

Solutions for *Analysis I&II* (Forth Edition) by Terence
Tao

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Chapter 1

Introduction

No exercises in this chapter.

Chapter 2

Starting at the beginning: the natural numbers

2.1 The Peano axioms

Axiom 2.1. *0 is a natural number.*

Axiom 2.2. *If n is a natural number, then $n++$ is also a natural number.*

Axiom 2.3. *0 is not the successor of any natural number; i.e., we have $n++ \neq 0$ for every natural number n .*

Axiom 2.4. *Different natural numbers must have different successors; i.e., if n, m are natural numbers and $n \neq m$, then $n++ \neq m++$. Equivalently, if $n++ = m++$, then we must have $n = m$.*

Axiom 2.5. (Principle of mathematical induction). *Let $P(n)$ be any property pertaining to a natural number n . Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n .*

Definition 2.1.3. *We define 1 to be the number $0++$, 2 to be the number $(0++)++$, etc.*

Proposition 2.1.4. *3 is a natural number.*

Proof. Given the Axiom 2.1, 0 is a natural number. $1 := 0++$ is a natural number by Axiom 2.2. Same as $2 := 1++$ and $3 := 2++$. \square

Proposition 2.1.6. *4 is not equal to 0.*

Proof. By Definition 2.1.3, $4 := 3++$, thus $3++ \stackrel{\neq}{=} 0$ what contradicts Axiom 2.3 \square

Proposition 2.1.8. *6 is not equal to 2*

Proof. Let's assume that $6 = 2$. By Definition 2.1.3, $6 = 5++$ and $2 = 1++$. That means, $5 = 1$ by Axiom 2.4. Repeating the procedure, we end up with $4 \stackrel{\neq}{=} 0$ what contradicts previously proven Proposition 2.1.6 \square

Proposition 2.1.16. *(Recursive definitions). Suppose for each natural number n , we have some function $f_n : \mathbb{N} \rightarrow \mathbb{N}$ from the natural numbers to the natural numbers. Let c be a natural number. Then we can assign a unique natural number to each natural number n , such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n .*

Proof. First, none of a_n redefines a_0 by Axiom 2.3. Second, let's assume that the procedure gives the unique value for a_n . Then it also provides the unique value for a_{n++} by Axiom 2.4 what closes the induction. \square

2.2 Addition

Definition 2.2.1. (*Addition of natural numbers*). Let m be a natural number. To add zero to m , we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m . Then we can add $n++$ to m by defining $(n++) + m := (n + m)++$.

Proposition 2.2.1.a. *The sum of two natural numbers is again a natural number.*

Proof. We induct on n . Using Definition 2.2.1, $0 + m := m$ is a natural number by the proposition's assumption. Let's assume that $n + m$ is a natural number. Then $(n++) + m := (n + m)++$ by Definition 2.2.1 and $(n + m)++$ is a natural number by Axiom 2.2. That closes the induction. \square

Lemma 2.2.2. *For any natural number n , $n + 0 = n$.*

Proof. Guess what, we use induction. Given that 0 is a natural number and Definition 2.2.1, $0 + 0 := 0$ what proves the base case. Let's assume that $n + 0 := n$, then $(n++) + 0 := (n + 0)++$ using Definition 2.2.1, and $(n + 0)++ = n++$ by the induction assumption what proves the step case. That closes the induction. \square

Lemma 2.2.3. *For any natural numbers n and m , $n + (m++) = (n + m)++$.*

Proof. Let's induct on n . To prove the base case, $0 + (m++) := m++ := (0 + m)++$ using Definition 2.2.1. For the step case, we need to prove $(n++) + (m++) = ((n++) + m)++$. Let's assume $n + (m++) = (n + m)++$, then

$$\begin{aligned} (n++) + (m++) &:= (n + (m++))++ && \text{Definition 2.2.1} \\ &= ((n + m)++)++ && \text{Inductive Hypothesis} \\ &= ((n++) + m)++ && \text{Definition 2.2.1} \end{aligned}$$

What closes the induction. \square

Corollary 2.2.3.a. *For any natural number n , $n++ = n + 1$.*

Proof. The base case:

$$\begin{aligned} 0++ &= (0 + 0)++ && \text{Definition 2.2.1} \\ &= 0 + (0++) && \text{Lemma 2.2.3} \\ &= 0 + 1 && \text{Definition 2.1.3} \end{aligned}$$

