Solutions for

Analysis I&II (Forth Edition)

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Contents

Chapter 1

Introduction

No exercises in this chapter.

Chapter 2

Starting at the beginning: the natural numbers

2.1 The Peano axioms

Axiom 2.1. 0 *is a natural number.*

Axiom 2.2. *If* n *is a natural number, then* $n{+}{+}$ *is also a natural number.*

Axiom 2.3. 0 *is not the successor of any natural number; i.e., we have* $n{+}{+} \neq 0$ *for every natural number* n*.*

Axiom 2.4. *Different natural numbers must have different successors; i.e., if* n, m *are natural numbers and* $n \neq m$*, then* $n{+}{+} \neq m{+}{+}$*. Equivalently, if* $n{+}{+} = m{+}{+}$*, then we must have* $n = m$*.*

Axiom 2.5. (Principle of mathematical induction)*. Let* $P(n)$ *be any property pertaining to a natural number* n*. Suppose that* $P(0)$ *is true, and suppose that whenever* $P(n)$ *is true,* $P(n{+}{+})$ *is also true. Then* $P(n)$ *is true for every natural number* n*.*

Definition 2.1.3. *We define* 1 *to be the number* $0{+}{+}$*,* 2 *to be the number* $(0{+}{+}){+}{+}$*, etc.*

Proposition 2.1.4. 3 *is a natural number.*

Proof. Given the Axiom 2.1, 0 is a natural number. $1 := 0{+}{+}$ is a natural number by Axiom 2.2. Same as $2 := 1{+}{+}$ and $3 := 2{+}{+}$. $\square$

Proposition 2.1.6. 4 *is not equal to* 0.

Proof. By Definition 2.1.3, $4 := 3{+}{+}$, thus $3{+}{+} \overset{\lightning}{=} 0$ what contradicts Axiom 2.3 $\square$

Proposition 2.1.8. 6 *is not equal to* 2

Proof. Let's assume that $6 = 2$. By Definition 2.1.3, $6 = 5{+}{+}$ and $2 = 1{+}{+}$. That means, $5 = 1$ by Axiom 2.4. Repeating the procedure, we end up with $4 \overset{\lightning}{=} 0$ what contradicts previously proven Proposition 2.1.6 $\square$

Lemma 2.1.8.a. *For any natural number n, $n \neq n{+}{+}$.*

Proof. For the base case, $0 \neq 0{+}{+}$ by Axiom 2.3. For the step case, we need to prove if $n \neq n{+}{+}$, then $n{+}{+} \neq (n{+}{+}){+}{+}$. Let's assume that $n{+}{+} = (n{+}{+}){+}{+}$. Then by Axiom 2.4, $n \overset{\lightning}{=} n{+}{+}$, what contradicts the inductive hypothesis. Now we have closed the induction. $\square$

Proposition 2.1.16. *(Recursive definitions). Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Let c be a natural number. Then we can assign a* <u>*unique*</u> *natural number to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.*

Proof. First, none of a_n redefines a_0 by Axiom 2.3. Second, let's assume that the procedure gives the unique value for a_n. Then it also provides the unique value for a_{n++} by Axiom 2.4 what closes the induction. $\square$

2.2 Addition

Definition 2.2.1. *(Addition of natural numbers). Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{+}{+}$ to m by defining $(n{+}{+}) + m := (n + m){+}{+}$.*

Proposition 2.2.1.a. *The sum of two natural numbers is again a natural number.*

Proof. We induct on n. Using Definition 2.2.1, $0 + m := m$ is a natural number by the proposition's assumption. Let's assume that $n + m$ is a natural number. Then $(n{+}{+}) + m := (n + m){+}{+}$ by Definition 2.2.1 and $(n + m){+}{+}$ is a natural number by Axiom 2.2. That closes the induction. □

