

**Problem 6**

- The second-largest element must be in one of the children of the root. Therefore, the second-largest element is at $A[2]$ or $A[3]$. The third-largest element has two values that are larger than it. Therefore, it could be a child of the root (in $A[2]$ or $A[3]$) or it could be a grand-child of the root (in either $A[4]$ or $A[5]$ or $A[6]$ or $A[7]$).
- In order to return the third-largest element in from the heap, we only need to verify which is the third-smallest element out of $A[1, \ldots, 7]$. This is a search of only 7 elements (a constant number), regardless of how many items are in the heap. Therefore the runtime is $O(1)$.

**Problem 7**

If we build a heap where each internal node has 3 children (instead of just 2) then the array would work as follows:
$A[1]$ contains the root
The parent of node $i$ is at index $\lfloor (i+1)/3 \rfloor$.
The four children of the node at position $i$ are $A[3i-1], A[3i], A[3i+1]$

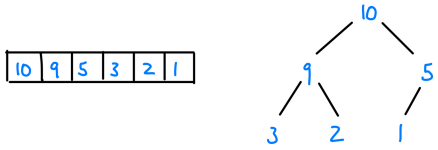



The height of this tree is $\Theta(\log_3(n))$, in fact it is exactly $\lfloor \log_3(2n) \rfloor$.

The bubble-up procedure works in the same way. We swap a node with its parent if and only if the parent is smaller. The bubble-down procedure would also be the same. We bubble-down as long as the current node has a child that is larger. We swap with the *largest* of the *three* children. The height of this new heap is still $O(\log n)$. Therefore both bubble up and bubble down still run in time $O(\log n)$. Using 3 children instead of 2 doesn't change the asymptotic runtime of the bubble-down or bubble-up procedures.

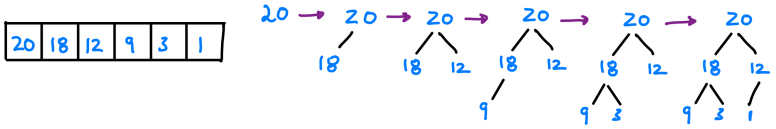
**Problem 8:**

- An array sorted in decreasing order represents a **max-heap**:

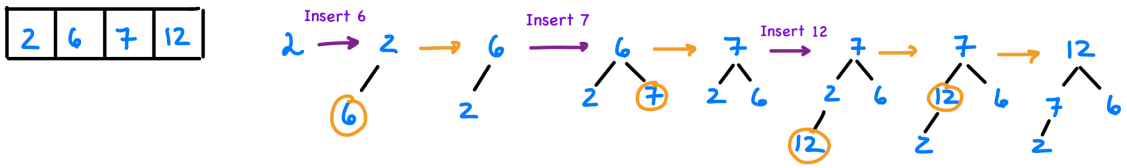



However, the converse is not true. A max-heap stored in an array is *not necessarily* in sorted order.

- **Iterative method:** Suppose the original input array is sorted in *decreasing* order. As each new leaf is inserted, there is *no work* to be done. The leaf is already in its final position, and no swaps are made with the parent node. The heap is built in $O(n)$ time.



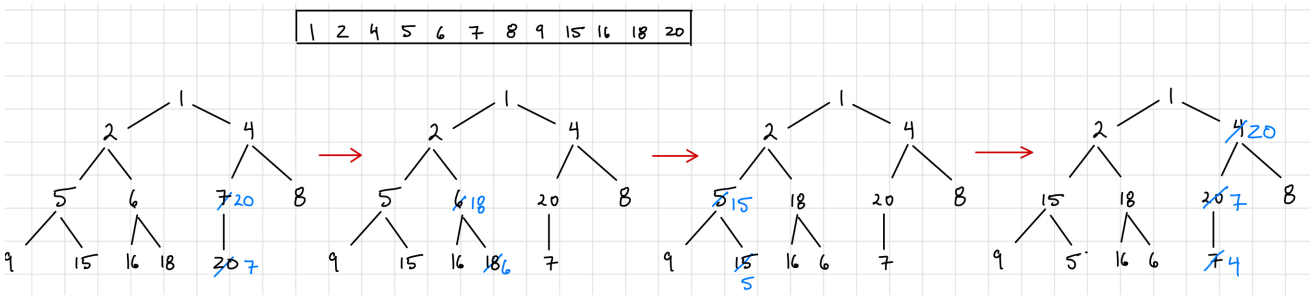

Suppose the original input array is sorted in *increasing* order. Then the iterative method will have to continue swapping each new leaf all the way up to the root position, taking time $O(\log n)$ for each insert. This represents the worst case of the iterative method. The overall run time is $O(n \log n)$, since there are a total of $n$ nodes inserted to build the heap.




**Bottom-up method:**

Suppose the original input array is sorted in *decreasing* order. Then the array is already a heap. The bottom-up method would go through each node and verify that it is the max of its children. No swaps would be done. The algorithm would take $O(n)$ time.

