Question 2: Filtering

A)

- given a noisy, but slowly shifting sensor signal
- filter it with a low-pass FIR filter
- What is the main delay and expected ~~SR~~ SNR boost for 2 types of filters below
- ~~do~~ plot results out

1.1) N filter taps, uniformly weighted

$$y[n] = (x[n] + \ldots + x[n-N+1])/N$$

Impulse response: $h[n] = \begin{cases} b_n, & 0 \le n \le \mu \\ 0, & \text{otherwise} \end{cases} \Rightarrow \begin{cases} \frac{1}{N}, & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$

Frequency Response: $H(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = \frac{1}{N} \sum_{n=0}^{N-1} e^{-jwn}$

★ geometric series summation $\Rightarrow \sum_{n=0}^{N-1} e^{-jwn} = \frac{1-e^{-jwN}}{1-e^{-jw}}$

$$H(w) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-jwn} = \frac{1}{N} \frac{1-e^{-jwN}}{1-e^{-jw}} = \frac{1}{N} \frac{e^{-jwN/2}}{e^{-jw/2}} \frac{e^{jwN/2} - e^{-jwN/2}}{e^{+jw/2} - e^{-jw/2}}$$

★ Euler's Identity $\Rightarrow \sin(w) = \frac{e^{jw} - e^{-jw}}{j2}$

★

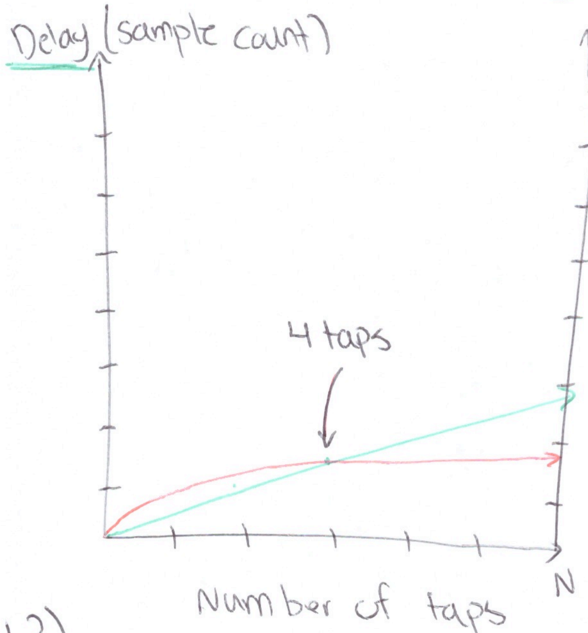
$$H(w) = \frac{1}{N} \frac{e^{-jwN/2}}{e^{-jw/2}} \frac{\cancel{j2}\sin\left(\frac{wN}{2}\right)}{\cancel{j2}\sin\left(\frac{w}{2}\right)} = \frac{1}{N} \underbrace{\frac{e^{-jwN/2}}{e^{-jw/2}}}_{\text{Phase Response}} \underbrace{\frac{\sin\left(\frac{wN}{2}\right)}{\sin\left(\frac{w}{2}\right)}}_{\text{Magnitude Response}}$$

$\boxed{\text{Delay} = \frac{N}{2}}$

$\boxed{\text{★ SNR Boost} = \sqrt{N}}$

↳ Found on the TomRoelandts.com/articles/moving-average-as-a-filter

☆N Filter Taps Uniformly Weighted



a) N=25 taps to boost SNR by factor of 5

b)

1.2)

~~N)~~ M Filter Taps Harmonically Weighted

$$z[n] = \frac{Mx[n] + (M-1)x[n-1] + \ldots + 1 \cdot x[n-(M-1)]}{\frac{1}{2}M(M+1)}$$

~~Impulse Response~~ $h(n) = \sum_{k=1}^{M} A_k (d_k)^n u[n]$

☆ delay $= \frac{1}{\pi}\int_0^{\pi} \tau(w)\,dw$

$\tau(w) = -\phi'(w)$

delay $= \frac{1}{\pi}\left[\phi(0) - \phi(\pi)\right]$

L impulse response: $h[n] = \begin{cases} \frac{2(M-n)}{M(M+1)}, & 0 \le n \le M \\ 0, & \text{else} \end{cases}$

L Frequency Response: $H(e^{jw}) = \frac{2}{M(M+1)} \sum_{n=0}^{M-1} (M-n)e^{-jnw} = \frac{2}{M(M+1)} e^{-jMw} \sum_{n=1}^{M} n e^{jnw}$

$$H(e^{jw}) = \frac{2}{M(M+1)} \frac{Me^{2jw} - (M+1)e^{jw} + e^{-j(M-1)w}}{(1-e^{jw})^2}, \quad w \neq 2k\pi, k \in \mathbb{Z}$$

☆ Wolfram Alpha

if $w = 2k\pi \Rightarrow H(e^{jw}) = H(e^{j2k\pi}) = H(1) = \frac{2}{M(M+1)} \sum_{n=1}^{M} n = 1$

☆ Euler's $\Rightarrow H(e^{jw}) = \frac{2}{M(M+1)} \frac{M(1-e^{jw}) + 1 - e^{-jMw}}{4\sin(w/2)}, \quad w \neq 2k\pi$ $\rightarrow \phi(w)$

$\phi(w) = \arg M(1-e^{-jw}) + 1 - e^{-jMw}, \quad w \neq 2k\pi$

$\phi'(w)$ $\phi(0) = 0 \quad \phi(\pi) = 0 \Rightarrow$ delay $(w) = 0$

From impulse Response & Faulhaber's formula

$$\text{delay} = \frac{\sum_{n=1}^{M-1} n\,h[n]}{\sum_{n=0}^{M-1} h[n]} \Rightarrow \boxed{\text{delay}(M) = \frac{M}{3}} \quad \text{Delay}$$