The pretty pretty pretty Priddy good complex

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Abstract

These are notes for the RTG learning seminar on Koszul algebras. Nothing here is original to me. I thank Eloísa Grifo and Adam LaClair for comments that changed these notes, usually for the better. *Dicatus melioribus mathematicis.*

My goals in this talk are to introduce quadratic algebras, Koszul algebras, quadratic duality and the Priddy complex, with an emphasis on examples.

See here for an explanation of the title.

1 Quadratic Algebras

Throughout k is a field and V is a finite dimensional k-vector space with basis $x_1, \dots, x_n$. The tensor algebra is the graded k-algebra

$$T(V) = \bigoplus_{j \geq 0} T^j(V) = \bigoplus_{j \geq 0} \overbrace{V \otimes_k \cdots \otimes_k V}^{j} = k\langle x_1, \dots, x_n \rangle,$$

the polynomial ring in n *non-commuting* variables. By a *k-algebra* we will always mean a quotient of $T(V)$ by a two-sided, homogeneous ideal $I = I_2 \oplus I_3 \oplus \cdots$:

$$A = T(A)/I.$$

So, A is an $\mathbb{N}$-graded k-algebra, with $A_0 = k$ and $A_1 = V$, that is generated as a k-algebra by $A_1 = V$. Note that we do *not* assume A is commutative.

Remark 1. Even though people in this room are mostly interested in commutative algebras, the quadratic dual of a commutative quadratic algebra is nearly always non-commutative.

We say A is *quadratic* if I is generated (as a two sided ideal) by its degree two part I_2; that is, if there is some vector subspace

$$W \subseteq V \otimes_k V = k\langle x_1, \dots, x_n \rangle_2$$

(which will be I_2) such that

$$A = T(V)/\langle W \rangle.$$

Here $\langle W \rangle$ is the smallest *two sided* ideal containing W. So the degree three part of $\langle W \rangle$ is

$$\langle W \rangle_3 = V \otimes_k W + W \otimes_k V \subseteq T^3(V),$$

the degree four part is

$$\langle W \rangle_4 = V \otimes_k V \otimes_k W + V \otimes_k W \otimes_k V + W \otimes_k V \otimes_k V \subseteq T^4(V),$$

and in general the degree d part is

$$\langle W \rangle_d = \sum_{i=0}^{d-2} T^i(V) \otimes_k W \otimes_k T^{d-i-2}(V) \subseteq T^d(V) = k\langle x_1, \ldots, x_n \rangle_d.$$

The following examples will be referred to throughout these notes.

Example (Symmetric algebra)**.** Let

$$W = k \cdot \{v \otimes v' - v' \otimes v \mid v, v' \in V\} = k \cdot \{x_i x_j - x_j x_i\} \subseteq V \otimes_k V.$$

We may identify W with $\Lambda^2(V)$; in general, $\Lambda^j(V)$ may be viewed as a subspace of $T^j(V)$ via the anti-symmetrization map; see (3.1) below. Then

$$T(V)/\langle W \rangle = S(V) = k[x_1, \ldots, x_n],$$

the symmetric algebra on V, aka the usual polynomial ring in n commuting variables.

Example (Exterior algebra)**.** Let

$$W = k\{v \otimes v' + v' \otimes v, v \otimes v \mid v, v' \in V\} =\subseteq V \otimes_k V.$$

In other words, $W = \Gamma^2(V) = T^2(V)^{\Sigma_2}$, the subspace of $T^2(V)$ invariant under the evident action of Σ_2. Since

$$v \otimes v' + v' \otimes v = (v + v') \otimes (v + v') - v \otimes v - v' \otimes v',$$

the first set of generators is actually redundant. Then

$$T(V)/\langle W \rangle = \Lambda(V) = \Lambda(x_1, \ldots, x_n)$$

the exterior algebra on V.

Example (W is Zero)**.** Taking things to an extreme: If $W = 0$ then $A = T(V) = k\langle x_1, \ldots, x_n \rangle$ is the tensor algebra.

Example (W is Everything)**.** Taking things to the opposite extreme: If $W = V \otimes_k V$ then $A = T(V)/\langle W \rangle = k \oplus V$ with (nearly) trivial multiplication: $v \cdot v' = 0$ for $v, v' \in V$ (and in general $(a, v) \cdot (a', v') = (aa', av' + a'v)$).

