Practice Set 7: solutions

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Problem 1

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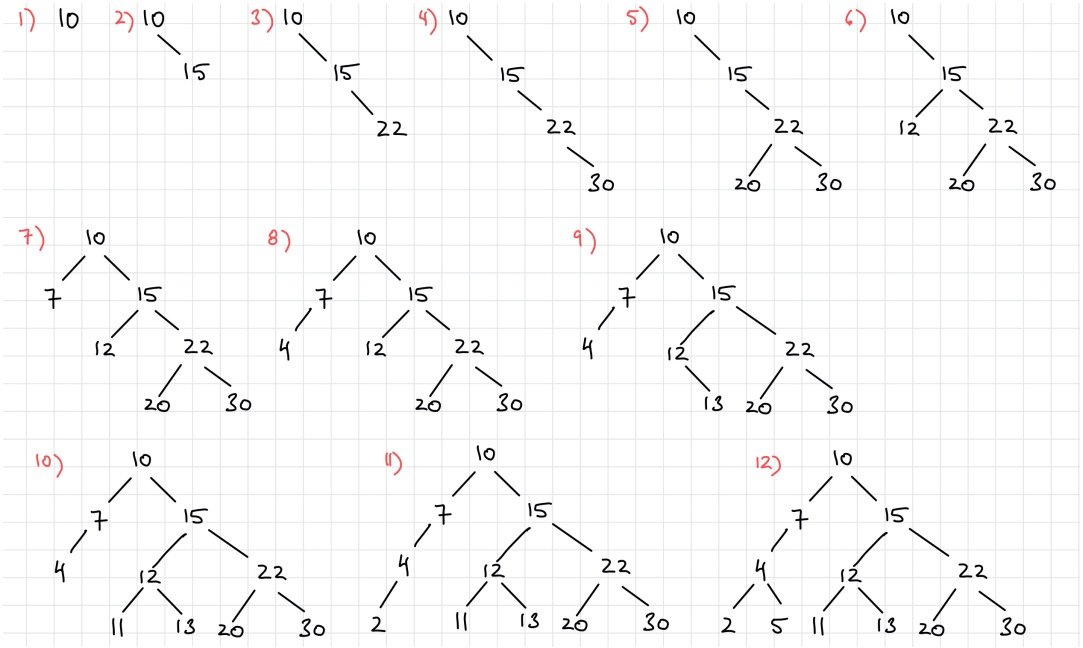

