Math 342W/642/742W

Recitation – Day #7 (2.25.25)

I. Linear Algebra Basics for Vectors

Define the following for a set of vectors $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \in \mathbb{R}^n$:

(i) span(V): $= \left\{ \sum_{j=1}^{k} a_j \cdot \boldsymbol{v}_j \;\middle|\; \forall j,\, a_j \in \mathbb{R} \right\}$ = set of all linear combinations of the $\boldsymbol{v}_j$'s

(ii) linearly independent set V: $\sum_{j=1}^{k} a_j \cdot \boldsymbol{v}_j = 0 \iff a_j = 0,\, j \in \{1, 2, \ldots, k\}$
i.e., no vector is in the span of the other vectors

II. Linear Algebra Basics for Matrices

Given an $n \times p$ matrix $A \in \mathbb{R}^{n \times p}$, define the following:

(i) coℓ(A):= span of the column vectors of A

(ii) row(A):= span of the row vectors of A

(iii) rank(A):= # of leading 1's in rref(A) = dim(coℓ(A)) = dim(row(A))

(iv) null(A):= $\{\boldsymbol{x} \in \mathbb{R}^p \mid A\boldsymbol{x} = \boldsymbol{0}\}$

Define the matrix multiplication of two matrices $A \in \mathbb{R}^{n \times p}$, and $B \in \mathbb{R}^{p \times m}$:

$$(AB)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \quad \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m$$

Define the **matrix-column representation** for the matrix product AB:

$$AB = A[\ \boldsymbol{b}_1 \mid \boldsymbol{b}_2 \mid \cdots \mid \boldsymbol{b}_m\ ] = [\ A\boldsymbol{b}_1 \mid A\boldsymbol{b}_2 \mid \cdots \mid A\boldsymbol{b}_m\ ]$$

Define the **matrix-row representation** for the matrix product AB:

$$AB = \begin{bmatrix} \boldsymbol{a}_1 \\ \hline \boldsymbol{a}_2 \\ \hline \vdots \\ \hline \boldsymbol{a}_n \end{bmatrix} B = \begin{bmatrix} \boldsymbol{a}_1 \cdot B \\ \hline \boldsymbol{a}_2 \cdot B \\ \hline \vdots \\ \hline \boldsymbol{a}_n \cdot B \end{bmatrix}$$

III. More on Rank

Provide justification for the following statements on rank:

1. Given $A \in \mathbb{R}^{n \times p}$, $\text{rank}(A) = \text{rank}(A^T)$.

   Pf: $\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A^T)) = \text{rank}(A^T)$ ■

2. Given $A \in \mathbb{R}^{n \times p}$, $\text{rank}(A) = \text{rank}(UA) = \text{rank}(AV)$ whenever U, V are invertible matrices.

   Pf: First, we show that $\text{rank}(A) = \text{rank}(UA)$. Since U is an invertible matrix it is a product of elementary matrices. Each elementary matrix represents an elementary row operation. Elementary row operations preserve the row space of a matrix. Hence, $\text{row}(A) = \text{row}(UA)$ which implies that $\text{rank}(A) = \text{rank}(UA)$. Now we show that $\text{rank}(A) = \text{rank}(AV)$ by noting that
   $$\text{rank}(AV) = \text{rank}(AV)^T = \text{rank}(V^T A^T) = \text{rank}(A^T) = \text{rank}(A) \quad ■$$

3. Given $A \in \mathbb{R}^{n \times p}$, and $B \in \mathbb{R}^{p \times m}$, (i) $\text{rank}(AB) \leq \text{rank}(A)$, and (ii) $\text{rank}(AB) \leq \text{rank}(B)$

   Pf: By applying the matrix-column representation for the matrix product AB, we have that $\text{col}(AB) \subseteq \text{col}(A)$ since the columns of AB are linear combinations of the columns of A. Hence, $\text{rank}(AB) \leq \text{rank}(A)$. Similarly, by applying the matrix-row representation for the matrix product AB, we have that $\text{row}(AB) \subseteq \text{row}(B)$ since the rows of AB are linear combinations of the rows of B. Hence, $\text{rank}(AB) \leq \text{rank}(B)$. ■

IV. Equivalent Statements

Complete the following list of equivalent statements for $A \in \mathbb{R}^{n \times p}$ (assume $n > p$):

1. $\text{rank}(A) = p$.
2. The columns of A span $\mathbb{R}^p$.
3. The columns of A are linearly independent in $\mathbb{R}^n$.
4. The $p \times p$ matrix $A^T A$ is invertible.
5. $CA = I_p$ for some $p \times n$ matrix C.
6. If $A\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{x} \in \mathbb{R}^n$, then $\boldsymbol{x} = \mathbf{0}$.