Example (Toy example)**.** The commutative algebra $A = k[x, y]/(x^2, y^2)$ is quadratic with $W = k \cdot \{x^2, xy - yx, y^2\} \subseteq k\langle x, y \rangle$.

Example (Toy example dual)**.** Say $W = k \cdot \{xy + yx\} \subseteq k\langle x, y \rangle$. Then $A = k\langle x, y \rangle / \langle W \rangle$ is the k-span of $\{x^i y^j\}$ but x and y anti-commute (and do not square to 0).

2 Koszul algebras

Let $A = \frac{T(V)}{I} = \frac{k\langle x_1, \ldots, x_n \rangle}{I}$ be a k-algebra. We regard k as an (left) A-module by identifying it with $A/\mathfrak{m}_A$ with $\mathfrak{m}_A := A_{\geq 1}$.

Definition 2. A k-algebra $A = T(V)/I$ is *Koszul* if the minimal graded free resolution of k (as a left A-module) is linear, in the sense that it has the form

$$\begin{tikzcd}
\cdots \arrow[r] & W_j \otimes_k A(-j) \arrow[r] \arrow[d, equal] & W_{j-1} \otimes_k A(-j+1) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & W_2 \otimes_k A(-2) \arrow[r] \arrow[d, equal] & W_1 \otimes_k A(-1) \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & A(-j)^{b_j} \arrow[r] & A(-j+1)^{b_{j-1}} \arrow[r] & \cdots \arrow[r] & A(-2)^{b_2} \arrow[r] & A(-1)^{b_1} \arrow[r] & A \arrow[r] & 0
\end{tikzcd} \tag{2.1}$$

for some vector spaces $W_1, W_2, \ldots$ which are regarded to be of degree 0.

Another way of defining this concept: A is Koszul iff each differential in the minimal graded free resolution of k is represented by a matrix of linear forms (i.e., elements of A_1). Yet another way: A is Koszul if and only if $\mathrm{Tor}_j^A(k,k)_i = 0$ unless $i = j$.

Proposition 3. *If A is Koszul then A is quadratic. In fact, so long as the minimal free resolution of k starts as $\cdots \to A(-2)^{b_2} \to A(-1)^{b_1} \to A$ then A is quadratic.*

Exercise 4. Prove the proposition. *Tips*: For $A = T(V)/I$, mess around with long exact sequences to prove that there are isomorphisms of graded k-vector spaces.

$$\mathrm{Tor}_2^A(k,k) \cong \mathrm{Ker}(\mathfrak{m}_A \otimes_k \mathfrak{m}_A \xrightarrow{\cdot} \mathfrak{m}_A) \cong I/\mathfrak{m}_{T(V)} I.$$

and hence $\dim_k \mathrm{Tor}_2^A(k,k)_j$ is the number of minimal generators of I of degree j.

Remark 5. The converse of Proposition 3 is false. (Justification: We would not be running this seminar if the converse were true.)

3 Examples and non-examples of Koszul algebras

Here are some examples of W's that give Koszul algebras:

Example (Symmetric algebra)**.** The symmetric algebra $S(V) = k[x_1, \ldots, x_n]$ is Koszul; see Example 1. We prove this by exhibiting a linear resolution.

In the beginning, God created the Koszul complex, and it was good.

The minimal free resolution of k is the Koszul[1] complex on $x_1, \cdots, x_n$, which has the form

$$0 \to A(-n) \to A(-n+1)^n \to \cdots \to A(-j)^{\binom{n}{j}} \to \cdots \to A(-2)^{\binom{n}{2}} \to A(-1)^n \to A.$$

This is a linear resolution of k and so A is Koszul.