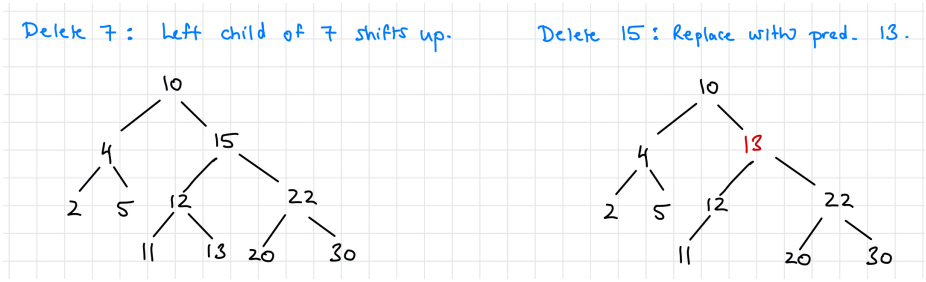

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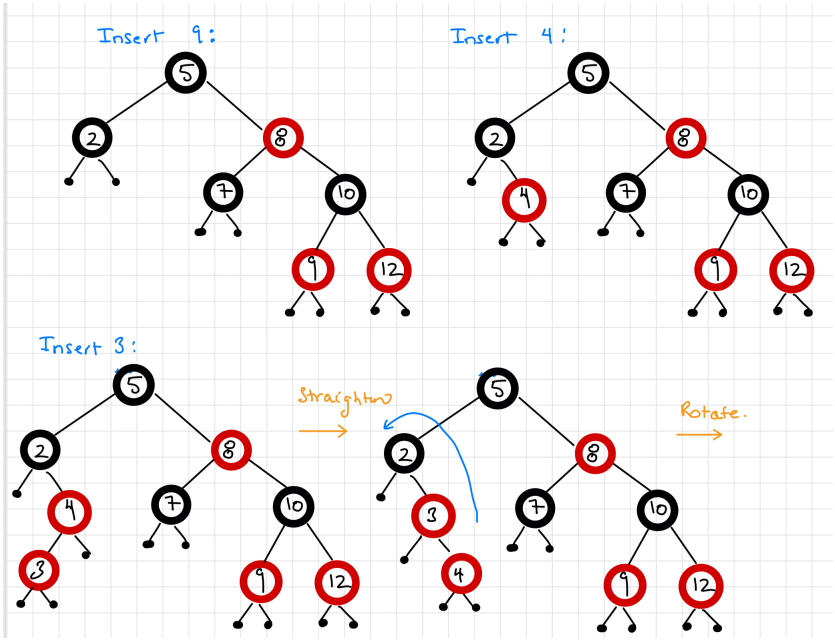
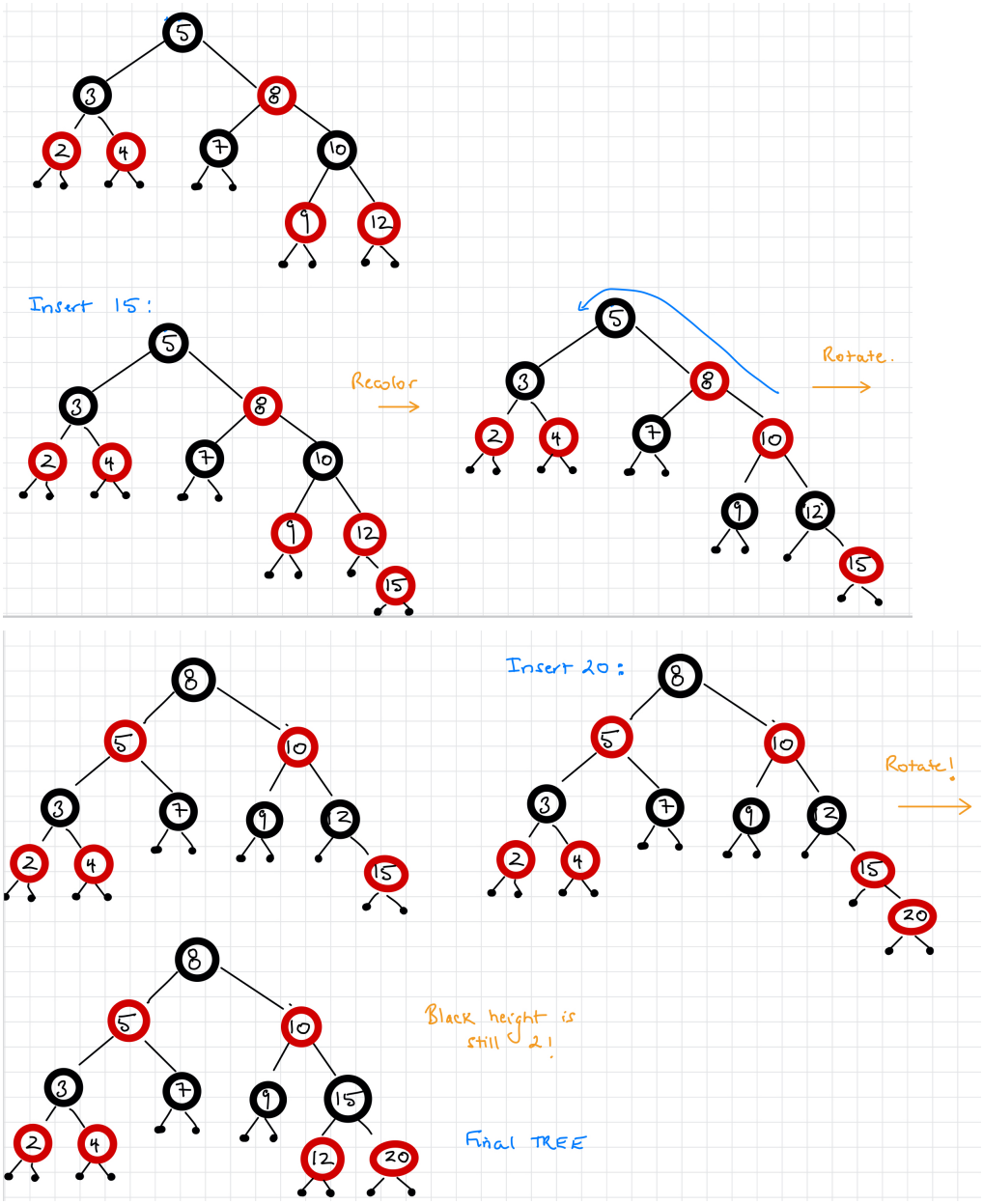




