The pretty pretty pretty Priddy good complex, Part Two

Mark E. Walker

February 2, 2026*

Abstract

These are notes for the RTG learning seminar on Koszul algebras. Nothing here is original to me. I thank Eloísa Grifo and Adam LaClair for comments that changed these notes, usually for the better. *Dicatus melioribus mathematicis.*

1 Brief review

Throughout, k is a field, V is a k-vector space with basis $x_1, \dots, x_n$. The *tensor algebra* on V is

$$T(V) = \bigoplus_{j \geq 0} T^j(V) = \bigoplus_{j \geq 0} \overbrace{V \otimes_k \cdots \otimes_k V}^{j} = k\langle x_1, \dots, x_n \rangle.$$

A *quadratic algebra* determined by a subspace $W \subseteq T^2(V) = V \otimes_k V$ is

$$A = T(A)/\langle W \rangle = k\langle x_1, \dots, x_n \rangle / \langle q_1, \dots q_m \rangle.$$

A standard graded k-algebra A is *Koszul* if the minimal graded free resolution of k is linear.

Proposition: Koszul implies quadratic. **Proof**: $\mathrm{Tor}_2^A(k,k) \cong I/\mathfrak{m}_{T(V)} I$ where $A = T(V)/I$.

Given $W \subseteq V \otimes_k V$ define $W^\perp := \mathrm{Ker}(V^* \otimes_k V^* \cong (V \otimes_k V)^* \twoheadrightarrow W^*)$. The *quadratic dual* of $A = T(V)/\langle W \rangle$ is

$$A^! = T(V^*)/\langle W^\perp \rangle.$$

Lemma: For any quadratic algebra $A = T(V)/\langle W \rangle$, for each d

$$(A^!_d)^* = \bigcap_i T^i(V) \otimes W \otimes T^{d-i-2}(V) \ \subseteq \ T^d(V).$$

The *Priddy complex* of a quadratic algebra A is the complex loosely described as $(A^!)^* \otimes_k A$; more precisely, it is the complex

$$\cdots \to (A^!_j)^* \otimes_k A \to (A^!_{j-1})^* \otimes_k A \to \cdots \to (A^!_3)^* \otimes_k A \to (A^!_2)^* \otimes_k A \to (A^!_1)^* \otimes_k A \to A \to 0.$$

where $(A^!_j)^*$ is interpreted as a vector space living in degree[1] j and the differential is given by the restriction of the multiplication map

$$T^j(V) \otimes_k A \to T^{j-1}(V) \otimes_k A, \qquad \text{given by } v_1 \otimes \cdots \otimes v_j \otimes a \mapsto v_1 \otimes \cdots \otimes v_{j-1} \otimes v_j \cdot a.$$

Theorem: For a quadratic algebra $A = T(V)/\langle W \rangle$, the Priddy complex is exact in all strictly positive degrees (and hence gives a resolution of k) if and only if A is Koszul.

---

*If a Priddy complex sees its shadow today, we'll have six more weeks of winter. Also, I worry we all wake up the morning after this with me still talking about Priddy complexes.

[1] I've changed by degree conventions a bit from last time.

2 A Toy example illustrating this Theorem

Example (Toy example)**.** Let $V = kx \oplus ky$ and $W = k \cdot (x \otimes y + y \otimes x) \subseteq V \otimes_k V$, and set

$$A = T(V)/\langle W \rangle = k\langle x, y \rangle / \langle xy + yx \rangle.$$

Then

$$W^! = k \cdot \{x^* \otimes y^* - y^* \otimes x^*, x^* \otimes x^*, y^* \otimes y^*\}$$

and so $A^!$ is the commutative k-algebra

$$A^! = k[x^*, y^*]/((x^*)^2, (y^*)^2).$$

We see that $(A^!)^*$ has k-basis $1, x, y, x \otimes y + y \otimes x$, of degrees 0, 1, 1, and 2. So the Priddy complex for A is

$$0 \longrightarrow (x \otimes y + y \otimes x) \otimes A \longrightarrow \begin{matrix} x \otimes A \\ \oplus \\ y \otimes A \end{matrix} \longrightarrow A \longrightarrow 0 \tag{2.1}$$

with differential given by $(x \otimes y + y \otimes x) \otimes a \mapsto x \otimes ya + y \otimes xa$ and $x \otimes a \mapsto xa$ and $y \otimes a \mapsto ya$. Using the evident bases, this is the complex

$$0 \to A(-2) \xrightarrow{\begin{bmatrix} y \\ x \end{bmatrix}} A(-1)^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} A \to 0.$$

One can verify directly that this is exact, proving A is Koszul.

