MA 3475 Calculus of Variations HW 1

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1)

Using the formula for length of a curve, we aim to minimize the functional:

$$I[y] = \int_a^b \sqrt{1 + y'^2} dx$$

Using the method of finite differences, we discterize this expression to

$$\varphi = \sum_{k=1}^{n+1} \sqrt{1 + y_k^{p'}} \Delta x_k$$

and then attempt to find critical points and minimize it:

$$\frac{\partial \varphi}{\partial x_j} = \sum_{k=1}^{n+1} \frac{\partial}{\partial x_j} (F(x_k, y_k, y_k^p) \Delta x_k) = 0$$

From the derivations in class, we arrive at the formula

$$F\left(x_j, y_j, y_j^p\right) - F_{y'}\left(x_j, y_j, y_j^p\right) y_j^p = F\left(x_{j+1}, y_{j+1}, y_{j+1}^p\right) - F_{y'}\left(x_{j+1}, y_{j+1}, y_{j+1}^p\right) y_{j+1}^p = C$$

Taking the limit as $n \to \infty$, we turn this back into a continuous function $F(y, y')$, and plug it and its first derivative into the formula above.

$$F(y, y') = \sqrt{1 + y'^2} \qquad F_{y'} = \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

$$\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}} = C$$

$$1 + y'^2 - y'^2 = C\sqrt{1 + y'^2}$$

$$\frac{1}{C} = \sqrt{1 + y'^2}$$

$$\frac{1}{C^2} - 1 = \beta = y'^2$$

$$y' = \frac{dy}{dx} = \sqrt{\beta} = \alpha$$

Finally, we solve the differential equation to obtain a linear function, which is the shortest path from point A to B.

$$\int dy = \int \alpha dx$$

$$\boldsymbol{y = \alpha x + C}$$

2)

We are given the functional:

$$I[y] = \int_a^b 2\pi y\sqrt{1+y'^2}dx$$

Using the simplified formula from the finite differences method, we arrive at an expression where we have to solve an ODE to find such y that will generate the minimal area of its surface revolution.

$$F\left(x_j, y_j, y_j^p\right) - F_{y'}\left(x_j, y_j, y_j^p\right)y_j^p = F\left(x_{j+1}, y_{j+1}, y_{j+1}^p\right) - F_{y'}\left(x_{j+1}, y_{j+1}, y_{j+1}^p\right)y_{j+1}^p = C$$

$$F(y, y') = 2\pi y \qquad F_{y'}(y, y') = \frac{2\pi y y'}{\sqrt{1+y'^2}}$$

$$2\pi y\sqrt{1+y'^2} - \frac{2\pi y y'^2}{\sqrt{1+y'^2}} = C$$

$$2\pi y(1+y'^2) - 2\pi y y'^2 = \sqrt{1+y'^2}C$$

$$2\pi y + 2\pi y y'^2 - 2\pi y y'^2 = \sqrt{1+y'^2}C$$

$$\frac{2\pi y}{\sqrt{1+y'^2}} = C$$

$$\frac{4\pi^2 y^2}{1+y'^2} = C^2 \Rightarrow y'^2 = \frac{4\pi^2 y^2}{C^2} - 1$$

Making the simplification that $c = \frac{2\pi}{C}$, this expression simplifies to

$$y' = \frac{dy}{dx} = \sqrt{c^2 y^2 - 1}$$

$$\int \frac{1}{\sqrt{(cy)^2 - 1}}dy = \int dx$$

$$u = cy \qquad \frac{du}{dy} = c \qquad du = c\,dy$$

$$\frac{1}{c}\int \frac{1}{\sqrt{u^2-1}}du = x + \beta$$

$$\frac{1}{c}\cosh^{-1}(u) = x + \beta \Rightarrow \frac{1}{c}\cosh^{-1}(cy) = x + \beta$$

$$cy = \cosh\left(\frac{x+\beta}{c}\right) \Rightarrow y = \frac{1}{c}\cosh\left(\frac{x+\beta}{c}\right)$$

$$\boldsymbol{y = \frac{C}{2\pi}\cosh\left(\frac{Cx+\beta}{2\pi}\right)}$$

which is our solution to the ODE.

3)

Given the multivariable function

$$\psi = \sum_{k=1}^{n} F(x_k, y_k, y_k^p)\Delta x_k$$

taking the partial derivative of ψ with respect to y_k, we get

$$\frac{\partial \psi}{\partial y_k} = \sum_{k=1}^{n} \left(F_y \frac{\partial y_k}{\partial y_j} + F_{y'} \frac{\partial y_k^p}{\partial y_j} \right) \Delta x_n = 0$$

We can simplify this massive sum by seeing that

$$y_k^p = \frac{y_k - y_{k-1}}{x_k - x_{k-1}}$$

$$\frac{\partial y_k^p}{\partial y_j} = \frac{1}{\Delta x_k}\left(\delta_{jk} - \delta_{j,k-1}\right)$$

which reduces the equation to

$$F_y\left(x_j, y_j, y_j^p\right)\Delta x_j - F_{y'}\left(x_j, y_j, y_j^p\right) - F_{y'}\left(x_{j_1}, y_{j+1}, y_{j+1}^p\right)$$

$$F_{y\left(x_j, y_j, y_j^p\right)} - \frac{F_{y'}\left(x_j, y_j, y_j^p\right) - F_{y'}\left(x_{j_1}, y_{j+1}, y_{j+1}^p\right)}{\Delta x_j}$$

