# Quantum Information Processing

## Lecture 2: Measurements and Composite Quantum Systems

# Outline

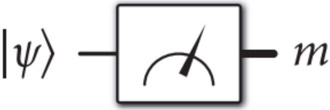
- Quantum Measurements
  - The Measurement Postulate (Born rule)
  - The Uncertainty Principle
- Composite Quantum Systems
  - Controlled Gates
  - The No-Cloning Theorem
  - Measurement of Composite Quantum Systems

Section 1

# Quantum Measurements

# Quantum Measurements

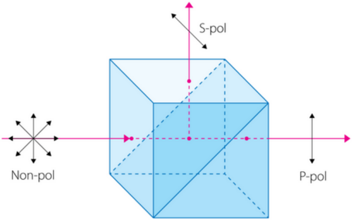
- The measurement is a type of quantum evolution that retrieves classical information from a quantum state $|\psi\rangle$.
- In nature we can only measure observables (physical variables like position, momentum,...)
- Observables are represented by Hermitian operators (which have real eigenvalues), and the measuring device outputs one of its eigenvalues (a real number $m$)



# Quantum Measurement

Physical Example of a Quantum Measurement: Measuring the state of polarization of a photon (e.g., vertical or horizontal)

Classical Interfacing: A polarizing beam-splitter changes the path of the photon depending on its polarization state and hits either a detector $D_1$ or a detector $D_2$. The measurement outcome is $i \in \{1, 2\}$ depending on which detector clicks.




Mathematical Model: Born rule

# The Measurement Postulate (Born rule)

## Measurement Postulate

Consider an observable with spectral decomposition

$$A = \lambda_1 |\lambda_1\rangle\langle\lambda_1| + \lambda_2 |\lambda_2\rangle\langle\lambda_2|$$

measured in a system with state $|\psi\rangle$, then the measurement is

$$m = \lambda_i \text{ with probability } |\langle\lambda_i|\psi\rangle|^2, \qquad i = \{1, 2\},$$

and the state collapses to $|\lambda_i\rangle$.

Example: In the computational basis, consider the observable

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

and the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$. The result of the measurement is

$$m = \begin{cases} 1 & \text{with probability } |\langle 0|\psi\rangle|^2 = |\alpha|^2 \\ -1 & \text{with probability } |\langle 1|\psi\rangle|^2 = |\beta|^2 \end{cases}$$

Remark: $\alpha$ and $\beta$ take the name of probability amplitudes.

# The Measurement Postulate (Exercise)

Exercise: Determine the measurement of the observable $X$ in the $+/-$ basis.
From Lecture 1, we know that

$$X = |+\rangle\langle+| - |-\rangle\langle-|$$

and that

$$\langle+|\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}, \qquad \langle-|\psi\rangle = \frac{\alpha-\beta}{\sqrt{2}}.$$

The measurement is

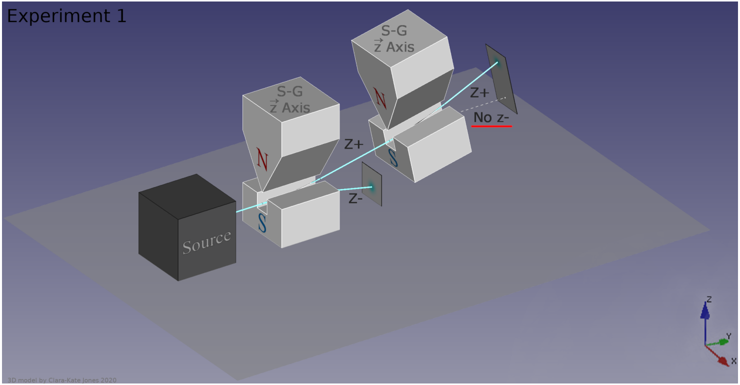
$$m = \begin{cases} 1 & \text{with probability } \frac{|\alpha+\beta|^2}{2} \\ -1 & \text{with probability } \frac{|\alpha-\beta|^2}{2} \end{cases}$$

Remarks:

- The superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ does not mean that the state was $|0\rangle$ with prob. $|\alpha|^2$ and $|1\rangle$ with prob. $|\beta|^2$ before the measurement. In fact, this probabilistic interpretation is classical.
- For example, suppose to measure state $|\psi\rangle$ with the $X$ operator. What is the measurement with the probabilistic interpretation and with the superposition interpretation?

