

Solutions for  
*Analysis I&II* (Forth Edition)  
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February 3, 2026

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# **Chapter 1**

## **Introduction**

No exercises in this chapter.

## Chapter 2

Starting at the beginning: the  
natural numbers

## 2.1 The Peano axioms

**Axiom 2.1.** 0 is a natural number.

**Axiom 2.2.** If  $n$  is a natural number, then  $n++$  is also a natural number.

**Axiom 2.3.** 0 is not the successor of any natural number; i.e., we have  $n++ \neq 0$  for every natural number  $n$ .

**Axiom 2.4.** Different natural numbers must have different successors; i.e., if  $n, m$  are natural numbers and  $n \neq m$ , then  $n++ \neq m++$ . Equivalently, if  $n++ = m++$ , then we must have  $n = m$ .

**Axiom 2.5.** (Principle of mathematical induction). Let  $P(n)$  be any property pertaining to a natural number  $n$ . Suppose that  $P(0)$  is true, and suppose that whenever  $P(n)$  is true,  $P(n++)$  is also true. Then  $P(n)$  is true for every natural number  $n$ .

**Definition 2.1.3.** We define 1 to be the number  $0++$ , 2 to be the number  $(0++)++$ , etc.

**Proposition 2.1.4.** *3 is a natural number.*

*Proof.* Given the Axiom 2.1, 0 is a natural number.  $1 := 0++$  is a natural number by Axiom 2.2. Same as  $2 := 1++$  and  $3 := 2++$ .  $\square$

**Proposition 2.1.6.** *4 is not equal to 0.*

*Proof.* By Definition 2.1.3,  $4 := 3++$ , thus  $3++ \not\equiv 0$  what contradicts Axiom 2.3  $\square$

**Proposition 2.1.8.** *6 is not equal to 2*

*Proof.* Let's assume that  $6 = 2$ . By Definition 2.1.3,  $6 = 5++$  and  $2 = 1++$ . That means,  $5 = 1$  by Axiom 2.4. Repeating the procedure, we end up with  $4 \not\equiv 0$  what contradicts previously proven Proposition 2.1.6  $\square$

**Lemma 2.1.8.a.** *For any natural number  $n$ ,  $n \neq n++$ .*

*Proof.* For the base case,  $0 \neq 0++$  by Axiom 2.3. For the step case, we need to prove if  $n \neq n++$ , then  $n++ \neq (n++)++$ . Let's assume that  $n++ = (n++)++$ . Then by Axiom 2.4,  $n \not\equiv n++$ , what contradicts the inductive hypothesis. Now we have closed the induction.  $\square$

**Proposition 2.1.16.** *(Recursive definitions). Suppose for each natural number  $n$ , we have some function  $f_n : \mathbb{N} \rightarrow \mathbb{N}$  from the natural numbers to the natural numbers. Let  $c$  be a natural number. Then we can assign a unique natural number to each natural number  $n$ , such that  $a_0 = c$  and  $a_{n++} = f_n(a_n)$  for each natural number  $n$ .*

*Proof.* First, none of  $a_n$  redefines  $a_0$  by Axiom 2.3. Second, let's assume that the procedure gives the unique value for  $a_n$ . Then it also provides the unique value for  $a_{n++}$  by Axiom 2.4 what closes the induction.  $\square$

## 2.2 Addition

**Definition 2.2.1.** (*Addition of natural numbers*). Let  $m$  be a natural number. To add zero to  $m$ , we define  $0 + m := m$ . Now suppose inductively that we have defined how to add  $n$  to  $m$ . Then we can add  $n++$  to  $m$  by defining  $(n++) + m := (n + m)++$ .

**Proposition 2.2.1.a.** *The sum of two natural numbers is again a natural number.*

*Proof.* We induct on  $n$ . Using Definition 2.2.1,  $0 + m := m$  is a natural number by the proposition's assumption. Let's assume that  $n + m$  is a natural number. Then  $(n++) + m := (n + m)++$  by Definition 2.2.1 and  $(n + m)++$  is a natural number by Axiom 2.2. That closes the induction.  $\square$