It will be helpful to exhibit this resolution in a basis-free way:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^n(V) \otimes_k S(V)(-n) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \Lambda^j(V) \otimes_k S(V)(-j) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \Lambda^2(V) \otimes_k S(V)(-2) \arrow[r] \arrow[d, equal] & V \otimes_k S(V)(-1) \arrow[r] \arrow[d, equal] & S(V) \arrow[r] \arrow[d, equal] & 0. \\
0 \arrow[r] & \Lambda^n(S(V)(-1)) \arrow[r] & \cdots \arrow[r] & \Lambda^j(S(V)(-1)^n) \arrow[r] & \cdots \arrow[r] & \Lambda^2(S(V)(-1)^n) \arrow[r] & S(V)(-1)^n \arrow[r] & S(V) \arrow[r] & 0.
\end{tikzcd}$$

One way of describing the differential uses that we can realize $\Lambda^j(V)$ as a subspace of $T^j(V)$ by the "anti-symmetrization" map

$$v_1 \wedge \cdots \wedge v_j \mapsto \sum_\sigma \mathrm{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)}. \tag{3.1}$$

Then the differential on the Koszul complex is the restriction of the "multiplication" map

$$T^j(V) \otimes_k S(V) \to T^{j-1}(V) \otimes_k S(V), \quad v_1 \otimes \cdots \otimes v_j \otimes f \mapsto v_1 \otimes \cdots \otimes v_{j-1} \otimes v_j \cdot f. \tag{3.2}$$

Exercise 6. Verify all the unproven claims made in the previous example.

Example (Exterior algebra)**.** The exterior algebra $\Lambda(V)$ (see Example 1) is Koszul.

Let $\Gamma^j(V)$ be the "symmetric tensors", that is, the subspace of $T^j(V)$ invariant under the evident action of the symmetric group Σ_j. E.g., $\Gamma^2(V)$ is the k-span of $\{v \otimes v' + v \otimes v'\} \cup \{v \otimes v\}$ (the first set is actually redundant). Notice that these are precisely the defining relations of the exterior algebra.

[1]Not a coincidence that the same word appears!

Then the minimal free resolution of k is the infinite linear resolution

$$\cdots \to \Gamma^2(V) \otimes_k \Lambda(V)(-2) \to V \otimes_k \Lambda(V)(-1) \to \Lambda(V) \to k.$$

As before, the differential is the restriction to $\Gamma^j(V)\otimes_k \Lambda(V) \to \Gamma^{j-1}(V)\otimes_k \Lambda(V)$ of the multiplication map 3.2 in the previous example.

Example (W is zero)**.** If $W = 0$ then $A = T(V)$. It is Koszul since

$$0 \to V \otimes_k T(V)(-1) \xrightarrow{\cdot} T(V) \to k$$

is exact. (Note that the projective dimension of k over $T(V)$ is finite — in fact, equal to 1 — to $T(V)$ is like a regular ring.)

Example (W is everything)**.** $W = T^2(V)$. Then $A = k \oplus V$ with (nearly) trivial multiplication. This is Koszul since the minimal resolution of k is

$$\cdots \to V \otimes_k V \otimes_k V \otimes_k A \xrightarrow{v_1\otimes v_2\otimes v_3\otimes a \mapsto v_1\otimes v_2\otimes v_3 a} V \otimes_k V \otimes_k A \xrightarrow{v_1\otimes v_2\otimes a\mapsto v_1\otimes v_2 a} V \otimes_k A \xrightarrow{\cdot} A \to k \to 0.$$

Remark 7. Do you notice that the previous four examples occur in "dual pairs"?

That is, the "totalization" of the minimal resolution of k over $S(V)$ is $\Lambda(V) \otimes_k S(V)$, whereas the totalization of the minimal resolution of k over $\Lambda(V)$ is $\Gamma(V) \otimes_k \Lambda(V)$, and we have

$$(\Gamma(V) \otimes_k \Lambda(V))^* \cong \Lambda(V^*) \otimes S(V^*).$$

Likewise, the totalization of the minimal resolution of k over $k \oplus V$ is $T(V) \otimes_k (k \oplus V)$ whereas the minimal resolution of k over $T(V)$ is $(V \oplus k) \otimes T(V)$, which are similarly dual.

This is no fluke!

Example (Toy example)**.** $A = k[x, y]/(x^2, y^2)$ is Koszul. This will be justified later.

Example (Toy example dual)**.** $k\langle x, y\rangle/\langle xy + yx\rangle$ is Koszul. This will be justified later.

