Solutions for *Analysis I&II* (Forth Edition) by Terence Tao

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Chapter 1

Introduction

No exercises in this chapter.

Chapter 2

Starting at the beginning: the natural numbers

2.1 The Peano axioms

Axiom 2.1. 0 *is a natural number.*

Axiom 2.2. *If n is a natural number, then $n{+}{+}$ is also a natural number.*

Axiom 2.3. 0 *is not the successor of any natural number; i.e., we have $n{+}{+} \neq 0$ for every natural number n.*

Axiom 2.4. *Different natural numbers must have different successors; i.e., if n, m are natural numbers and $n \neq m$, then $n{+}{+} \neq m{+}{+}$. Equivalently, if $n{+}{+} = m{+}{+}$, then we must have $n = m$.*

Axiom 2.5. *(Principle of mathematical induction). Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.*

Definition 2.1.3. *We define* 1 *to be the number* $0{+}{+}$, 2 *to be the number* $(0{+}{+}){+}{+}$, *etc.*

Proposition 2.1.4. 3 *is a natural number.*

Proof. Given the Axiom 2.1, 0 is a natural number. $1 := 0{+\!+}$ is a natural number by Axiom 2.2. Same as $2 := 1{+\!+}$ and $3 := 2{+\!+}$. $\square$

Proposition 2.1.6. 4 *is not equal to* 0.

Proof. By Definition 2.1.3, $4 := 3{+\!+}$, thus $3{+\!+} \overset{\lightning}{=} 0$ what contradicts Axiom 2.3 $\square$

Proposition 2.1.8. 6 *is not equal to* 2

Proof. Let's assume that $6 = 2$. By Definition 2.1.3, $6 = 5{+\!+}$ and $2 = 1{+\!+}$. That means, $5 = 1$ by Axiom 2.4. Repeating the procedure, we end up with $4 \overset{\lightning}{=} 0$ what contradicts previously proven Proposition 2.1.6 $\square$

Proposition 2.1.16. *(Recursive definitions). Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Let c be a natural number. Then we can assign a $\underline{\text{unique}}$ natural number to each natural number n, such that $a_0 = c$ and $a_{n{+\!+}} = f_n(a_n)$ for each natural number n.*

Proof. First, none of a_n redefines a_0 by Axiom 2.3. Second, let's assume that the procedure gives the unique value for a_n. Then it also provides the unique value for $a_{n{+\!+}}$ by Axiom 2.4 what closes the induction. $\square$

2.2 Addition

Definition 2.2.1. *(Addition of natural numbers). Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{+}{+}$ to m by defining $(n{+}{+}) + m := (n + m){+}{+}$.*

Proposition 2.2.1.a. *The sum of two natural numbers is again a natural number.*

Proof. We induct on n. Using Definition 2.2.1, $0 + m := m$ is a natural number by the proposition's assumption. Let's assume that $n + m$ is a natural number. Then $(n{+}{+}) + m := (n + m){+}{+}$ by Definition 2.2.1 and $(n + m){+}{+}$ is a natural number by Axiom 2.2. That closes the induction. $\square$

Lemma 2.2.2. *For any natural number n, $n + 0 = n$.*

Proof. Guess what, we use induction. Given that 0 is a natural number and Definition 2.2.1, $0 + 0 := 0$ what proves the base case. Let's assume that $n + 0 := n$, then $(n{+}{+}) + 0 := (n + 0){+}{+}$ using Definition 2.2.1, and $(n + 0){+}{+} = n{+}{+}$ by the induction assumption what proves the step case. That closes the induction. $\square$

Lemma 2.2.3. *For any natural numbers of n and m, $n + (m{+}{+}) = (n + m){+}{+}$*

Proof. Let's induct on n. To prove the base case, $0+(m{+}{+}) := m{+}{+} := (0+m){+}{+}$ using Definition 2.2.1. For the step case, we need to prove $(n{+}{+})+(m{+}{+}) = ((n{+}{+})+m){+}{+}$. Let's assume $n + (m{+}{+}) = (n + m){+}{+}$, then

$$\begin{aligned}
(n{+}{+}) + (m{+}{+}) &:= (n + (m{+}{+})){+}{+} && \text{Definition 2.2.1} \\
&= ((n + m){+}{+}){+}{+} && \text{induction assumption} \\
&= ((n{+}{+}) + m){+}{+} && \text{Definition 2.2.1}
\end{aligned}$$

What closes the induction. $\square$