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1) $$\begin{bmatrix}2&-1&0\\1&2&5\\1&1&3\end{bmatrix} \xrightarrow{(R_1\cdot\frac{1}{2})} \begin{bmatrix}1&-\frac{1}{2}&0\\1&2&5\\1&1&3\end{bmatrix} \xrightarrow{(R_2-R_1),\ (R_3-R_1)} \begin{bmatrix}1&-\frac{1}{2}&0\\0&\frac{5}{2}&5\\0&\frac{3}{2}&3\end{bmatrix} \xrightarrow{(R_2\cdot\frac{2}{5}),\ (R_3\cdot\frac{2}{3})} \begin{bmatrix}1&-\frac{1}{2}&0\\0&1&2\\0&1&2\end{bmatrix} \xrightarrow{(R_3-R_2)} \begin{bmatrix}1&-\frac{1}{2}&0\\0&1&2\\0&0&0\end{bmatrix}$$

* As the row echelon matrix has 2 leading 1's, the dimension of the columnspace is 2.

* Since the third row is [0 0 0], the third ~~row~~ of b must be 0 for the system to have a solution.

2) $$A=\begin{bmatrix}1&-2&1\\1&2&2\\2&-2&2\end{bmatrix} \xrightarrow{(R_2-R_1),\ (R_3-2R_1)} \begin{bmatrix}1&-2&1\\0&4&1\\0&2&0\end{bmatrix} \xrightarrow{(R_2-2R_3),\ (\text{then } R_3\cdot\frac{1}{2})} \begin{bmatrix}1&-2&1\\0&0&1\\0&1&0\end{bmatrix} \xrightarrow{(R_2 \text{ and } R_3 \text{ swap})} \begin{bmatrix}1&-2&1\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{(R_1+2R_2),\ (R_1-R_3)} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$b=\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

a) b is in the columnspace if the system is consistent

$$b=\begin{bmatrix}-1\\1\\1\end{bmatrix}\to\begin{bmatrix}-1\\2\\3\end{bmatrix}\to\begin{bmatrix}-1\\-4\\3/2\end{bmatrix}\to\begin{bmatrix}-1\\3/2\\-4\end{bmatrix}\to\begin{bmatrix}6\\3/2\\-4\end{bmatrix}$$ (same row operations) $$6\begin{bmatrix}1\\1\\2\end{bmatrix}+\frac{3}{2}\begin{bmatrix}-2\\2\\-2\end{bmatrix}-4\begin{bmatrix}1\\2\\2\end{bmatrix}$$

b) There are 3 leading 1's in the solution of $Ax=0$, so $\text{rank}(A)=3$

c) Nullity of the matrix is equal to the parameters or $n-\text{rank}(A)$ which is 0.

d) $$\left[\begin{array}{ccc|c}1&-2&1&0\\1&2&2&0\\2&-2&2&0\end{array}\right]$$ (same row operations) $$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right] \quad x=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

* The reduced row echelon form of the matrix is $$\begin{bmatrix}1&0&0&6\\0&1&0&3/2\\0&0&1&-4\end{bmatrix}\to$$ The general solution is; $$\begin{bmatrix}6\\3/2\\-4\end{bmatrix}$$

3) a) $$\left[\begin{array}{cccc|c}2&2&0&4&0\\1&0&1&3&0\\2&4&-2&2&0\end{array}\right] \xrightarrow{(R_3-R_1),\ (\text{then } R_1\cdot\frac{1}{2}),\ (R_2-\text{new } R_1)} \left[\begin{array}{cccc|c}1&1&0&2&0\\0&-1&1&1&0\\0&2&-2&-2&0\end{array}\right] \xrightarrow{(R_2\cdot -1),\ (R_3\cdot\frac{1}{2}),\ (\text{then } R_3-\text{new } R_2)} \left[\begin{array}{cccc|c}1&1&0&2&0\\0&1&-1&-1&0\\0&0&0&0&0\end{array}\right] \xrightarrow{(R_1-R_2)} \left[\begin{array}{cccc|c}1&0&1&3&0\\0&1&-1&-1&0\\0&0&0&0&0\end{array}\right]$$

$\Rightarrow$ $x_1+x_3+3x_4=0$
$x_2-x_3-x_4=0$

$\downarrow$

$x_4=s,\ x_3=v$

$x_1=-v-3s$

$x_2=v+s$

* Basis for the nullspace of A $\Leftarrow$ $$\begin{bmatrix}-v-3s\\v+s\\v\\s\end{bmatrix}=v\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}-3\\1\\0\\1\end{bmatrix}$$ $\leftarrow$

