PROBLEMS

Due Friday Feb. 10, 2023 @ 4pm

1. Consider the cost function
$$J(\boldsymbol{\beta}) = \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2 + \boldsymbol{\beta}^\top \mathbf{R}\boldsymbol{\beta}$$
for a fixed matrix $\mathbf{R}$ for which $\mathbf{R}+\mathbf{R}^\top$ is positive definite. Show $\boldsymbol{\beta}_{\mathsf{opt}} = \arg\min_{\boldsymbol{\beta}} J(\boldsymbol{\beta})$ equals
$$\boldsymbol{\beta}_{\mathsf{opt}} = (\mathbf{X}^\top\mathbf{X} + \tfrac{1}{2}\mathbf{R} + \tfrac{1}{2}\mathbf{R}^\top)^{-1}\mathbf{X}^\top\mathbf{y}.$$
Hint: Use the gradient-zeroing technique that was used to derive $\boldsymbol{\beta}_{\mathsf{ls}}$ in the lecture.

2. Suppose that we have a model
$$y = \widehat{f}(x;\theta) + \epsilon$$
with zero-mean σ^2-variance Gaussian noise ϵ and
$$\widehat{f}(x;\theta) = e^\theta x$$
with unknown $\theta \in \mathbb{R}$. We would like to fit θ to a dataset $\{(x_i, y_i)\}_{i=1}^n$ of independent draws from this model.
   (a) Derive an expression for the negative log likelihood function. Simplify as much as possible
   (b) Find the maximum-likelihood estimate of θ. *Hint:* Set the gradient of the negative loglikelihood to zero.

3. Say that $\mathbf{y} = [y_1, \ldots, y_n]^\top$ are independent samples of a Gaussian random variable Y with mean 1 and unknown variance σ^2. That is, the pdf of Y equals
$$p_Y(y\,|\,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(y-1)^2\Big).$$
   (a) Derive an expression for the maximum-likelihood estimate of the standard deviation σ from $\mathbf{y}$, i.e., $\widehat{\sigma} = \arg\max_\sigma p(\mathbf{y}\,|\,\sigma^2)$.
   (b) Derive an expression for the maximum-likelihood estimate of the variance σ^2 from $\mathbf{y}$, i.e., $\widehat{\sigma^2} = \arg\max_{\sigma^2} p(\mathbf{y}\,|\,\sigma^2)$. Does it agree with your answer to part (a)?

   In your derviations, be very careful to distinguish scalar quantities like y from vector quantities like $\mathbf{y}$. (When writing by hand, it's typical to underline vector quantities.)