# Solutions for *Analysis I&II* (Forth Edition) by Terence Tao

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# Contents

# Chapter 1

# Introduction

No exercises in this chapter.

# Chapter 2

# Starting at the beginning: the natural numbers

## 2.1 The Peano axioms

**Axiom 2.1.** 0 *is a natural number.*

**Axiom 2.2.** *If* $n$ *is a natural number, then* $n{+}{+}$ *is also a natural number.*

**Axiom 2.3.** 0 *is not the successor of any natural number; i.e., we have* $n{+}{+} \neq 0$ *for every natural number* $n$.

**Axiom 2.4.** *Different natural numbers must have different successors; i.e., if* $n, m$ *are natural numbers and* $n \neq m$, *then* $n{+}{+} \neq m{+}{+}$. *Equivalently, if* $n{+}{+} = m{+}{+}$, *then we must have* $n = m$.

**Axiom 2.5.** (Principle of mathematical induction). *Let* $P(n)$ *be any property pertaining to a natural number* $n$. *Suppose that* $P(0)$ *is true, and suppose that whenever* $P(n)$ *is true,* $P(n{+}{+})$ *is also true. Then* $P(n)$ *is true for every natural number* $n$.

**Definition 2.1.3.** *We define* 1 *to be the number* $0{+}{+}$, 2 *to be the number* $(0{+}{+}){+}{+}$, *etc.*

**Proposition 2.1.4.** 3 *is a natural number.*

*Proof.* Given the Axiom 2.1, 0 is a natural number. $1 := 0{+}{+}$ is a natural number by Axiom 2.2. Same as $2 := 1{+}{+}$ and $3 := 2{+}{+}$. $\square$

**Proposition 2.1.6.** 4 *is not equal to* 0*.*

*Proof.* By Definition 2.1.3, $4 := 3{+}{+}$, thus $3{+}{+} \overset{\lightning}{=} 0$ what contradicts Axiom 2.3 $\square$

**Proposition 2.1.8.** 6 *is not equal to* 2

*Proof.* Let's assume that $6 = 2$. By Definition 2.1.3, $6 = 5{+}{+}$ and $2 = 1{+}{+}$. That means, $5 = 1$ by Axiom 2.4. Repeating the procedure, we end up with $4 \overset{\lightning}{=} 0$ what contradicts previously proven Proposition 2.1.6 $\square$

**Proposition 2.1.16.** *(Recursive definitions). Suppose for each natural number $n$, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Let $c$ be a natural number. Then we can assign a unique natural number to each natural number $n$, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number $n$.*

*Proof.* First, none of $a_n$ redefines $a_0$ by Axiom 2.3. Second, let's assume that the procedure gives the unique value for $a_n$. Then it also provides the unique value for $a_{n++}$ by Axiom 2.4 what closes the induction. $\square$

## 2.2 Addition

**Definition 2.2.1.** *(Addition of natural numbers). Let $m$ be a natural number. To add zero to $m$, we define $0 + m := m$. Now suppose inductively that we have defined how to add $n$ to $m$. Then we can add $n{+}{+}$ to $m$ by defining $(n{+}{+}) + m := (n + m){+}{+}$.*

**Proposition 2.2.1.a.** *The sum of two natural numbers is again a natural number.*

*Proof.* We induct on $n$. Using Definition 2.2.1, $0 + m := m$ is a natural number by the proposition's assumption. Let's assume that $n + m$ is a natural number. Then $(n{+}{+}) + m := (n + m){+}{+}$ by Definition 2.2.1 and $(n + m){+}{+}$ is a natural number by Axiom 2.2. That closes the induction. □

**Lemma 2.2.2.** *For any natural number $n$, $n + 0 = n$.*

*Proof.* Guess what, we use induction. Given that 0 is a natural number and Definition 2.2.1, $0 + 0 := 0$ what proves the base case. Let's assume that $n + 0 := n$, then $(n{+}{+}) + 0 := (n + 0){+}{+}$ using Definition 2.2.1, and $(n + 0){+}{+} = n{+}{+}$ by the induction assumption what proves the step case. That closes the induction. □

