Quantum Information Processing

Lecture 3: Entanglement

Outline

Section 1

Entanglement

Some History

- Composite quantum systems give rise to a uniquely quantum phenomenon: entanglement
- Entanglement was first observed by Schrödinger
- Schrödinger coined the German word "Verschränkung"[1] to describe the phenomenon, which means "little parts that, though far from one another, always keep the exact same distance from each other."
- Einstein described entanglement as "spukhafte Fernwirkung," which translates as "long-distance ghostly effect" or more commonly stated as "spooky action at a distance".

[1] It literally translates to interconnection.

Unentangled vs. Entangled States

Suppose two parties Alice and Bob share the state:

- (Unentangled State)

$$|0\rangle_A |0\rangle_B$$

  Note that it is clear how to determine the state of both Alice and Bob. Alice definitely is in the state $|0\rangle_A$ and Bob is in the state $|0\rangle_B$.

- (Entangled State) Consider

$$\left|\Phi^+\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right),$$

  where Alice has possession of the first qubit in the system and Bob has possession of the second qubit. The above state is a uniform superposition of states $|0\rangle_A |0\rangle_B$ and $|1\rangle_A |1\rangle_B$. Note that we cannot describe the state $\left|\Phi^+\right\rangle_{AB}$ as a product state of the form $|\phi\rangle_A |\psi\rangle_B$, for any states $|\phi\rangle_A$ and $|\psi\rangle_B$, i.e., we cannot determine the individual state of Alice or the individual state of Bob.

Definition of Entanglement

Pure-State Entanglement

A pure bipartite state $|\psi\rangle_{AB}$ is entangled if it cannot be written as a product state $|\phi\rangle_A |\psi\rangle_B$ for any choices of states $|\phi\rangle_A$ and $|\psi\rangle_B$.

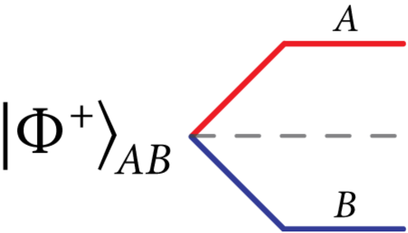
Exercise: Show that $|\Phi^+\rangle_{AB}$ defined in the previous slide can be written as follows in the $+/-$ basis

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|+\rangle_A |+\rangle_B + |-\rangle_A |-\rangle_B \right)$$

Hint: start by writing $|0\rangle$ and $|1\rangle$ in $+/-$ basis.

Graphical Depiction of Entanglement

We depict entanglement as follows:



The diagram indicates that

- A source location creates entanglement and distributes one subsystem to A and one to B.
- A and B are spatially separated.

Generating Shared Randomness

Consider the following protocol:

- Suppose that Alice and Bob share an ebit (entangled qubit)
- Suppose that they decided to measure their qubits in the computational basis
- Suppose Alice performs a measurement first
- Alice performs a measurement of the Z_A operator (i.e., she measures $Z_A \otimes I_B$)
- Just before observing the measurement outcome, Alice's knowledge can be described as the following ensemble of states:

$$|0\rangle_A |0\rangle_B \text{ with probability } \frac{1}{2}$$

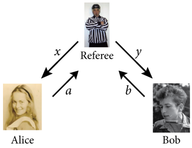
$$|1\rangle_A |1\rangle_B \text{ with probability } \frac{1}{2}$$

Remarks:

- Note that Bob's state is determined after Alice's measurement.
- If Alice measures $|0\rangle_A$, then Alice also knows that Bob's state is $|0\rangle_B$.
- Suppose Alice's measurement is $|0\rangle_A$, then when Bob measures, he obtains state $|0\rangle_B$. In addition, Bob knows that Alice's state is $|0\rangle_A$.
- Thus, this protocol is a method for them to generate one bit of shared randomness.
- An ebit can generate one classical bit of shared randomness. However, it is not possible to generate one ebit exclusively from shared randomness

CHSH Game

CHSH game (after Clauser, Horne, Shimony, and Holt) is one of the simplest examples that demonstrate the power of entanglement.