**Lemma 2.2.2.** *For any natural number  $n$ ,  $n + 0 = n$ .*

*Proof.* Guess what, we use induction. Given that 0 is a natural number and Definition 2.2.1,  $0 + 0 := 0$  what proves the base case. Let's assume that  $n + 0 := n$ , then  $(n++) + 0 := (n + 0)++$  using Definition 2.2.1, and  $(n + 0)++ = n++$  by the induction assumption what proves the step case. That closes the induction.  $\square$

**Lemma 2.2.3.** *For any natural numbers  $n$  and  $m$ ,  $n + (m++) = (n + m)++$ .*

*Proof.* Let's induct on  $n$ . To prove the base case,  $0 + (m++) := m++ := (0 + m)++$  using Definition 2.2.1. For the step case, we need to prove  $(n++) + (m++) = ((n++) + m)++$ . Let's assume  $n + (m++) = (n + m)++$ , then

$$\begin{aligned} (n++) + (m++) &:= (n + (m++))++ && \text{Definition 2.2.1} \\ &= ((n + m)++)++ && \text{Inductive Hypothesis} \\ &= ((n++) + m)++ && \text{Definition 2.2.1} \end{aligned}$$

What closes the induction.  $\square$

**Corollary 2.2.3.a.** *For any natural number  $n$ ,  $n++ = n + 1$ .*

*Proof.* The base case:

$$\begin{aligned} 0++ &= (0 + 0)++ && \text{Definition 2.2.1} \\ &= 0 + (0++) && \text{Lemma 2.2.3} \\ &= 0 + 1 && \text{Definition 2.1.3} \end{aligned}$$

For the step case, we need to prove  $(n++)++ = (n++) + 1$ , assuming  $n++ = n + 1$ :

$$\begin{aligned}
 (n++)++ &= ((n + 0)++)++ && \text{Definition 2.2.1} \\
 &= (n + 0++)++ && \text{Lemma 2.2.3} \\
 &= (n + 1)++ && \text{Definition 2.1.3} \\
 &= (n++) + 1 && \text{Definition 2.2.1}
 \end{aligned}$$

□

**Proposition 2.2.4.** (Addition is commutative). *For any natural numbers  $n$  and  $m$ ,  $n + m = m + n$ .*

*Proof.* Let's (again) induct on  $n$ . For the base case we need to prove  $0 + m = m + 0$ . The left term equals to  $m$  by Definition 2.2.1. The right term equals also to  $m$  by Lemma 2.2.2. Thus, both terms are equal. For the step case, we need to prove  $(n++) + m = m + (n++)$ , assuming  $n + m = m + n$ :

$$\begin{aligned}
 (n++) + m &= (n + m)++ && \text{Definition 2.2.1} \\
 &= (m + n)++ && \text{Inductive Hypothesis} \\
 &= (m + (n++)) && \text{Lemma 2.2.3}
 \end{aligned}$$

What closes the induction. □

**Proposition 2.2.5.** (Addition is associative). *For any natural numbers  $a, b, c$ , we have  $(a + b) + c = a + (b + c)$ .*

*Proof.* Let's induct on  $a$ . For the base case, we need to prove

$$\begin{aligned}
 (0 + b) + c &= 0 + (b + c) \\
 \text{Definition 2.2.1} && b + c = b + c && \text{Lemma 2.2.2}
 \end{aligned}$$

For the step case, we need to prove  $((a++) + b) + c = (a++) + (b + c)$ , assuming  $(a + b) + c = a + (b + c)$ :

$$\begin{aligned}
 ((a++) + b) + c &= ((a + b)++) + c && \text{Definition 2.2.1} \\
 &= ((a + b) + c)++ && \text{Definition 2.2.1} \\
 &= (a + (b + c))++ && \text{Inductive Hypothesis} \\
 &= (a++) + (b + c) && \text{Definition 2.2.1}
 \end{aligned}$$

what closes the induction. □

**Proposition 2.2.6.** (Cancellation Law). *Let  $a, b, c$  be natural numbers such  $a + b = a + c$ . Then we have  $b = c$ .*

*Proof.* Let's induct on  $a$ . The base case, we need to prove  $0 + b = 0 + c$  what leads to  $b = c$  by Definition 2.2.1. For the step case, we need to prove if  $(a++) + b = (a++) + c$ , assuming  $a + b = a + c$ , then  $b = c$ .

$$\begin{array}{ll} (a++) + b = (a++) + c & \\ (a + b)++ = (a + c)++ & \text{Definition 2.2.1} \\ (a + b) = (a + c) & \text{Axiom 2.4} \\ b = c & \text{Inductive Hypothesis} \end{array}$$