**Lemma 2.2.3.** *For any natural numbers $n$ and $m$, $n + (m{+}{+}) = (n + m){+}{+}$.*

*Proof.* Let's induct on $n$. To prove the base case, $0+(m{+}{+}) := m{+}{+} := (0+m){+}{+}$ using Definition 2.2.1. For the step case, we need to prove $(n{+}{+})+(m{+}{+}) = ((n{+}{+})+m){+}{+}$. Let's assume $n + (m{+}{+}) = (n + m){+}{+}$, then

$$\begin{aligned}
(n{+}{+}) + (m{+}{+}) &:= (n + (m{+}{+})){+}{+} && \text{Definition 2.2.1} \\
&= ((n + m){+}{+}){+}{+} && \text{Inductive Hypothesis} \\
&= ((n{+}{+}) + m){+}{+} && \text{Definition 2.2.1}
\end{aligned}$$

What closes the induction. □

**Corollary 2.2.3.a.** *For any natural number $n$, $n{+}{+} = n + 1$.*

*Proof.* The base case:

$$\begin{aligned}
0{+}{+} &= (0 + 0){+}{+} && \text{Definition 2.2.1} \\
&= 0 + (0{+}{+}) && \text{Lemma 2.2.3} \\
&= 0 + 1 && \text{Definition 2.1.3}
\end{aligned}$$

For the step case, we need to prove $(n{+}{+}){+}{+} = (n{+}{+}) + 1$, assuming $n{+}{+} = n + 1$:

$$
\begin{aligned}
(n{+}{+}){+}{+} &= ((n+0){+}{+}){+}{+} && \text{Definition 2.2.1} \\
&= (n + 0{+}{+}){+}{+} && \text{Lemma 2.2.3} \\
&= (n+1){+}{+} && \text{Definition 2.1.3} \\
&= (n{+}{+}) + 1 && \text{Definition 2.2.1}
\end{aligned}
$$

$\square$

**Proposition 2.2.4.** (Addition is commutative). *For any natural numbers $n$ and $m$, $n + m = m + n$.*

*Proof.* Let's (again) induct on $n$. For the base case we need to prove $0 + m = m + 0$. The left term equals to $m$ by Definition 2.2.1. The right term equals also to $m$ by Lemma 2.2.2. Thus, both terms are equal. For the step case, we need to prove $(n{+}{+}) + m = m + (n{+}{+})$, assuming $n + m = m + n$:

$$
\begin{aligned}
(n{+}{+}) + m &= (n+m){+}{+} && \text{Definition 2.2.1} \\
&= (m+n){+}{+} && \text{Inductive Hypothesis} \\
&= (m + (n{+}{+})) && \text{Lemma 2.2.3}
\end{aligned}
$$

What closes the induction. $\square$

**Proposition 2.2.5.** (Addition is associative). *For any natural numbers $a, b, c$, we have $(a + b) + c = a + (b + c)$.*

*Proof.* Let's induct on $a$. For the base case, we need to prove

$$(0 + b) + c = 0 + (b + c)$$

$$\text{Definition 2.2.1} \qquad b + c = b + c \qquad \text{Lemma 2.2.2}$$

For the step case, we need to prove $((a{+}{+}) + b) + c = (a{+}{+}) + (b + c)$, assuming $(a + b) + c = a + (b + c)$:

$$
\begin{aligned}
((a{+}{+}) + b) + c &= ((a+b){+}{+}) + c && \text{Definition 2.2.1} \\
&= ((a+b)+c){+}{+} && \text{Definition 2.2.1} \\
&= (a + (b+c)){+}{+} && \text{Inductive Hypothesis} \\
&= (a{+}{+}) + (b+c) && \text{Definition 2.2.1}
\end{aligned}
$$

what closes the induction. $\square$

**Proposition 2.2.6.** (Cancellation Law). *Let $a, b, c$ be natural numbers such $a + b = a + c$. Then we have $b = c$.*