Lemma 2.2.2. *For any natural number n, $n + 0 = n$.*

Proof. Guess what, we use induction. Given that 0 is a natural number and Definition 2.2.1, $0 + 0 := 0$ what proves the base case. Let's assume that $n + 0 := n$, then $(n{+}{+}) + 0 := (n + 0){+}{+}$ using Definition 2.2.1, and $(n + 0){+}{+} = n{+}{+}$ by the induction assumption what proves the step case. That closes the induction. □

Lemma 2.2.3. *For any natural numbers n and m, $n + (m{+}{+}) = (n + m){+}{+}$.*

Proof. Let's induct on n. To prove the base case, $0+(m{+}{+}) := m{+}{+} := (0+m){+}{+}$ using Definition 2.2.1. For the step case, we need to prove $(n{+}{+})+(m{+}{+}) = ((n{+}{+})+m){+}{+}$. Let's assume $n + (m{+}{+}) = (n + m){+}{+}$, then

$$
\begin{aligned}
(n{+}{+}) + (m{+}{+}) &:= (n + (m{+}{+})){+}{+} && \text{Definition 2.2.1} \\
&= ((n + m){+}{+}){+}{+} && \text{Inductive Hypothesis} \\
&= ((n{+}{+}) + m){+}{+} && \text{Definition 2.2.1}
\end{aligned}
$$

What closes the induction. □

Corollary 2.2.3.a. *For any natural number n, $n{+}{+} = n + 1$.*

Proof. The base case:

$$
\begin{aligned}
0{+}{+} &= (0 + 0){+}{+} && \text{Definition 2.2.1} \\
&= 0 + (0{+}{+}) && \text{Lemma 2.2.3} \\
&= 0 + 1 && \text{Definition 2.1.3}
\end{aligned}
$$

For the step case, we need to prove $(n{+}{+}){+}{+} = (n{+}{+}) + 1$, assuming $n{+}{+} = n + 1$:

$$\begin{aligned}
(n{+}{+}){+}{+} &= ((n+0){+}{+}){+}{+} && \text{Definition 2.2.1}\\
&= (n+0{+}{+}){+}{+} && \text{Lemma 2.2.3}\\
&= (n+1){+}{+} && \text{Definition 2.1.3}\\
&= (n{+}{+})+1 && \text{Definition 2.2.1}
\end{aligned}$$

□

Proposition 2.2.4. (Addition is commutative). *For any natural numbers n and m, $n + m = m + n$.*

Proof. Let's (again) induct on n. For the base case we need to prove $0 + m = m + 0$. The left term equals to m by Definition 2.2.1. The right term equals also to m by Lemma 2.2.2. Thus, both terms are equal. For the step case, we need to prove $(n{+}{+}) + m = m + (n{+}{+})$, assuming $n + m = m + n$:

$$\begin{aligned}
(n{+}{+})+m &= (n+m){+}{+} && \text{Definition 2.2.1}\\
&= (m+n){+}{+} && \text{Inductive Hypothesis}\\
&= (m+(n{+}{+})) && \text{Lemma 2.2.3}
\end{aligned}$$

What closes the induction. □

Proposition 2.2.5. (Addition is associative). *For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

Proof. Let's induct on a. For the base case, we need to prove

$$(0+b)+c = 0+(b+c)$$

$$\text{Definition 2.2.1} \qquad b+c = b+c \qquad \text{Lemma 2.2.2}$$

For the step case, we need to prove $((a{+}{+}) + b) + c = (a{+}{+}) + (b + c)$, assuming $(a + b) + c = a + (b + c)$:

$$\begin{aligned}
((a{+}{+})+b)+c &= ((a+b){+}{+})+c && \text{Definition 2.2.1}\\
&= ((a+b)+c){+}{+} && \text{Definition 2.2.1}\\
&= (a+(b+c)){+}{+} && \text{Inductive Hypothesis}\\
&= (a{+}{+})+(b+c) && \text{Definition 2.2.1}
\end{aligned}$$

what closes the induction. □

Proposition 2.2.6. (Cancellation Law). *Let a, b, c be natural numbers such $a + b = a + c$. Then we have $b = c$.*

