1. Quadratic algebras.

   (a) **Definitions**: $T(V)$ is the *tensor algebra*; A is *quadratic* if $A = T(V)/\langle W\rangle$ for $W \subseteq V \otimes_k V$.

   (b) Running Examples:

      i. symmetric algebra

      ii. exterior algebra

      iii. tensor algebra

      iv. trivial algebra

      v. $\text{Toy} = k[x,y]/(x^2, y^2) = k\langle x, y\rangle/\langle xy - yx, x^2, y^2\rangle$

      vi. $\text{Toy}^! = k\langle x, y\rangle/\langle xy + yx\rangle$

   (c) Why are we looking at non-commutative algebras?

2. Koszul algebras.

   (a) **Definition:** A graded algebra A is *Koszul* if the minimal resolution of k is linear:

   $$\cdots \to W_j \otimes_k A(-j) \to \cdots \to W_3 \otimes_k A(-3) \to W_2 \otimes_k A(-2) \to W_1 \otimes_k A(-1) \to A.$$

   (b) **Proposition:** Koszul implies quadratic. (Proof: $\mathrm{Tor}_2^A(k,k) \cong I/\mathfrak{m}_{T(V)}I$.)

   (c) Each of the running examples is Koszul; for each we can exhibit a linear resolution of k.

   (d) The running examples occur in dual pairs.

   (e) A non-example (Conca): The commutative algebra $A = k[x,y,z,t]/(x^2, y^2, z^2, t^2, xy + zt)$ is not Koszul since $b^R_{3,4}(k) = 5$. (W_3 needs a component of degree 1.)

3. Quadratic duals.

   (a) **Definitions**: $W^\perp = \mathrm{Ker}(V^* \otimes V^* \twoheadrightarrow W^*)$ and $A^! = T(V^*)/\langle W^\perp\rangle$.

   (b) For the running examples: $S(V)^! = \Lambda(V^*)$, $T(V)^! = k \oplus V$, etc.

4. The Priddy complex.

   (a) **Lemma:** For all d we have $(A^!_d)^* = \bigcap_i T^i(V) \otimes W \otimes T^{d-i-2}(V)$ as subspaces of $T^d(V)$.

   (b) **Defintion**: For any quadratic algebra A, the Priddy complex is

   $$\cdots \to (A^!_j)^* \otimes_k A(-j) \to \cdots \to (A^!_3)^* \otimes_k A(-3) \to W \otimes_k A(-2) \to V \otimes_k A(-1) \to A \to 0.$$

   with differential $v_1 \otimes \cdots \otimes v_j \otimes a \mapsto v_1 \otimes \cdots \otimes v_{j-1} \otimes v_j \cdot a$.

   (c) **Theorem:** A quadratic algebra is Koszul iff its Priddy complex is a resolution of k.

   (d) Sketch of proof: Show $W_j = (A^!_j)^*$ and the differential is as claimed by induction on j.

   (e) The running examples illustrate the Priddy complex

   (f) **Corollary:** A quadratic algebra A is Koszul iff $A^!$ is Koszul. (Proof: The Priddy complexes for A and $A^!$ are dual.)

   (g) **Corollary:** For A Koszul, $A^! = \mathrm{Ext}^*_A(k,k)$ and $A = \mathrm{Ext}^*_{\mathrm{Ext}^*_A(k,k)}(k,k)$.

   (h) BGG: For A Koszul, $\mathrm{Thick}_A(k) \cong \mathrm{Perf}(A^!)$ and $\mathrm{Thick}_{A^!}(k) \cong \mathrm{Perf}(A)$.

A	linear resolution of k	$A^!$
$S(V) = \dfrac{T(V)}{\langle v \otimes v' - v' \otimes v \rangle}$	$\cdots \to \Lambda^3(V) \otimes_k A(-3) \to \Lambda^2(V) \otimes_k A(-2) \to V \otimes_k A(-1) \to A$	$\Lambda(V)$
$\Lambda(V) = \dfrac{T(V)}{\langle v \otimes v' + v' \otimes v, v \otimes v \rangle}$	$\cdots \to \Gamma^3(V) \otimes_k A(-3) \to \Gamma^2(V) \otimes_k A(-2) \to V \otimes_k A(-1) \to A$	$S(V)$
$T(V) = \dfrac{T(V)}{\langle 0 \rangle}$	$0 \to V \otimes_k A(-1) \to A$	$k \oplus V$
$k \oplus V = \dfrac{T(V)}{\langle V \otimes V \rangle}$	$\cdots \to T^3(V) \otimes_k A(-3) \to T^2(V) \otimes_k A(-2) \to V \otimes_k A(-1) \to A$	$T(V)$
$\mathrm{Toy} = \dfrac{k[x,y]}{(x^2, y^2)} = \dfrac{k\langle x, y \rangle}{\langle xy - yx, x^2, y^2 \rangle}$	$\cdots \to A(-3)^5 \to A(-2)^3 \xrightarrow{\begin{bmatrix} -y & x & 0 \\ x & 0 & y \end{bmatrix}} A(-1)^2 \xrightarrow{(x,y)} A$	$\mathrm{Toy}^!$
$\mathrm{Toy}^! = \dfrac{k\langle x, y \rangle}{\langle xy + yx \rangle}$	$0 \to A(-2) \xrightarrow{\begin{bmatrix} y \\ x \end{bmatrix}} A(-1)^2 \xrightarrow{(x,y)} A$	Toy