*Proof.* Let's induct on $a$. The base case, we need to prove $0 + b = 0 + c$ what leads to $b = c$ by Definition 2.2.1. For the step case, we need to prove if $(a{+}{+}) + b = (a{+}{+}) + c$, assuming $a + b = a + c$, then $b = c$.

$$
\begin{aligned}
(a{+}{+}) + b &= (a{+}{+}) + c && \\
(a + b){+}{+} &= (a + c){+}{+} && \text{Definition 2.2.1} \\
(a + b) &= (a + c) && \text{Axiom 2.4} \\
b &= c && \text{Inductive Hypothesis}
\end{aligned}
$$

□

**Definition 2.2.7.** (Positive natural numbers). *A natural number $n$ is said to be positive iff it is not equal to* $0$.

**Proposition 2.2.8.** *If $a$ is positive and $b$ is a natural number, then $a + b$ is positive.*

*Proof.* Let's induct on $b$. For the base case, $a + 0 = a$ by Definition 2.2.1. The result is positive by the propositional assumption. For the step case, we need to prove $a + (b{+}{+})$ is positive, assuming $a + b$ is positive. $a + (b{+}{+}) = (a + b){+}{+}$ by Lemma 2.2.3. Given Axiom 2.3, $(a + b){+}{+}$ is positive as well, what closes the induction. □

**Corollary 2.2.9.** *If $a$ and $b$ are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$.*

*Proof.* Let's assume that $a \neq 0$ and induct on $b$. For the base case, we'll need to prove $a + 0 = 0$. The left term equals to $a$ by Lemma 2.2.2, what contradicts our assumption $a \neq 0$. Doing the same for $b$, we'll end up with the same outcome. What leads to the conclusion that $a$ and $b$ must be both 0. □

**Lemma 2.2.10.** *Let $a$ be a positive number. Then there exists exactly one natural number $b$ such that $b{+}{+} = a$.*

*Proof.* Suppose for sake of contradiction, there is another natural number $c$ such that $c{+}{+} = a$. Then $b{+}{+} = c{+}{+}$, what makes $b = c$ by Axiom 2.4. □

**Definition 2.2.11.** (Ordering of the natural numbers). *Let $n$ and $m$ be natural numbers. We say that $n$ is greater than or equal to $m$, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number $a$. We say that $n$ is strictly greater than $m$, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.*

**Lemma 2.2.11.a.** *For any natural number $n$, $n{+}{+} > n$.*

*Proof.* Proof by induction. For the base case, we need to prove $0{+}{+} > 0$. The left term $n$ can be represented in the form of $m + a$ as $0 + 1$ by Corollary 2.2.3.a which is equal to 1 by Definition 2.2.1. Then we can conclude that $0{+}{+} > 0$ because $1 \neq 0$ using Definition 2.2.11. For the step case, we need to prove $(n{+}{+}){+}{+} > n{+}{+}$ assuming $n{+}{+} > n$:

$$\begin{aligned} (n{+}{+}){+}{+} &> n{+}{+} \\ n{+}{+} &> n && \text{Axiom 2.4} \end{aligned}$$

what closes the induction using the inductive hypothesis. □

**Proposition 2.2.12.** (Basic properties of order for natural numbers). *Let $a$, $b$, $c$ be natural numbers. Then*

**Proposition 2.2.12.a.** (Order is reflexive) *$a \geq a$.*

*Proof.* The base case $0 \geq 0$: The left term can be represented as $0 = 0 + 0$ by Definition 2.2.1. Thus, $0 \geq 0$ by Definition 2.2.11. The step case $a{+}{+} \geq a{+}{+}$, assuming $a \geq a$. The equation can be transformed into $a \geq a$ by Axiom 2.4 what closes the induction by the inductive hypothesis. □