Taking the limit as $n \to \infty$, we get the Euler-Lagrange ODE

$$F_y - \frac{d}{dx}F_{y'} = 0$$

If we make the assumption that $F = F(y, y')$, the expression $\frac{d}{dx}F_{y'}$ is no longer ambiguous, and we can expand the equation it using the multivariable chain rule:

$$F_y - \frac{d}{dx}F_{y'} = F_y - F_{y'y}y' - F_{y'y'}y'' = 0$$

and multiplying this equation by y', we arrive at an expression that can be expanded by doing the reverse-product rule.

$$F_y y' - F_{y'y}y'^2 - F_{y'y'}y'y' = \frac{d}{dx}\left(F - y'F_{y'}\right) = 0$$

which has the first integral of

$$\boldsymbol{F - y'F_{y'} = C}$$

4)

We are asked to show that a set $\varsigma(a, b)$ of all continuous functions defined on the interval [a,b] with the norm $\|y\| = \left\{\int_a^b |y(x)|^2 \; dx\right\}^{\frac{1}{2}}$ is a normed linear space. For this to be true, the following conditions have to be satisfied:

1. $\|v\| = 0$ iff $v = 0$
2. $\|av\| = |a|\|v\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

Property 1

Assuming we input 0 into the norm function,

$$\|0\| = \left\{ \int_a^b 0^2 dx \right\}^{\frac{1}{2}} = 0^{\frac{1}{2}} = 0$$

we get a result of 0.

Property 2

$$\|ay\| = \left\{ \int_a^b a^2 y^2 dx \right\}^{\frac{1}{2}} = \left\{ a^2 \int_a^b |y|^2 \ dx \right\}^{\frac{1}{2}} = |a| \ \left\{ \int_a^b |y|^2 \ dx \right\}^{\frac{1}{2}} = |a| \ \|y\|$$

Property 3

$$\|x\| + \|y\| = \left\{ \int_a^b x^2 dx \right\}^{\frac{1}{2}} + \left\{ \int_a^b |y(x)|^2 \ dx \right\}^{\frac{1}{2}}$$

$$\|x + y\| = \left\{ \int_a^b (x + y)^2 dx \right\}^{\frac{1}{2}}$$

Squaring both values and plugging them into the original inequality, we obtain

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x\|^2 + 2\left\{ \int_a^b x^2 dx \right\}^{\frac{1}{2}} \left\{ \int_a^b |y(x)|^2 \ dx \right\}^{\frac{1}{2}} + \|y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x\|^2 + 2 \ \|x\| \ \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

$$(\|x\| + \|y\|)^2 = (\|x\| + \|y\|)^2$$

using the Cauchy-Schwarz Inequality.

5)

A functional $J : S \to \mathbb{R}$ is continuous at the point (or function) $y_0 \varepsilon S$ if $\forall \varepsilon > 0 : \exists \delta > 0 :$ If $\|y - y_0\| < \delta$ then $|J[y] - J[y_o]| < \varepsilon$

We are given two norms:

$$\|y\| = \max_{a \leq x \leq b} \sum_{t=0}^{n} |y^{(t)}(x)| \qquad (1)$$

$$\|y\| = \max_{a \leq x \leq b} \left\{ |y(x)|, |y'(x)|, ..., |y^{(n)}(x)| \right\} \qquad (2)$$

In this problem, we are not explicitly given a functional definition $J[y]$. Assuming that this functional is continuous with respect to norm (1), we get the inequalities:

$$\|y - y_0\| = \sum_{t=0}^{n} \max_{a \leq x \leq b} |y^{(t)} - y_0^{(t)}| < \delta$$

$$|J[y] - J[y_0]| < \varepsilon$$

for some positive value δ and ε. If our norm for this space changes to norm (2) however, the only thing that changes is the norm-delta inequality. The epsilon inequality remains the same because we are not changing the definition of the functional by switching the norm definition. This gives us a new norm-delta inequality of:

$$\|y - y_0\| = \max_{a \leq x \leq b} \left\{ |y - y_0|, |y' - y_0'|, ..., |y^{(n)} - y_0^{(n)}| \right\}$$

This new norm inequality only returns one difference of n^{th} derivatives for y and y_0, unlike the original norm inequality which returns the total sum of all derivatives. This means that

$$\max_{a \leq x \leq b} \left\{ |y - y_0|, |y' - y_0'|, ..., |y^{(n)} - y_0^{(n)}| \right\} < \sum_{t=0}^{n} \max_{a \leq x \leq b} |y^{(t)} - y_0^{(t)}| < \delta$$

which means the epsilon-delta inequalities are still satisfied, and any functional on $\varsigma(a, b)$ which is continuous w.r.t norm (1) is also continuous w.r.t norm(2).