Quantum Information Processing 101

Lecture 5: Three Unit Quantum Protocols

What Will We Learn?

- Non-Local Unit Resources
- The Three Unit Quantum Protocols
  - Entanglement Distribution
  - Super-Dense Coding
  - Teleportation

Introduction

- We introduce unit quantum communication protocols that involve a single sender Alice and a single receiver Bob
- The protocols are ideal and noiseless (perfect classical and quantum communication, perfect entanglement)
- Alice and Bob may wish to perform one of several quantum information-processing tasks, such as the transmission of
  - classical information
  - quantum information
  - entanglement
- Noiseless entanglement is an important resource in quantum Shannon theory because it enables Alice and Bob to perform other protocols that are not possible with classical resources only.
- Elementary coding is a trivial method Alice can use to communicate classical information to Bob

Introduction

The three unit quantum protocols:

- Entanglement distribution: it is a simple and idealized protocol for generating entanglement
- Super-dense coding: it allows for transmitting more classical information that would be possible with a noiseless qubit channel alone
- Quantum teleportation: it can transmit quantum information without using a noiseless qubit channel (which is difficult to engineer in practice)

Non-Local Unit Resources

- The concept of non-local resource counting is introduced to quantify the communication cost of achieving a certain task
- Next we define what we mean by
  - noiseless qubit channel
  - noiseless classical bit channel
  - noiseless entanglement
- Each of the above resources is a non-local, unit resource.
- A resource is non-local if two spatially separated parties share it or if one party uses it to communicate to another
- A resource is said to be unit if it comes in some "gold standard" form, such as:
  - quantum bits (qubit)
  - classical bits (cbit)
  - entangled bits (ebit)

Noiseless Qubit Channel

Definition of Noiseless Qubit Channel $[q \to q]$

A noiseless qubit channel is any mechanism such that

$$|i\rangle_A \to |i\rangle_B \,, \qquad i \in \{0, 1\}$$

where $\{|0\rangle_A , |1\rangle_A\}$ and $\{|0\rangle_B , |1\rangle_B\}$ are some preferred orthonormal basis on Alice's and Bob's system, respectively.

Remarks:

- The bases $\{|0\rangle_A , |1\rangle_A\}$ and $\{|0\rangle_B , |1\rangle_B\}$ do not have to be the same, but it must be clear which basis each party is using
- The map is linear so that it preserves arbitrary superposition states (it preserves any qubit):

$$\alpha |0\rangle_A + \beta |1\rangle_A \to \alpha |0\rangle_B + \beta |1\rangle_B$$

Noiseless Qubit Channel

- It can be written as the following isometry:

$$\sum_{i=0}^{1} |i\rangle_B \langle i|_A = |0\rangle_B \langle 0|_A + |1\rangle_B \langle 1|_A$$

For example: $(|0\rangle_B \langle 0|_A + |1\rangle_B \langle 1|_A) |0\rangle_A = |0\rangle_B \langle 0|_A |0\rangle_A = |0\rangle_B$

- Any information-processing protocol that implements the above map simulates a noiseless qubit channel.
- We label the communication resource of a noiseless qubit channel as

$$[q \to q],$$

where the notation indicates one forward use of a noiseless qubit channel.

Noiseless Classical Bit Channel

Definition of Noiseless Classical Bit Channel $[c \to c]$

A noiseless classical bit channel is any mechanism such that

$$\begin{aligned} |i\rangle\langle i|_A &\to |i\rangle\langle i|_B\,, \qquad i \in \{0,1\}, \\ |i\rangle\langle j|_A &\to 0, \qquad i \neq j, \end{aligned}$$

where $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ are some preferred orthonormal basis on Alice's and Bob's system, respectively.

Remarks:

- Alice prepares either the classical state $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$, send it through the classical channel, and Bob performs a computational basis measurement to determine the message
- This resource is weaker than a noiseless qubit channel because it does not require Alice and Bob to maintain arbitrary superposition states, i.e., it merely transfers classical information

Noiseless Classical Bit Channel

- The channel maintains the diagonal elements of a density operator in the basis $\left\{|0\rangle_A, |1\rangle_A\right\}$, but it eliminates the off-diagonal elements.
- We can write it as the following linear map acting on a density operator ρ_A:

$$\rho_A \to \sum_{i=0}^{1} |i\rangle_B \langle i_A|\rho_A|i_A\rangle \langle i|_B .$$

- The communication resource of a noiseless classical bit channel is denoted as follows:

$$[c \to c] ,$$

  where the notation indicates one forward use of a noiseless classical bit channel.
- A noiseless qubit channel can simulate a noiseless classical bit channel, and we denote this fact with the resource inequality:

$$[q \to q] \geq [c \to c].$$

Noiseless Classical Bit Channel

Example: Suppose Alice tosses a fair coin and, based on the outcome, she prepares the state $|0\rangle$ or $|1\rangle$.

