Mathematical Logic

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Contents

1 Proving Tautology using Identical Transformations

Using identical transformations prove that the following formula is a tautology:

$$P \to (Q \to ((P \lor Q) \to (P \land Q)))$$

1.1 Understanding Tautology

A formula is a tautology if it evaluates to true for all possible truth value assignments. We'll use logical equivalences to transform the formula.

1.2 Applying Transformations

We begin with the implication equivalence: $A \to B \equiv \neg A \lor B$
Starting with the formula:

$$P \to (Q \to ((P \lor Q) \to (P \land Q)))$$

Apply implication elimination to the outermost implication:

$$\equiv \neg P \lor (Q \to ((P \lor Q) \to (P \land Q)))$$

Apply implication elimination to the next implication:

$$\equiv \neg P \lor (\neg Q \lor ((P \lor Q) \to (P \land Q)))$$

Apply implication elimination to the innermost implication:

$$\equiv \neg P \lor (\neg Q \lor (\neg (P \lor Q) \lor (P \land Q)))$$

By De Morgan's law: $\neg (P \lor Q) \equiv \neg P \land \neg Q$

$$\equiv \neg P \lor (\neg Q \lor ((\neg P \land \neg Q) \lor (P \land Q)))$$

Applying associativity of disjunction:

$$\equiv \neg P \vee \neg Q \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

By absorption law: $A \vee (A \wedge B) \equiv A$, we observe that $\neg P \vee (\neg P \wedge \neg Q) \equiv \neg P$:

$$\equiv \neg P \vee \neg Q \vee (P \wedge Q)$$

Now we can factor this expression. Notice that:

$$\neg P \vee \neg Q \vee (P \wedge Q) \equiv (\neg P \vee \neg Q) \vee (P \wedge Q)$$

By De Morgan's law in reverse: $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

$$\equiv \neg(P \wedge Q) \vee (P \wedge Q)$$

This is the law of excluded middle: $\neg A \vee A \equiv \top$ (always true)

$$\equiv \top$$

Therefore, the formula is a **tautology**.

$$\boxed{P \to (Q \to ((P \vee Q) \to (P \wedge Q))) \equiv \top}$$

2 Finding the ANF (Zhegalkin Polynomial)

Find the ANF (Zhegalkin polynomial) of the following Boolean function and determine whether it can be expressed as a linear Zhegalkin polynomial:

$$f(x,y,z) = xyz \vee xy'z \vee x'yz' \vee x'y'z'$$

2.1 Constructing the Truth Table

First, we construct the truth table for the given function:

x	y	z	$f(x,y,z)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

2.2 Computing the ANF using Pascal's Triangle Method

To find the ANF, we use the XOR-based method. We create an auxiliary table where each row is computed by XORing adjacent values:

Row	000	001	010	011	100	101	110	111
0	1	0	1	0	0	1	0	1
1	1	1	1	0	1	1	1	
2	0	0	1	1	0	0		
3	0	1	0	1				
4	1	1	1					
5	0	0						
6	0							

2.3 Reading the ANF Coefficients

The ANF coefficients correspond to the first column of each row. Reading from top to bottom, we get the coefficients for terms: $1, x, y, xy, z, xz, yz, xyz$

From our table: coefficients are $1, 1, 0, 0, 1, 0, 0, 0$

Therefore, the ANF is:

$$\boxed{f(x, y, z) = 1 \oplus x \oplus z}$$

2.4 Determining Linearity

A Zhegalkin polynomial is linear if it contains only terms of degree 0 (constant) and degree 1 (single variables), with no products of variables.

Our ANF contains only the constant term 1 and the linear terms x and z. Therefore:

$$\boxed{\text{The function IS a linear Zhegalkin polynomial}}$$

3 Constructing Truth Table of Boolean Function

Construct the truth table of the following boolean function:

$$f(x,y,z) = x' \to (x \leftrightarrow (y \oplus (xz)))$$

3.1 Understanding the Operators

- $\oplus$ is XOR (exclusive or): $A \oplus B$ is true when exactly one of A or B is true
- $\leftrightarrow$ is biconditional (XNOR): $A \leftrightarrow B$ is true when A and B have the same truth value
- $\to$ is implication: $A \to B$ is false only when A is true and B is false