Rules of the game:

1. Referee selects two bits x and y uniformly at random and sends x to Alice and y to Bob
2. Alice and Bob are spatially separated and are not allowed to communicate (but they can agree on a strategy before the game starts)
3. Alice sends back bit a and Bob sends back bit b. (Note that a can only depend on x and b can only depend on y)
4. Winning Condition: Alice and Bob win if and only if

$$x \cdot y = a \oplus b$$

Goal: Find the winning probability of the CHSH game.

Best Classical Strategy

The Winning Strategy: Alice and Bob always return $a = 0$ and $b = 0$. The winning probability is $\frac{3}{4}$.

x	y	$x \wedge y$	$= a_x \oplus b_y$
0	0	0	$= a_0 \oplus b_0$
0	1	0	$= a_0 \oplus b_1$
1	0	0	$= a_1 \oplus b_0$
1	1	1	$= a_1 \oplus b_1$

Quantum Strategy

A Winning Strategy:

The winning probability for the strategy $p_{AB|XY}(a, b|x, y)$ is given by

$$\frac{1}{4}\sum_{a,b,x,y} 1_{\{x\cdot y=a\oplus b\}} p_{AB|XY}(a, b|x, y) =$$

$$= \frac{1}{4}\left(p_{AB|XY}(0, 0|0, 0) + p_{AB|XY}(1, 1|0, 0) + p_{AB|XY}(0, 0|0, 1) + p_{AB|XY}(1, 1|0, 1)\right.$$

$$\left. + p_{AB|XY}(0, 0|1, 0) + p_{AB|XY}(1, 1|1, 0) + p_{AB|XY}(0, 1|1, 1) + p_{AB|XY}(1, 0|1, 1)\right)$$

Description of a general quantum strategy:

- Assume that Alice and Bob share a joint state

$$\left|\Phi^+_{AB}\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$$

- Choose basis vectors

$$|\nu_0(\theta)\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$
$$|\nu_1(\theta)\rangle = \sin\theta\,|0\rangle - \cos\theta\,|1\rangle$$

- Alice uses θ_{A0} on receiving 0 from the referee and θ_{A1} on receiving 1;
- Bob uses θ_{B0} on receiving 0 from the referee and θ_{B1} on receiving 1;
- Alice and Bob measure the Bell state $\left|\Phi^+_{AB}\right\rangle$ according to the basis vectors $\{|\nu_i(\theta)\rangle\}_{i\in\{0,1\}}$ and send back to the referee the outcome of the measurement

Analysis of the Quantum Strategy

If the referee sends $x = 0, y = 0$, Alice and Bob win if they answer $(a, b) = (0, 0)$ or $(a, b) = (1, 1)$. Now

$$
\begin{aligned}
& p_{AB|XY}(0,0|0,0) + p_{AB|XY}(1,1|0,0) \\
&= \left|(\langle \nu_{0,A}(\theta_{A0})| \otimes \langle \nu_{0,B}(\theta_{B0})|) \left|\Phi^+_{AB}\right\rangle\right|^2 + \left|(\langle \nu_{1,A}(\theta_{A0})| \otimes \langle \nu_{1,B}(\theta_{B0})|) \left|\Phi^+_{AB}\right\rangle\right|^2 \\
&= \left|(\cos\theta_{A0} \langle 0| + \sin\theta_{A0} \langle 1|) \otimes (\cos\theta_{B0} \langle 0| + \sin\theta_{B0} \langle 1|) \left|\Phi^+_{AB}\right\rangle\right|^2 \\
&\quad + \left|(\sin\theta_{A0} \langle 0| - \cos\theta_{A0} \langle 1|) \otimes (\sin\theta_{B0} \langle 0| - \cos\theta_{B0} \langle 1|) \left|\Phi^+_{AB}\right\rangle\right|^2 \\
&= \frac{1}{2}\left|\cos\theta_{A0}\cos\theta_{B0} + \sin\theta_{A0}\sin\theta_{B0}\right|^2 + \frac{1}{2}\left|\sin\theta_{A0}\sin\theta_{B0} + \cos\theta_{A0}\cos\theta_{B0}\right|^2 \\
&= \cos^2(\theta_{A0} - \theta_{B0})
\end{aligned}
$$