Proof. Let's induct on a. The base case, we need to prove $0 + b = 0 + c$ what leads to $b = c$ by Definition 2.2.1. For the step case, we need to prove if $(a{+\!+}) + b = (a{+\!+}) + c$, assuming $a + b = a + c$, then $b = c$.

$$
\begin{aligned}
(a{+\!+}) + b &= (a{+\!+}) + c && \\
(a + b){+\!+} &= (a + c){+\!+} && \text{Definition 2.2.1} \\
(a + b) &= (a + c) && \text{Axiom 2.4} \\
b &= c && \text{Inductive Hypothesis}
\end{aligned}
$$

□

Definition 2.2.7. (Positive natural numbers). *A natural number n is said to be positive iff it is not equal to 0.*

Proposition 2.2.8. *If a is positive and b is a natural number, then $a + b$ is positive.*

Proof. Let's induct on b. For the base case, $a + 0 = a$ by Definition 2.2.1. The result is positive by the propositional assumption. For the step case, we need to prove $a + (b{+\!+})$ is positive, assuming $a + b$ is positive. $a + (b{+\!+}) = (a + b){+\!+}$ by Lemma 2.2.3. Given Axiom 2.3, $(a + b){+\!+}$ is positive as well, what closes the induction. □

Corollary 2.2.9. *If a and b are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$.*

Proof. Let's assume that $a \neq 0$ and induct on b. For the base case, we'll need to prove $a + 0 = 0$. The left term equals to a by Lemma 2.2.2, what contradicts our assumption $a \neq 0$. Doing the same for b, we'll end up with the same outcome. What leads to the conclusion that a and b must be both 0. □

Lemma 2.2.10. *Let a be a positive number. Then there exists exactly one natural number b such that $b{+\!+} = a$.*

Proof. Suppose for sake of contradiction, there is another natural number c such that $c{+\!+} = a$. Then $b{+\!+} = c{+\!+}$, what makes $b = c$ by Axiom 2.4. □

Definition 2.2.11. (Ordering of the natural numbers). *Let n and m be natural numbers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.*

Lemma 2.2.11.a. *For any natural number n, $n{+}{+} > n$.*

Proof. For any natural number, we know that $n{+}{+} = n + 1$ from Corollary 2.2.3.a. Then using the fact that $n \neq n{+}{+}$ from Lemma 2.1.8.a, we can conclude $n{+}{+} > n$ by Definition 2.2.11. $\square$

Proposition 2.2.12. (Basic properties of order for natural numbers). *Let a, b, c be natural numbers. Then*

Proposition 2.2.12.a. (Order is reflexive) *$a \geq a$.*

Proof. Any natural number a can be represented $a = a + 0$ using Lemma 2.2.2 what proves by Definition 2.2.11 that $a \geq a$. $\square$

Proposition 2.2.12.b. (Order is transitive) *If $a \geq b$ and $b \geq c$, then $a \geq c$.*

Proof. Given Definition 2.2.11 and the propositional assumption, b can be represented as $c+\tilde{c}$ and a can be represented as $b+\tilde{b}$ where $\tilde{b}$ and $\tilde{c}$ are natural numbers. Then combining both representations, $a = b+\tilde{b} = (c+\tilde{c})+\tilde{b} = c+(\tilde{c}+\tilde{b})$ using Proposition 2.2.5. $\tilde{c}+\tilde{b}$ is a natural number by Proposition 2.2.1.a what leads us to $a \geq c$ by Definition 2.2.11. $\square$

Proposition 2.2.12.c. (Order is antisymmetric) *$a \geq b$ and $b \geq a$ iff $a = b$.*
(Original) if $a \geq b$ and $b \geq a$, then $a = b$.