Some more examples:

Example 1. (McCullough-Peeva) Let $A = k[x, y, z]/(x^2, y^2, xy + xz, xy + yz)$; that is, take W to be the k-span of $xy-yx, xz-zx, yz-zy, x^2, y^2, xy+xz, xy+yz$. Then A is Koszul. This can be justified by the fact that its "Koszul dual" $A^!$ admits a quadratic Gröbner basis.

Example 2. (McCullough-Peeve) Here is a non-example:

$$A = k[x_1, x_2, x_3, x_4]/(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 + x_1x_3 + x_1x_4),$$

so that W the k-span of

$$\{x_ix_j - x_jx_i\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 + x_1x_3 + x_1x_4\}.$$

This can be shown to not be Koszul by using that if A were Koszul then we would have $P_A(t) \cdot H_A(-t) = 1$; this will be proven by a tentative Eloísa next week. But

$$1/H_A(-t) = \frac{1}{1 + 4t + 5t^2 + t^3} = 1 + 4t + 11t^2 + 25t^3 + 49t^4 + 82t^5 + 108t^6 + 71t^7 - 174t^8 + \cdots$$

cannot be the Poincaré series of k.

Example 3. (Conca) Here is another non-example: Let R be the quadratic algebra

$$R = k[x, y, z, t]/(x^2, y^2, z^2, t^2, xy + zt).$$

(So W is the k-span of all the commutators $xy - yx, xz - zx, \ldots$ along with $x^2, y^2, z^2, t^2, xy + zt$.) Then $b^R_{3,4}(k) = 5$ and hence R is not Koszul.

Exercise 8. Compute the quadratic dual of Conca's example (once you learn what that is).

4 Quadratic duals

Recall that we write V^* for $\mathrm{Hom}_k(V,k)$ and that $x_1,\ldots,x_n$ is a basis of V. By the *dual basis* of V^* we mean the basis $x_1^*,\ldots,x_n^*$ of V^* with $x_i^*(x_j)=\delta_{i,j}$. (Roughly, $x_i^*=\partial/\partial x_i$.)

The inclusion $W\subseteq V\otimes_k V$, induces a surjection by taking k-linear duals

$$V^*\otimes_k V^*\cong (V\otimes_k V)^*\twoheadrightarrow W^*$$

and we write the kernel of this surjection as $W^\perp\subseteq V^*\otimes_k V^*$. The *quadratic dual* of the quadratic algebra $A=T(V)/\langle W\rangle$ is the quadratic algebra

$$A^!=T(V^*)/(W^\perp).$$

Remark 9. We have identified $V^*\otimes_k V^*$ with $(V\otimes_k V)^*$, but there are multiple ways of doing so. Let's agree to use the following one: $\gamma_1\otimes\gamma_2\in V^*\otimes_k V^*$ corresponds to the map $V\otimes_k V\to k$ given by $v_1\otimes v_2\mapsto\gamma_1(v_2)\gamma_2(v_1)$. If we think of this as a pairing $V^*\otimes_k V^*\otimes_k V\otimes_k V\to k$, the convention we use is to combine adjacent terms: $\gamma_1\otimes\gamma_2\otimes v_1\otimes v_2\mapsto\gamma_2(v_1)\gamma_1\otimes v_2\mapsto\gamma_1(v_2)\gamma_2(v_1)$.

Example (Symmetric algebra)**.** If $W=\Lambda^2(V)$, the k-span of $\{v_1\otimes v_2-v_2\otimes v_1\}$, so that $A=S(V)$, then $W^\perp$ is the k-span of $\{\gamma\otimes\gamma\mid\gamma\in V^*\}$. (Recall that this span includes $\gamma_1\otimes\gamma_2+\gamma_2\otimes\gamma_1$.) Thus

$$S(V)^!=\Lambda(V^*).$$

Example (Exterior algebra)**.** If W is the k-span of $\{v\otimes v\mid v\in V\}$, so that $A=\Lambda(V)$, then $W^\perp$ is the k-span of $\{\gamma_1\otimes\gamma_2-\gamma_2\otimes\gamma_1\mid\gamma_1,\gamma_2\in V^*\}$. Thus

$$\Lambda(V)^!=S(V^*).$$

This is an example of quadratic duality:

Proposition 10. *For any quadratic algebra A, we have $(A^!)^!\cong A$. We call A and $A^!$* quadratic duals*.*

Exercise 11. Prove the previous Proposition.