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b) Nullity of A is the dimension of the nullspace and therefore is 2.

c) rank is n-nullity or the number of leading 1's which is 2

d) $\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 5 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 4 & -2 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 5/2 \\ 0 & -1 & 1 & 1 & 1/2 \\ 0 & 2 & -2 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 5/2 \\ 0 & 1 & -1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(same row operations)

$x_1 + x_2 + 2x_4 = 5/2$

$x_2 - x_3 - x_4 = -1/2$

$\downarrow$

$x_3 = d \quad , x_4 = e$

$x_1 = 3 - 3e - d$

$x_2 = -\frac{1}{2} + d + e$

$$X = \underbrace{\underbrace{d \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Homogeneous solution}} + \begin{bmatrix} 3 \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}}_{\text{general solution}}$$

e) The third row is $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ and b_3 must be equal to 0 for the system to be consistent. Otherwise, solutions for any b can not be determined.

4) $A = \begin{bmatrix} 1 & 0 & 1 & -4 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 4 & 6 \end{bmatrix} \xrightarrow[(R_3 - 2R_1)]{(R_2 - 2R_1)} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 2 & 14 \end{bmatrix} \xrightarrow[(R_3 - 2R_2)]{(R_1 - R_2)} \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Columnspace base $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$

Rowspace base $= \left\{ \begin{bmatrix} 1 & 0 & 1 & -4 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 3 & -1 \end{bmatrix} \right\}$

5) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow[(R_3 - R_1)]{(R_2 - 2R_1)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow[(R_4 + R_3)]{(R_1 + R_3)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$

Columnspace basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix} \right\}$

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6) $E \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Columnspace of $EA = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x+2y \\ y \end{bmatrix}$

Not equal.

Columnspace of $A = x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x+y \\ -x-y \end{bmatrix}$

7) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -4 & 6 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 6 & -2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -8 & -2 & 1 \\ 0 & -2 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/4 & -\frac{1}{8} \\ 0 & -2 & -6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/4 & -\frac{1}{8} \\ 0 & 0 & -\frac{11}{2} & \frac{11}{4} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{8} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$

$(R_2 - 2R_1)$ $(R_4 - 4R_1)$ $\quad (R_2 \cdot -1/8) \quad (R_3 + 2R_2) \quad (R_3 \cdot -\frac{2}{11})$

independent columns

$\begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$

$(R_2 - R_3 \cdot \frac{1}{4})$
$(R_1 - 2R_2)$
$(R_1 - R_4)$

Rowspace basis of $A = \{ [1\ 2\ 4], [2\ -4\ 6], [1\ 0\ -2] \}$

8) $t = \frac{x}{2} = y = 2z \rightarrow x - 2y = 0 , x - 4z = 0 , y - 2z = 0$

$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$(R_3 - R_1) \quad (R_3 - 2R_2)$

* Any matrix with this reduced row echelon form satisfies the condition

* Rank is equal to the number of leading 1's, therefore;

Rank = 2

9) $\begin{bmatrix} 1 & 1 & -1 & b1 \\ 2 & -1 & 3 & b2 \\ -1 & 3 & 1 & b3 \\ 0 & 2 & -1 & b4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & b1 \\ 0 & -3 & 5 & b2+b1 \\ 0 & 4 & 0 & b3+b1 \\ 0 & 2 & -1 & b4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & b1 - \frac{b3+b1}{4} \\ 0 & 0 & 5 & b2+b1 - \frac{b3+b1}{4} \cdot 3 \\ 0 & 1 & 0 & (b3+b1)/4 \\ 0 & 0 & 1 & -b4 + \frac{b3+b1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & b1 - \frac{b3+b1}{4} - b4 + \frac{b3+b1}{2} \\ 0 & 0 & 0 & b2+b1 - \frac{b3+b1}{4} \cdot 3 + 5b4 - \frac{5(b3+b1)}{2} \\ 0 & 1 & 0 & (b3+b1)/4 \\ 0 & 0 & 1 & -b4 + \frac{b3+b1}{2} \end{bmatrix}$

* So, for the system to be consistent $-\frac{15}{4} b1 + b2 - \frac{7}{4} b3 + 5 b4 = 0$

* The existing unique solution is, $x1 = \frac{5b1}{4} + \frac{b3}{4} + b4 \quad x2 = \frac{b1}{4} + \frac{b3}{4} \quad x3 = \frac{b1}{2} + \frac{b3}{2} - b4$

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10) 3x6 matrix, Rank = 3 so free variable number is $6-3=3$, nullity = 3
The system has infinitely many solutions as the nullity is 3 and not zero. Dimension of the solution space shall be equal to the row number, therefore 3.