Suppose the original input array is sorted in *increasing* order. Then the bottom-up method will need to carry out a *Bubble-down* operation *all the way down* to the leaf during each iteration of the bottom-up procedure. This represents the worst-case run time of the bottom-up method, and has runtime $O(n)$ as shown in class.




**Problem 9:**

If the new item to insert is smaller than the current min, we simply insert it at the last position in the heap. Otherwise, we temporarily insert it in the last position, and carry out a bubble-up to position it in its correct position. Then we put the actual minimum back in the last spot of the heap.

```
NewHeapInsert(A,x)
    if x < A[A.heapsize]
        A.heapsize = A.heapsize + 1
        A[A.heapsize] = x
    else
        min = A[A.heapsize]
        A[A.heapsize] = x
        Bubble-up(A,A.heapsize)
        A.heapsize += 1
        A[A.heapsize] = min
```

**Problem 10:**

The partitioning algorithm operates as shown in our class video, by using an external array. When the partitioning is complete, we call ArrayCopy(B,A,s,f) which copies the elements from $B[1, \ldots, n]$ into array $A[s, \ldots f]$.

```
Partition(A,s,f)
```

```
p = Random(s,f).        #A random number between s and f inclusively.
n = f − s + 1
initialize B[1.. n]
i = 1, j = n            #index i inserts to the left of B, index j inserts to the right
for count = s to f do:  #loop through the elements of A and insert them into B
    if count ≠ p
        if A[count] < A[p]
            B[i] = A[count]
            i = i + 1
        else
            B[j] = A[count]
            j = j − 1
B[i] = A[p]             #insert the pivot element into the correct position
ArrayCopy(B,A,s,f)      #copy B into A
return j #return the rank of the pivot
```

**Problem 11:**

Recall that the Partition(A,s,f) algorithm from problem 10 returns the **rank** of the randomly chosen pivot (not the exact index).

```
Randomized-Select(A,s,f,k)
    if s = f and k = 1.     #If the array has size 1, and the rank is 1, then return the first element
        return A[s]
    else
        p = Partition(A,s,f)        #p is the RANK of the pivot element (not the array index, the actual rank)
        if p = k
            return A[p + s − 1]     #return the pivot element by adjusting the rank p to the start index
        else if k < p
            return Randomized-Select(A, s, s + p − 2, k)    #Recursive call to the left subarray
        else return Randomized-Select(A, s + p, f, k − p)   #Recursive call to the right subarray
```

**Problem 12:**

- Given an array of size $n$, we can use the Select algorithm to determine which element represents the median. Once we have this element, we can loop through the array and output all elements that are larger than the median. The Select algorithm runs in $\Theta(n)$ time. Therefore the entire algorithm runs in $O(n)$.

```
k = Select((n + 1)/2)
for i = 1 to n
    if A[i] > k
        Print A[i]
```

- In order to print out the largest 10 elements, we can run the Select algorithm to determine the element of rank $n - 9$. Once we have this element, we can loop through the array and output all elements that are larger than that element.

```
k = Select(n − 9)
for i = 1 to n
    if A[i] ≥ k
        Print A[i]
```

  The two main steps are the same as in the previous algorithm, and therefore the runtime is the same.

- In order to print out the largest $n/4$ elements, we can run the Select algorithm to determine the element of rank $n - n/4 + 1$. Once we have this element, we can loop through the array and output all elements that are larger than that element.

```
k = Select(3n/4 + 1)
for i = 1 to n
    if A[i] ≥ k
        Print A[i]
```

The two main steps are the same as in the previous algorithm, and therefore the runtime is the same.

- We can use the Select algorithm to find the median. The create two variables called *close1* and *close2* which will represent the elements that are closest to the median. Note that the solution below assumes the array has size at least 3.

```
k = Select((n + 1)/2)
close1 = A[1]
close2 = A[1]
for i = 1 to n
    if A[i] ≠ k
        if abs(close1 - k) > abs(A[i] − k)
            close2 = close1
            close1 = A[i]
        else if abs(close2 - k)> abs(A[i] − k)
            close2 = A[i]
Print close1, close2
```

Note that all four options have the same runtime. However, option 2 is the only one that could be solved in time $O(n)$ using brute-force.

**Problem 13:**
We demonstrate the three different approaches using $k = 3$ and the array:

$$\{3, 6, 7, 15, 8, 1, 9, 2, 18, 13\}$$

The three different approaches to solving this problem are:
*Option 1:* If we sort the entire array, using Mergesort for example, this takes time $\Theta(n \log n)$.