□

**Definition 2.2.7.** (Positive natural numbers). *A natural number  $n$  is said to be positive iff it is not equal to 0.*

**Proposition 2.2.8.** *If  $a$  is positive and  $b$  is a natural number, then  $a + b$  is positive.*

*Proof.* Let's induct on  $b$ . For the base case,  $a + 0 = a$  by Definition 2.2.1. The result is positive by the propositional assumption. For the step case, we need to prove  $a + (b++)$  is positive, assuming  $a + b$  is positive.  $a + (b++) = (a + b)++$  by Lemma 2.2.3. Given Axiom 2.3,  $(a + b)++$  is positive as well, what closes the induction. □

**Corollary 2.2.9.** *If  $a$  and  $b$  are natural numbers such that  $a + b = 0$ , then  $a = 0$  and  $b = 0$ .*

*Proof.* Let's assume that  $a \neq 0$  and induct on  $b$ . For the base case, we'll need to prove  $a + 0 = 0$ . The left term equals to  $a$  by Lemma 2.2.2, what contradicts our assumption  $a \neq 0$ . Doing the same for  $b$ , we'll end up with the same outcome. What leads to the conclusion that  $a$  and  $b$  must be both 0. □

**Lemma 2.2.10.** *Let  $a$  be a positive number. Then there exists exactly one natural number  $b$  such that  $b++ = a$ .*

*Proof.* Suppose for sake of contradiction, there is another natural number  $c$  such that  $c++ = a$ . Then  $b++ = c++$ , what makes  $b = c$  by Axiom 2.4. □

**Definition 2.2.11.** (Ordering of the natural numbers). Let  $n$  and  $m$  be natural numbers. We say that  $n$  is greater than or equal to  $m$ , and write  $n \geq m$  or  $m \leq n$ , iff we have  $n = m + a$  for some natural number  $a$ . We say that  $n$  is strictly greater than  $m$ , and write  $n > m$  or  $m < n$ , iff  $n \geq m$  and  $n \neq m$ .

**Lemma 2.2.11.a.** For any natural number  $n$ ,  $n++ > n$ .

*Proof.* For any natural number, we know that  $n++ = n + 1$  from Corollary 2.2.3.a. Then using the fact that  $n \neq n++$  from Lemma 2.1.8.a, we can conclude  $n++ > n$  by Definition 2.2.11.  $\square$

**Proposition 2.2.12.** (Basic properties of order for natural numbers). Let  $a, b, c$  be natural numbers. Then

**Proposition 2.2.12.a.** (Order is reflexive)  $a \geq a$ .

*Proof.* Any natural number  $a$  can be represented  $a = a + 0$  using Lemma 2.2.2 what proves by Definition 2.2.11 that  $a \geq a$ .  $\square$

**Proposition 2.2.12.b.** (Order is transitive) If  $a \geq b$  and  $b \geq c$ , then  $a \geq c$ .

*Proof.* Given Definition 2.2.11 and the propositional assumption,  $b$  can be represented as  $c + \tilde{c}$  and  $a$  can be represented as  $b + \tilde{b}$  where  $\tilde{b}$  and  $\tilde{c}$  are natural numbers. Then combining both representations,  $a = b + \tilde{b} = (c + \tilde{c}) + \tilde{b} = c + (\tilde{c} + \tilde{b})$  using Proposition 2.2.5.  $\tilde{c} + \tilde{b}$  is a natural number by Proposition 2.2.1.a what leads us to  $a \geq c$  by Definition 2.2.11.  $\square$

**Proposition 2.2.12.c.** (Order is antisymmetric)  $a \geq b$  and  $b \geq a$  iff  $a = b$ .

(Original) if  $a \geq b$  and  $b \geq a$ , then  $a = b$ .

*Proof.* The forward pass: by the propositional assumption and Definition 2.2.11,  $a = b + \tilde{b}$  and  $b = a + \tilde{a}$ . Combining both terms,  $a = b + \tilde{b} = (a + \tilde{a}) + \tilde{b}$  what equals to  $a + (\tilde{a} + \tilde{b})$  using Proposition 2.2.5. Recalling Cancellation Law from Proposition 2.2.6,  $(\tilde{a} + \tilde{b})$  must be equal to 0 since  $a = a + 0$  by Definition 2.2.1. By Corollary 2.2.9,  $\tilde{a} = 0$ . Plugging it back to  $b = a + \tilde{a} = a + 0 = a$ .