Proof. The forward pass: by the propositional assumption and Definition 2.2.11, $a = b + \tilde{b}$ and $b = a + \tilde{a}$. Combining both terms, $a = b + \tilde{a} = (a + \tilde{a}) + \tilde{a}$ what equals to $a + (\tilde{a} + \tilde{a})$ using Proposition 2.2.5. Recalling Cancellation Law from Proposition 2.2.6, $(\tilde{a}+\tilde{a})$ must be equal to 0 since $a = a+0$ by Definition 2.2.1. By Corollary 2.2.9, $\tilde{a} = 0$. Plugging it back to $b = a + \tilde{a} = a + 0 = a$.

The reverse pass: if $a = b$, then $a \geq b$ and $b \geq a$. If $a = b$, we can use the reflexivity of order from Proposition 2.2.12.a to transform $a \geq a$ and $b \geq b$ to $a \geq b$ and $b \geq a$ respectively. That closes the proof in both directions. $\square$

Proposition 2.2.12.d. (Addition preserves order) *$a \geq b$ if and only if $a + c \geq b + c$.*

Proof. Let's start with the true case: if $a \geq b$, then $a + c \geq b + c$. If $a \geq b$, then we can represent $a = b + \tilde{b}$ by Definition 2.2.11. Plugging the expression into $a + c$ results in $(b + \tilde{b}) + c$. Using Proposition 2.2.5 and Proposition 2.2.4, we can transform the expression into $(b + c) + \tilde{b}$ what proves $a + c \geq b + c$ by Definition 2.2.11.

Now the false case: if $a < b$, then $a + c < b + c$. If $a < b$, then $b = a + \tilde{a}$ and $b \neq a$ by Definition 2.2.11. Using the expression in $b + c$ results in $(a + \tilde{a}) + c$. Again using Propositions 2.2.5 and 2.2.4, we end up with $(a + c) + \tilde{a}$, what proves $b + c \geq a + c$ by Definition 2.2.11. Now we need to show that $(a + c) \neq (b + c)$. Let's assume they are equal. Then by Cancellation Law from Proposition 2.2.6, we can get rid of c, what results in $a \overset{\lightning}{=} b$. That contradicts the antecedent $a < b$, from which we know that $a \neq b$. Thus, $a + c < b + c$, what closes the proof for all cases. ∎

Proposition 2.2.12.e. *$a < b$ if and only if $b = a + (d{+}{+})$ for any natural number d.*

Proof. For the true case we need to prove that if $a < b$, then $b = a + d$ and $d \neq 0$. If $a < b$, then $b = a + d$ and $b \neq a$ by Definition 2.2.11. Let's assume that d is not a positive natural number (i.e. $d = 0$). Then $a + d = a + 0 = a \overset{\lightning}{=} b$ by the antecedent $b \neq a$.

For the false case, we need to prove that if $a \geq b$, then $b \neq a + d$ or $d = 0$. If $a \geq b$, then $a = b + d$ and d is a natural number (including zero) by Definition 2.2.11, what contradicts $d = 0$. That closes both cases. ∎

Proposition 2.2.12.f. *$a < b$ if and only if $a{+}{+} \leq b$.*

Proof. For true case, we need to prove if $a{+}{+} \leq b$, then $a < b$. Combining Definition 2.2.11 and associativity from Proposition 2.2.5, we can infer $b = (a{+}{+}) + d = a + (d{+}{+})$ where $d{+}{+}$ is a positive natural number by Axiom 2.3 and Definition 2.2.7. Thus, we can conclude that $b > a$ by Proposition 2.2.12.e.

For the false case, we need to prove if $a{+}{+} > b$[1], then $a \geq b$. If $a{+}{+} > b$, then $a{+}{+} = b + (d{+}{+}) = (b + d){+}{+}$ using Proposition 2.2.12.e and Definition 2.2.1, where $d{+}{+}$ is a positive number by Axiom 2.3. Then by Axiom 2.4, we can infer $a = b + d$ what proves $a \geq b$ by Definition 2.1.3. ∎

Lemma 2.2.12.g. $0 \leq b$ *for all natural numbers.*

Proof. b can be represented as $b = 0 + b$ by Definition 2.2.1, what makes it $0 \leq b$ by Definition 2.2.11. ∎

[1] Do we need to prove that this is the false case of $a{+}{+} \leq b$?

