Yoneda's lemma proves that $0.9... \neq 1$

1 Yoneda completion yields number systems in which $0.9... \neq 1$

The limitologists like to claim that $0.9... = 1$. But here we will reveal a secret that they don't want you to know about. This secret is called "enriched Yoneda completion" and it allows you to construct a version of the real numbers in which $0.9... \neq 1$. Get ready for the utter anihilation of your dogmatic preconceptions:

Let's start slowly with the basics.
An *unsigned decimal* is a function $f : \mathbb{Z} \to \{0, \dots, 9\}$ such that
$\exists n \in \mathbb{Z}, \forall k > n, f(k) = 0$.

If n is as in the above statement, then we may also denote f by the string $f(n)f(n-1)\dots f(0).f(-1)f(-2)\dots$.

For example $12121.21212\dots$ is an unsigned decimal. The string $000012121.21212\dots$ is also an unsigned decimal, which is equal to the first example (at least they're equal if the ambiguous "$\dots$" dots are interpreted correctly). They're equal because they have the same digits, i.e. they define the same function $\mathbb{Z} \to \{0, \dots, 9\}$.

Another two examples are $1.000\dots$ and $0.999\dots$. These two unsigned decimals are not equal. In fact their -1st digits are different. I hope you're not triggered yet.

Dec_+ denotes the set of all unsigned decimals.

Now let's get a bit more advanced.

Let $\{0 \to 1\}$ be the category that has one initial object 0 and one terminal object 1 that is not isomorphic to 0.
With respect to the cartesian product $\wedge$ this category forms a Benabou-cosmos, i.e. a bicomplete symmetric closed monoidal category.

Let $\mathbb{N}[\frac{1}{10}]$ be the set of non-negative rational numbers whose denominators are of the form $2^n \cdot 5^m$ for $n, m \in \mathbb{N}$.
With the usual $\leq$-ordering $\mathbb{N}[\frac{1}{10}]$ forms a preorder, and hence a $\{0 \to 1\}$-enriched category.
Its enriched Yoneda completion $Y(\mathbb{N}[\frac{1}{10}])$ is the $\{0 \to 1\}$-enriched category of $\{0 \to 1\}$-enriched functors $\mathbb{N}[\frac{1}{10}]^{op} \to \{0 \to 1\}$, and all but two of its objects can be identified with unsigned decimals via the map
$\Phi : Dec_+ \to Y(\mathbb{N}[\frac{1}{10}])$,
$\Phi(f)(x) := \exists k < n.(x \leq \sum_{i=k}^{n} f(i) \cdot 10^i))$

Here we of course identify the objects 0 and 1 from $\{0 \to 1\}$ with the truth values False and True. So the proposition $\exists k < n.(x \leq \sum_{i=k}^{n} f(i) \cdot 10^i)$ denotes an object of $\{0 \to 1\}$.

So for example $\Phi(0.9...)(x) = \exists k < -1.(x \leq \sum_{i=k}^{-1} 9 \cdot 10^i)) = (x < 1)$,
while $\Phi(1)(x) = (x \leq 1)$.
The two presheaves $\Phi(0.9...)$ and $\Phi(1)$ are different, since $\Phi(0.9...)(1) =$ False $\neq$ True $= \Phi(1)(1)$.

The map Φ is in fact order-preserving and injective.

If we denote the two constant functors in $Y(\mathbb{N}[\frac{1}{10}])$ by $-\infty$ and ∞ we get that

$$Y(\mathbb{N}[\frac{1}{10}]) \cong \{\pm\infty\} \cup Dec_+$$

where we order the right hand side by letting $\pm\infty$ be top and bottom elements.

So the Yoneda completion consists precisely out of the unsigned decimals and $\pm\infty$. We have $0.9... \neq 1$ in the Yoneda completion.

"Ok", some of you will now smugly try to reply, "but we can add, subtract, multiply and divide in the $0.9... = 1$ real numbers, while we cannot do arithmetic with your enriched presheaves, can we?"

