Math 342W/642/742W

Recitation – Day #9 (3.4.25)

I. Geometric Proof for R^2

(i) Define the following terms:

- SST $= \sum_{i=1}^{n} (y_i - \bar{y})^2 = \|\boldsymbol{y} - \bar{y} \cdot \mathbf{1}_n\|^2$ *(total sum of squares)*

- SSE $= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} e_i^2 = \boldsymbol{e}^T \boldsymbol{e} = \|\boldsymbol{e}\|^2$
  (residual/error sum of squares a.k.a. RSS)

- SSR $= \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 = \|\hat{\boldsymbol{y}} - \hat{y} \cdot \mathbf{1}_n\|^2$
  (regression sum of squares a.k.a. explained sum of squares, ESS)

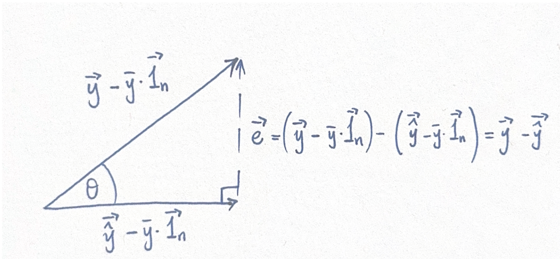
- $R^2 = \dfrac{\text{SSR}}{\text{SST}} = 1 - \dfrac{\text{SSE}}{\text{SST}}$ *(coefficient of determination)*

(ii) Define what ***mean-centering*** is: $\boldsymbol{y} - \bar{y} \cdot \mathbf{1}_n$

(iii) Project the mean-centered vector onto the column space of X:

$$\text{proj}_{\text{colsp}[X]} (\boldsymbol{y} - \bar{y}\mathbf{1}_n) = H(\boldsymbol{y} - \bar{y} \cdot \mathbf{1}_n) = H\boldsymbol{y} - H(\bar{y} \cdot \mathbf{1}_n) = \hat{\boldsymbol{y}} - \bar{y} \cdot H\mathbf{1}_n = \hat{y} - \bar{y} \cdot \mathbf{1}_n$$

(iv) Draw an illustration of this projection:



(v) Using geometric principles, show that $R^2 \in [0, 1]$.

By the Pythagorean Theorem and applying it to the illustration in part (iv) we have,

$$\|\boldsymbol{y} - \bar{y} \cdot \mathbf{1}_n\|^2 = \|\hat{\boldsymbol{y}} - \hat{y} \cdot \mathbf{1}_n\|^2 + \|\boldsymbol{e}\|^2 \implies \text{SST} = \text{SSR} + \text{SSE}$$
$$\implies \cos^2(\theta) = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}} \in [0, 1].$$

II. QR-Decomposition in Action

Let W be a subspace of $\mathbb{R}^4$ defined as $W = \text{span}\left(\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}\right)$. Let V be a 4×3 matrix whose columns are the vectors that span W.

(i) Construct an orthogonal basis for W.

Apply the Gram-Schmidt to the given basis for W:

$$\boldsymbol{u}_1 = \boldsymbol{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\boldsymbol{u}_2 = \boldsymbol{v}_2 - \text{proj}_{\boldsymbol{u}_1}(\boldsymbol{v}_1) = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{\boldsymbol{v_1}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\cdot\boldsymbol{u}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$$

$$\boldsymbol{u}_3 = \boldsymbol{v}_3 - \left(\text{proj}_{\boldsymbol{u_1}}(\boldsymbol{v}_3) + \text{proj}_{\boldsymbol{u_2}}(\boldsymbol{v}_3)\right) = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{\boldsymbol{v_3}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\cdot\boldsymbol{u}_1 - \frac{\boldsymbol{v_3}\cdot\boldsymbol{u}_2}{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}\cdot\boldsymbol{u}_2$$

$$= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{\frac{1}{2}}{\frac{3}{4}}\begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix} - \frac{2}{3}\begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix} = \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$$

Now we normalize the orthogonal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ into $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$ which will comprise the columns of Q where each $\boldsymbol{q}_i = \frac{1}{\|\boldsymbol{u}\|}\cdot\boldsymbol{u}_i$ for $i \in \{1,2,3\}$.

(ii) Find the QR-decomposition(factorization) of V.

$$Q = \begin{bmatrix}\frac{1}{2} & -\frac{3}{\sqrt{12}} & 0\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}}\end{bmatrix} \implies R = Q^TA = \begin{bmatrix}2 & \frac{3}{2} & 1\\ 0 & \frac{3}{\sqrt{12}} & \frac{2}{\sqrt{12}}\\ 0 & 0 & \frac{2}{\sqrt{6}}\end{bmatrix}$$

The $QR-$decomposition of V: $\underbrace{\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\\1&1&1\end{bmatrix}}_{V} = \underbrace{\begin{bmatrix}\frac{1}{2} & -\frac{3}{\sqrt{12}} & 0\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}}\\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}2 & \frac{3}{2} & 1\\ 0 & \frac{3}{\sqrt{12}} & \frac{2}{\sqrt{12}}\\ 0 & 0 & \frac{2}{\sqrt{6}}\end{bmatrix}}_{R}$