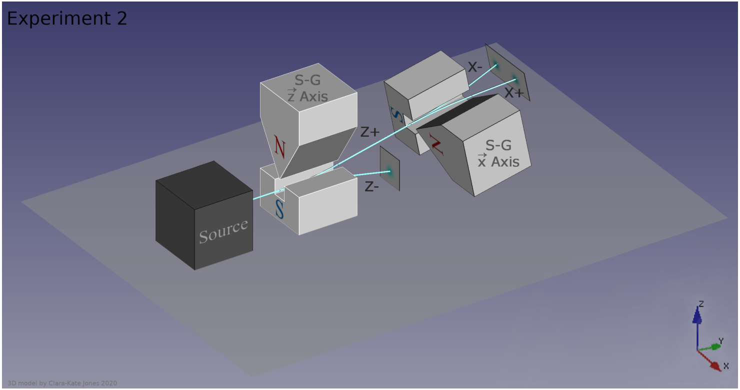
# The Measurement Postulate: Successive Measurements

Example: Stern-Gerlach experiment 1.



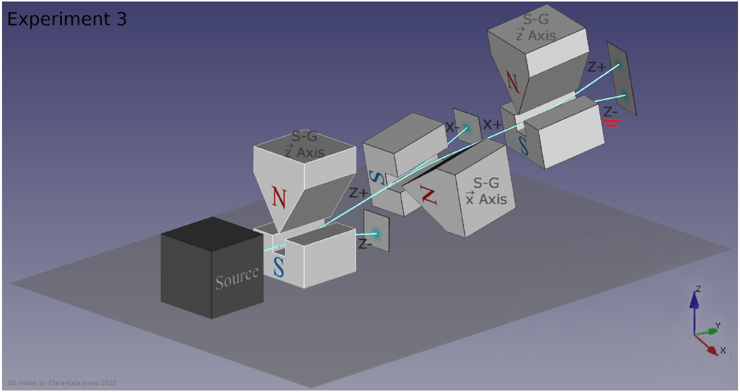
# The Measurement Postulate: Successive Measurements

Example: Stern-Gerlach experiment 2.



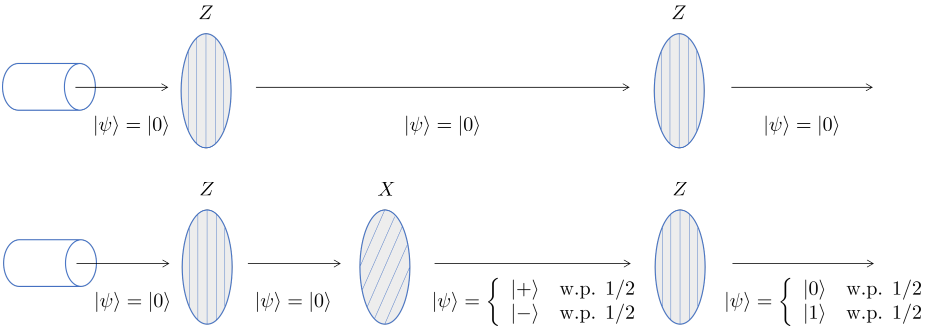
# The Measurement Postulate: Successive Measurements

Example: Stern-Gerlach experiment 3.



# The Measurement Postulate: Successive Measurements

Example: Stern-Gerlach experiment.



# The Uncertainty Principle

- Heisemberg's uncertainty principle is a fundamental feature of quantum theory
- There are some physical observables which are not compatible, for example position and velocity of a particle
- Measuring velocity of a particle will perturb the position. The final state of the particle will depend on the order of the measurements.

## Section 2

## Composite Quantum Systems

# Composite Quantum Systems

In quantum information processing the only interesting tasks involve many qubits at the same time. We need to introduce mathematical objects and notation for composite quantum systems.

A pair of classical bits $c_1 c_0$ can be represented with the following two-qubit state:

$$c_1 c_0 \mapsto |c_1\rangle\,|c_0\rangle \equiv |c_1 c_0\rangle\,.$$

By the superposition principle, a possible two-qubit state is:

$$|\xi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$$

where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.
Linear algebra is needed to describe all possible linear combinations between composite states.

# Tensor Product

The tensor product $A \otimes B$ of two operators $A$ and $B$ is

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}
\end{aligned}
$$

Example: Recalling that $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, if we use the following two-qubit basis

$$
|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
$$

then we can recognize that $|c_1 c_0\rangle = |c_1\rangle \otimes |c_0\rangle$, for $c_i \in \{0, 1\}$.