Example (Toy example)**.** $k[x,y]/(x^2,y^2)$ and $k\langle x^*,y^*\rangle/\langle x^*y^*+y^*x^*\rangle$ are quadratic duals.

Example (Extreme examples)**.** The tensor algebra $T(V)$ and the trivial algebra $k\oplus V^*$ are quadratic duals:

$$T(V)^!=(k\oplus V^*)\ \text{ and }\ (k\otimes V)^!=T(V^*).$$

Here is another example:

Example 4. Let $A=k[x,y,z]/(x^2,y^2,xy+xz,xy+yz)$; that is, take

$$W=\text{ the }k\text{-span of}\{xy-yx,xz-zx,yz-zy,x^2,y^2,xy+xz,xy+yz\}.$$

Then

$$A^!=k\langle x^*,y^*,z^*\rangle/(z^*z^*,x^*y^*+y^*x^*-x^*z^*-z^*x^*-y^*z^*-z^*y^*)$$

Exercise 12. Verify the previous example.

Exercise 13. Prove that if $\mathrm{char}(k)\neq 2$ and $n\geq 2$, then for a quadratic algebra A, A and $A^!$ cannot both be commutative.

What is the story when $\mathrm{char}(k)=2$?

5 The pretty pretty pretty Priddy good complex

We come to the Priddy[2] complex of a quadratic algebra $A = T(V)/\langle W\rangle$. You should think of this as a generalization of the Koszul complex and indeed some authors refer to the Priddy complex as the "(generalized) Koszul complex".

Let us note that the degree d component of $A^!$ is a quotient of $T^d(V^*)$ and so its dual $(A^!_d)^*$ can be identified with a subspace of $T^d(V)$. For instance $(A^!_1)^* = V$, $(A^!_2)^* = W$, and

$$(A^!_3)^* = W \otimes_k V \cap V \otimes_k W \subseteq T^3(V).$$

In general,

Lemma 14. *For any quadratic algebra $A = T(V)/\langle W\rangle$, we have*

$$(A^!_d)^* = \bigcap_i T^i(V) \otimes W \otimes T^{d-i-2}(V)$$

for each d, as subspaces of $T^d(V)$.

For example

$$(A^!_4)^* = (W \otimes_k V \otimes_k V) \cap (V \otimes_k W \otimes_k V) \cap (V \otimes_k V \otimes_k W) \subseteq T^4(V).$$

Definition 15. For a quadratic algebra $A = T(V)/\langle W\rangle$, the *Priddy complex* of A (also known as the *generalized Koszul complex*) is the complex

$$\cdots \to (A^!_j)^* \otimes_k A(-j) \to (A^!_{j-1})^* \otimes_k A(-j+1) \to \cdots \to (A^!_2)^* \otimes_k A(-2) \to W \otimes_k A(-2) \to V \otimes_k A(-1) \to A \to 0.$$

where the differential is given by the restriction of the multiplication map

$$T^j(V) \otimes_k A(-j) \to T^{j-1}(V) \otimes_k A(-j+1), \quad v_1 \otimes \cdots \otimes v_j \otimes a \mapsto v_1 \otimes \cdots \otimes v_{j-1} \otimes v_j \cdot a.$$

Exercise 16. Verify that this really is a complex (i.e., that the differential squares to 0).

Example (Toy example)**.** Recall that if $A = k\langle x, y\rangle/\langle xy + yx\rangle$ then $A^! = k[x^*, y^*]/(x^*x^*, y^*y^*) = k\langle x^*, y^*\rangle/\langle x^*x^*, y^*x^* - x^*y^*, y^*y^*\rangle$. So $A^!$ has k-basis $1, x^*, y^*, y^*x^* + x^*y^*$ and $(A^!)^*$ has k-basis $1, x, y, xy + yx$. So the priddy complex for A is

$$0 \to (xy + yx) \otimes A(-2) \to x \otimes A(-1) \oplus y \otimes A(-1) \to A \to 0 \tag{5.1}$$

with differential $(xy + yx) \otimes a \mapsto x \otimes ya + t \otimes xa$ etc. Using the evident bases, this is the complex

$$0 \to A(-2) \xrightarrow{\begin{bmatrix} y \\ x \end{bmatrix}} A(-1)^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} A \to 0.$$

Exercise 17. Verify the details of the previous example.