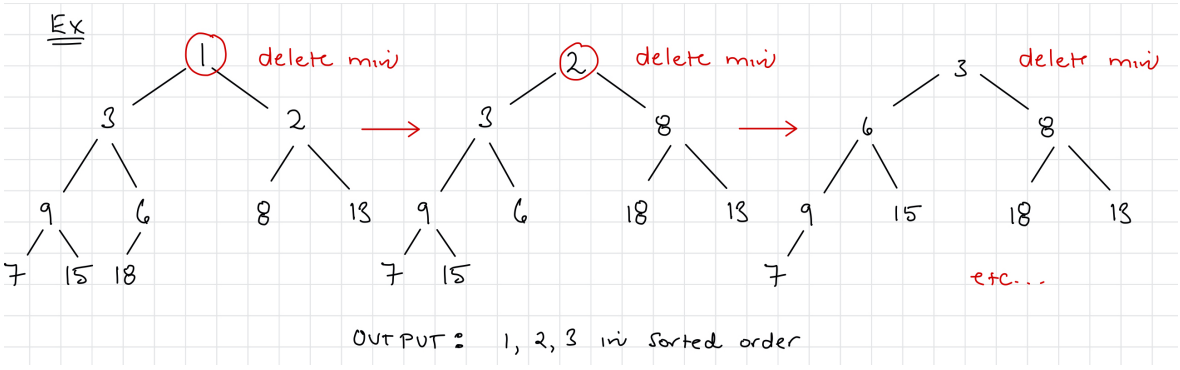
$$1, 2, 3, 6, 7, 8, 9, 13, 15, 18$$

Then we could simply pass through the sorted list from back to front and output the first $k$ items from the list.

$$1, 2, 3, 6, 7, 8, 9, 13, 15, 18$$

The step would take time $\Theta(k)$. The total runtime would then be $\Theta(n \log n)$.

*Option 2:* We could build a min-heap in time $O(n)$. Then we could call *Delete-Min* exactly $k$ times. Each call to Delete-min takes time $O(\log n)$, and so deleting the min $k$ times takes $O(k \log n)$. The entire procedure (building the heap and deleting the min's) takes time $O(k \log n + n)$. Depending on the value of $k$, this results in a variety of runtimes. For example, if $k = 3$, then the runtime is $O(\log n + n) = O(n)$, but if $k = \sqrt{n}$, then the runtime is $O(\sqrt{n} \log n + n) = O(n)$.




*Option 3:* We could use the Select algorithm to find the element of rank $k$. This would take time $O(n)$. For $k = 3$ for example, the result is element 3. Next, we would loop through the array to find all numbers that are *at most* 3. This takes time $O(n)$ and results in the set:

$$\{3, 1, 2\}$$

Finally we sort this set using Merge-sort, resulting in the final output $\{1, 2, 3\}$. Since there are only $k$ items in this set, the time is $O(k \log k)$. Thus the total runtime of this third option is $O(n + k \log k)$. Again, depending on the value of $k$, this results in different runtimes. If $k = 3$, the runtime is $O(n)$. If $k = \sqrt{n}$, the runtime is $O(n + \sqrt{n} \log n) = O(n)$.

**Problem 14:**

If we divide our numbers into groups of size $\sqrt{n}$:

There would be exactly $\sqrt{n}$ groups. Recall that the first step of the Select algorithm is to find the median of each group. If we find the median of each group using insertion sort (as in the original algorithm), then this takes time $O((\sqrt{n})^2) = O(n)$, and since there are a total of $\sqrt{n}$ groups, then sorting all of them takes $\sqrt{n} \cdot O(n) = O(n^{3/2})$. If we find the median of each group using mergesort, then sorting each group takes time $\Theta(\sqrt{n} \log \sqrt{n})$, but again since there are $\sqrt{n}$ groups, the total time is $\sqrt{n} \cdot \Theta(\sqrt{n} \log n) = \Theta(n \log n)$. Unfortunately, regardless of what sorting algorithm we use, we cannot find the median of the medians in $O(n)$ time. Therefore this algorithm is not as efficient as the Select algorithm from class, which runs in time $O(n)$.

If we used groups of 3 instead of groups of 5:

1. We can find the median of each mini-group using Insertion-Sort, in constant time per column. We repeat this for all $n/3$ groups. Therefore all medians are found in time $O(n)$:

2. Next we make a recursive call to find the median of medians (called $x$), which takes time $T(n/3)$.

3. Next, we partition the elements into those that that are larger than $x$ and those that are smaller than $x$. This takes time $\Theta(n)$

4. Finally we make a recursive call to one of the above sets. The best-case scenario would be a recursive call on input of size at least

$$\frac{1}{2} \cdot \frac{n}{3} \cdot 2 = \frac{n}{3}$$

and thus the worst-case scenario is to a set of size at most $2n/3$. So the recursive call takes time $T(2n/3)$ in the worst case.

The worst-case runtime of this recursive algorithm is :

$$T(n) = T(n/3) + T(2n/3) + cn$$

Unfortunately, this runtime is $\Omega(n \log n)$, so not as fast as the original Select algorithm.

**Problem 15**

The worst-case runtime of the Randomized-Select is when the algorithm picks the maximum element as the random pivot. In this case the recursive call is made to an array of size $n - 1$. The runtime recurrence is $T(n) = T(n - 1) + cn$. This is the same recurrence as for the recursive version of Insertion Sort, which we already showed is $O(n^2)$. In the best-case scenario, the random pivot is actually the element $k$. In this case, the algorithm terminates after the first call to the partition procedure. Since the partition procedure runs in time $O(n)$, the best case runtime for Randomized-Select is $O(n)$.