Problem 2

This problem has been updated from the original solution in class. The original problem assumed the data was already partitioned around the root. This solution here is correct for any binary tree Suppose the tree is empty. Certainly, this is a valid tree, and the procedure should return true. If both the left and right subtrees are also valid, it remains to check that the right child is greater than (or equal to) the root and the left child is less than (or equal to) the root. In this updated version, we also check if **all the nodes** in the right subtree are greater than the root. We can do this by making a call to FindMin of the right subtree. Similarly, all the nodes in the left subtree should be smaller than the root.

```
VerifyBST(T)
    If T = Nil
        return true
    else if VerifyBST(T.left) = true and Verify(T.right) = true   Both left and right subtrees are valid
        if T.right != Nil
            if T.right.key < T.key or FindMin(T.right) < T.key      Right tree has invalid entry
                return FALSE
        if T.left ≠ Nil
            if T.left.key > T.key or FindMax(T.left) > T.key      Left node has invalid entry
```

```
                return FALSE
        return TRUE
else return FALSE
```

Problem 3

The minimum is the left-most node in the tree, and the maximum is the right-most node in the tree.

```
Find-Max(T)
  x = T
  while x.right != NIL
     x = x.right
  return x
```

```
Find-Min(T)
  x = T
  while x.left!= NIL
     x = x.left
  return x
```

In order to find the successor of a node x, we must consider two cases: it may be that the node x has a right child. In this case, the successor is the minimum node in the right subtree. If the node x has no right child, then the successor is somewhere on the path from the root to x. The algorithm below searches down at most one path in the tree. In the first case, it searches down a path from node x. In the second case, it searches down a path from the root to x. Therefore in either case, the algorithm runs in time $O(h)$.

```
Successor(T,x)
    If x.right is not NIL     # x has a right child
        return Find-Min(x.right)
    else     # z has no right child
        y = T
        while y is not x
            if y.key > x.key     # x is on the left path
                success = y
                y = y.left
            else     # x is on the right path
                y = y.right
        return success     # z has only a right child
```

Problem 4

We need to link the parent of x to the new node y. This is done by checking if x is a left or right child of its parent.

```
Replace(x,y)
    If x.parent.left = x     # x is a left child
        x.parent.left = y
    else x.parent.right = y     # x is a right child
    if y != NIL
        y.parent = x.parent
```

The above function runs in constant time, as it only re-assigned reference pointers.