3.2 Building the Truth Table

We evaluate the function step by step for each combination:

x	y	z	x'	xz	$y \oplus (xz)$	$x \leftrightarrow (y \oplus (xz))$	$f(x,y,z)$
0	0	0	1	0	0	1	1
0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	0	0	0	1
1	0	1	0	1	1	1	1
1	1	0	0	0	1	1	1
1	1	1	0	1	0	0	1

3.3 Analysis

The function evaluates to 0 when $x = 0$ and $y = 1$ (rows 3 and 4), and evaluates to 1 otherwise. Therefore:

$$\boxed{f(x,y,z) \not\equiv 1 \text{ (not a tautology)}}$$

The function can be expressed in DNF (Disjunctive Normal Form) by taking the minterms where $f = 1$:

$$f(x,y,z) = \bar{x}\bar{y}\bar{z} \vee \bar{x}\bar{y}z \vee x\bar{y}\bar{z} \vee x\bar{y}z \vee xy\bar{z} \vee xyz$$

4 Using K-maps to Simplify Boolean Expression

Using K-maps simplify the following boolean expression:

$f(w,x,y,z) = \bar{w}xy\bar{z} + w\bar{x}yz + wx\bar{y}\bar{z} + \bar{w}xy\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}x\bar{y}z + w\bar{x}\bar{y}\bar{z} + \bar{w}xyz + w\bar{x}yz + wx\bar{y}z + \bar{w}xy\bar{z} + \bar{w}x\bar{y}\bar{z}$

4.1 Identifying the Minterms

First, we identify which minterms are present (removing duplicates):

- $\bar{w}xy\bar{z}$ → 0110 → minterm 6
- $w\bar{x}yz$ → 1011 → minterm 11
- $wx\bar{y}\bar{z}$ → 1100 → minterm 12
- $\bar{w}\bar{x}y\bar{z}$ → 0010 → minterm 2
- $\bar{w}x\bar{y}z$ → 0101 → minterm 5
- $w\bar{x}\bar{y}\bar{z}$ → 1000 → minterm 8
- $\bar{w}xyz$ → 0111 → minterm 7
- $wx\bar{y}z$ → 1101 → minterm 13
- $\bar{w}x\bar{y}\bar{z}$ → 0100 → minterm 4

Minterms: 2, 4, 5, 6, 7, 8, 11, 12, 13

4.2 Constructing the K-map

We fill in a 4-variable Karnaugh map:

$wx \backslash yz$	00	01	11	10
00	0	0	0	1
01	1	1	1	1
11	1	1	0	0
10	1	0	1	0

Where rows represent wx: 00, 01, 11, 10 and columns represent yz: 00, 01, 11, 10.

4.3 Grouping the Ones

We identify the prime implicants by grouping adjacent 1s in the K-map. The goal is to form the largest possible groups (powers of 2) to minimize the final expression.

Group 1 (size 4): Covers minterms 4, 5, 6, 7

- Row 01, all columns
- Variables that remain constant: $w = 0, x = 1$
- Prime implicant: $\bar{w}x$

Group 2 (size 2): Covers minterms 12, 13

- Row 11, columns 00 and 01
- Variables that remain constant: $w = 1, x = 1, y = 0$
- Prime implicant: $wx\bar{y}$

Group 3 (size 2): Covers minterms 2, 6

- Column 10, rows 00 and 01
- Variables that remain constant: $w = 0, y = 1, z = 0$
- Prime implicant: $\bar{w}y\bar{z}$

Group 4 (size 2): Covers minterms 8, 12

- Column 00, rows 10 and 11
- Variables that remain constant: $w = 1, y = 0, z = 0$
- Prime implicant: $w\bar{y}\bar{z}$

Group 5 (size 1): Covers minterm 11

- This minterm cannot be grouped with others
- Prime implicant: $w\bar{x}yz$

4.4 Simplified Expression

Combining all prime implicants that are necessary to cover all minterms:

$$\boxed{f(w,x,y,z) = \bar{w}x + wx\bar{y} + \bar{w}y\bar{z} + w\bar{y}\bar{z} + w\bar{x}yz}$$

Note: $\bar{w}y\bar{z}$ is essential for minterm 2, $w\bar{y}\bar{z}$ is essential for minterm 8, and $w\bar{x}yz$ is essential for minterm 11.