Quantum Information Processing 101

Lecture 4: Quantum State Preparation, Outer Product, and Density Operator

Who am I?

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- Office: Building 20, 3rd floor
- Research interests:
  - Classical and Quantum Information Theory
  - Communication Theory
  - Terrestrial and Non-Terrestrial Communications
  - Molecular Communications
  - Body Area Networks
- Courses:
  - Sistemi di Comunicazione (3rd year Bachelor degree)
  - Information Theory (Master degree)
  - Communication in Green Infrastructures (PoliMI Ambassador in Green Technologies and Smart Infrastructures)
  - Quantum Information Theory (with Prof. L. Barletta, Doctoral degree)

What Will We Learn?

- Quantum State Preparation
- Outer Product
- Density Operator

Introduction

So far, we have made the assumption of perfect knowledge of the state preparation.

In the first part of the lecture we relax this assumption and introduce the concept of density operator to characterize such an imprecise knowledge.

The density operator is used to characterize imprecise knowledge in the preparation of quantum states, but also to describe the evolution or the measurement of quantum states and to develop a theory that incorporates an imprecise knowledge of these states.

The concept of density operator allows us to fuse the probability theory and the quantum theory into a single formalism.

Quantum States Preparation

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State Preparation

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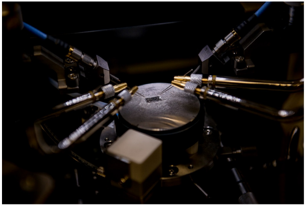
Desired
state $|\psi\rangle$

Quantum States Preparation

State Preparation

Desired state $|\psi\rangle$



Quantum lab

Quantum States Preparation

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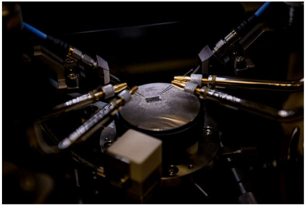

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Prepared State $|\psi'\rangle \neq |\psi\rangle$.

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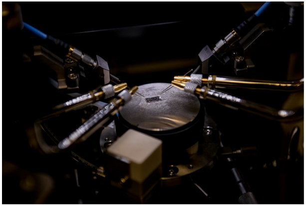

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Quantum States Preparation

State Preparation

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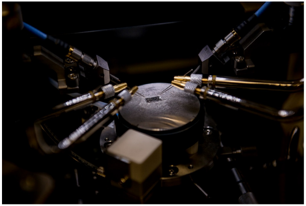

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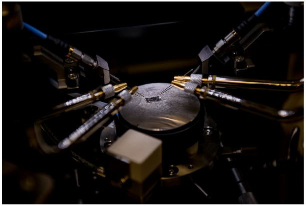

Quantum lab

Prepared State $|\psi'\rangle \neq |\psi\rangle$.

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$\vdots$

Quantum States Preparation

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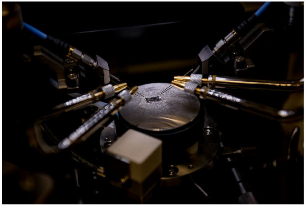

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How do we describe and deal with all these effects?

Quantum States Preparation

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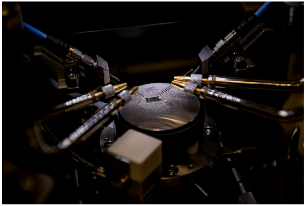

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How do we describe and deal with all these effects?

The description of the state is given by an ensemble $\mathcal{E}$ of quantum states

$$\mathcal{E} = \{p_X(x), |\psi_x\rangle\}_{x \in \mathcal{X}},$$

where X is an RV with distribution $p_X(x)$ whose realization x, belonging to an alphabet $\mathcal{X}$, merely acts as index to identify the quantum state $|\psi_x\rangle$.