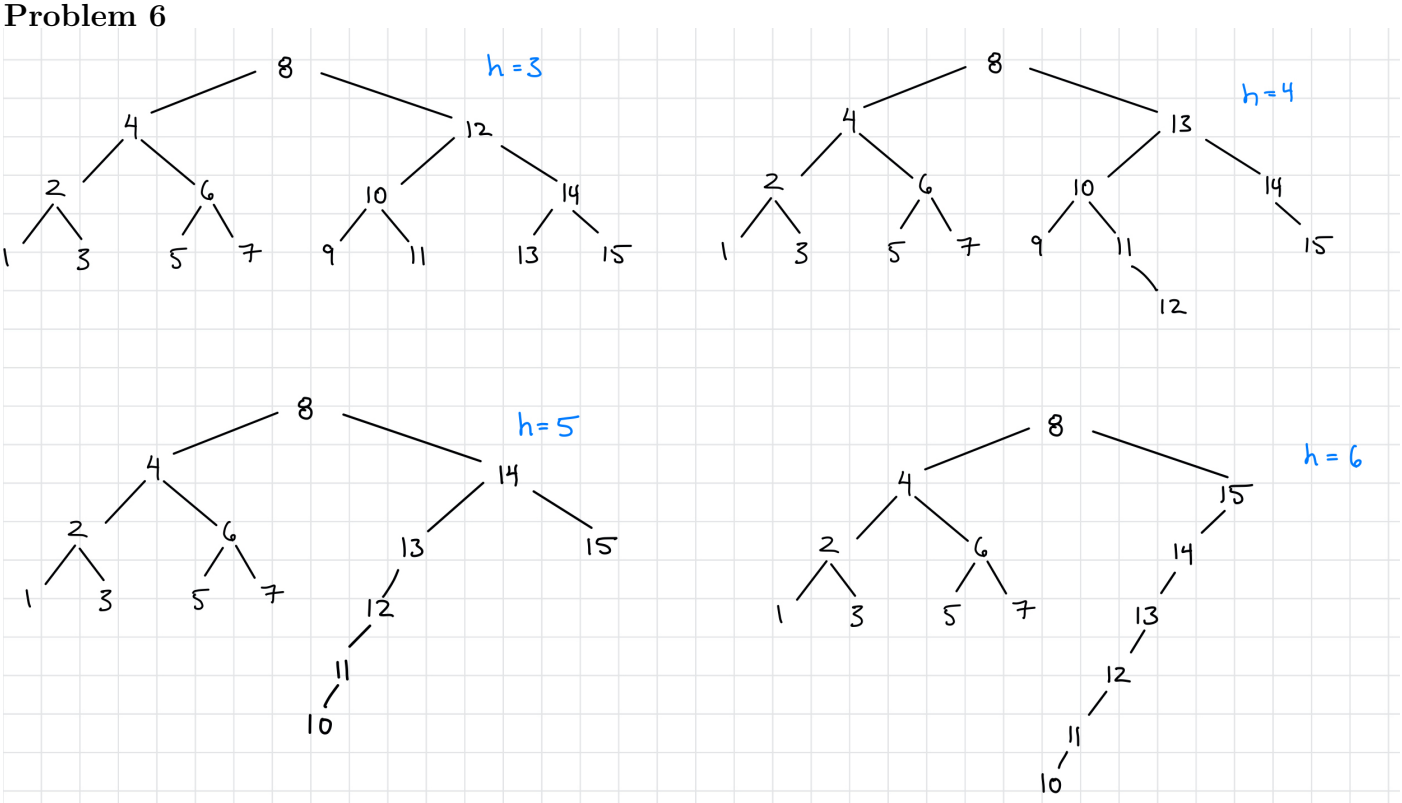
Problem 5

The algorithm BST-Delete(T,z) deletes node z from the tree T with the assumption that z is a node in the tree, and returns a reference to the new tree.

```
BST-Delete(T,z)
    If z = T and z.left = NIL     # z is the root, return right child
        z.right.parent = NIL
        return z.right
    else if z = T and z.right = NIL     # z is the root, return left child
        z.left.parent = NIL
        return z.left
    else:
        If z.left = NIL and z.right = NIL     # z is a leaf
            If z = z.parent.left
```

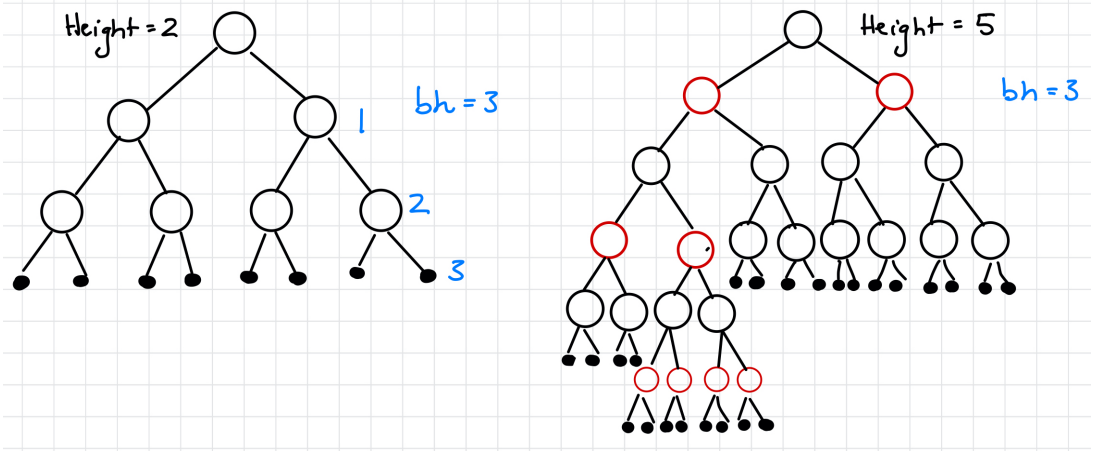
```
            z.parent.left = NIL
        else z.parent.right = NIL
    else if z.left = NIL    # z has only a right child
        Replace(z,z.right)
    else if z.right= NIL    # z has only a left child
        Replace(z,z.left)
    else    # z has two children
        y = Find-Min(z.right)
        z.key = y.key
        Replace(y,y.right)
    return T
```