# Tensor Product

Example: The vector representation of the superposition state $|\xi\rangle$ is

$$|\xi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$$

$$= \alpha\,|0\rangle \otimes |0\rangle + \beta\,|0\rangle \otimes |1\rangle + \gamma\,|1\rangle \otimes |0\rangle + \delta\,|1\rangle \otimes |1\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}.$$

In the rest of the course we will use the following equivalent notations for two qubits that are local to one party:

$$|c_1 c_0\rangle \qquad |c_1\rangle\,|c_0\rangle \qquad |c_1\rangle \otimes |c_0\rangle\,.$$

If two or more parties, such as $A$ and $B$, are involved, then we can put subscripts on the qubits:

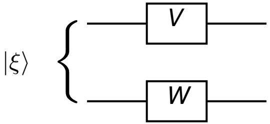
$$|c_1 c_0\rangle_{AB} \qquad |c_1\rangle_A\,|c_0\rangle_B \qquad |c_1\rangle_A \otimes |c_0\rangle_B\,.$$

# Evolution of Composite Quantum Systems

- In general, we can apply any unitary evolution $U$ on the composite quantum state $|\xi\rangle$.
- Suppose we want to apply transformation $V$ to the first qubit and transformation $W$ to the second qubit of $|\xi\rangle$, then

$$U = V \otimes W$$

Block diagram representation:




$$V_1 W_2 |\xi\rangle = (V \otimes W)|\xi\rangle$$

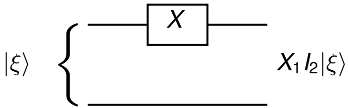
# Evolution of Composite Quantum Systems

Example: We want to apply a NOT gate on the first qubit. Then $U = X \otimes I \equiv X_1 I_2$, where the subscript indicates the qubit. The matrix representation of the operator $X_1 I_2$ is

$$X_1 I_2 = \begin{bmatrix} \langle 00|X_1 I_2|00\rangle & \langle 00|X_1 I_2|01\rangle & \langle 00|X_1 I_2|10\rangle & \langle 00|X_1 I_2|11\rangle \\ \langle 01|X_1 I_2|00\rangle & \langle 01|X_1 I_2|01\rangle & \langle 01|X_1 I_2|10\rangle & \langle 01|X_1 I_2|11\rangle \\ \langle 10|X_1 I_2|00\rangle & \langle 10|X_1 I_2|01\rangle & \langle 10|X_1 I_2|10\rangle & \langle 10|X_1 I_2|11\rangle \\ \langle 11|X_1 I_2|00\rangle & \langle 11|X_1 I_2|01\rangle & \langle 11|X_1 I_2|10\rangle & \langle 11|X_1 I_2|11\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 00|10\rangle & \langle 00|11\rangle & \langle 00|00\rangle & \langle 00|01\rangle \\ \langle 01|10\rangle & \langle 01|11\rangle & \langle 01|00\rangle & \langle 01|01\rangle \\ \langle 10|10\rangle & \langle 10|11\rangle & \langle 10|00\rangle & \langle 10|01\rangle \\ \langle 11|10\rangle & \langle 11|11\rangle & \langle 11|00\rangle & \langle 11|01\rangle \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = X \otimes I$$

Block diagram representation:



# Probability Amplitudes for Composite Systems

We can make alternative choices for the basis. Suppose we have states $|\phi_0\rangle, |\phi_1\rangle$ for the first qubit and states $|\psi_0\rangle, |\psi_1\rangle$ for the second qubit.

Consider the two-qubit states

$$|\phi_0\rangle \otimes |\psi_0\rangle \equiv |\phi_0\psi_0\rangle, \qquad |\phi_1\rangle \otimes |\psi_1\rangle \equiv |\phi_1\psi_1\rangle.$$

A probability amplitude can be calculated by

$$\langle\phi_1\psi_1|\phi_0\psi_0\rangle = \langle\phi_1|\phi_0\rangle\langle\psi_1|\psi_0\rangle.$$

Example: In the computational basis, we have

$$\langle ij|kl\rangle = \langle i|k\rangle\langle j|l\rangle \qquad \forall i, j, k, l \in \{0, 1\}.$$

# Controlled-NOT (CNOT) Gate

- The CNOT gate acts on two qubits. If the first qubit is $|0\rangle$, then it does nothing; Otherwise, it flips the second qubit.