Exercise 1. Write down the first few terms of the Priddy complex for the commutative k-algebra $k[x, y]/(x^2, y^2)$ using the description of its quadratic dual given above, and then compare this to the resolution coming from Tate (using that this ring is a complete intersection).

3 Proof of the Theorem

Let $A = T(V)/\langle W \rangle$ be any quadratic algebra. Since the Priddy complex of A is clearly a linear complex, one direction is immediate: If the Priddy complex of A is a resolution k, then A is Koszul.

Assume A is Koszul. This means we have an exact sequence of graded A-modules of the form

$$\cdots \longrightarrow W_j \otimes_k A \longrightarrow \cdots \longrightarrow W_2 \otimes_k A \longrightarrow W_1 \otimes_k A \longrightarrow A \longrightarrow k \longrightarrow 0. \tag{3.1}$$

for some finite dimensional vector spaces W_j, $j \geq 1$, with W_j viewed as being concentrated in degree j. (The degree conventions used here differ from last time.) For any d, the degree d part of $W_j \otimes_k A$ is $W_j \otimes_k A_{d-j}$ and the degree d part of (3.1) is the exact sequences of finite dimensional k-vector spaces

$$0 \to W_d \to W_{d-1} \otimes_k A_1 \to W_{d-2} \otimes_k A_2 \to \cdots \to W_1 \otimes_k A_{d-1} \to A_d \to 0.$$

We can use this to recursively identify each W_d and what the maps are. In detail:

For $d = 1$ we get the exact sequence

$$0 \to W_1 \to A_1 \to 0,$$

and since $I_1 = 0$ we have $A_1 = V$ and thus (up to isomorphism) we deduce that

- $W_1 = V$ and
- the differential $W_1 \otimes_k A \to A$ is given by multiplication: $v \otimes a \mapsto v \cdot a$.

This confirms what we already knew: the minimal resolution of k starts as $V \otimes_k A \xrightarrow{\cdot} A$ (with V of degree one) or in other words as $A(-1)^n \xrightarrow{(x_1,\ldots,x_n)} A$.

For $d = 2$ we get the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & W_2 \arrow[r] \arrow[d, equal] & W_1 \otimes_k A_1 \arrow[r] \arrow[d, equal] & A_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_2 \arrow[r] & V \otimes V \arrow[r] & \frac{V \otimes_k V}{W} \arrow[r] & 0
\end{tikzcd}$$

and thus (up to isomorphism) we have

- $W_2 = W \subseteq T^2(V)$ and
- the differential $W_2 \otimes_k A \to W_1 \otimes_k A$ is multiplication, in the sense that it is the restriction of the map on $T^2(V) \otimes_k A \to V \otimes_k A$ given by $v_1 \otimes v_2 \otimes a \mapsto v_1 \otimes v_2 a$.

For $d = 3$ we get

$$\begin{tikzcd}
0 \arrow[r] & W_3 \arrow[r] \arrow[d, equal] & W_2 \otimes_k A_1 \arrow[r] \arrow[d, equal] & W_1 \otimes_k A_2 \arrow[r] \arrow[d, equal] & A_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_3 \arrow[r] & W \otimes V \arrow[r] & \dfrac{V \otimes_k V \otimes_k V}{V \otimes_k W} \arrow[r] & \dfrac{V \otimes_k V \otimes_k V}{W \otimes_k V + V \otimes_k W} \arrow[r] & 0
\end{tikzcd}$$

and thus (up to isomorphism) we have

- $W_3 = (W \otimes_k V \cap V \otimes_k W) = (A^!_3)^* \subseteq T^3(V)$ and
- the differential $W_3 \otimes_k A \to W_2 \otimes_k A$ is multiplication: $v_1 \otimes v_2 \otimes v_3 \otimes a \mapsto v_1 \otimes v_2 \otimes v_3 a$.

For $d = 4$ we get

$$\begin{tikzcd}
0 \arrow[r] & W_4 \arrow[r] \arrow[d, equal] & W_3 \otimes_k A_1 \arrow[r] \arrow[d, equal] & W_2 \otimes_k A_2 \arrow[r] \arrow[d, equal] & W_1 \otimes_k A_3 \arrow[r] \arrow[d, equal] & A_4 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_4 \arrow[r] & (W \otimes_k V \cap V \otimes_k W) \otimes V \arrow[r] & W \otimes \frac{V \otimes_k V}{W} \arrow[r] & \frac{V \otimes_k V \otimes_k V \otimes_k V}{V \otimes_k W \otimes_k V + V \otimes_k V \otimes_k W} \arrow[r] & A_4 \arrow[r] & 0
\end{tikzcd}$$

and thus (up to isomorphism) we have

- $W_4 = (W \otimes_k V \otimes_k V \cap V \otimes_k W \otimes_k V \cap V \otimes_k V \otimes_k W) = (A^!_4)^* \subseteq T^4(V)$ and
- the differential $W_4 \otimes_k A \to W_3 \otimes_k A$ is multiplication.