For the step case, we need to prove $(n++)++ = (n++) + 1$, assuming $n++ = n + 1$:

$$\begin{aligned}
(n++)++ &= ((n+0)++)++ && \text{Definition 2.2.1} \\
&= (n+0++)++ && \text{Lemma 2.2.3} \\
&= (n+1)++ && \text{Definition 2.1.3} \\
&= (n++) + 1 && \text{Definition 2.2.1}
\end{aligned}$$

□

Proposition 2.2.4. (Addition is commutative). *For any natural numbers n and m , $n + m = m + n$.*

Proof. Let's (again) induct on n . For the base case we need to prove $0 + m = m + 0$. The left term equals to m by Definition 2.2.1. The right term equals also to m by Lemma 2.2.2. Thus, both terms are equal. For the step case, we need to prove $(n++) + m = m + (n++)$, assuming $n + m = m + n$:

$$\begin{aligned}
(n++) + m &= (n + m)++ && \text{Definition 2.2.1} \\
&= (m + n)++ && \text{Inductive Hypothesis} \\
&= (m + (n++)) && \text{Lemma 2.2.3}
\end{aligned}$$

What closes the induction.

□

Proposition 2.2.5. (Addition is associative). *For any natural numbers a, b, c , we have $(a + b) + c = a + (b + c)$.*

Proof. Let's induct on a . For the base case, we need to prove

$$\begin{aligned}
(0 + b) + c &= 0 + (b + c) \\
\text{Definition 2.2.1} \quad b + c &= b + c \quad \text{Lemma 2.2.2}
\end{aligned}$$

For the step case, we need to prove $((a++) + b) + c = (a++) + (b + c)$, assuming $(a + b) + c = a + (b + c)$:

$$\begin{aligned}
((a++) + b) + c &= ((a + b)++) + c && \text{Definition 2.2.1} \\
&= ((a + b) + c)++ && \text{Definition 2.2.1} \\
&= (a + (b + c))++ && \text{Inductive Hypothesis} \\
&= (a++) + (b + c) && \text{Definition 2.2.1}
\end{aligned}$$

what closes the induction.

□

Proposition 2.2.6. (Cancellation Law). *Let a, b, c be natural numbers such $a + b = a + c$. Then we have $b = c$.*

Proof. Let's induct on a . The base case, we need to prove $0 + b = 0 + c$ what leads to $b = c$ by Definition 2.2.1. For the step case, we need to prove if $(a++) + b = (a++) + c$, assuming $a + b = a + c$, then $b = c$.

$$\begin{array}{ll}
 (a++) + b = (a++) + c & \\
 (a + b)++ = (a + c)++ & \text{Definition 2.2.1} \\
 (a + b) = (a + c) & \text{Axiom 2.4} \\
 b = c & \text{Inductive Hypothesis}
 \end{array}$$

□

Definition 2.2.7. (Positive natural numbers). *A natural number n is said to be positive iff it is not equal to 0.*

Proposition 2.2.8. *If a is positive and b is a natural number, then $a + b$ is positive.*

Proof. Let's induct on b . For the base case, $a + 0 = a$ by Definition 2.2.1. The result is positive by the propositional assumption. For the step case, we need to prove $a + (b++)$ is positive, assuming $a + b$ is positive. $a + (b++) = (a + b)++$ by Lemma 2.2.3. Given Axiom 2.3, $(a + b)++$ is positive as well, what closes the induction. □

Corollary 2.2.9. *If a and b are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$.*

Proof. Let's assume that $a \neq 0$ and induct on b . For the base case, we'll need to prove $a + 0 = 0$. The left term equals to a by Lemma 2.2.2, what contradicts our assumption $a \neq 0$. Doing the same for b , we'll end up with the same outcome. What leads to the conclusion that a and b must be both 0. □

Lemma 2.2.10. *Let a be a positive number. Then there exists exactly one natural number b such that $b++ = a$.*

Proof. Suppose for sake of contradiction, there is another natural number c such that $c++ = a$. Then $b++ = c++$, what makes $b = c$ by Axiom 2.4. □