11) The column number (n) is equal to the sum of rank and nullity.

$$\underbrace{\text{rank}}_{\substack{\text{common dimension of row} \\ \text{and column spaces}}}$$

• The nullspace, rowspace and columnspace must be 1 dimensional to be a line. However the sum of rank and nullspace is not equal to the column number in this case ($1+1 \neq 3$). Thus such thing can not happen.

– 2x4 matrix → n (column number) = 4. This time the elements must be 2 dimensional to be a plane. Thus the equation holds $(2+2=4)$
So it is possible.

$$\underbrace{2}_{\text{rank}} + \underbrace{2}_{\text{nullspace}} = \underbrace{4}_{\text{column number}}$$

12)

$$\begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \rightarrow \underset{(R_1 \cdot \frac{1}{4})}{\begin{bmatrix} 1 & 1/2 \\ t & 1 \\ 3 & t \end{bmatrix}} \rightarrow \underset{\substack{(R_2 - tR_1) \\ (R_3 - 3R_1)}}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1-\frac{t}{2} \\ 0 & t-\frac{3}{2} \end{bmatrix}} \rightarrow \underset{(R_2 + \frac{R_3}{2})}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1/4 \\ 0 & t-\frac{3}{2} \end{bmatrix}} \rightarrow \underset{\substack{(R_1 - 2R_2) \\ (R_2 \cdot 4)}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & t-\frac{3}{2} \end{bmatrix}} \rightarrow \underset{(R_3 - (t-\frac{3}{2})R_2)}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

• The reduced row echelon form of the matrix does not contain t, therefore, row and column spaces are same for all values of t.

$$\begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \cdot x = b \rightarrow \begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 4x_1 + 2x_2 \\ tx_1 + x_2 \\ 3x_1 + tx_2 \end{bmatrix}}^{b}$$

$$\left[\begin{array}{cc|c} 4 & 2 & 4x_1+2x_2 \\ t & 1 & tx_1+x_2 \\ 3 & t & 3x_1+tx_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & x_1+x_2/2 \\ t & 1 & tx_1+x_2 \\ 3 & t & 3x_1+tx_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & x_1+x_2/2 \\ 0 & 1-t/2 & (1-t/2)x_2 \\ 0 & t-3/2 & (t-3/2)x_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & x_1+x_2/2 \\ 0 & 1/4 & x_2/4 \\ 0 & t-\frac{3}{2} & (t-3/2)x_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & x_1+x_2/2 \\ 0 & 1 & x_2 \\ 0 & t-\frac{3}{2} & (t-3/2)x_2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 0 \end{array}\right]$$

• Solution of the equation does not include t, hence the solution spaces are same for all values of t.

13) a) $Ax = \lambda x$, $(A - I\lambda)x = 0$, $|A - I\lambda| = 0$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 1 + \lambda^2 - 2\lambda - 9 = \underbrace{\lambda^2 - 2\lambda - 8 = 0}_{\text{c. equation}} \to (\lambda-4)(\lambda+2) = 0 \to 4, -2 \text{ are eigen values}$$

For $\lambda = 4$

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -3v_1 + 3v_2 = 0 \quad v_1 = k \quad v_2 = k \to \underbrace{V = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{eigen vector}}$$

- Rank of $(\lambda I - A) = 1$

basis of eigenspace $\{[1, 1]\}$

for $\lambda = -2$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow 3v_1 + 3v_2 = 0 \quad v_1 = k, v_2 = -k \to \underbrace{V = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{eigen vector}}$$

$\{[1, -1]\}$ basis of eigenspace

- Rank of $(\lambda I - A) = 1$

b)

$$\begin{bmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} \to \underset{\substack{(R_1 + \lambda R_3) \\ (R_2 + R_3)}}{\begin{bmatrix} 0 & -\lambda & 1-\lambda^2 \\ 0 & -\lambda & -1-\lambda \\ 1 & -1 & -\lambda \end{bmatrix}} \quad \det(A) = (-\lambda)\cdot(-1-\lambda) - (1-\lambda^2)\cdot(-\lambda) = \lambda^2 + 2\lambda - \lambda^3 \Rightarrow \underbrace{-\lambda^3 + \lambda^2 + 2\lambda = 0}_{\text{c. equation}}$$

