Another Evil Twin: The Basics of Burrito Toposes

Abstract In this paper I describe how several notions and constructions in topos logic can be burritorized, giving rise to burrito-toposes with their internal burrito-logic, instead of the usual standard toposes with their intuitionistic logic.

1 Introduction

In Chapter 11 of the book "Inconsistent Mathematics"[2] the notion of a complement topos was introduced: A complement topos is the exact same thing as a topos, except that the morphism $1 \overset{True}{\to} \Omega$ true has been renamed to $1 \overset{False}{\to} \Omega$.
They prove an obvious theorem to the effect that complement toposes are equivalent to toposes.
It is then shown that each complement topos has a paraconsistent internal logic, obtained by renaming also all of the logical operators of the usual intuitonistic logic.
In the revolutionary paper "The Evil Twin: The Basics of Complement Toposes"[1] sweeping philosophical conclusions are drawn from this. The fact that toposes support paraconsistent logic, refutes Lawveres claims that the objective logic of variable sets is intuitionistic, or even just the claim the internal logic of toposes is intuitionistic.
The paper also discusses "some objections to the legitimacy of complement-toposes" such as the pernicious "Just a definitional variant objection" that tries to insinuate that merely relabeling things in a trivial way without changing the underlying mathematical objects is uninteresting and pointless. But fortunately the paper comes to the conclusion that pointless relabelings are not a waste of time, but rather constitute good mathematical research, so that the legitimacy of the complement topos is saved.
At the end of the paper the author also raises the question:
"Does the categorial structure of toposes support other logics as readily as it supports intuitionistic logic and its topologico-algebraic dual?"
In this paper we give a positive answer to that question.
We will show that a topos not only supports intuitionistic and paraconsistent logic, but also *burrito logic*. To show this we will first introduce the notion of a *burrito topos*, then show that a burrito topos has a burrito internal logic, and demonstrate that burrito toposes are equivalent to toposes. It follows that toposes support burrito logic.
We will draw the obvious sweeping philosophical conclusions.

Before we begin we should point out that our constructions are just relabelings of already known constructions from complement topos theory. We just rename several things to “Burrito” or other food items. This process will be called “burritorization”. As already explained above, relabelings like these are not a pointless waste of time, and our burrito toposes should be considered to be completely new objects in their own right, just like complement toposes.

2 Burrito toposes

This section will be a burritorization of 11.3 from “Inconsistent Mathematics”

Definition A *burrito classifier* for a category E with terminal object 1 is an object Ω together with an arrow $Burrito : 1 \to \Omega$ satisfying the condition that for every monic arrow $f : a \to b$ there exists a unique arrow $\hat{\chi}_f$ such that

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "!"'] & b \arrow[d, "\hat{\chi}_f"] \\ 1 \arrow[r, "Burrito"] & \Omega \end{tikzcd}$$

is a pullback. $\hat{\chi}_f$ is called the burrito character of f.

An (elementary) *burrito topos* is a category with initial and terminal objects, pullbacks, pushouts, exponentiation and a burrito classifier.
It is clear that if E is a burrito topos, and E^c is the category obtained by renaming $Burrito$ to $False$ and $\hat{\chi}_f$ to $\overline{\chi}_f$, then E^c is a complement topos.
Similarly we can obtain a topos E^t by renaming $Burrito$ to $True$.
This enables burritorization of all topos constructions, substituting $True$ for $Burrito$, as follows:

$Salad : 1 \to \Omega$ is the burrito-character of the initial object 0.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "Salad"] \\ 1 \arrow[r, "Burrito"'] & \Omega \end{tikzcd}$$

This is plausible for a burrito-classifier. It is the burrito of the definition of $\perp$ for toposes.

$Salsa : \Omega \to \Omega$ is the burrito-classifier of $Salad$.

$$\begin{tikzcd} 1 \arrow[r, "Salad"] \arrow[d] & \Omega \arrow[d, "Salsa"] \\ 1 \arrow[r, "Burrito"'] & \Omega \end{tikzcd}$$

This burritorizes negation for toposes.