Well I am about to end your whole career.
Or should I say, *co-end* your career?

$$(X \otimes_{Day} Y)(c) := \int^{a,b \in C^{op}} C^{op}(a \otimes b, c) \otimes X(a) \otimes Y(b)$$

That's right, we can do arithmetic with enriched presheaves by using co-ends and day convolution.
Because what I haven't told you yet is that $\mathbb{N}[\frac{1}{10}]$ is not just a $\{0 \to 1\}$-enriched category, it is in fact a *monoidal* $\{0 \to 1\}$-enriched category. In fact it has not just one, but two monoidal structures, given by addition and multiplication respectively.
Via day convolution we then get two symmetric closed monoidal structures on $Y(\mathbb{N}[\frac{1}{10}])$, and the day multiplication still distributes over the day addition.
So we have associate and commutative addition and multiplication operators satisfying the distributive law, and right adjoint subtraction and division operators in a number system whose construction is far more natural and categorical than the usual ad hoc Dedekind cuts and Cauchy sequences, and we have all this while still maintaining that $0.9... \neq 1$. Checkmate!

Explicitly the addition works like this: For $y, z \in Y(\mathbb{N}[\frac{1}{10}])$ and $x \in \mathbb{N}[\frac{1}{10}]$ we have

$$(x \leq y + z) = \int^{a,b \in \mathbb{N}[\frac{1}{10}]^{op}} \mathbb{N}[\frac{1}{10}](x, a+b) \otimes y(a) \otimes z(b) =$$

$$= \exists a, b \in \mathbb{N}[\frac{1}{10}], x \leq a + b \wedge a \leq y \wedge b \leq z$$

So for example $-\infty + \infty = -\infty$ because
$\exists a, b \in \mathbb{N}[\frac{1}{10}], x \leq a + b \wedge \text{False} \wedge \text{True}$
is false for any x.

We will now go over various "Proofs" of $0.9... = 1$ and show how they fail in our structure.

Non-proof 1:
$1 = \frac{3}{3} = 3 \cdot \frac{1}{3} = 3 \cdot 0.3... = 0.9...$

Refutation 1:
$\frac{3}{3} \neq 3 \cdot \frac{1}{3}$, because the division is not inverse but merely adjoint to the multiplication. All we have are the unit and counit inequalities $a \cdot \frac{b}{a} \leq b \leq \frac{a \cdot b}{a}$, which tell us that $\frac{3}{3} \geq 3 \cdot \frac{1}{3}$, but an equality is not given.

Non-proof 2:
$x = 0.9...$
$10x = 9.9...$
$10x - x = 9$
$9x = 9$
$x = 1$
Refutation 2:
$10x - x \neq 9x$
The multiplication distributes over the addition, but it need not distribute over the subtraction. Also subtraction is not inverse but only adjoint to addition, which means that we have the unit and counit inequalities $(b-a)+a \leq b \leq (b+a) - a$ which allows us to deduce that $10x - x = (9x + x) - x \geq 9x$, but we do not get an equality.

Non-proof 3:
$1 - 0.9... = 0$, therefore $1 = 0.9...$
Refutation 3:
The equation $1 - 0.9... = 0$ is correct, but one cannot deduce that $1 = 0.9...$ because, once again, subtraction is only adjoint to addition. The only thing you can do here is apply the counit inequality to get $1 \geq (1-0.9...)+0.9... = 0 + 0.9... = 0.9...$

Summary:

$0.9... \neq 1$ is valid in the very reasonable number system $Y(\mathbb{N}[\frac{1}{10}])$. Also the construction of our number system is much more categorically straightforward than the ad hoc constructions of $\mathbb{R}$ via Dedekind cuts or Cauchy sequences. So $\mathbb{R}$ ought to be abandoned in favor of our superior $Y(\mathbb{N}[\frac{1}{10}])$.

2 Overview of equations in $Y(\mathbb{N}[\frac{1}{10}])$

Each element in $Y(\mathbb{N}[\frac{1}{10}])$ is either an unsigned decimal, or ∞ or $-\infty$.
The following laws are all satisfied, for all a, b, c:

Neutral elements:
$0 + a = a$
$1 \cdot a = a$

Associative laws:
$(a + b) + c = a + (b + c)$
$a \cdot (b \cdot c) = a \cdot (b \cdot c)$

Commutative laws:
$a + b = b + a$
$a \cdot b = b \cdot a$

Distributive law:
$a \cdot (b + c) = a \cdot b + a \cdot c$

Adjunctions:
$(a + b \leq c) \Leftrightarrow (a \leq b - c)$
$(a \cdot b \leq c) \Leftrightarrow (a \leq \frac{b}{c})$

Left adjoints preserve colimits:
$-\infty + a = -\infty$
$-\infty \cdot a = -\infty$
$\max(b, c) + a = \max(b + a, c + a)$
$\max(b, c) \cdot a = \max(b \cdot a, c \cdot a)$