Problem 6



The shortest RB tree we can build is when all the nodes are black. For example, when then black-height of the tree is 3, the height of the tree is 2. Therefore for a tree with black-heigh b, the shortest tree has height $b - 1$.

The longest red-black path we can build in a tree is a path that alternates red and black nodes. For example, when the black-height is 3, the maximum height of the tree is 5. An example of each case is shown below. Therefore when the black height is b, the tree with the maximum height has height $2b - 1$.



Problem 7

In class we showed the INORDER(x) runs in time $T(n)$ where $T(n) \leq T(|L_x|) + T(|R_x|) + c$ where L_x and R_x denote the left and right subtrees of x, and $|L_x|$ refers to its size. We can prove this is $O(n)$ using substitution. Assume that $L_x = k$ and $R_x = m$. Then $m + k + 1 = n$.

Goal: $T(n) \leq dn$ for all n

Assume: $T(k) \leq dk$ and $T(m) \leq mk$

Therefore:

$$\begin{aligned} T(n) &= T(k) + T(m) + c \\ &\leq dk + dm + c \\ &= d(k+m) + c \\ &= d(n-1) + c = dn - d + c \\ &\leq dn \end{aligned}$$

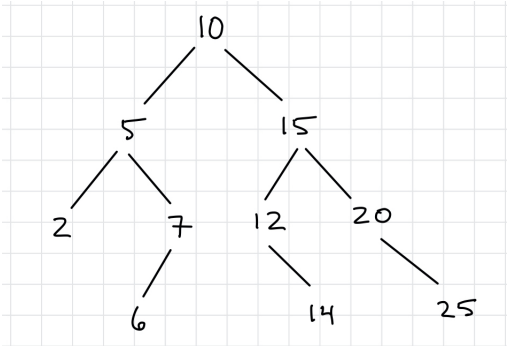
as long as $d > c$

The pseudo-code for pre and post order are almost the same, we just alter the step at which we print the node's key. Both pre-order(x) and post-order(x) have a runtime recurrence of $T(n) = T(L_n) + T(R_n) + c$, which is the same as that of Inorder. Therefore all three algorithms run in time $O(n)$

```
Preorder(x)
  if x is not NIL
     Print(x.key)
     Preorder(x.left)
     Preorder(x.right)
```

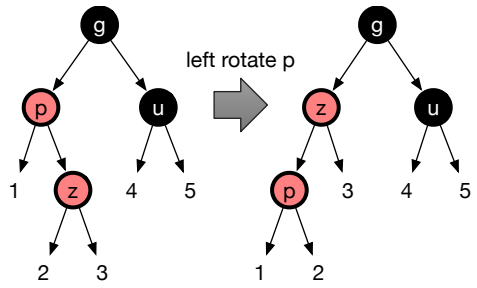
```
Postorder(x)
  if x is not NIL
     Postorder(x.left)
     Postorder(x.right)
     Print(x.key)
```

Given either the pre-order or the post-order, we can determine the exact shape of the tree. For example, given the pre-order $10, 5, 2, 7, 6, 15, 12, 14, 20, 25$ we know that 10 must be the root, and everything less than 10 is on the left subtree, and everything larger than 10 is in the right subtree.