$\lambda_1 = 0$
$\lambda_2 = -1$ → eigen values
$\lambda_3 = 2$

For $\lambda = 0$, $Ax = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \to \underset{(R_2 + R_3)}{\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]} \to \underset{(R_1 + R_2)}{\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]} \Rightarrow \begin{matrix} x_1 - x_2 = 0 \\ x_3 = 0 \end{matrix} \quad \begin{matrix} x_1 = k \quad x_2 = k \\ x_3 = 0 \end{matrix}$$

$$\underbrace{x = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{eigen vector}}$$, base of eigenspace $= \{[1, 1, 0]\}$

- Rank of $\overset{\to 0}{\lambda} I - A \to -A = 2$

$$-A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

for $\lambda = -1$, $A + I = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \to \underset{\substack{(R_2 + R_1) \\ (R_3 - R_1)}}{\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]} \to \underset{\substack{(R_2 \cdot 1/2) \\ (R_3 + R_2)}}{\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \quad \begin{matrix} x_2 = 0 \\ x_1 + x_3 = 0 \end{matrix} \quad \begin{matrix} x_1 = k \quad x_2 = 0 \\ x_3 = -k \end{matrix}$$

- Rank of $\overset{-1}{\lambda} I - A = -I - A \to 2$

$$x = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$ → eigen vector base of eigenspace $= \{[1, 0, -1]\}$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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for $\lambda = 2$, $\lambda - 2I = 0$

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ 0 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -3 \\ 1 & 0 & -1/2 \end{bmatrix}$$

$x_1 - \frac{x_3}{2} = 0$ $\quad$ $-x_2 - x_3 = 0$ $\quad$ $x_1 = \frac{k}{2}$ $\quad$ $x_2 = \frac{-3k}{2}$ $\quad$ $x_3 = k$

$\underbrace{x = \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix}}_{\text{eigen vector}}$, base of eigenspace $= \{[1/2, -3/2, 1]\}$ $\quad$ • Rank of $\lambda I - A \rightarrow 2I - A = 2$

$$\begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

c) $\begin{bmatrix} 1-\lambda & -1 & -2 \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{bmatrix} \rightarrow \begin{vmatrix} 0 & (2-\lambda)(1-\lambda)-1 & 3-5\lambda \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0 \rightarrow [(2-\lambda)(1-\lambda)-1].(3-\lambda) - (3-5\lambda) = 0$

$= -\lambda^3 + 6\lambda^2 - 5\lambda = 0$ $\quad$ $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 5$ $\rightarrow$ eigen values

for $\lambda = 0$, $Ax = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 2 & 5 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{(R_2+R_1)} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{(R_1+R_2),\ (R_3-R_2)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$ $\quad$ $x_2 + 3x_3 = 0$ $\quad$ $x_1 = -k$ $\quad$ $x_2 = -3k$ $\quad$ $x_3 = k$

$\underbrace{x = k\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}}_{\text{eigen vector}}$, base of eigenspace $= \{[-1, -3, 1]\}$ $\quad$ • Rank of $\lambda I - A \rightarrow -A = 2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

For $\lambda = 1$, $(A - I)x = 0$

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ -1 & 1 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{(R_1+R_3),\ (R_2+R_3)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$-x_1 + 3x_3 = 0$ $\quad$ $x_2 + 2x_3 = 0$ $\quad$ $x_1 = 3k$ $\quad$ $x_2 = -2k$ $\quad$ $x_3 = k$

eigen vector

$x = k\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, base of eigenspace $= \{[3, -2, 1]\}$ $\quad$ • Rank of $\lambda I - A \rightarrow I - A = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -3 \\ 0 & -1 & -1 \end{bmatrix}$$

For $\lambda = 5$, $(A - 5I)x = 0$

$$\left[\begin{array}{ccc|c} -4 & -1 & -2 & 0 \\ -1 & -3 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 11 & -22 & 0 \\ 1 & 3 & -5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$ $\quad$ $x_2 - 2x_3 = 0$ $\quad$ $x_1 = -k$ $\quad$ $x_2 = 2k$ $\quad$ $x_3 = k$

