FINITE ELEMENTS

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ABSTRACT. Study the

INTRODUCTION

TRIANGLE

Let $p, q, r \in \mathbb{R}^2$ be the vertices of a triangle $T \subset \mathbb{R}^2$ with area A_T. Let θ_p be the interior angle at vertex p, and likewise θ_q, θ_r. Let P be the length of the side opposite p, likewise with Q and R. (We may sometimes abuse notation and denote by P, Q, R the sides themselves.)

Area is computed by

$$\begin{aligned} A_T &= \frac{1}{2} \det \left[\, p - r \mid q - r \, \right] \\ &= \frac{1}{2} \det \left[\, p - q \mid r - q \, \right] \\ &= \frac{1}{2} \det \left[\, q - p \mid r - p \, \right] \end{aligned}$$

BARYCENTRIC COORDINATES

Let $s = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Then we can write $s = \alpha_p p + \alpha_q q + \alpha_r r$, with $\alpha_p + \alpha_q + \alpha_r = 1$. Define

$$B = \left[\begin{array}{c|c|c} p & q & r \\ 1 & 1 & 1 \end{array} \right]$$

and

$$\alpha = \begin{bmatrix} \alpha_p \\ \alpha_q \\ \alpha_r \end{bmatrix}$$

so that $x = B\alpha$.

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Observe that

$$\begin{aligned}\det B &= \det\left[\begin{array}{c|c} q & r \end{array}\right] - \det\left[\begin{array}{c|c} p & r \end{array}\right] + \det\left[\begin{array}{c|c} p & q \end{array}\right] \\ &= (q_x r_y - q_y r_x) - (p_x r_y - p_y r_x) + (p_x q_y - p_y q_x) \\ &= p_x q_y - p_x r_y - r_x q_y - p_y q_x + p_y r_x + r_y q_x - r_x r_y + r_x r_y \\ &= (p_x - r_x)(q_y - r_y) - (p_y - r_y)(q_x - r_x) \\ &= 2A_T\end{aligned}$$

Using cofactor expansion we calculate

$$B^{-1} = \frac{1}{2A_T}\begin{bmatrix} q_y - r_y & -(q_x - r_x) & \det\left[\begin{array}{c|c} q & r \end{array}\right] \\ -(p_y - r_y) & p_x - r_x & -\det\left[\begin{array}{c|c} p & r \end{array}\right] \\ p_y - q_y & -(p_x - q_x) & \det\left[\begin{array}{c|c} p & q \end{array}\right] \end{bmatrix}$$

Building blocks of elements

Define $v_p : T \to \mathbb{R}$ to be the restriction of the unique affine-linear function with $v_p(p) = 1$, $v_p(q) = 0$, $v_p(r) = 0$. Define v_q and v_r similarly.

In fact $v_p(s)$ is the p coordinate in the barycentric expansion of $s = (x, y)$. (This is because linear functions are uniquely determined by their values.) If e_i is the i^{th} standard basis element of $\mathbb{R}^3$ we have

$$v_p(s) = e_1^T B^{-1} s = \frac{1}{2A_T}\Big((q_y - r_y)x - (q_x - r_x)y + \det\left[\begin{array}{c|c} q & r \end{array}\right]\Big)$$

and its gradient is constant and equal to

$$\nabla v_p = \frac{1}{2A_T}\begin{bmatrix} q_y - r_y \\ -(q_x - r_x) \end{bmatrix}$$

Integrals. We now compute L^2 norm of v_p and the L^2 inner product of v_p, v_q.

To compute the squared norm $||v_p||^2$ of v_p:

$$||v_p||^2 = \int_T (v_p)^2 = \int_0^1 t^2 |v_p^{-1}(t)|\, dt$$

Observing that $|v_p^{-1}(t)| = (1-t)P$ we have

$$\begin{aligned}\int_T (v_p)^2 &= \int_0^1 t^2(1-t)P\, dt \\ &= P\Big(\frac{1}{3} - \frac{1}{4}\Big) \\ &= \frac{1}{12}P\end{aligned}$$

Likewise $||v_q||^2 = \frac{1}{12}Q$ and $||v_r||^2 \frac{1}{12}R$.

To compute the inner product $\langle v_q, v_r\rangle_2$ of v_p and v_q we foliate T by lines parallel to P, which are level sets of v_p.

$$\langle v_q, v_r\rangle = \int_0^1 \int_{v_p^{-1}(t)} v_q v_r \, dt$$

We use a linear parametrization of $v_p^{-1}(t)$ with independent variable s, traversing from $tp + (1-t)q$ when $s = 0$ to $tp + (1-t)r$ when $s = 1$:

$$s \mapsto s(tp + (1-t)r) + (1-s)(tp + (1-t)q) = tp + s(1-t)r + (1-s)(1-t)q$$

As the coefficients sum to one, they are barycentric coordinates. In abuse of notation re-use the symbols v_q, v_r for their restrictions to $v_p^{-1}(t)$. Then $v_q(s) = (1-s)(1-t)$ and $v_r(s) = s(1-t)$.

Thus

$$\int_{v_p^{-1}(t)} v_q v_r = \int_0^1 s(1-s)(1-t)^2 \, ds = \frac{1}{6}(1-t)^2$$

and so

$$\begin{aligned} \langle v_q, v_r\rangle_2 &= \int_0^1 \frac{1}{6}(1-t)^2 \, dt \\ &= \frac{1}{18} \end{aligned}$$

The L^2 norm of ∇v_p is nothing more than

$$||\nabla v_p||^2 = \int_T \frac{(q_y - r_y)^2 + (q_x - r_x)^2}{4A_T^2} = \frac{P^2}{4A_T}$$

The inner product of ∇v_p and ∇v_q is

$$\langle \nabla v_p, \nabla v_q\rangle = -\frac{1}{4A_T^2}\Big((q_y - r_y)(p_y - r_y) + (q_x - r_x)(p_x - r_x)\Big) = -\frac{\langle q - r, p - r\rangle}{4A_T^2}$$

This can also be interpreted as proportional to the cosine of the interior angle at r:

$$\langle \nabla v_p, \nabla v_q\rangle = -\frac{1}{4A_T} PQ \cos\theta_r$$

Triangulations and elements

Let $\Omega \subset \mathbb{R}^2$ be a precompact domain with piecewise linear boundary. Let $\mathcal{T}$ be a triangulation of Ω such that:

- Each triangle is nondegenerate
- No triangle has a vertex in the interior of its face
- ...

Such a triangulation is "regular."

For a vertex p of the triangulation, define

$$\delta_p(x) = \begin{cases} v_p(x), & x \text{ in the interior of a triangle adjacent to } p \\ 0, & \text{else} \end{cases}$$

It is easily checked that δ_p is piecewise linear.

The space $V_{\mathcal{T}}$ spanned by the δ_p is a finite-dimensional subspace of $H^1(\Omega)$. Problems in $H^1(\Omega)$ can be weakly approximated by linear algebra in $V_{\mathcal{T}}$.

The H^1 norm of δ_p can be computed as follows:

$$||\nabla\delta_p||^2 = \textstyle\sum_{T\ni p} ||\nabla v_p||^2 \tag{1}$$

$$= \textstyle\sum_{T\ni p} \frac{1}{12}(\text{length of far side of } T) \tag{2}$$

Computations with triangles

Triangles admit a self-similar triangulation.

Proof of Weyl's law

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