Theorem 18. *A quadratic algebra A is Koszul if and only if the Priddy complex is a resolution of k.*

Example. The Priddy complex is a resolution of k for the symmetric algebra, the exterior algebra, the tensor algebra and the trivial algebra.

Example (Toy example)**.** One[3] may verify directly that the Priddy complex for $A = k\langle x, y\rangle/\langle xy+yx\rangle$ shown in (5.1) above is a resolution of k. So, A is Koszul.

[2] I knew Stuart Priddy back in the day

[3] i.e., you, dear reader

Proof of Theorem. Since the Priddy complex is clearly a linear complex, one direction is immediate: If the Priddy complex resolution k, then A is Koszul.

Assume A is Koszul. This means we have an exact sequence of graded A-modules of the form

$$\cdots \longrightarrow W_j \otimes_k A(-j) \longrightarrow \cdots \longrightarrow W_2 \otimes_k A(-2) \longrightarrow W_1 \otimes_k A(-1) \longrightarrow A \longrightarrow k \longrightarrow 0. \tag{5.2}$$

for some finite dimensional vector spaces W_j, $j \geq 1$, each viewed as being concentrated in degree zero. For any d, the degree d part of $W_j \otimes_k A(-j)$ is $W_j \otimes_k A_{d-j}$ and the degree d part of (5.2) is the exact sequences of finite dimensional k-vector spaces

$$0 \to W_d \to W_{d-1} \otimes_k A_1 \to W_{d-2} \otimes_k A_2 \to \cdots \to W_1 \otimes_k A_{d-1} \to A_d \to 0.$$

We can use this to recursively identify each W_d and what the maps are. In detail:

For $d = 1$ we get the exact sequence

$$0 \to W_1 \to A_1 \to 0,$$

and since $I_1 = 0$ we have $A_1 = V$ and thus (up to isomorphism) we deduce that

- $W_1 = V$ and
- the differential $W_1 \otimes_k A(-1) \to A$ is given by multiplication: $v \otimes a \mapsto v \cdot a$.

This confirms what we already knew: the minimal resolution of k starts as $V \otimes_k A(-1) \dot{\to} A$ or in other words as $A(-1)^n \xrightarrow{(x_1,\ldots,x_n)} A$.

For $d = 2$ we get the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & W_2 \arrow[r] \arrow[d, equal] & W_1 \otimes_k A_1 \arrow[r] \arrow[d, equal] & A_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_2 \arrow[r] & V \otimes V \arrow[r] & \frac{V \otimes_k V}{W} \arrow[r] & 0
\end{tikzcd}$$

and thus (up to isomorphism) we have

- $W_2 = W \subseteq T^2(V)$ and
- the differential $W_2 \otimes_k A(-2) \to W_1 \otimes_k A(-1)$ is multiplication, in the sense that it is the restriction of the map on $T^2(V) \otimes_k A(-2) \to V \otimes_k A(-1)$ given by $v_1 \otimes v_2 \otimes a \mapsto v_1 \otimes v_2 a$.

For $d = 3$ we get

$$\begin{tikzcd}
0 \arrow[r] & W_3 \arrow[r] \arrow[d, equal] & W_2 \otimes_k A_1 \arrow[r] \arrow[d, equal] & W_1 \otimes_k A_2 \arrow[r] \arrow[d, equal] & A_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_3 \arrow[r] & W \otimes V \arrow[r] & \frac{V \otimes_k V \otimes_k V}{V \otimes_k W} \arrow[r] & \frac{V \otimes_k V \otimes_k V}{W \otimes_k V + V \otimes_k W} \arrow[r] & 0
\end{tikzcd}$$

and thus (up to isomorphism) we have

- $W_3 = (W \otimes_k V \cap V \otimes_k W) = (A^!_3)^* \subseteq T^3(V)$ and
- the differential $W_3 \otimes_k A(-3) \to W_2 \otimes_k A(-2)$ is multiplication: $v_1 \otimes v_2 \otimes v_3 \otimes a \mapsto v_1 \otimes v_2 \otimes v_3 a$.