$\underbrace{x = k\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}_{\text{eigen vector}}$, base of eigenspace $= \{[-1, 2, 1]\}$ $\quad$ • Rank of $\lambda I - A \rightarrow 5I - A = 2$

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14) So, $(A^5 - I\lambda)x = 0$ Eigen values are $\lambda_1^5 = 0$, $\lambda_2^5 = -1$, $\lambda_3^5 = 32$

Eigen vectors are the same with Q13-b, $v_1 = k\begin{bmatrix}1\\1\\0\end{bmatrix}$ basis $= \{[1,1,0]\}$, dimension $=1$

$v_2 = k\begin{bmatrix}1\\0\\-1\end{bmatrix}$ basis $= \{[1,0,-1]\}$, dimension $=1$

$v_3 = k\begin{bmatrix}1/2\\-3/2\\1\end{bmatrix}$ basis $= \{[1/2,-3/2,1]\}$, dimension $=1$

15) A matrix has n linearly independent eigen vectors if it is diagonalizable. And an eigen value can account for many eigen vectors, however not necessarily n. Thus it can't always be diagonalized in the case of not having n amount of eigen vectors.

16) a) Since the polynomial is of 2nd degree, the matrix has a dimension of 2x2.

b) determinant of $(\lambda I - A)$ is equal to the determinant of characteristic polynomial.

$\lambda = 0$, $\det(-A) = \det(A).(-1)^2 = 0^2 - 9 = -9$

c) $A = \begin{bmatrix}k & l\\ m & n\end{bmatrix}$ determinant $= kn - lm$, $A^{-1} = \frac{1}{kn-lm}\begin{bmatrix}n & -l\\ -m & k\end{bmatrix}$ $|\lambda I - A| = 0$

$$\begin{vmatrix}\lambda-k & l\\ m & \lambda-n\end{vmatrix} = \lambda^2 - (k+n)\lambda + kn - lm = \lambda^2 - 9 \rightarrow k+n=0 \quad kn-lm=-9$$

$$A^{-1}\cdot A\cdot x = A^{-1}\cdot\lambda\cdot x \rightarrow Ix = A^{-1}\lambda x \rightarrow \frac{I}{\lambda}.x = A^{-1}x \rightarrow \left(\frac{I}{\lambda} - A^{-1}\right).x = 0$$

$$\left|\frac{I}{\lambda} - A^{-1}\right| = \begin{vmatrix}\frac{1}{\lambda} - \frac{n}{kn-lm} & \frac{l}{kn-lm}\\ \frac{m}{kn-lm} & \frac{1}{\lambda} - \frac{k}{kn-lm}\end{vmatrix} = \frac{1}{\lambda^2} - \frac{1}{\lambda}\left(\frac{k+n}{kn-lm}\right) + \frac{kn-lm}{(kn-lm)^2}$$

$$= \frac{1}{\lambda^2} - \frac{1}{9}$$

17) $|A - \lambda I| = 0$ $\begin{vmatrix}1-\lambda & 1 & 2\\ 0 & 3-\lambda & -4\\ 0 & 0 & -1-\lambda\end{vmatrix} = (1-\lambda).(3-\lambda)(-1-\lambda) = 0 \rightarrow (\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -1)$

* As the system has no 0 eigen value, it is diagonalizable.

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For $\lambda=1$; $\begin{bmatrix}0&1&2\\0&2&-4\\0&0&-2\end{bmatrix} \rightarrow \begin{bmatrix}0&1&2\\0&0&-8\\0&0&1\end{bmatrix} \rightarrow \begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}$ $x_1=0$ $x_2=k$ $x_3=0$ $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=k\begin{bmatrix}0\\1\\0\end{bmatrix}$

(R_2-2R_1) $(R_3\cdot -\frac{1}{2})$ (R_1-2R_3) (R_2+8R_3)

For $\lambda=3$; $\begin{bmatrix}-2&1&2\\0&0&-4\\0&0&-4\end{bmatrix} \rightarrow \begin{bmatrix}2&-1&2\\0&0&0\\0&0&1\end{bmatrix} \rightarrow \begin{bmatrix}2&-1&0\\0&0&0\\0&0&1\end{bmatrix}$ $2x_1-x_2=0$ $x_3=0$ $x_1=k$ $x_2=k$ $x_3=0$