Outer Product

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In previous lessons we have seen the inner product between two quantum states

$$\langle\psi|\phi\rangle = [\alpha^*\beta^*]\begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \alpha^*\gamma + \beta^*\delta, \quad \text{(complex scalar)}.$$

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By changing the order we get the outer product

$$|\phi\rangle \langle\psi| = \left[\begin{array}{c} \gamma \\ \delta \end{array} \right] [\alpha^* + \beta^*] = \left[\begin{array}{cc} \gamma\alpha^* & \gamma\beta^* \\ \delta\alpha^* & \delta\beta^* \end{array} \right], \quad \text{(complex matrix)}.$$

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Outer products are useful to:

1. describe measurements;
2. lead to new description of quantum states $\rightarrow$ Density operator.

Outer Product: Measurements

Given the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$.

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- Then consider the action of $|1\rangle\langle 1|$

$$\begin{aligned} |1\rangle\langle 1\,|\psi\rangle &= |1\rangle\langle 1|\,(\alpha\,|0\rangle + \beta\,|1\rangle) \\ &= \alpha\,|1\rangle\langle 1\,|0\rangle + \beta\,|1\rangle \\ &= \beta\,|1\rangle\,. \end{aligned}$$

Outer Product: Measurements

Given the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

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\end{aligned}
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$|0\rangle\langle 0|$ "projects" onto $|0\rangle \quad \Rightarrow \quad$ Measurement in Pauli Z basis with output $+1$.

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Outer Product: Measurements

What about measures in other bases?

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\begin{aligned}
|+\rangle \langle + |\psi\rangle &= \frac{\alpha + \beta}{\sqrt{2}} |+\rangle \qquad \text{outcome} + 1, \\
|-\rangle \langle - |\psi\rangle &= \frac{\alpha - \beta}{\sqrt{2}} |-\rangle \qquad \text{outcome} - 1.
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- Pauli Y basis

$$|i\rangle \langle i |\psi\rangle = \frac{\alpha - i\beta}{\sqrt{2}} |i\rangle \qquad \text{outcome} + 1,$$
$$|-i\rangle \langle -i |\psi\rangle = \frac{\alpha + i\beta}{\sqrt{2}} |-i\rangle \qquad \text{outcome} - 1.$$

Density Operator

Suppose we have the ability to perform a perfect, projective measurement of a system with ensemble description $\mathcal{E}$.

Let Π_j be the elements of this projective measurement so that $\sum_j \Pi_j = I$, and let J be the random variable corresponding to the measurement outcome j.

Given that state is equal to $|\psi_x\rangle$, then the Born rule states that the conditional probability $p_{J|X}(j|x)$ of obtaining measurement result j is

$$p_{J|X}(j|x) = \langle\psi_x|\,\Pi_j\,|\psi_x\rangle$$

and the post-measurement state is $\Pi_j\,|\psi_x\rangle \,/\, \sqrt{p_{J|X}(j|x)}$.

By the *law of total probability*, the unconditional probability $p_J(j)$ of obtaining measurement result j for the ensemble description $\mathcal{E}$ is

$$\begin{aligned} p_J(j) &= \sum_{x\in\mathcal{X}} p_{J|X}(j|x)\, p_X(x) \\ &= \sum_{x\in\mathcal{X}} \langle\psi_x|\,\Pi_j\,|\psi_x\rangle\, p_X(x)\,. \end{aligned}$$

Density Operator

Definition

(Trace). The trace $\mathrm{Tr}\{A\}$ of a square operator A acting on a Hilbert space $\mathcal{H}$ is defined as follows:

$$\mathrm{Tr}\{A\} = \sum_i \langle i| A |i\rangle ,$$

where $\{|i\rangle\}$ is some complete, orthonormal basis for $\mathcal{H}$.

The trace operator is *linear* and *independent* of which orthonormal basis we choose.
Exercise:

$$\begin{aligned}
\mathrm{Tr}\{A\} &= \sum_i \langle i| A |i\rangle = \sum_i \langle i| A \left(\sum_j |\phi_j\rangle \langle\phi_j| \right) |i\rangle \\
&= \sum_i \sum_j \langle i| A |\phi_j\rangle \langle\phi_j|i\rangle = \sum_i \sum_j \langle\phi_j|i\rangle \langle i| A |\phi_j\rangle \\
&= \sum_j \langle\phi_j| \left(\sum_i |i\rangle \langle i| \right) A |\phi_j\rangle = \sum_j \langle\phi_j| A |\phi_j\rangle ,
\end{aligned}$$

where $\{|\phi_j\rangle\}$ is some other orthonormal basis for $\mathcal{H}$ and we made use of the completeness relation: $I = \sum_j |\phi_j\rangle \langle\phi_j| = \sum_i |i\rangle \langle i|$.