**Proposition 2.2.12.b.** (Order is transitive) *If $a \geq b$ and $b \geq c$, then $a \geq c$.*

*Proof.* Given Definition 2.2.11 and the propositional assumption, $b$ can be represented as $c+\tilde{c}$ and $a$ can be represented as $b+\tilde{b}$ where $\tilde{b}$ and $\tilde{c}$ are natural numbers. Then combining both representations, $a = b+\tilde{b} = (c+\tilde{c})+\tilde{b} = c+(\tilde{c}+\tilde{b})$ using Proposition 2.2.5. $\tilde{c}+\tilde{b}$ is a natural number by Proposition 2.2.1.a what leads us to $a \geq c$ by Definition 2.2.11. □

**Proposition 2.2.12.c.** (Order is antisymmetric) *If $a \geq b$ and $b \geq a$, then $a = b$.*

*Proof.* By the propositional assumption and Definition 2.2.11, $a = b + \tilde{b}$ and $b = a + \tilde{a}$. Combining both terms, $a = b + \tilde{a} = (a + \tilde{a}) + \tilde{a}$ what equals to $a + (\tilde{a} + \tilde{a})$ using Proposition 2.2.5. Recalling the Cancellation Law from Proposition 2.2.6, $(\tilde{a} + \tilde{a})$ must be equal to 0 since $a = a + 0$ by Definition 2.2.1. By Corollary 2.2.9, $\tilde{a} = 0$. Plugging it back to $b = a + \tilde{a} = a + 0 = a$ what closes the proof. □

**Proposition 2.2.12.d.** (Addition preserves order) $a \geq b$ *if and only if* $a + c \geq b + c$.

*Proof.* □

**Proposition 2.2.12.e.** $a < b$ *if and only if* $a{+}{+} \leq b$.

**Proposition 2.2.12.f.** $a < b$ *if and only if* $b = a + d$ *for some positive number* $d$.

## Appendix A

# The basics of mathematical logic

## A.1 Mathematical statements

**Exercise A.1.1.** *What is the negation of the statement "either $X$ is true or $Y$ is true, but not both"?*

*Solution.* $X$ and $Y$ are false or $X$ and $Y$ are true. □

**Exercise A.1.2.** *What is the negation of the statement "$X$ is true if and only if $Y$ is true"?*

*Solution.* "$X$ is false iff $Y$ is false" □

**Exercise A.1.3.** *Suppose that you have shown that whenever $X$ is true, then $Y$ is true, and whenever $X$ is false, then $Y$ is false. Have you demonstrated that $X$ and $Y$ are logically equivalent? Explain.*

*Solution.* I assume, yes. A **well-defined** statement can be either true or false only (all possible values). Now, we consider the assumptions: (1) if $X$ is false, then $Y$ is false ($Y$ is never true in that case), and (2) if $X$ is true, then $Y$ is false ($Y$ is never true here). Thus, both statements coincides in all possible values what makes them "equivalent". □

**Exercise A.1.4.** *Suppose that you have shown that whenever $X$ is true iff $Y$ is true, and you know that $Y$ is true iff $Z$ is true. Is this enough to show that $X, Y, Z$ are logically equivalent? Explain.*

*Solution.* Yes. We know that the pairs $(X, Y)$ and $(Y, Z)$ are logically equivalent, but we don't know whether $X$ and $Z$ are equivalent. If $X$ is true/false, then $Y$ is true/false, and if $Y$ is true/false, then $Z$ is also true/false. We can also prove the same, but going from left to right ($X \to Y \to Z$) and from right-to-left ($Z \to Y \to X$). □

**Exercise A.1.5.** *Suppose that you have shown that whenever $X$ is true, then $Y$ is true; that if $Y$ is true, then $Z$ is true; and if $Z$ is true, then $X$ is true. Is this enough to show that $X, Y, Z$ are logically equivalent? Explain.*

*Solution.* Yes, for all pairs of statements, we can construct the following. Let's start at $X$. (Forward direction) If $X$ is true, then $Y$ is also true. (Reverse direction) If $Y$ is true, then $Z$ is also true, and if $Z$ is true, then $X$ is also true. That closes the cycle in both ways for all pairs, thus the statements logically equivalent. □

# Appendix B

# The decimal system