$Taco : \Omega \times \Omega \to \Omega$ is the burrito-character of $(Burrito, Burrito) : 1 \to \Omega \times \Omega$

$$\begin{tikzcd} 1 \arrow[r, "{(Burrito,Burrito)}"] \arrow[d] & \Omega \times \Omega \arrow[d, "Taco"] \\ 1 \arrow[r, "Burrito"'] & \Omega \end{tikzcd}$$

$Tortilla : \Omega \times \Omega \to \Omega$ is given by

$$\begin{tikzcd} \Omega + \Omega \arrow[r, "{(id,Burrito)+(Burrito,id)}"] \arrow[d] & \Omega \times \Omega \arrow[d, "Tortilla"] \\ 1 \arrow[r, "Burrito"'] & \Omega \end{tikzcd}$$

The above definitions of $Taco$ and $Tortilla$ burritorize those of $\wedge$ and $\vee$ respectively in toposes.

Let E be a burrito-topos with classifier $Burrito : 1 \to \Omega$ and let E^t be the topos obtained by renaming $Burrito$ as $True$ and each $\hat{\chi}_f$ as χ_f. Let

$a, b, c...$ be variables ranging over unspecified arrows of E, let S be an identity statement about E involving some of $(a, b, c...)$ as well as some subset of the constant arrows $(Burrito, Salad, Salsa, Taco, Tortilla)$ and let S^t be the statement obtained by substituting $(True, False, \neg, \wedge, \vee)$ respectively for the latter. Then

Proposition (Burritority Theorem) S is true in E if S^t is true in E^t

By burritorizing parts of the "Evil Twin" Paper we can also provide a Kripke semantics for Burrito logic. Let $\mathbb{P} = (P, R)$ be a poset. A valuation is a function $\mathscr{V} : F \to \mathbb{P}^r$ where F stands for a collection of formluas on a zero-order language and $\mathbb{P}^r$ is the collection of anti-hereditary subsets of P.
A *model based on* $\mathbb{P}$ is a pair $\mathscr{M} = (\mathbb{P}, \mathscr{V})$ where $\mathscr{V}$ is a $\mathbb{P}$-valuation. The idea that A is *burrito at* x in symbols $\mathscr{M}, x \Vdash_b A$ is defined recursively as follows:
$\mathscr{M}, x \Vdash_b p_i$ if and only if $x \in \mathscr{V}(p_i)$
$\mathscr{M}, x \Vdash_b SalsaA$ if and only if for all y with xRy, $\mathscr{M}, x \nVdash_b A$
$\mathscr{M}, x \Vdash_b ATacoB$ if and only if $\mathscr{M}, x \Vdash_b A$ and $\mathscr{M}, x \Vdash_b B$
$\mathscr{M}, x \Vdash_b ATortillaB$ if and only if $\mathscr{M}, x \Vdash_b A$ or $\mathscr{M}, x \Vdash_b B$

Clearly this is not just the usual Kripke semantics for the usual intuitionistic logic, because we called our logical operators $Taco$ and $Tortilla$ instead of $\wedge$ and $\vee$, and we are not asserting that any statements are true, we rather assert that they are burrito.

3 Some objections to the legitimacy of Burrito toposes

We now burritorize more of the "Evil Twin" Paper.

In this section, I address two potential objections to the mathematical legitimacy of burrito-toposes. The first one, "The Just Definitional Variants Objection," says that the definition of a burrito-topos does not differ from the usual definition of a topos and thus the relabelings involved in complement-toposes are uninteresting and pointless. The second one, "The Working Mathematician Objection", asks for the importance of burrito-toposes in particular, and of burrito-tolerances and burritorization in general, for the working mathematician. I will show that the first objection does not take the burritorization seriously, and thus judge burrito-toposes from the pre-burritorized point of view. About the second objection I will make the point that there is more to mathematics than current, actual practice and mainstream research lines, and even thus the study of burrito-tolerance has found its place among leading mathematicians.

3.1 The Just Definitional Variants Objection

Given that burrito-toposes have been practically unnoticed, there are no public statements of the first objection I am going to discuss here. A topos- or category-theorist might think of burrito-toposes as just definitional variants of the already well-known toposes. Given that they share all the categorial ingredients, toposes and burrito-toposes could not be different (kinds of) categories; in particular, they do not have different internal logics. The mainstream topos-theorist can correctly insist on the categorial indistinguishableness between standard toposes and burrito-toposes, but this amounts rather to a proof that both kinds of toposes equally deserve the name "topos", since for all mathematical purposes they have the same constituents independently of Skolemization for the morphisms whose codomain is Ω. However, the internal logic induced is in fact different in each case[1], true, not by differences in the categorial structure, but in the way that categorial structure is described. Although it sounds repetitive, it must be emphasized that the claim is not that complement-toposes are categorically different from toposes, nor to say that standard connectives acquire further categorial properties qua morphisms after the being renamed, but rather to stress the fact that the