Proposition 2.2.13. (Trichotomy of order for natural numbers). *Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b, a = b, a > b$.*

Proof. Let's start that **no more than one** statement holds at a time. If $a = b$, then neither $a < b$ nor $a > b$ holds by Definition 2.2.11. If $a > b$ and $a < b$, then by Definition 2.2.11, $a \geq b$, $a \leq b$ and $a \neq b$. If $a \geq$b and $a \leq b$, then $a \overset{\lightning}{=} b$ using Proposition 2.2.12.c, what contradicts the antecedent. Thus, no more than one statement holds at a time.

Now let's prove that there is **at least one** statement holds inducting on a. For the base case, $0 \leq b$ by Lemma 2.2.12.g. For the step case, suppose that one of the trichotomy statements holds for a. We need to prove whether any holds for a++. If $a < b$, then $a\text{++} \leq b$ by Proposition 2.2.12.f. If $a = b$, then $a\text{++} > b$ by Lemma 2.2.11.a. If $a > b$, then $a = b + (d\text{++})$ by Proposition 2.2.12.e. Then we can take the successor $a\text{++} = (b + (d\text{++}))\text{++}$ by Axiom 2.4, what equals to $b + (d\text{++})\text{++}$ by Definition 2.2.1. $(d\text{++})\text{++}$ is a positive number by Definition 2.2.7 and Axiom 2.3. Thus, $a\text{++} > b$ still holds by Proposition 2.2.12.e. That covers all possible cases and closes the induction. □

Proposition 2.2.14. (Strong principle of induction). *Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $n \geq m_0$, we have the following implication: if $P(m)$ is true for all natural numbers $m_0 \leq m < n$, then $P(n)$ is also true (in particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous). Then we can conclude that $P(n)$ is true for all natural numbers $n \geq m_0$.*

Proof. We'll induct on n. For the base case m_0: using Definition 2.2.11, $P(m)$ is true for $m_0 \leq m \leq m_0$ and $m \neq m_0$. If $m_0 \leq m$ and $m_0 \geq m$, then $m_0 \overset{\lightning}{=} m$ by Proposition 2.2.12.c, what contradicts the fact that $m \neq m_0$. That means that there are no such m that satisfy $m_0 \leq m < m_0$. Thus, $P(m)$ is vacuously true for $m_0 \leq m < m_0$.

For the step case, suppose $P(m)$ for $m_0 \leq m < n \implies P(n)$, we need to prove $P(m)$ for $m_0 \leq m < n\text{++} \implies P(n\text{++})$. Since $P(m)$ holds for $m_0 \leq m < n$ and for $m = n$ (due to the given implication on $P(n)$), $P(m)$ holds for $m_0 \leq m \leq n$. The latter term is equivalent to $m_0 \leq m < n\text{++}$ by Proposition 2.2.12.f. That means $P(m)$ is true for $m_0 \leq m < n\text{++}$, and it means $P(n\text{++})$ is true. That closes the induction. □

Proposition 2.2.15. (Principle of backwards induction). *Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever*

$P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is true. Prove that $P(m)$ is true for all natural numbers $m \leq n$.

Proof. Let's induct on n. If $n = 0$, then $m \leq 0$. Combining with Lemma 2.2.12.g, we can conclude that $m = 0$ by Proposition 2.2.12.c. Since $P(n)$ and $n = m$ are true, we proved $P(m)$ for $m \leq n$.