$$|00\rangle \mapsto |00\rangle\,, \quad |01\rangle \mapsto |01\rangle\,, \quad |10\rangle \mapsto |11\rangle\,, \quad |11\rangle \mapsto |10\rangle\,.$$

- By linearity, we have

$$\alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle \xrightarrow{\text{CNOT}} \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|11\rangle + \delta\,|10\rangle\,.$$

# Controlled-NOT (CNOT) Gate

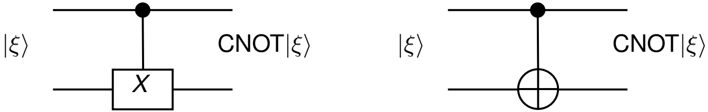
- The operator representation of the CNOT gate is

$$\mathrm{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

- The matrix representation of the CNOT gate (in computational basis) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Block diagram representation:



# The No-Cloning Theorem

## No-Cloning Theorem

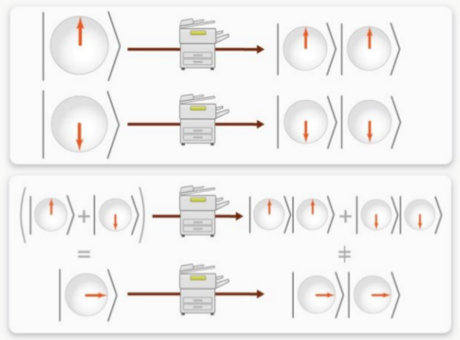
It is physically impossible to build a device that can copy any arbitrary quantum state that is input to it.

$$\nexists U : \; U |\psi\rangle |0\rangle = |\psi\rangle |\psi\rangle \qquad \text{for all } |\psi\rangle \in \mathbb{C}^2$$

Remarks:

- Copying classical information is possible and very easy
- It is possible to build an ad-hoc device to copy certain quantum states, but not a universal copier. For example, if we would like to copy only the states $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$, then a CNOT gate will do.

# The No-Cloning Theorem (Proof Sketch)



# Measurement of Composite Systems

Consider the superposition state $|\xi\rangle$ and the amplitudes in computational basis

$$\langle 00|\xi\rangle = \alpha, \quad \langle 01|\xi\rangle = \beta, \quad \langle 10|\xi\rangle = \gamma, \quad \langle 11|\xi\rangle = \delta.$$

Define the projection operators

$$\Pi_{00} = |00\rangle\langle 00|, \quad \Pi_{01} = |01\rangle\langle 01|, \quad \Pi_{10} = |10\rangle\langle 10|, \quad \Pi_{11} = |11\rangle\langle 11|.$$

The Born rule gives the probabilities for each result:

$$\langle\xi|\Pi_{00}|\xi\rangle = |\alpha|^2, \quad \langle\xi|\Pi_{01}|\xi\rangle = |\beta|^2, \quad \langle\xi|\Pi_{10}|\xi\rangle = |\gamma|^2, \quad \langle\xi|\Pi_{11}|\xi\rangle = |\delta|^2.$$

# Measurement of Composite Systems

Example: Suppose we want to measure the observable $Z_1 I_2$, i.e., to measure the $Z$ operator on the first qubit only.

The set of measurement operators is

$$\{\Pi_0 \otimes I,\ \Pi_1 \otimes I\} = \{|0\rangle\langle 0| \otimes I,\ |1\rangle\langle 1| \otimes I\}\,.$$

After the measurement, the state collapses to

$$\frac{(\Pi_0 \otimes I)\,|\xi\rangle}{\sqrt{\langle\xi|(\Pi_0 \otimes I)|\xi\rangle}} = \frac{\alpha\,|00\rangle + \beta\,|01\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}} \qquad \text{with probability } \langle\xi|(\Pi_0 \otimes I)|\xi\rangle = |\alpha|^2 + |\beta|^2,$$

and to

$$\frac{(\Pi_1 \otimes I)\,|\xi\rangle}{\sqrt{\langle\xi|(\Pi_1 \otimes I)|\xi\rangle}} = \frac{\gamma\,|10\rangle + \delta\,|11\rangle}{\sqrt{|\gamma|^2 + |\delta|^2}} \qquad \text{with probability } \langle\xi|(\Pi_1 \otimes I)|\xi\rangle = |\gamma|^2 + |\delta|^2.$$