Density Operator

We can then show the following useful property:

$$p_{J|X}(j|x) = \langle\psi_x|\,\Pi_j\,|\psi_x\rangle = \langle\psi_x|\left(\sum_i |i\rangle\langle i|\right)\Pi_j\,|\psi_x\rangle = \sum_i \langle\psi_x|i\rangle\langle i|\,\Pi_j\,|\psi_x\rangle$$
$$= \sum_i \langle i|\,\Pi_j\,|\psi_x\rangle\langle\psi_x|i\rangle = \mathrm{Tr}\left\{\Pi_j\,|\psi_x\rangle\langle\psi_x|\right\}.$$

It is possible to show that

$$p_J(j) = \sum_{x\in\mathcal{X}} \mathrm{Tr}\left\{\Pi_j\,|\psi_x\rangle\langle\psi_x|\right\} p_X(x) = \mathrm{Tr}\left\{\Pi_j \sum_{x\in\mathcal{X}} p_X(x)\,|\psi_x\rangle\langle\psi_x|\right\}.$$

The last equation can be rewritten as

$$p_J(j) = \mathrm{Tr}\left\{\Pi_j\rho\right\},$$

where

$$\rho = \sum_{x\in\mathcal{X}} p_X(x)\,|\psi_x\rangle\langle\psi_x|$$

is the *density operator* corresponding to the ensemble $\mathcal{E}$, which is the quantum generalization of a probability density operator.

Density Operator

We sometimes refer to the density operator as the *expected density operator* because there is a sense in which we are taking the expectation over all of the states in the ensemble in order to obtain the density operator.

The density operator can be equivalently written as

$$\rho = \mathbb{E}_X \left\{ |\psi_X\rangle \langle\psi_X| \right\}.$$

Density Operator as the State

- Every ensemble has a unique density operator, but the opposite does not necessarily hold: every density operator does not correspond to a unique ensemble and could correspond to many ensembles.
- The density operator can also be referred to as the state of a given quantum system because it is possible to use it to calculate probabilities for any measurement performed on that system.
- Any state $|\psi\rangle$ can be represented by means of a density operator $|\psi\rangle \langle\psi|$, and all calculations with this density operator give the same results as using the state $|\psi\rangle$.
- For the above reasons, we say that the state of a given quantum system is a density operator.

Outer Product: States

In previous lessons we saw that quantum states are described by kets

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

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The description of quantum states by outer products is as follows

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad |\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}.$$

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Spectral Decomposition

By the spectral theorem, it follows that every density operator ρ has a spectral decomposition in terms of eigenstates $\{|\phi_x\rangle\}_{x\in\{0,\dots,d-1\}}$ because every ρ is Hermitian

$$\rho = \sum_{x=0}^{d-1} \lambda_x |\phi_x\rangle\langle\phi_x|,$$

where the coefficients λ_x are the eigenvalues.

Density Operator

Suppose Alice tosses a fair coin and, based on the outcome, she prepares the state $|0\rangle$ or $|1\rangle$.

- What Alice prepares is not a pure quantum state $|\psi\rangle$
- Rather, she prepares an ensemble of pure quantum states

$$\mathcal{E} = \{p_X(x), |x\rangle\}_{x\in\{0,1\}}$$

- The outer products of the pure states $|0\rangle$ and $|1\rangle$ are useful to compactly describe this ensemble

$$\rho \equiv \sum_{x\in\{0,1\}} p_X(x)\, |x\rangle\langle x|\,,$$

  which is the previously introduced density operator.
- If $p_X(x) = 1$ for some x, then we say that the state ρ is pure; Otherwise, we say that the state ρ is mixed

In this example, Alice has prepared the following density operator

$$\frac{1}{2}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A),$$

which is a mixed state.