Right adjoints preserve limits:

$\infty - a = \infty$

$\frac{\infty}{a} = \infty$

$\min(b, c) - a = \min(b - a, c - a)$

$\frac{\min(b,c)}{a} = \min(\frac{b}{a}, \frac{c}{a})$

Random sample of equations:

$1 + 1 = 2$

$0.9... + 1 = 1.9...$

$0.9... + 0.9... = 1.9...$

$2 - 0.9.. = 1$

$2 - 1 = 1$

$1.9... - 0.9... = 1$

$1.9... - 1 = 0.9...$

$0.9... \cdot 2 = 1.9...$

$0.9... \cdot 1.9.. = 1.9...$

$\frac{1.9...}{0.9...} = 2$

$\frac{2}{0.9...} = 2$

$\frac{1.9...}{2} = 0.9...$

$\frac{1}{0} = \infty$

$\frac{1}{\infty} = 0$

$\frac{a}{-\infty} = \infty, \forall a$

$\frac{0}{0} = \infty$

$\frac{\infty}{\infty} = \infty$ (already mentioned in right ajdoints preserve limits)

$0 \cdot \infty = 0$

$0 \cdot -\infty = -\infty$ (already mentioned in left ajdoints preserve colimits)

$1 - 2 = -\infty$ (we only have have non-negative numbers except $-\infty$)

3 Why negative numbers are a bad idea

You might try to do the same construction with $\mathbb{Z}[\frac{1}{10}]$ instead of $\mathbb{N}[\frac{1}{10}]$. It works a little bit, but we ultimately decided not to go for it. In this section we discuss what works and what doesn't.

A *signed decimal* a pair consisting out of an unsigned decimal f and a sign $\sigma \in \{+, -\}$.

$Dec_{\pm}$ denotes the set of signed decimals.

There is a fully faithful embedding $Dec_{\pm} \to Y(\mathbb{Z}[\frac{1}{10}])$, and we do again have $Y(\mathbb{Z}[\frac{1}{10}]) \cong \{\pm\infty\} \cup Dec_{\pm}$.
Note that $+0 \neq -0$ in $Dec_{\pm}$. That is questionable, but required to make this isomorphism work.

$\mathbb{Z}[\frac{1}{10}]$ is a monoidal enriched category with respect to addition, but it is not monoidal with respect to multiplication, because multiplication is not order preserving and hence not functorial.

So we do get an associative commutative addition on $Y(\mathbb{Z}[\frac{1}{10}])$, with an adjoint subtraction, but we get no multiplication or division.

Also if you compute around with the day addition you quickly notice things like $2 + (-1) = 0.9... \neq 1$ which nobody asked for.
In $Y(\mathbb{N}[\frac{1}{10}])$ the arithmetic tends to agree with the intuitive arithmetic one would expect on Dec_{+}, and only calculations involving $-\infty$ yield slightly unexpected results. In $Y(\mathbb{Z}[\frac{1}{10}])$ on the other hand the arithmetic does not agree at all with any intuitive expectations of how arithmetic on $Dec_{\pm}$ should work.

Due to $+0 \neq -0$, $2 + (-1) = 0.9...$ and the complete lack of multiplication we have decided that $Y(\mathbb{Z}[\frac{1}{10}])$ is not worthy of further consideration.

4 Some proof sketches

We have claimed that $Y(\mathbb{N}[\frac{1}{10}]) \cong \{\pm\infty\} \cup Dec_+$, but did not really justify that claim. In this section we will do that. At least a bit.

Throughout this section let Q be a dense subring of $\mathbb{R}$. Examples of such Q are given by $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{10}]$.
Theorem:
i) $\mathbb{R}$ is order-isomorphic to the full subcategory of non-representable non-constant $\{0 \to 1\}$-enriched presheaves on Q.
ii) Under this isomorphism the usual addition of $\mathbb{R}$ coincides with the Day convolution with respect to the addition on Q.

Proof:
i) Since $Q \subseteq \mathbb{R}$ is dense, we know that $\mathbb{R}$ is the set of Dedekind cuts of Q. A Dedekind cut of Q is by definition a pair (A, B) where $A, B \subseteq Q$ are subsets forming a partition of Q such that the following conditions are satisfied:
1.) $A \neq \emptyset$
2.) $A \neq Q$
3.) If $x \in A$ and $y < x$ then $y \in A$
4.) A has no greatest element.