The reverse pass: if  $a = b$ , then  $a \geq b$  and  $b \geq a$ . If  $a = b$ , we can use the reflexivity of order from Proposition 2.2.12.a to transform  $a \geq a$  and  $b \geq b$  to  $a \geq b$  and  $b \geq a$  respectively. That closes the proof in both directions.  $\square$

**Proposition 2.2.12.d.** (Addition preserves order)  $a \geq b$  if and only if  $a + c \geq b + c$ .

*Proof.* Let's start with the true case: if  $a \geq b$ , then  $a + c \geq b + c$ . If  $a \geq b$ , then we can represent  $a = b + \tilde{b}$  by Definition 2.2.11. Plugging the expression into  $a + c$  results in  $(b + \tilde{b}) + c$ . Using Proposition 2.2.5 and Proposition 2.2.4, we can transform the expression into  $(b + c) + \tilde{b}$  what proves  $a + c \geq b + c$  by Definition 2.2.11.

Now the false case: if  $a < b$ , then  $a + c < b + c$ . If  $a < b$ , then  $b = a + \tilde{a}$  and  $b \neq a$  by Definition 2.2.11. Using the expression in  $b + c$  results in  $(a + \tilde{a}) + c$ . Again using Propositions 2.2.5 and 2.2.4, we end up with  $(a + c) + \tilde{a}$ , what proves  $b + c \geq a + c$  by Definition 2.2.11. Now we need to show that  $(a + c) \neq (b + c)$ . Let's assume they are equal. Then by Cancellation Law from Proposition 2.2.6, we can get rid of  $c$ , what results in  $a \not\equiv b$ . That contradicts the antecedent  $a < b$ , from which we know that  $a \neq b$ . Thus,  $a + c < b + c$ , what closes the proof for all cases.  $\square$

**Proposition 2.2.12.e.**  *$a < b$  if and only if  $b = a + (d++)$  for any natural number  $d$ .*

*Proof.* For the true case we need to prove that if  $a < b$ , then  $b = a + d$  and  $d \neq 0$ . If  $a < b$ , then  $b = a + d$  and  $b \neq a$  by Definition 2.2.11. Let's assume that  $d$  is not a positive natural number (i.e.  $d = 0$ ). Then  $a + d = a + 0 = a \not\equiv b$  by the antecedent  $b \neq a$ .

For the false case, we need to prove that if  $a \geq b$ , then  $b \neq a + d$  or  $d = 0$ . If  $a \geq b$ , then  $a = b + d$  and  $d$  is a natural number (including zero) by Definition 2.2.11, what contradicts  $d = 0$ . That closes both cases.  $\square$

**Proposition 2.2.12.f.**  *$a < b$  if and only if  $a++ \leq b$ .*

*Proof.* For true case, we need to prove if  $a++ \leq b$ , then  $a < b$ . Combining Definition 2.2.11 and associativity from Proposition 2.2.5, we can infer  $b = (a++) + d = a + (d++)$  where  $d++$  is a positive natural number by Axiom 2.3 and Definition 2.2.7. Thus, we can conclude that  $b > a$  by Proposition 2.2.12.e.

For the false case, we need to prove if  $a++ > b$ <sup>1</sup>, then  $a \geq b$ . If  $a++ > b$ , then  $a++ = b + (d++) = (b + d)++$  using Proposition 2.2.12.e and Definition 2.2.1, where  $d++$  is a positive number by Axiom 2.3. Then by Axiom 2.4, we can infer  $a = b + d$  what proves  $a \geq b$  by Definition 2.1.3.  $\square$

**Proposition 2.2.13.** (Trichotomy of order for natural numbers). *Let  $a$  and  $b$  be natural numbers. Then exactly one of the following statements is true:  $a < b$ ,  $a = b$ ,  $a > b$ .*

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<sup>1</sup>Do we need to prove that this is the false case of  $a++ \leq b$ ?