Using similar steps we can compute

$$
\begin{aligned}
p_{AB|XY}(0,0|0,1) + p_{AB|XY}(1,1|0,1) &= \cos^2(\theta_{A0} - \theta_{B1}) \\
p_{AB|XY}(0,0|1,0) + p_{AB|XY}(1,1|1,0) &= \cos^2(\theta_{A1} - \theta_{B0})
\end{aligned}
$$

and

$$
p_{AB|XY}(0,1|1,1) + p_{AB|XY}(1,0|1,1) = \sin^2(\theta_{A1} - \theta_{B1})
$$

Analysis of the Quantum Strategy (Con't)

Combining the probabilities, the probability of winning is given by

$$P(\text{win}) = \frac{\cos^2(\theta_{A0} - \theta_{B0}) + \cos^2(\theta_{A0} - \theta_{B1}) + \cos^2(\theta_{A1} - \theta_{B0}) + \sin^2(\theta_{A1} - \theta_{B1})}{4}$$

Now if we choose $\theta_{A0} = 0, \theta_{A1} = \frac{\pi}{4}, \theta_{B0} = \frac{\pi}{8}, \theta_{B1} = -\frac{\pi}{8}$, then

$$P(\text{win}) = \cos^2\left(\frac{\pi}{8}\right) \approx 0.85355$$

Note that this is larger than 0.75 which is attained by the best classical strategy.

The Bell States

The most important entangled states for a two-qubit system are the Bell states defined as

$$\left|\Phi^+\right\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$$
$$\left|\Phi^-\right\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B)$$
$$\left|\Psi^+\right\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)$$
$$\left|\Psi^-\right\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$$

A few facts:

- the Bell states form an orthonormal basis
- We can also use the following labeling:

$$\left|\Phi^{zx}\right\rangle = Z_A^z X_A^x \left|\Phi^+\right\rangle_{AB}$$

where the two binary numbers zx indicate whether Alice applies I_A, X_A, Z_A or $Z_A X_A$ (e.g., $\left|\Phi^{00}\right\rangle = \left|\Phi^+\right\rangle_{AB}$).

Section 2

Extension to Qudit States

Extension to Qudit States

Qudits

A qudit state $|\psi\rangle$ is an arbitrary superposition of some set of orthonormal basis states $\{|j\rangle\}_{j\in\{0,1,\ldots,d-1\}}$ for a d-dimensional quantum system

$$|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle .$$

The amplitudes $\alpha_j \in \mathbb{C}$ obey the normalization condition $\sum_{j=0}^{d-1} |\alpha_j|^2 = 1$.

Measuring Qudits

The measurement of qudits is similar to the measurement of qubits.
Suppose that we have

- a state $|\psi\rangle$
- a Hermitian operator A with the diagonalization

$$A = \sum_{j=1}^{d} f(j)\Pi_j,$$

where $\Pi_j \Pi_k = \Pi_j \delta_{jk}$ and $\sum_j \Pi_j = I$.

Then, a measurement of the operator A returns j with the following probability:

$$p(j) = \langle\psi| \Pi_j |\psi\rangle ,$$

and the resulting state is

$$\frac{\Pi_j |\psi\rangle}{\sqrt{p(j)}}$$

Composite Systems of Qudits

We can define a system of multiple qudits by employing the tensor product. A general two-qudit state on two subsystems A and B has the following form:

$$|\xi\rangle_{AB} = \sum_{j,k=0}^{d-1} \alpha_{j,k} |j\rangle_A |k\rangle_B$$

For example: The maximally entangled qudit state is

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B$$