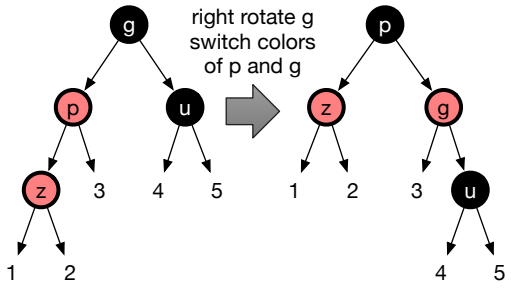


Problem 8

- The original RB tree (before the new item is inserted) is a valid RB tree. Therefore in the figure below, we know that the number of black nodes on paths 1,2,3 is the same. Likewise the number of black nodes on paths 4,5 is the same. Now let's see what changes after the rotation. Node U has the same subtrees, so its black height is the same. Node P has paths through 1,2,3 before the rotation, and through paths 1,2 after, all of which have the same number of black nodes. Node Z had paths through 2,3 before, and through 1,2,3 after, all of which have the same number of black nodes. Node G had paths through 1,2,3 before and after, (so no change there) AND through paths U+ 4,5 before and after, also no change there. So the black heights of all nodes is the same after the rotation.



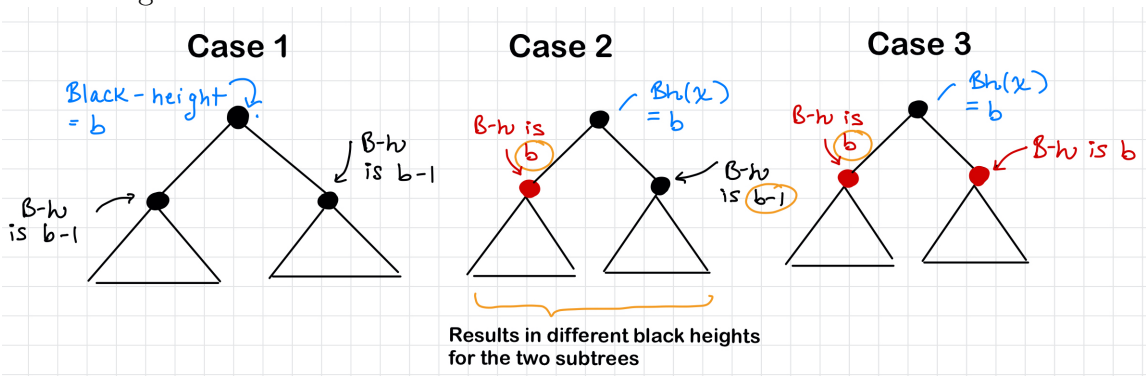
Case 2 may also involve another type of rotation, shown below. Node Z and node U have identical subtrees before and after the rotations, so their black heights are the same. Node P had paths through 1,2,3 before the rotation and through 1,2,3 *and* U+ 4,5 after the rotation. Note that the number of black nodes through U to 4,5 is the *same* as the number of black nodes through paths 1,2,3 (since the original tree was red-black). Therefore the black height of P has not changed. Finally, node G had paths through 1,2,3, and U+4,5 before the rotation, and still has paths through 3 and U+4,5 after the rotation, so the black height is the same.



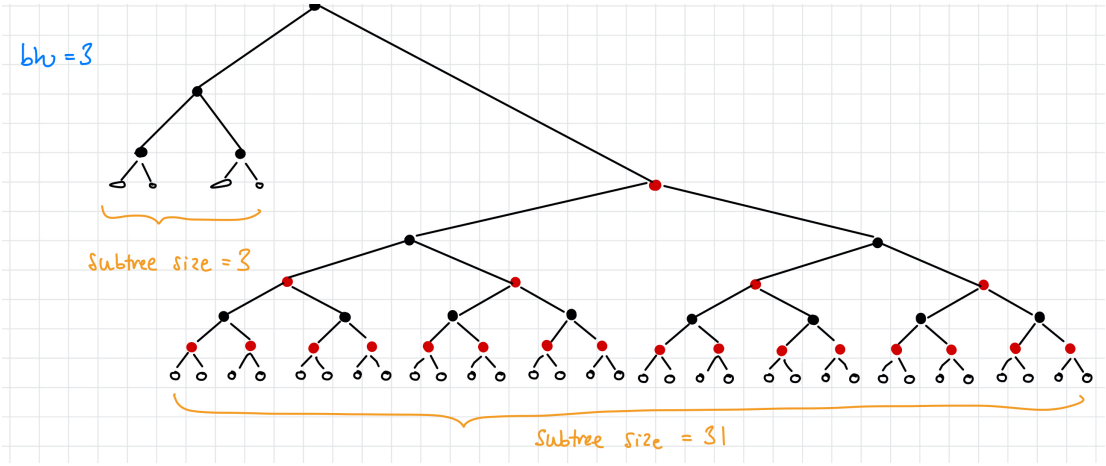
- A new node does not always change the black-height of tree. The RB-repair algorithm performs rotations in Case 2(which do not alter the number of black nodes on a path). RB-repair performs recolorings in Case1 (the parent and the uncle are recolored black, and the grandparent is colored red). Generally this recoloring does *not* change the number of black nodes on the path from the root. However, it may be that the grandparent is in fact the *root* node. In this case, the grandparent is left as a black node. The black height of the entire tree has increased by one.

