Practice Problem Set 14

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Problem 1

RSA is an example of public-key cryptography, and it a popular method used to encrypt and decrypt messages. If you are unfamiliar with RSA, it is based on the simple concept of creating a number n that is the product of two large prime numbers. Explain why it is important that the primes are *large*, and why it is that someone who overhears the transferred encrypted message is unable to decrypt it. What happens if/when quantum computers hit the market?

Problem 2

A student has concluded that they have just won one million dollars, because they have come up with a polynomial-time algorithm to solve the Dominating set problem. Their algorithm works as follows: pick the vertex with the largest degree in G. Add that vertex to the dominating set. Then eliminate that vertex and all it's neighbours from V and repeat until there are no vertices left.

Explain why this student is unfortunately not wining anything...

Problem 3

For each of the following, give an example of a connected graph G with 10 vertices that satisfies the statement:

- The min vertex cover has size 4 and the min dominating set has size 4
- The min edge cover has size 5 and the min vertex cover has size 5
- The min vertex cover has size 6 and the min dominating set has size 4.

Problem 4

A student tries to win a million dollars by proving they can solve the Travelling Salesman problem in polynomial time.

The solution works like this: Given a weighted graph G and a maximum tour length L for the TSP problem, the algorithm first computes the maximum edge-weight in G, and calls it m. Next, the minimum spanning tree of G is computed, and if its total weight is less than $L - m$, the algorithm returns TRUE. Otherwise it returns FALSE.

Explain why this algorithm does not solve the TSP problem.

Problem 5

A set of n people is such that any two people are either *friends* or *not friends*. Given as input such a set, and an integer k, the problem **FINDTEAMS** is the decision problem of determining if there are two disjoint sets of people, T_1 and T_2, of *equal size* such that everyone in T_1 is friends with everyone else in T_1 and similarly everyone in T_2 is friends with everyone else in T_2. The size of each time must be at least k. Show that this problem is NP-complete.

Problem 6

In class we saw that the decision problem Hamiltonian Cycle is NP-complete. Another similar problem is "Hamiltonian Path". In this decision problem, we are given as input an undirected graph G on n vertices, and the goal is to determine if there is a simple *path* through all the vertices. The Hamiltonian Path problem is also NP-complete. Suppose you were asked to show that Hamiltonian Path was NP-complete using a reduction from Hamiltonian cycle. You came up with the following idea: take the input graph G to Hamiltonian cycle. Since we are looking for a cycle in G, we can simply *remove* any edge from G and call it graph G_P, and then pass this graph as input to Hamiltonian Path. If there is a Hamiltonian path in the new graph, then there was a Hamiltonian Cycle in the original graph, and vice versa.

Is this reduction correct?

Problem 7

For each of the following problems, decide whether or not a polynomial-time algorithm is known. You must justify your answer using material from class.

- A museum consists of n display rooms, each room has several doors. The director would like to install a set of cameras that are capable of observing the entire museum collection. The cameras will be installed in the doorways. Assume that any camera in a doorway is able to observe both rooms adjacent to that door. The problem is to determine the minimum number of cameras needed so that all rooms can be observed.
- The new director of the museum (from above) decides it might be better to place the cameras in the display rooms. A camera in a display room can observe that room, and all rooms adjacent to that rooms. The problem is to determine the minimum number of cameras needed so that all rooms are observed.

Problem 8

For each of the following problems, decide whether or not a polynomial-time algorithm is known. You must justify your answer using material from class.

- The previous two museum directors are fired. The replacement decides to convert the museum into a move theatre. The display rooms can be converted into movie screening rooms. The only problem is that noise travels easily, and no two adjacent rooms can be used to show movies. The problem is to determine the maximum number of movies he can play at the same time.
- A jury is to be selected from a set of n individuals. Unfortunately any two individuals who are married or who have been married are not allowed to both be on the jury.

The problem is to determine if a jury of size at least k can be selected from the n individuals.

Problem 9

For each of the following problems, decide whether or not a polynomial-time algorithm is known. You must justify your answer using material from class.

- Given a set of n webpages, assume that each page may or may not have links to one or some of the other pages. Supposed we want to find a way to click from one page to the next, such that we visit each page exactly once.
- A set of m nurses are available to administer vaccines. The vaccines will be administered over a period of n days. Each nurse must list exactly **two** dates (out of the n) on which they are available to work. The city would like to hire the minimum number of nurses possible to ensure that there is **at least one** nurse working on each of the n days.

Problem 10

For each of the following problems, decide whether or not a polynomial-time algorithm is known. You must justify your answer using material from class.

- Given an undirected graph on n vertices, determine if it there is a simple path of length $\geq k$ in the graph.
- Given an undirected graph on n vertices, determine if there is a simple path of length ≥ 5 in the graph.

Problem 11

For each of the following problems, decide whether or not a polynomial-time algorithm is known. You must justify your answer using material from class.

- Carol and Bob go shopping for flour. At the shop, there are n bags of flour, where each flour bag has a certain weight. Determine if it is possible for Carol and Bob to purchase exactly one of each type of flour bag, in such a way that on the way home they can each share the load exactly. In other words, the total weight of flour carried by Carol is exactly that of Bob. Assume the flour bag weights are given in $w[1, \ldots, n]$.
- Santa Claus is handing out presents to a group of n school children. Certain pairs of the children are friends, where a friendship is a bi-directional relationship between two children. A child may be friends with 0 to $n-1$ other children, but cannot be friends with themselves. Santa Claus is running low on funds. So he can't hand out a present to each child. He wants to ensure that every child either gets a present, or is friends with someone who get a present. Determine the minimum number of toys he can hand out.

Problem 12

The following problem should seem familiar to you. We provided a Dynamic Programming solution to this problem with runtime $O(nT)$:

Given a set of n water bottles, where each water bottle has a given volume of water in litres and cost, determine if it is possible to select a set of water bottles that has total volume exactly T litres and whose total cost is at most C. Assume the water bottle sizes are given in $w[1, \dots, n]$ and the corresponding prices are given in $v[1, \dots, n]$

Show that this problem is NP-complete using a reduction from Subset Sum. Does this mean that our Dynamic programming solution was actually incorrect?

Problem 13

In a particular village there are m different potential vaccine locations. There are n people living in village, and each person has listed the vaccine locations they would be able to visit. Assume that each person lists at least one vaccine location. The problem is to determine the if the city can open at most k vaccine sites, such that each person has access to a vaccine site location, according to their listed options. Show that this problem is NP-complete.

Problem 14

A set of n children attend a summer camp. The camp offers offers m activities, where each child may register for any one (or none) of the activities. The camp has a certain number of first-aid kits they would like to distribute to the children. There are only k first-aid kits, and therefore not all children can carry first-aid kits. The director decides that the camp will be safe if each child is either carrying a first-aid kit themselves, or is in an activity where someone in that activity has a first-aid kit. Call this problem "FirstAid". Show that this problem is NP-complete using a reduction from a problem from class.