In other words, Alice has prepared the following density operator

$$\frac{1}{2}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A).$$

If she sends the above state over a noiseless classical channel, then Bob receives the following density operator:

$$\frac{1}{2}(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B),$$

i.e., he receives a classical fair coin toss.

Remark

However, the above classical bit channel map does not preserve off-diagonal elements of a density operator. Thus, it is impossible for a noiseless classical channel to simulate a noiseless qubit channel because it cannot maintain arbitrary superposition states. This can be seen in the example that follows.

Noiseless Classical Bit Channel

Example: Suppose instead Alice prepares a superposition state

$$|\psi\rangle = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}.$$

The density operator corresponding to this state is

$$|\psi\rangle\langle\psi| = \frac{|0\rangle\langle 0|_A + |0\rangle\langle 1|_A + |1\rangle\langle 0|_A + |1\rangle\langle 1|_A}{2}.$$

Suppose Alice then transmits this state through the above classical channel. The classical channel eliminates the elements $|0\rangle\langle 1|_A$ and $|1\rangle\langle 0|_A$ of the state, and Bob receives the state

$$\frac{|0\rangle\langle 0|_B + |1\rangle\langle 1|_B}{2},$$

which corresponds to a classical fair coin toss.

Shared Entanglement

Definition of Noiseless Ebit

An ebit is the following state

$$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$$

where Alice possesses the first qubit and Bob possesses the second.

- The ebit is the "gold standard" resource for pure bipartite (two-party) entanglement.
- Next we show how a noiseless qubit channel can generate a noiseless ebit through entanglement distribution.
- An ebit cannot simulate a noiseless qubit channel. Therefore, noiseless quantum communication is the strongest of all three resources, and entanglement and classical communication are in some sense "orthogonal" to one another because neither can simulate the other.

Entanglement Distribution

The entanglement distribution protocol is the most basic of the three unit protocols.

It exploits one use of a noiseless qubit channel to establish one shared noiseless ebit.

It consists of the following two steps:

1. Alice prepares a Bell state locally in her lab: starting from the state $|0\rangle_A |0\rangle_{A'}$, she performs a Hadamard gate on qubit A to get

$$\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \right) |0\rangle_{A'} ,$$

   and then performs a CNOT gate with control qubit A and target qubit A' to get the Bell state

$$\left| \Phi^+ \right\rangle_{AA'} = \frac{|00\rangle_{AA'} + |11\rangle_{AA'}}{\sqrt{2}}$$

2. Alice sends qubit A' to Bob with one use of a noiseless qubit channel. Alice and Bob share the ebit $\left| \Phi^+ \right\rangle_{AB}$

Entanglement Distribution

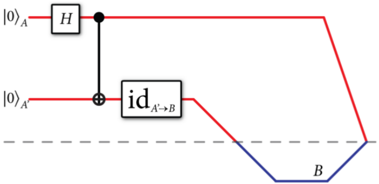


Figure: Pictorial description of the entanglement distribution protocol.

- The following resource inequality quantifies the non-local resources consumed or generated in the above protocol:

$$[q \to q] \geq [qq],$$

  where $[q \to q]$ denotes one forward use of a noiseless qubit channel and $[qq]$ denotes a shared, noiseless ebit.
- The meaning of the resource inequality is that there exists a protocol that consumes the resource on the left to generate the resource on the right[1].
- The resource count involves nonlocal resources only. We factor out all local operations (e.g. Hadamard gate, CNOT)

[1] The best analogy is to think of a resource inequality as a “chemical reaction”-like formula, where the protocol is like a chemical reaction that transforms one resource into another.

Entanglement and Quantum Communication

- Can entanglement enable two parties to communicate quantum information? It is natural to wonder if there is a protocol corresponding to the following resource inequality:
$$[qq] \geq [q \to q],$$
- Unfortunately, it is physically impossible to construct a protocol that implements the above resource inequality.
- The argument against such a protocol arises from the theory of relativity.
- Specifically, the theory of relativity prohibits information transfer or signaling at a speed greater than the speed of light.
- If a protocol were to exist that implements the above resource inequality, it would imply that two parties could communicate quantum information faster than the speed of light, because they would be exploiting the entanglement for the instantaneous transfer of quantum information.