For the step case, suppose the following implication holds: $P(n) \implies P(m)$ is true for $m \leq n$. Now we need to prove $P(n{+}{+}) \implies P(m)$ is true for $m \leq n{+}{+}$. Let's assume that $P(n{+}{+})$ is true, then $P(n)$ is also true by the "backward" property. If $P(n)$ is true, then $P(m)$ is true for $m \leq n$. Since $P(m)$ is true for $m \leq n$ and for $m = n{+}{+}$ (by our assumption), then $P(m)$ is true for $m \leq n{+}{+}$. That proves the implication and closes the induction. □

Proposition 2.2.16. (Principle of induction starting from the base case n). *Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m)$ is true, $P(m{+}{+})$ is true. Show that if $P(n)$ is true, then $P(m)$ is true for all $m \geq n$.*

Proof. Since $m \geq n$, then we can represent $m = n + k$. So, let's induct on k. For the base case $k = 0$, $m = n$. Since $P(n)$ is true and $m = n$, then $P(m)$ is also true.

For the step case, we assume if $P(n)$ is true, then $P(n+k)$ is true for all $n+k \geq n$. Now we need to prove that the same holds for $P(n + (k{+}{+}))$. Given the fact that $m = n + k$, $n + (k{+}{+}) = (n + k){+}{+}$ by Definition 2.2.1, and the property assumption, if $P(m)$ is true, then $P(m{+}{+})$ is also true, we can conclude that $P(n + (k{+}{+}))$ is true. That closes the induction. □

2.3 Multiplication

Definition 2.3.1. (Multiplication of natural numbers). *Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n{+}{+}$ to m by defining $(n{+}{+}) \times m := (n \times m) + m$.*

Appendix A

The basics of mathematical logic

A.1 Mathematical statements

Exercise A.1.1. *What is the negation of the statement "either X is true or Y is true, but not both"?*

Solution. X and Y are false or X and Y are true. $\square$

Exercise A.1.2. *What is the negation of the statement "X is true if and only if Y is true"?*

Solution. "X is false iff Y is false" $\square$

Exercise A.1.3. *Suppose that you have shown that whenever X is true, then Y is true, and whenever X is false, then Y is false. Have you demonstrated that X and Y are logically equivalent? Explain.*

Solution. I assume, yes. A **well-defined** statement can be either true or false only (all possible values). Now, we consider the assumptions: (1) if X is false, then Y is false (Y is never true in that case), and (2) if X is true, then Y is false (Y is never true here). Thus, both statements coincides in all possible values what makes them "equivalent". $\square$

Exercise A.1.4. *Suppose that you have shown that X is true iff Y is true, and you know that Y is true iff Z is true. Is this enough to show that X, Y, Z are logically equivalent? Explain.*

Solution. Yes. We know that the pairs (X, Y) and (Y, Z) are logically equivalent, but we don't know whether X and Z are equivalent. If X is true/false, then Y is true/false, and if Y is true/false, then Z is also true/false. We can also prove the same, but going from left to right ($X \to Y \to Z$) and from right-to-left ($Z \to Y \to X$). $\square$

Exercise A.1.5. *Suppose that you have shown that whenever X is true, then Y is true; that if Y is true, then Z is true; and if Z is true, then X is true. Is this enough to show that X, Y, Z are logically equivalent? Explain.*

Solution. Yes, for all pairs of statements, we can construct the following. Let's start at X. (Forward direction) If X is true, then Y is also true. (Reverse direction) If Y is true, then Z is also true, and if Z is true, then X is also true. That closes the cycle in both ways for all pairs, thus the statements logically equivalent. $\square$

A.2 Implication

The main ideas in the chapter are the following:

- Vacuous hypothesis is a hypothesis where an antecedent is false. Since false implies false/true is always true, it does not provide any new information.
- We can also draw the following conclusions:

$$X \implies Y \iff \neg Y \implies \neg X \qquad \text{Contrapositive}$$
$$X \implies Y \not\Longleftrightarrow Y \implies X \qquad \text{Converse}$$
$$X \implies Y \not\Longleftrightarrow \neg X \implies \neg Y \qquad \text{Inverse}$$

- Although, vacuous hypothesis can be viewed as "useless", it can still be used for a proof by contradiction, for instance. For that case, we need to have a consequent that is knows to be false.

A.3 The structure of proofs

Appendix B

The decimal system