

CHEAT SHEET

This is a *listeners* guide to basic topics from Homological Algebra and Commutative Algebra II that appear in seminar. The only purpose of this is so that you might get a better chance of understanding the big picture of a talk that uses these concepts.

Equal / mixed characteristic. A local ring (R, \mathfrak{m}, k) has:

- *equal characteristic zero* if $\text{char}(R) = \text{char}(k) = 0$
- *equal characteristic p* if $\text{char}(R) = \text{char}(k) = p$ for some prime $p > 0$
- *mixed characteristic $(0, p)$* if $\text{char}(R) = 0$ and $\text{char}(k) = p$ for some prime $p > 0$.

Any *reduced* local ring satisfies one of the three conditions above; *any* local ring satisfies one of the above or $\text{char}(R) = p^n$ and $\text{char}(k) = p$ for some prime p and $n > 1$.

Regular rings. When you hear *regular*, you should think of localization of a polynomial ring or a power series ring. A Noetherian local ring (R, \mathfrak{m}, k) is *regular* if $\dim(R) = \dim_{k\text{-vectorspace}}(\mathfrak{m}/\mathfrak{m}^2)$. Note that the inequality \leq always holds by Krull height and NAK. A not-necessarily-local Noetherian ring is *regular* if all of its localizations are. The basic examples are

- $K[X_1, \dots, X_n]$ is regular for a field K .
- $K[[X_1, \dots, X_n]]$ is regular for a field K .

Complete / completion. When you hear *complete*, you should think of quotient of power series ring: the quintessential example of a complete local ring is $K[[X_1, \dots, X_n]]/I$ for some field K and ideal I .

In short, the *completion* of a local ring (R, \mathfrak{m}, k) is the metric completion of R equipped with the (pseudo)metric given by $\text{dist}(r_1, r_2) = 2^{-\sup\{n \mid r_1 - r_2 \in \mathfrak{m}^n\}}$; i.e., two elements are close if their difference is in a large power of \mathfrak{m} . A ring is *complete* if it is complete with respect to this (pseudo)metric. One can do a similar thing with R -modules.

Here are some key points:

- The completion of $K[X_1, \dots, X_n]_{(X_1, \dots, X_n)}/I$ is $K[[X_1, \dots, X_n]]/I$ for a field K .
- Every complete Noetherian local ring of equal characteristic is isomorphic to a ring of the form $K[[X_1, \dots, X_n]]/I$ for some field K , and something similar is true in mixed characteristic.
- A Noetherian local ring is “closely connected” to its completion (the map is faithfully flat, and with regular fibers in many cases).
- Complete local rings have good technical properties (like stronger versions of NAK, Hensel’s Lemma).

Exact sequences. An *exact sequence* of R -modules is a (finite or infinite) collection of modules and maps

$$\cdots \rightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \rightarrow \cdots$$

such that the kernel of any map d_i “out” of a module is equal to the image of the previous map d_{i+1} “in” to the module. Special examples are *short exact sequences*: exact sequences of the form

$$0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$$

the definition of exact in this case says that α is injective, β is surjective, and $N \cong M/L$.

The main tricks to beware of in exact sequences are:

- if $\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$ appears, then $M = 0$.
- if $\cdots \rightarrow 0 \rightarrow M \rightarrow N \rightarrow 0 \rightarrow \cdots$ appears, then $M \cong N$.
- if $\cdots \rightarrow 0 \rightarrow M \xrightarrow{\alpha} N \rightarrow \cdots$ appears, then α is injective.
- if $\cdots \rightarrow M \xrightarrow{\beta} N \rightarrow 0 \rightarrow \cdots$ appears, then β is surjective.

Complexes and homology. A *complex* of R -modules is a weakening of exact sequence: a complex is a (finite or infinite) collection of modules and maps

$$\cdots \rightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \rightarrow \cdots$$

such that the composition of any two maps in a row is zero.

The i th *homology* or *cohomology* (you can think of these words as interchangeable at first) of the complex above is the module $\ker(d_i)/\text{im}(d_{i+1})$; that is, the kernel of the map “out” modulo the image of the map “in”.

Free resolutions. A *free resolution* of an R -module M is a (finite or infinite) exact sequence of the form

$$\cdots \rightarrow R^{b_2} \rightarrow R^{b_1} \rightarrow R^{b_0}(\rightarrow M) \rightarrow 0$$

for some b_i (also possibly infinite).

Tensor products. The *tensor product* of two R -modules M and N is another module $M \otimes_R N$ that satisfies a particular universal property. The key examples of tensor products are

- $R/I \otimes_R R/J = R/(I + J)$.
- If S is an R -algebra and M is an R -module with presentation matrix B , then $S \otimes_R M$ is the S -module with presentation matrix B (image of B in S).

Ext and Tor. To any pair M, N of R -modules, there is a sequence of Ext modules $\text{Ext}_R^i(M, N)$. In short, they are defined by taking a free resolution of M , computing the module of homomorphisms into N at each step to get a complex, and taking the homologies. One of the main tricks with Ext is that given a short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$, there is a long exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(M, N') \rightarrow \text{Ext}_R^i(M, N) \rightarrow \text{Ext}_R^i(M, N'') \rightarrow \text{Ext}_R^{i+1}(M, N') \rightarrow \cdots$$

Also, $\text{Ext}_R^0(M, N) = \text{Hom}_R(M, N)$, the module of R -linear homomorphisms from M to N .

There is a sequence of Tor modules $\text{Tor}_i^R(M, N)$. In short, they are defined by taking a free resolution of M , tensoring with N at each step to get a complex, and taking the homologies. One of the main tricks with Tor is that given a short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$, there is a long exact sequence

$$\cdots \rightarrow \text{Tor}_i^R(M, N') \rightarrow \text{Tor}_i^R(M, N) \rightarrow \text{Tor}_i^R(M, N'') \rightarrow \text{Tor}_i^R(M, N') \rightarrow \cdots$$

The other main trick with Tor is that $\text{Tor}_i^R(M, N) \cong \text{Tor}_i^R(N, M)$. Also, $\text{Tor}_R^0(M, N) = M \otimes_R N$.