(R_2-R_3) $(R_2\cdot -\frac{1}{4})$ (R_1-2R_3)

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=k\begin{bmatrix}1\\2\\0\end{bmatrix}$

For $\lambda=-1$; $\begin{bmatrix}2&1&2\\0&4&-4\\0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}2&1&2\\0&1&-1\\0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}2&0&3\\0&1&-1\\0&0&0\end{bmatrix}$ $2x_1+3x_3=0$ $x_2-x_3=0$

$(R_2\cdot\frac{1}{4})$ (R_1-R_2)

$x_1=-3/2k$ $x_2=k$ $x_3=k$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=k\begin{bmatrix}-3/2\\1\\1\end{bmatrix}$

$P=\begin{bmatrix}1&1&-3\\0&2&2\\0&0&2\end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc}1&1&-3&1&0&0\\0&2&2&0&1&0\\0&0&2&0&0&1\end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc}1&1&-3&1&0&0\\0&2&0&0&1&-1\\0&0&1&0&0&1/2\end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc}1&1&0&1&0&3/2\\0&1&0&0&1/2&-1/2\\0&0&1&0&0&1/2\end{array}\right] \rightarrow$

$(R_2-R_3,\ R_3\cdot\frac{1}{2})$ $(R_1+3R_3,\ R_2\cdot\frac{1}{2})$

$\left[\begin{array}{ccc|ccc}1&0&0&1&-1/2&2\\0&1&0&0&1/2&-1/2\\0&0&1&0&0&1/2\end{array}\right]$, $D=\begin{bmatrix}1&-1/2&2\\0&1/2&-1/2\\0&0&1/2\end{bmatrix}\cdot\begin{bmatrix}1&1&2\\0&3&-4\\0&0&-1\end{bmatrix}\cdot\begin{bmatrix}1&1&-3\\0&2&0\\0&0&2\end{bmatrix}=\begin{bmatrix}1&3&3\\0&6&-2\\0&0&-2\end{bmatrix}$

(R_1-R_2) P^{-1}

$D=P^{-1}.A.P=\begin{bmatrix}1&0&0\\0&3&0\\0&0&-1\end{bmatrix}$ $A=P.D.P^{-1}$ $A=P.D.P^{T}$

$A=\begin{bmatrix}1&1&-3\\0&2&2\\0&0&2\end{bmatrix}\cdot\left(\begin{bmatrix}2&0&0\\0&0&0\\0&0&0\end{bmatrix}\cdot\lambda_1+\begin{bmatrix}0&0&0\\1&2&0\\0&0&0\end{bmatrix}\cdot\lambda_2+\begin{bmatrix}0&0&0\\0&0&0\\3/2&1&1\end{bmatrix}\cdot\lambda_3\right)$

- Written as linear combination of rank 1 matrices made from the eigenvectors.

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18) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda).(3-\lambda).(2-\lambda) - (3-\lambda) + (3-\lambda) = 0$

$= (\lambda^2 - 5\lambda + 4).(3-\lambda) = 0 \rightarrow \lambda_1 = 3 \quad \lambda_2 = 4 \quad \lambda_3 = 1$

for $\lambda = 4$; $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \underset{(R1+R2)}{\begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}} \rightarrow \underset{(R1+R3)}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}} \rightarrow \underset{(R2+R3)}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}$ $\quad x_1 - x_3 = 0 \quad x_2 - x_3 = 0 \quad x_1 = k \quad x_2 = k \quad x_3 = k$

For $\lambda = 1$; $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \underset{\substack{(R1-R2)\\(R2-R3)}}{\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}} \rightarrow \underset{(R1-R2)}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}}$ $\quad x_1 - x_3 = 0 \quad x_2 + 2x_3 = 0 \quad x_1 = k \quad x_2 = -2k \quad x_3 = k$

For $\lambda = 3$; $\begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \underset{\substack{(R1-2R3)\\(R2-R3)}}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$ $\quad x_1 + x_3 = 0 \quad x_2 = 0 \quad x_1 = -k \quad x_2 = 0 \quad x_3 = k$

• So, the eigenspaces are $(1,1,1)$, $(1,-2,1)$, $(1,0,-1)$

And by applying Gram Schmidt;

$u_1 = v_1 = (1,-2,-1) \qquad e_1 = \frac{(1,-2,-1)}{\sqrt{6}} = \left(\frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}, \frac{-\sqrt{6}}{6}\right)$