*Proof.* Let's start that **no more than one** statement holds at a time. If  $a = b$ , then neither  $a < b$  nor  $a > b$  holds by Definition 2.2.11. If  $a > b$  and  $a < b$ , then by Definition 2.2.11,  $a \geq b$ ,  $a \leq b$  and  $a \neq b$ . If  $a \geq b$  and  $a \leq b$ , then  $a \not\equiv b$  using Proposition 2.2.12.c, what contradicts the antecedent. Thus, no more than one statement holds at a time.

Now let's prove that there is **at least one** statement holds inducting on  $a$ . For the base case,  $b$  can be represented as  $b = 0 + b$  by Definition 2.2.1, what makes it  $0 \leq b$  by Definition 2.2.11. For the step case, suppose that one of the trichotomy statements holds for  $a$ . We need to prove whether any holds for  $a++$ . If  $a < b$ , then  $a++ \leq b$  by Proposition 2.2.12.f. If  $a = b$ , then  $a++ > b$  by Lemma 2.2.11.a. If  $a > b$ , then  $a = b + (d++)$  by Proposition 2.2.12.e. Then we can take the successor  $a++ = (b + (d++))++$  by Axiom 2.4, what equals to  $b + (d++)++$  by Definition 2.2.1.  $(d++)++$  is a positive number by Definition 2.2.7 and Axiom 2.3. Thus,  $a++ > b$  still holds by Proposition 2.2.12.e. That covers all possible cases and closes the induction.  $\square$

**Proposition 2.2.14.** (Strong principle of induction). *Let  $m_0$  be a natural number, and let  $P(m)$  be a property pertaining to an arbitrary natural number  $m$ . Suppose that for each  $m \geq m_0$ , we have the following implication: if  $P(m')$  is true for all natural numbers  $m_0 \geq m' < m$ , then  $P(m)$  is also true (in particular, this means that  $P(m_0)$  is true, since in this case the hypothesis is vacuous). Then we can conclude that  $P(m)$  is true for all natural numbers  $m \geq m_0$ .*

## Appendix A

### The basics of mathematical logic

## A.1 Mathematical statements

**Exercise A.1.1.** *What is the negation of the statement “either  $X$  is true or  $Y$  is true, but not both”?*

*Solution.*  $X$  and  $Y$  are false or  $X$  and  $Y$  are true. □

**Exercise A.1.2.** *What is the negation of the statement “ $X$  is true if and only if  $Y$  is true”?*

*Solution.* “ $X$  is false iff  $Y$  is false” □

**Exercise A.1.3.** *Suppose that you have shown that whenever  $X$  is true, then  $Y$  is true, and whenever  $X$  is false, then  $Y$  is false. Have you demonstrated that  $X$  and  $Y$  are logically equivalent? Explain.*

*Solution.* I assume, yes. A **well-defined** statement can be either true or false only (all possible values). Now, we consider the assumptions: (1) if  $X$  is false, then  $Y$  is false ( $Y$  is never true in that case), and (2) if  $X$  is true, then  $Y$  is false ( $Y$  is never true here). Thus, both statements coincides in all possible values what makes them “equivalent”. □

**Exercise A.1.4.** *Suppose that you have shown that  $X$  is true iff  $Y$  is true, and you know that  $Y$  is true iff  $Z$  is true. Is this enough to show that  $X, Y, Z$  are logically equivalent? Explain.*

*Solution.* Yes. We know that the pairs  $(X, Y)$  and  $(Y, Z)$  are logically equivalent, but we don't know whether  $X$  and  $Z$  are equivalent. If  $X$  is true/false, then  $Y$  is true/false, and if  $Y$  is true/false, then  $Z$  is also true/false. We can also prove the same, but going from left to right ( $X \rightarrow Y \rightarrow Z$ ) and from right-to-left ( $Z \rightarrow Y \rightarrow X$ ). □

**Exercise A.1.5.** *Suppose that you have shown that whenever  $X$  is true, then  $Y$  is true; that if  $Y$  is true, then  $Z$  is true; and if  $Z$  is true, then  $X$  is true. Is this enough to show that  $X, Y, Z$  are logically equivalent? Explain.*

*Solution.* Yes, for all pairs of statements, we can construct the following. Let's start at  $X$ . (Forward direction) If  $X$  is true, then  $Y$  is also true. (Reverse direction) If  $Y$  is true, then  $Z$  is also true, and if  $Z$  is true, then  $X$  is also true. That closes the cycle in both ways for all pairs, thus the statements logically equivalent. □

## Appendix B

### The decimal system