Elementary Coding

It is also possible to send classical information using a noiseless qubit channel.

Elementary coding is a simple protocol for doing so that consists of the following steps:

1. Alice prepares either $|0\rangle$ or $|1\rangle$, depending on the classical bit she would like to send
2. Alice uses a noiseless qubit channel and Bob receives the qubit
3. Bob performs a measurement in the $|0\rangle$ / $|1\rangle$ basis

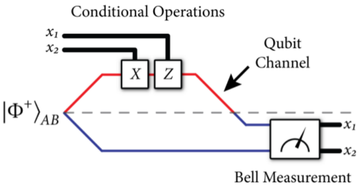
Remarks:

- The following resource inequality applies to elementary coding:

$$[q \to q] \geq [c \to c]$$

- Again, we are only counting non-local resources in the resource count — we do not count the state preparation at the beginning or the measurement at the end.
- It can be proved that this protocol is optimal, i.e., one cannot do better than to transmit one classical bit of information per use of a noiseless qubit channel!
- We cannot access the information in the continuous degrees of freedom using any measurement scheme

Quantum Super-Dense Coding



Super-dense coding has the striking property that noiseless entanglement can double the classical communication ability of a noiseless qubit channel.

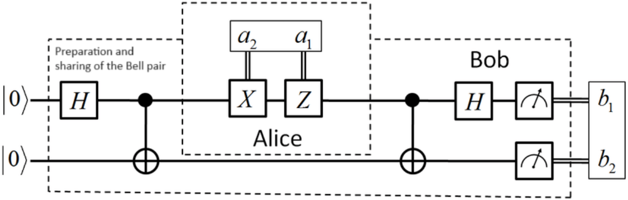
Protocol:

- Alice and Bob share an ebit $|\Phi^+\rangle_{AB}$. Alice applies one of four unitary operations $\{I, X, Z, XZ\}$ to her share of the ebit. The state becomes one of the following Bell states

$$|\Phi^+\rangle_{AB}, \quad |\Phi^-\rangle_{AB}, \quad |\Psi^+\rangle_{AB}, \quad |\Psi^-\rangle_{AB}$$

- Alice transmits her qubit to Bob with one use of a noiseless qubit channel
- Bob performs a Bell measurement in the basis $\{|\Phi^+\rangle_{AB}, |\Phi^-\rangle_{AB}, |\Psi^+\rangle_{AB}, |\Psi^-\rangle_{AB}\}$ to distinguish the four states perfectly.
- A number of $\log_2 4 = 2$ bit is transmitted.

Quantum Super-Dense Coding



Remarks:

- This is implemented by first applying CNOT with A as control qubit and B as target qubit, and then performing $H \otimes I$ unitary operation on the entangled qubit A.
- Super-dense coding performs the following resource inequality

$$[qq] + [q \to q] \geq 2[c \to c]$$

- Super-dense coding is more powerful than elementary coding, because we consume an ebit to help transmit two classical bits, instead of consuming the stronger resource of an extra noiseless qubit channel:

$$2[q \to q] \geq 2[c \to c]$$

Quantum Super-Dense Coding: Example

- Suppose Alice intends to transmits the two-bits 10 to Bob. She applies the quantum phase-flip gate Z to her qubit so that the resulting entangled state is

$$\left|\Phi^-\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0_A\rangle\,|0\rangle_B - |1\rangle_A\,|1\rangle_B\right).$$

- Alice transmits her entangled qubit to Bob using a quantum channel.
- At the receiver Bob performs a CNOT with A as control bit and B as target bit and this will give the state

$$\frac{1}{\sqrt{2}}\left(|0_A\rangle\,|0\rangle_B - |1\rangle_A\,|0\rangle_B\right).$$

The Hadamard gate is applied to A to obtain

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle_A + |1\rangle_A\right)\otimes|0\rangle_B - \frac{1}{\sqrt{2}}\left(|0\rangle_A - |1\rangle_A\right)\otimes|0\rangle_B\right) = \tag{1}$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|0\rangle_B\right) - \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B - |1\rangle_A|0\rangle_B\right)\right) = |1\rangle_A|0\rangle_B \tag{2}$$

The basis state is $|1\rangle_A|0\rangle_B$ and therefore Bob knows Alice intended to transmit the two bits 10.