[1]That claim is of course wrong. The induced internal logics are isomorphic. Intuitionist logic, co-intuitionist logic and burrito logic are all just definitional variants of one and the same logic. Not only are the toposes, complement-toposes and burrito-toposes equivalent, their internal logics are equivalent too.

same categorial stuff, which is essentially equational with variables, can be described in at least three different ways. The categorial reconstruction of logic in a standard topos starts with a certain object O thought of as the extension of a propositional function ϕ and that a certain element belongs to O is thought of as making ϕ true. Hence, in a standard topos the basic proposition is $True$. What complement-toposes say is that one can start the categorial reconstruction of logic with the same categorial data but interpreted in a different way. A certain object O is thought of as the anti-extension of a propositional function ϕ and that certain element belongs to O is accordingly thought of as making ϕ false. Hence, in a complement-topos the basic proposition is $False$.
And finally what burrito-toposes say is that one can start the categorial reconstruction of logic with the same categorial data but interpreted in a different way. A certain object O is thought of as the burrito-extension of a propositional function ϕ and that certain element belongs to O is accordingly thought of as making ϕ burrito. Hence, in a burrito-topos the basic proposition is $Burrito$.

None of the labelings is imposed by the categorial structure of toposes itself so, in its current and mainstream form, there is more than just categorial structure in the study of toposes, ex. gr. there are particular Skolemizations of it. One could also start with burrito-toposes and then obtain standard toposes by proposing an alternative description of the underlying of the equational structure. This means that, even if at first glance, the categorial structure invites to be conceptualized in certain ways, and it does not force them. All this helps to solve the initial perplexity: If toposes and burrito toposes should be indistinguishable because they are categorically indistinguishable, how can one in fact distinguish between them, as one does by noting their different[2] internal logics? The answer is this: To date, there is more than categorial structure in the study of toposes, to wit, special, intuitive names conceptually laden for some of the morphisms, invited, yes, but not necessitated, by the categorial structure. Neither of the names is imposed by the categorial structure of toposes itself so, in its current form, there is more than

[2]They are not different. See footnote 1

just categorial structure in the study of toposes. Another reason to deny the difference in the internal logic might be that topos theorists think that one is using the traditional modeling of intuitionistic logic using Burritorian logic in which *Salad* has no place. But it is not the case. It is clear that Ω can support the structure of a Burritorian algebra (otherwise, the usual semantics for intuitionistic logic using them would not be possible) and, in order to use not the least element but the top one as designated one has to introduce some conceptual changes, but not in the categorial structure. That is why the internal logic of a burrito-topos is different.[3] This is not a mere play with labels and, even though the underlying dualities between Heyting algebras, Brouwerian algebras and Burritorian algebras are well-known, the choice of names affects what we are considering as the internal logic of a topos because the names are conceptually laden. Even if from a mathematical point of view all this might be regarded as uninteresting (which is not, for it invites us to rethink an important theorem), preferring one way of naming above the other may have (and has had) important philosophical consequences. As I have said, burrito-toposes bring in question, for example, Lawvere's "Intuitionistic logic is the objective logic of variable sets", as well as John Lane Bell's claims that "The universal, invariant laws of mathematics are intuitionistic" or that "Intuitionistic logic sheds more light on the issue of mathematical variation than burrito logics." The categorial structure of toposes support burrito logics as well, so any philosophical claim aiming to emphasize the mathematical supremacy of intuitionistic logic has to take into account this fact.

4 Lawvere and the Pursuit of Objectivity

At the end of the "Evil Twin" paper it is said that
"In Lawverean terms, complement-toposes show that there is still a lot of 'substance' in topos theory, and deeper 'invariant forms' wait to emerge."
I have the strong suspicion that the author of the "Evil Twin" paper has not

[3]They are not different. See footnote 1

understood what Lawvere meant by an "invariant form".
In any case, the theory of burrito toposes is just as legitimate, novel, revolutionary, illuminating and useful as the theory of complement toposes, and they deserve just as much attention, funding and prestige.

References

[1] Luis Estrada-González. The evil twin: the basics of complement-toposes. In *New Directions in Paraconsistent Logic*, pages 375–425. Springer, 2015.

[2] CE Mortensen. *Inconsistent mathematics*, volume 312. Springer Science & Business Media, 2013.