CHEAT SHEET

This is a *listener's* guide to basic topics from Homological Algebra and Commutative Algebra II that appear in seminar. The only purpose of this is so that you might get a better chance of understanding the big picture of a talk that uses these concepts.

Regular rings. When you hear *regular*, for most purposes you should think of localization of a polynomial ring or a power series ring. A Noetherian local ring $(R, \mathfrak{m}, k)$ is *regular* if $\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Note that the inequality $\leq$ always holds by Krull height and NAK. A not-necessarily-local Noetherian ring is *regular* if all of its localizations are. The basic examples are

- $K[X_1, \dots, X_n]$ is regular for a field K.
- $K[\![X_1, \dots, X_n]\!]$ is regular for a field K.

Exact sequences. An *exact seqeunce* of R-modules is a (finite or infinite) collection of modules and maps

$$\cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \cdots$$

such that the kernel of any map d_i "out" of a module is equal to the image of the previous map d_{i+1} "in" to the module. Special examples are *short exact sequences*: exact sequences of the form

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

the definition of exact in this case says that α is injective, β is surjective, and $N \cong M/L$.

The main tricks to beware of in exact sequences are:

- if $\cdots \to 0 \to M \to 0 \to \cdots$ occurs, then $M = 0$.
- if $\cdots \to 0 \to M \to N \to 0 \to \cdots$ occurs, then $M \cong N$.
- if $\cdots \to 0 \to M \xrightarrow{\alpha} N \to \cdots$ occurs, then α is injective.
- if $\cdots \to M \xrightarrow{\beta} N \to 0 \to \cdots$ occurs, then β is surjective.

Complexes and homology. A *complex* of R-modules is a weakening of exact sequence: a complex is a (finite or infinite) collection of modules and maps

$$\cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \cdots$$

such that the composition of any two maps in a row is zero.

The ith *homology* or *cohomology* (you can think of these words as interchangable at first) of the complex above is the module $\ker(d_i)/\mathrm{im}(d_{i+1})$; that is, the kernel of the map "out" modulo the image of the map "in".

Free resolutions. A *free resolution* of an R-module M is a (finite or infinite) exact sequence of the form

$$\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} (\to M) \to 0$$

for some b_i (also possibly infinite).

Ext. To any pair M, N of R-modules, there is a sequence of Ext modules $\mathrm{Ext}^i_R(M, N)$. In short, they are defined by taking a free resolution of M, computing the module of homomorphisms into N at each step to get a complex, and taking the homologies. One of the main tricks with Ext is that given a short exact sequence $0 \to N' \to N \to N'' \to 0$, there is a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(M, N') \to \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N'') \to \mathrm{Ext}^{i+1}_R(M, N') \to \cdots$$

Also, $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$, the module of R-linear homomorphisms from M to N.