Quantum Teleportation

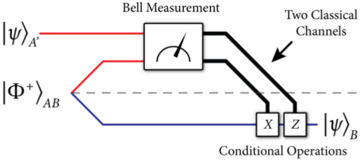
Preparation:

- Alice possesses a qubit
$$|\psi\rangle_{A'} = \alpha\,|0\rangle_{A'} + \beta\,|1\rangle_{A'}$$
and shares an ebit $\left|\Phi^+\right\rangle_{AB}$ with Bob
- The joint state of the systems A', A, and B can be rewritten as follows:

$$
\begin{aligned}
|\psi\rangle_{A'}\left|\Phi^+\right\rangle_{AB} &= \left(\alpha\,|0\rangle_{A'} + \beta\,|1\rangle_{A'}\right)\left(\frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle_{A'AB} + \beta\,|100\rangle_{A'AB} + \alpha\,|011\rangle_{A'AB} + \beta\,|111\rangle_{A'AB}\right) \\
&= \frac{1}{2}\Big(\alpha\left(\left|\Phi^+\right\rangle_{A'A} + \left|\Phi^-\right\rangle_{A'A}\right)|0\rangle_B + \beta\left(\left|\Psi^+\right\rangle_{A'A} - \left|\Psi^-\right\rangle_{A'A}\right)|0\rangle_B \\
&\qquad + \alpha\left(\left|\Psi^+\right\rangle_{A'A} + \left|\Psi^-\right\rangle_{A'A}\right)|1\rangle_B + \beta\left(\left|\Phi^+\right\rangle_{A'A} - \left|\Phi^-\right\rangle_{A'A}\right)|1\rangle_B\Big) \\
&= \frac{1}{2}\Big(\left|\Phi^+\right\rangle_{A'A}\left(\alpha\,|0\rangle_B + \beta\,|1\rangle_B\right) + \left|\Phi^-\right\rangle_{A'A}\left(\alpha\,|0\rangle_B - \beta\,|1\rangle_B\right) \\
&\qquad + \left|\Psi^+\right\rangle_{A'A}\left(\alpha\,|1\rangle_B + \beta\,|0\rangle_B\right) + \left|\Phi^-\right\rangle_{A'A}\left(\alpha\,|1\rangle_B - \beta\,|0\rangle_B\right)\Big) \\
&= \frac{\left|\Phi^+\right\rangle_{A'A}|\psi\rangle_B + \left|\Phi^-\right\rangle_{A'A}Z\,|\psi\rangle_B + \left|\Psi^+\right\rangle_{A'A}X\,|\psi\rangle_B + \left|\Psi^-\right\rangle_{A'A}XZ\,|\psi\rangle_B}{2}
\end{aligned}
$$

where we have used the Bell basis of the joint system $A'A$

Quantum Teleportation



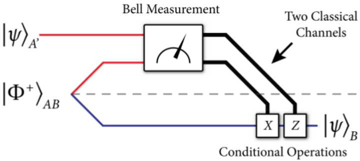
Protocol:

- Alice performs a Bell measurement on her system $A'A$. The state collapses to one of the four states

$$\begin{array}{cc} |\Phi^+\rangle_{A'A} |\psi\rangle_B & |\Phi^-\rangle_{A'A} Z |\psi\rangle_B \\ |\Psi^+\rangle_{A'A} X |\psi\rangle_B & |\Psi^-\rangle_{A'A} XZ |\psi\rangle_B \end{array}$$

  with uniform probability
- At this point, Alice knows the measurement and whether Bob's state is $|\psi\rangle_B$, $Z|\psi\rangle_B$, $X|\psi\rangle_B$ or $XZ|\psi\rangle_B$. Bob does not know anything about his state
- Alice transmits two classical bits to Bob that indicate which of the four measurement outcomes occurred
- Depending on the classical information received from Alice, Bob performs the restoration operation on its system B

Quantum Teleportation



Remarks:

- Teleportation destroys the quantum state $|\psi\rangle$ at Alice location and recreates it at Bob's, with the help of shared entanglement
- Teleportation is universal: it works independently of $|\psi\rangle$
- Teleportation does not violate the no-cloning theorem, because the state $|\psi\rangle$ is not copied, but it is destroyed due to the Bell measurement
- Teleportation implements the following resource inequality:

$$[qq] + 2[c \to c] \geq [q \to q]$$