Of course the B is completely redundant information here. A Dedekind cut can equivalently be thought of simply being a subset $A \subseteq Q$ satisfying the 4 conditions.

Now a subset $A \subseteq Q$ has a characteristic function $\chi_A : Q \to \{0, 1\}$. Taking characteristic functions is in fact an isomorphism between the power set $P(Q)$ and the set of all functions $\mathrm{Hom}_{\mathrm{Set}}(Q, \{0, 1\})$.

So a Dedekind cut induces a function $Q \to \{0, 1\}$. The condition 3 of Dedekind cuts says that this function is in fact an order-preserving function $Q^{op} \to \{0 \leq 1\}$. There is in general an isomorphism $\{A \subseteq Q | A \text{ satisfies condition } 3\} \cong \mathrm{Hom}_{\mathrm{PoSet}}(Q^{op}, \{0 \leq 1\})$

Now an order-preserving functionr $Q^{op} \to \{0 \leq 1\}$ is of course the exact same thing as a functor $Q^{op} \to \{0 \leq 1\}$ if one regards both sides as categories, and that is the exact same thing as a $\{0 \leq 1\}$-enriched functor $Q^{op} \to \{0 \leq 1\}$ if one regards both sides as $\{0 \leq 1\}$-categories. But that is the exact same thing as a $\{0 \leq 1\}$-enriched presheaf on Q.
So we in fact have
$\{A \subseteq Q | A \text{ satisfies condition 3}\} \cong Y(Q)$, where $Y(Q)$ is the enriched Yoneda completion of Q.

We then look at what the conditions 1, 2 and 4 mean under this translation. Condition 1 and 2 precisely say that a Dedekind cut defines a non-constant sheaf. Condition 4 precisely says that a Dedekind cut defines a non-representable sheaf.

So all in all, real numbers are precisely the non-constant non-representable presheaves on Q.

One must wonder why Dedekind included condition 4 in his definition at all. The most likely explanation is that he simply couldn't handle the fact that $0.9... \neq 1$ and included condition 4 only to get rid of that obvious inequality.

ii) The definition of Day convolution gives us for all $x \in Q$ and $y, z \in \mathbb{R}$ that

$$(x \leq y + z) \Leftrightarrow \int^{a,b \in Q^{op}} Q(x, a+b) \otimes y(a) \otimes z(b) \Leftrightarrow$$
$$\Leftrightarrow \exists a, b \in Q, x \leq a + b \wedge a \leq y \wedge b \leq z$$

From this description one can easily see that this coincides with the usual addition of real numbers. □

<u>Corollary</u>
The underlying set of $Y(Q)$ is isomorphic to $\{\pm\infty\} \cup \mathbb{R} \cup Q$
<u>Proof</u>
$\{\pm\infty\}$ are the two constant presheaves.

Q are the representable preshaves.
$\mathbb{R}$ are all the other presheaves. by the previous theorem. $\square$

<u>Theorem</u>

$Y(\mathbb{Z}[\frac{1}{10}]) \cong \{\pm\infty\} \cup Dec_{\pm}$ as sets.

<u>Proof</u>

$Dec_{\pm} \cong \mathbb{R} \cup \mathbb{Z}[\frac{1}{10}]$ because each real number that does not lie in $\mathbb{Z}[\frac{1}{10}]$ has a unique decimal representation, while each number in $x \in \mathbb{Z}[\frac{1}{10}]$ can be represented in exactly two ways by decimals: If $x \neq 0$ then there is one representation ending in 000..., and one representation ending in 999.... If on the other hand $x = 0$ then x can be represented by $+0$ and -0, which are two distinct elements in $Dec_{\pm}$. So $Dec_{\pm} \cong \mathbb{R} \cup \mathbb{Z}[\frac{1}{10}]$.
With our previous corollary we may now conclude that $Y(\mathbb{Z}[\frac{1}{10}]) \cong \{\pm\infty\} \cup \mathbb{R} \cup \mathbb{Z}[\frac{1}{10}] \cong \{\pm\infty\} \cup Dec_{\pm}$. $\square$
Remark: If one keeps track of the order then one notices that the order of $Y(\mathbb{Z}[\frac{1}{10}])$ coincides with the intuitive ordering of $\{\pm\infty\} \cup Dec_{\pm}$.

The theorem relating $Y(\mathbb{N}[\frac{1}{10}])$ and Dec_{+} works quite similarly, but working out the details is left as an exercise for the reader.