

FINITE ELEMENTS

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ABSTRACT. Study the

INTRODUCTION

TRIANGLE

Let $p, q, r \in \mathbb{R}^2$ be the vertices of a triangle $T \subset \mathbb{R}^2$ with area A_T . Let θ_p be the interior angle at vertex p , and likewise θ_q, θ_r . Let P be the length of the side opposite p , likewise with Q and R . (We may sometimes abuse notation and denote by P, Q, R the sides themselves.)

Area is computed by

$$\begin{aligned} A_T &= \frac{1}{2} \det \begin{bmatrix} p - r & q - r \end{bmatrix} \\ &= \frac{1}{2} \det \begin{bmatrix} p - q & r - q \end{bmatrix} \\ &= \frac{1}{2} \det \begin{bmatrix} q - p & r - p \end{bmatrix} \end{aligned}$$

BARYCENTRIC COORDINATES

Let $s = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Then we can write $s = \alpha_p p + \alpha_q q + \alpha_r r$, with $\alpha_p + \alpha_q + \alpha_r = 1$. Define

$$B = \left[\begin{array}{c|c|c} p & q & r \\ \hline 1 & 1 & 1 \end{array} \right]$$

and

$$\alpha = \begin{bmatrix} \alpha_p \\ \alpha_q \\ \alpha_r \end{bmatrix}$$

so that $s = B\alpha$.

2000 *Mathematics Subject Classification.* Primary 51R10; Secondary 51R12.

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Observe that

$$\begin{aligned}
 \det B &= \det \begin{bmatrix} q & r \end{bmatrix} - \det \begin{bmatrix} p & r \end{bmatrix} + \det \begin{bmatrix} p & q \end{bmatrix} \\
 &= (q_x r_y - q_y r_x) - (p_x r_y - p_y r_x) + (p_x q_y - p_y q_x) \\
 &= p_x q_y - p_x r_y - r_x q_y - p_y q_x + p_y r_x + r_y q_x - r_x r_y + r_x r_y \\
 &= (p_x - r_x)(q_y - r_y) - (p_y - r_y)(q_x - r_x) \\
 &= 2A_T
 \end{aligned}$$

Using cofactor expansion we calculate

$$B^{-1} = \frac{1}{2A_T} \begin{bmatrix} q_y - r_y & -(q_x - r_x) & \det \begin{bmatrix} q & r \end{bmatrix} \\ -(p_y - r_y) & p_x - r_x & -\det \begin{bmatrix} p & r \end{bmatrix} \\ p_y - q_y & -(p_x - q_x) & \det \begin{bmatrix} p & q \end{bmatrix} \end{bmatrix}$$

BUILDING BLOCKS OF ELEMENTS

Define $v_p : T \rightarrow \mathbb{R}$ to be the restriction of the unique affine-linear function with $v_p(p) = 1$, $v_p(q) = 0$, $v_p(r) = 0$. Define v_q and v_r similarly.

In fact $v_p(s)$ is the p coordinate in the barycentric expansion of $s = (x, y)$. (This is because linear functions are uniquely determined by their values.) If e_i is the i^{th} standard basis element of \mathbb{R}^3 we have

$$v_p(s) = e_1^T B^{-1} s = \frac{1}{2A_T} \left((q_y - r_y)x - (q_x - r_x)y + \det \begin{bmatrix} q & r \end{bmatrix} \right)$$

and its gradient is constant and equal to

$$\nabla v_p = \frac{1}{2A_T} \begin{bmatrix} q_y - r_y \\ -(q_x - r_x) \end{bmatrix}$$

Integrals. We now compute L^2 norm of v_p and the L^2 inner product of v_p, v_q .

To compute the squared norm $\|v_p\|^2$ of v_p :

$$\|v_p\|^2 = \int_T (v_p)^2 = \int_0^1 t^2 |v_p^{-1}(t)| dt$$

Observing that $|v_p^{-1}(t)| = (1-t)P$ we have

$$\begin{aligned}
 \int_T (v_p)^2 &= \int_0^1 t^2 (1-t)P dt \\
 &= P \left(\frac{1}{3} - \frac{1}{4} \right) \\
 &= \frac{1}{12}P
 \end{aligned}$$

Likewise $\|v_q\|^2 = \frac{1}{12}Q$ and $\|v_r\|^2 = \frac{1}{12}R$.

To compute the inner product $\langle v_p, v_q \rangle_2$ of v_p and v_q we foliate T by lines parallel to R , which are level sets of v_r .

$$\langle v_p, v_q \rangle = \int_0^1 \int_{v_r^{-1}(t)} v_p v_q dt$$

We use a linear parametrization of $v_r^{-1}(t)$ with independent variable s , traversing from P when $s = 0$ to Q when $s = 1$.

In abuse of notation re-use the symbols v_p, v_q for their restrictions to $v_r^{-1}(t)$. Then $v_p(0) = 0$ and $v_p(1) = (1-t)$, while $v_q(0) = 1-t$ and $v_q(1) = 0$. So $v_p(s) = s(1-t)$ and $v_q(s) = (1-s)(1-t)$.

Thus

$$\int_{v_r^{-1}(t)} v_p v_q = \int_0^1 s(1-s)(1-t)^2 R ds = \frac{1}{6} R (1-t)^2$$

and so

$$\begin{aligned} \langle v_p, v_q \rangle_2 &= \int_0^1 \frac{1}{6} R (1-t)^2 dt \\ &= \frac{1}{18} R \end{aligned}$$

The L^2 norm of ∇v_p is nothing more than

$$\|\nabla v_p\|^2 = \frac{(q_y - r_y)^2 + (q_x - r_x)^2}{4A_T^2} = \frac{P^2}{4A_T^2}$$

The inner product of ∇v_p and ∇v_q is

$$\langle \nabla v_p, \nabla v_q \rangle = -\frac{1}{4A_T^2} \left((q_y - r_y)(p_y - r_y) + (q_x - r_x)(p_x - r_x) \right) = -\frac{\langle q - r, p - r \rangle}{4A_T^2}$$

This can also be interpreted as proportional to the cosine of the interior angle at r :

$$\langle \nabla v_p, \nabla v_q \rangle = -\frac{1}{4A_T^2} PQ \cos \theta_r$$

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