For $d = 4$ we get

$$\begin{tikzcd}
0 \arrow[r] & W_4 \arrow[r] \arrow[d, equal] & W_3 \otimes_k A_1 \arrow[r] \arrow[d, equal] & W_2 \otimes_k A_2 \arrow[r] \arrow[d, equal] & W_1 \otimes_k A_3 \arrow[r] \arrow[d, equal] & A_4 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_4 \arrow[r] & (W \otimes_k V \cap V \otimes_k W) \otimes V \arrow[r] & W \otimes \frac{V \otimes_k V}{W} \arrow[r] & \frac{V \otimes_k V \otimes_k V \otimes_k V}{V \otimes_k W \otimes_k V + V \otimes_k V \otimes_k W} \arrow[r] & A_4 \arrow[r] & 0
\end{tikzcd}$$

and thus (up to isomorphism) we have

- $W_4 = (W \otimes_k V \otimes_k V \cap V \otimes_k W \otimes_k V \cap V \otimes_k V \otimes_k W) = (A^!_4)^* \subseteq T^4(V)$ and
- the differential $W_4 \otimes_k A(-4) \to W_3 \otimes_k A(-3)$ is multiplication.

By the Freshman's Principle of Mathematical Induction, we have proven that (5.2) is isomorphic to the Priddy complex. □

Corollary 19. *A quadratic algebra A is Koszul if and only if its quadratic dual $A^!$ is Koszul.*

Proof. This holds since the k-dual of the Priddy complex for A is the Priddy complex for $A^!$:

$$\left((A^!)^* \otimes_k A\right)^* \cong A^! \otimes_k A^* \cong A^* \otimes_k A^! \cong ((A^!)^!)^* \otimes_k A^!.$$

□

Example (Toy example)**.** $k[x, y]/(x^2, y^2)$ is Koszul since, as we saw above, its quadratic dual is.

Exercise 20. Write down the first few terms of the Priddy complex for $k[x, y]/(x^2, y^2)$.

Corollary 21. *If A is Koszul,*

$$\mathrm{Ext}^*_A(k, k) \cong A^!.$$

Proof. Since the Priddy complex $(A^!)^* \otimes_k A$ resolves k, we have

$$\mathrm{Ext}^*_A(k, k) = H^* \operatorname{Hom}_A((A^!)^* \otimes_k A, k) = H^* \operatorname{Hom}_k((A^!)^*, k) = A^!$$

as graded k-vector spaces. To complete the proof one should verify the multiplication rules match up; I leave that as an exercise. □

Corollary 22. *If A is Koszul,*

$$A \cong \mathrm{Ext}^*_{\mathrm{Ext}^*_A(k,k)}(k, k).$$

Remark 23. The previous Corollary admits a vast generalization: If R is any complete local ring, or any reasonable graded ring, we have

$$R \sim \mathbb{R}\operatorname{Hom}_{\mathbb{R}\operatorname{Hom}_R(k,k)}(k, k).$$

(If A is Koszul, then the dga $\mathbb{R}\operatorname{Hom}_A(k, k)$ is formal: it is quasi-isomorphic, as a dga, to its homology.)

Remark 24. In general, $\mathrm{Ext}^*_A(k, k)$ is the k-linear dual of $\mathrm{Tor}^A_*(k, k)$. If A is commutative, $\mathrm{Tor}^A_*(k, k)$ is a graded commutative ring. So, when A is commutative both $\mathrm{Ext}^*_A(k, k)$ and its k-linear dual are algebras; in fact, $\mathrm{Ext}^*_A(k, k)$ is what is known as a Hopf algebra in this case.

So, if A is Koszul and commutative, $A^!$ is actually a Hopf algebra.

Exercise 25. Assume A is Koszul. Prove k has finite projective dimension over A if and only if $A^!$ is artinian.

Remark 26. If $pd_A(k) < \infty$ then every finitely generated graded A-module has finite projective dimension. This holds since $\mathrm{Tor}^A_*(M, k)$ can be computed by either resolving M or k. So if A is a Koszul algebra such that $A^!$ is artinian, then A is like a regular ring.

Exercise 27. Show that if A is commutative and Koszul and $A^!$ is artinian, then A is a polynomial ring.