CHEAT SHEET

This is a *listeners* guide to basic topics from Homological Algebra and Commutative Algebra II that appear in seminar. The only purpose of this is so that you might get a better chance of understanding the big picture of a talk that uses these concepts.

Equal / mixed characteristic. A local ring $(R, \mathfrak{m}, k)$ has:

- *equal characteristic zero* if $\operatorname{char}(R) = \operatorname{char}(k) = 0$
- *equal characteristic* p if $\operatorname{char}(R) = \operatorname{char}(k) = p$ for some prime $p > 0$
- *mixed characteristic* $(0, p)$ if $\operatorname{char}(R) = 0$ and $\operatorname{char}(k) = p$ for some prime $p > 0$.

Any *reduced* local ring satisfies one of the three conditions above; *any* local ring satisfies one of the above or $\operatorname{char}(R) = p^n$ and $\operatorname{char}(k) = p$ for some prime p and $n > 1$.

Regular rings. When you hear *regular*, you should think of localization of a polynomial ring or a power series ring. A Noetherian local ring $(R, \mathfrak{m}, k)$ is *regular* if $\dim(R) = \dim_{k-\text{vectorspace}}(\mathfrak{m}/\mathfrak{m}^2)$. Note that the inequality $\leq$ always holds by Krull height and NAK. A not-necessarily-local Noetherian ring is *regular* if all of its localizations are. The basic examples are

- $K[X_1, \dots, X_n]$ is regular for a field K.
- $K[\![X_1, \dots, X_n]\!]$ is regular for a field K.

Complete / completion. When you hear *complete*, you should think of quotient of power series ring: the quintessential example of a complete local ring is $K[\![X_1, \dots, X_n]\!]/I$ for some field K and ideal I.

In short, the *completion* of a local ring $(R, \mathfrak{m}, k)$ is the metric completion of R equipped with the (pseudo)metric given by $\operatorname{dist}(r_1, r_2) = 2^{-\sup\{n \mid r_1 - r_2 \in \mathfrak{m}^n\}}$; i.e., two elements are close if their difference is in a large power of $\mathfrak{m}$. A ring is *complete* if it is complete with respect to this (pseudo)metric. One can do a similar thing with R-modules.

Here are some key points:

- The completion of $K[X_1, \dots, X_n]_{(X_1, \dots, X_n)}/I$ is $K[\![X_1, \dots, X_n]\!]/I$ for a field K.
- Every complete Noetherian local ring of equal characteristic is isomorphic to a ring of the form $K[\![X_1, \dots, X_n]\!]/I$ for some field K, and something similar is true in mixed characteristic.
- A Noetherian local ring is "closely connected" to its completion (the map is faithfully flat, and with regular fibers in many cases).
- Complete local rings have good technical properties (like stronger versions of NAK, Hensel's Lemma).

Exact sequences. An *exact seqeunce* of R-modules is a (finite or infinite) collection of modules and maps

$$\cdots \to M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \to \cdots$$

such that the kernel of any map d_i is equal to the image of the previous map d_{i-1}. Special examples are *short exact sequences*: exact sequences of the form

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

the definition of exact in this case says that α is injective, β is surjective, and $N \cong M/L$.

The main tricks to beware of in exact sequences are:

- if $\cdots \to 0 \to M \to 0 \to \cdots$ appears, then $M = 0$.
- if $\cdots \to 0 \to M \to N \to 0 \to \cdots$ appears, then $M \cong N$.
- if $\cdots \to 0 \to M \xrightarrow{\alpha} N \to \cdots$ appears, then α is injective.
- if $\cdots \to M \xrightarrow{\beta} N \to 0 \to \cdots$ appears, then β is surjective.

Complexes and homology. A *complex* of R-modules weakening of exact sequence: a complex is a (finite or infinite) collection of modules and maps

$$\cdots \to M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \to \cdots$$

such that the composition of any two maps in a row is zero.

The ith *homology* or *cohomology* (you can think of these words as interchangable at first) of the complex above is the module $\ker(d_i)/\mathrm{im}(d_{i-1})$.

Free resolutions. A *free resolution* of n R-module M is a (finite or infinite) exact sequence of the form

$$\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0}(\to M) \to 0$$

for some b_i (also possibly infinite).

Tensor products. The *tensor product* of two R-modules M and N is another module $M \otimes_R N$ that satisfies a particular universal property. The key examples of tensor products are

- $R/I \otimes_R R/J = R/(I+J)$
- If S is an R-algebra and M is an R-module with presentation matrix B, then $S \otimes_R M$ is the S-module with presentation matrix B (image of B in S).

Ext and Tor. To any pair M, N of R-modules, there is a sequence of Ext modules $\mathrm{Ext}^i_R(M,N)$. In short, they are defined by taking a free resolution of M, computing the module of homomorphisms into N at each step to get a complex, and taking the homologies. One of the main tricks with Ext is that given a short exact sequence $0 \to N' \to N \to N'' \to 0$, there is a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(M,N') \to \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,N'') \to \mathrm{Ext}^{i+1}_R(M,N') \to \cdots$$

Also, $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$, the module of R-linear homomorphisms from M to N.

There is a sequence of Tor modules $\mathrm{Tor}^R_i(M,N)$. In short, they are defined by taking a free resolution of M, computing the module of homomorphisms into N at each step to get a complex, and taking the homologies. One of the main tricks with Tor is that given a short exact sequence $0 \to N' \to N \to N'' \to 0$, there is a long exact sequence

$$\cdots \to \mathrm{Tor}^R_i(M,N') \to \mathrm{Tor}^R_i(M,N) \to \mathrm{Tor}^R_i(M,N'') \to \mathrm{Tor}^R_i(M,N') \to \cdots$$

The other main trick with Tor is that $\mathrm{Tor}^R_i(M,N) \cong \mathrm{Tor}^R_i(N,M)$.