$u_2 - \frac{\langle u_2, u_1 \rangle}{\|u_1\|^2} v_1 = v_2 \qquad (1,0,-1) - \frac{1}{3}.(1,-2,-1) = (1,0,-1) - \frac{(1,-2,-1)}{3} = \left(\frac{2}{3}, \frac{2}{3}, \frac{-2}{3}\right)$

$\dfrac{\frac{2}{3}, \frac{2}{3}, \frac{-2}{3}}{\frac{2}{\sqrt{3}}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = e_2$

$v_3 = (1,1,1) - \frac{\langle u_3, u_1 \rangle (1,2,-1)}{\|u_1^2\|} - \frac{\langle v_2, u_2 \rangle (1,0,-1)}{\|v_2^2\|} = (1,1,1) - \left(\frac{-1}{3}, \frac{2}{3}, \frac{1}{3}\right) - (0,0,0)$

$e_3 = \frac{(1,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \qquad P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$

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$A = UDU^T$ can be written when U is orthogonal and D is diagonal because A is symmetric matrix. $U^{-1} = U^T$, columns of U are orthonormal basis with eigenvectors of A.

$$A = (v_1, v_2, v_3) \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \cdot \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} \qquad A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T$$

- Thus, A can be written as linear combination of rank 1 matrices made from its eigenvectors

19) Row 2 − 2 Row 1 is a 0 row. Therefore determinant of A is 0 and A is non invertible. And determinant $= \lambda_1 \lambda_2 \dots \lambda_n$. Thus there is a zero eigen value

20)
$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$V_1 = U_1 = (1, -1, 0)$$

$$V_2 = U_2 - \frac{\langle U_1, V_2 \rangle U_1}{\|U_1\|^2} - \frac{\langle V_2, V_2 \rangle U_2}{\|U_2\|^2} = \left(\tfrac{1}{2}, \tfrac{1}{2}, 1\right)$$

$$V_3 = U_3 - \frac{\langle U_1, V_3 \rangle U_1}{\|U_1\|^2} - \frac{\langle V_2, V_3 \rangle U_2}{\|U_2\|^2} = (0, 0, 0)$$

$$V_4 = U_4 - \frac{\langle U_1, V_4 \rangle U_1}{\|U_1\|^2} - \frac{\langle U_2, V_4 \rangle U_2}{\|U_2\|^2} - \frac{\langle U_3, V_4 \rangle U_3}{\|U_3\|^2} = \left(\tfrac{-2}{3}, \tfrac{-2}{3}, \tfrac{2}{3}\right)$$

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad q_4 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\text{Basis} = \{ q_1^T, q_2^T, q_4^T \}$$

$$A^T = QR \qquad A = R^T Q^T$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 0 & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix} \quad R = \begin{bmatrix} \sqrt{2} & -3/\sqrt{2} & -2/\sqrt{2} & -1/\sqrt{2} \\ 0 & 3/\sqrt{6} & \sqrt{6} & 7/\sqrt{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ -3/\sqrt{2} & 3/\sqrt{6} & 0 & 0 \\ -2/\sqrt{2} & \sqrt{6} & 0 & 0 \\ 1/\sqrt{2} & 7/\sqrt{6} & 0 & 2/\sqrt{3} \end{bmatrix}}_{R^T} \cdot \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & 0 \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}}_{Q^T}$$

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21)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -3 \\ -6 & 15 & 11 \\ -3 & 11 & 14 \end{bmatrix}$$

$$A^T . b = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -6 & 3 & 2 \\ -6 & 15 & 11 & -4 \\ 3 & 11 & 14 & -2 \end{array}\right] \xrightarrow{\substack{(R_2+2R_1) \\ (R_3-R_1)}} \left[\begin{array}{ccc|c} 3 & -6 & 3 & 2 \\ 0 & 3 & 5 & 0 \\ 0 & 5 & 11 & 0 \end{array}\right] \xrightarrow{\substack{(R_1 \cdot \frac{1}{3}) \\ (R_2 \cdot \frac{1}{3}) \\ (R_3 - 5R_2)}} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2/3 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 8/3 & 0 \end{array}\right] \xrightarrow{\substack{(R_1+2R_2) \\ (R_2 - \frac{5}{8}R_3) \\ (R_3 \cdot \frac{3}{8}) \\ (R_1 - \frac{13}{3}R_4)}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = 2/3$

$x_2 = 0$

$x_3 = 0$