By the Freshman's Principle of Mathematical Induction, we have proven that (3.1) is isomorphic to the Priddy complex.

4 Some consequences of the Theorem

Corollary 2. *A quadratic algebra A is Koszul if and only if its quadratic dual $A^!$ is Koszul.*

Proof. This holds since the k-dual of the Priddy complex for A is the Priddy complex for $A^!$:

$$\left((A^!)^* \otimes_k A\right)^* \cong A^! \otimes_k A^* \cong A^* \otimes_k A^! \cong ((A^!)^!)^* \otimes_k A^!.$$

$\square$

Example (Toy example)**.** Let $V = kx \oplus ky$ and $W = k \cdot \{x \otimes y - y \otimes x, x \otimes x, y \otimes y\}$ and set $A = T(V)/\langle W \rangle$. Then
$$A = k\langle x, y\rangle/\langle xy - yx, x^2, y^2\rangle = k[x, y]/(x^2, y^2).$$
Note that $W^\perp = k \cdot (x^* \otimes y^* + y^* x^*)$ and so

$$A^! = k\langle x^*, y^*\rangle/\langle x^* y^* + y^* x^*\rangle$$

We exhibited a linear resolution for $A^!$ above and concluded that it is Koszul. By Corollary 2, A itself is Koszul.

Corollary 3. *If A is Koszul,*
$$\mathrm{Ext}^*_A(k, k) \cong A^!.$$

Proof. Since the Priddy complex $(A^!)^* \otimes_k A$ resolves k, we have

$$\mathrm{Ext}^*_A(k, k) = H^* \operatorname{Hom}_A((A^!)^* \otimes_k A, k) = H^* \operatorname{Hom}_k((A^!)^*, k) = A^!$$

as graded k-vector spaces. To complete the proof one should verify the multiplication rules match up; I leave that as an exercise. □

Corollary 4. *If A is Koszul,*
$$A \cong \mathrm{Ext}^*_{\mathrm{Ext}^*_A(k,k)}(k, k).$$

Remark 5. The previous Corollary admits a vast generalization: If R is any complete local ring, or any reasonable graded ring, we have a quasi-isomorphism of dgas

$$R \sim \mathbb{R}\operatorname{Hom}_{\mathbb{R}\operatorname{Hom}_R(k,k)}(k, k).$$

If A is Koszul, then the dga $\mathbb{R}\operatorname{Hom}_A(k, k)$ is formal: it is quasi-isomorphic, as a dga, to its homology.

Remark 6. In general, $\mathrm{Ext}^*_A(k, k)$ is the k-linear dual of $\mathrm{Tor}^A_*(k, k)$. If A is commutative, $\mathrm{Tor}^A_*(k, k)$ is a graded commutative ring. So, when A is commutative both $\mathrm{Ext}^*_A(k, k)$ and its k-linear dual are algebras; in fact, $\mathrm{Ext}^*_A(k, k)$ is what is known as a Hopf algebra in this case.

So, if A is Koszul and commutative, $A^!$ is actually a Hopf algebra.

Exercise 7. Assume A is Koszul. Prove k has finite projective dimension over A if and only if $A^!$ is artinian.

Remark 8. If $pd_A(k) < \infty$ then every finitely generated graded A-module has finite projective dimension. This holds since $\mathrm{Tor}^A_*(M, k)$ can be computed by either resolving M or k. So if A is a Koszul algebra such that $A^!$ is artinian, then A is like a regular ring.

Exercise 9. Show that if A is commutative and Koszul and $A^!$ is artinian, then A is a polynomial ring.

Remark 10. For those who know about derived categories, if A is Koszul then $\mathrm{Thick}_k(A) \cong \mathrm{Perf}(A^!)$ and $\mathrm{Thick}_k(A^!) \cong \mathrm{Perf}(A)$. When $A = S(V)$ and $A^! = \Lambda(A)$, this recovers the famed Berstein-Gelfand-Gelfand correspondence: in this case $\mathrm{Thick}_k(\Lambda(V)) = D^b(\Lambda(V))$ and $\mathrm{Perf}(S(V)) = D^b(S(V))$.