Problem 9

If the children of the root, x, have different colors, then their black heights will differ by one. Otherwise the black heights of each child is the same.



If a red-black tree has black-height 3, the left subtree could be completely full of black nodes, in which case there would be 3 nodes. The right subtree could alternate red and black nodes. If we alternate as long as possible the result is 31 nodes in the right subtree. Therefore red-black trees can have subtree-sizes that are quite different. The height of the left subtree is 1 and the height of the right subtree is 4.



Problem 10

```
BlackHeight(x)
    if (x is NilNode) return 0
    else
        a = BlackHeight(x.left)
        if a = −1 return -1
        if (x.left.color = black)
            a++
        b = BlackHeight(x.right)
        if b = −1 return -1
        if (x.right.color = black)
            b++
        if (a ≠ b)
            return -1
        else
            return a
```

Problem 11:

If we rewrite TREE-INSERT this as a recursive algorithm, we must ensure that we write something that allows us to link z to its parent. Therefore, we write the recursive algorithm so that it "follows" the parent node down the tree. The algorithm below inserts z into T. If the tree is empty, it returns node z as the root.

OPTION 1:

```
TREE-INSERT(T,z)
    if T is NIL
        return z
    else
        if z.key < T.key
            if T.left = NIL
                T.left = z
                z.parent = T
            else TREE-INSERT(T.left, z)
        else
            if T.right = NIL
                T.right = z
                z.parent = T
            else TREE-INSERT(T.right, z)
        return T
```

OPTION 2:

```
TREE-INSERT(T,z)
    if T is NIL
        return z
    else
        if z.key < T.key
            T.left = TREE-INSERT(T.left, z)
            T.left.parent = T
        else
            T.right = TREE-INSERT(T.right, z)
            T.right.parent = T
        return T
```

Problem 12

If the tree is an AVL tree, then the algorithm below returns the height of the tree (which can be interpreted as true) and if it is NOT an AVL tree, it returns -2 (interpreted as false). The idea is to use recursion to determine the heights of the left and right subtrees, and return false if either of these subtrees are not AVL trees, **or** if the difference in the heights of the subtrees is more than 1. For the purpose of recursion, we use a height of -1 for any empty tree so that we can distinguish it from the height of a single node, which is 0.

```
IsAVL(x)

    if x = NIL return -1
    a = IsAVL(x.left)
    b = IsAVL(x.right)
    if (a = −2 OR b = −2 OR |a − b| > 1)
        return -2
    else
        return max(a,b) +1
```

Problem 13

The code below is copied from the Tree-Insert algorithm, with updates shown in purple.

```
BST-Delete(T,z)
    If z.count >1.
        z.count = z.count - 1.
        return T
    If z = T and z.left = NIL
        return z.right
    else if z = T and z.right = NIL
        return z.left
    else:
        If z.left = NIL and z.right = NIL
            If z = z.parent.left
                z.parent.left = NIL
            else z.parent.right = NIL
        else if z.left = NIL
            Replace(z,z.right)
        else if z.right= NIL
            Replace(z,z.left)
        else
            y = Find-Min(z.right)
            z.key = y.key
            z.count = y.count
            Replace(y,y.right)
        return T
```

```
TREE-INSERT(T,z)
        if T = NIL
          z.count = 1
          return z
        else x = T
        While x ≠ NIL
            y = x
            if z.key < x.key
                x = x.left
            else if z.key = x.key
                x.count ++
                return T
            else x = x.right
        z.count = 1
        z.parent = y
        if z.key < y.key
            y.left = z
        else y.right = z
        return T
```