

ՀՀ ԿԳՄՍ ՆԱԽԱՐԱՐՈՒԹՅՈՒՆ  
ՀԱՅԱՍՏԱՆԻ ԵՎՐՈՊԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ  
ՏԵՂԵԿԱՏՎԱԿԱՆ ՀԱՄԱԿԱՐԳԵՐ

## ԱՆՀԱՏԱԿԱՆ ԱՇԽԱՏԱՆՔ

ԱՌԱՐԿԱ՝ ՄԱԹԵՄԱԹԻԿԱԿԱՆ ՏՐԱՄԱԲԱՆՈՒԹՅՈՒՆ  
ԿՈՒՐՍ՝ ԵՐԿՐՈՐԴ  
ՈՒՍԱՆՈՂ՝ ՏԻԳՐԱՆՅԱՆ ՍԵՐՈԲ  
ԴԱՍԱԽՈՍ՝ Ն. ՀԱԿՈԲՅԱՆ

ԵՐԵՎԱՆ 2025

# Mathematical Logic

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December 18, 2025

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# 1 Proving Tautology using Identical Transformations

Using identical transformations prove that the following formula is a tautology:

$$P \rightarrow (Q \rightarrow ((P \vee Q) \rightarrow (P \wedge Q)))$$

## 1.1 Understanding Tautology

A formula is a tautology if it evaluates to true for all possible truth value assignments. We'll use logical equivalences to transform the formula.

## 1.2 Applying Transformations

We begin with the implication equivalence:  $A \rightarrow B \equiv \neg A \vee B$

Starting with the formula:

$$P \rightarrow (Q \rightarrow ((P \vee Q) \rightarrow (P \wedge Q)))$$

Apply implication elimination to the outermost implication:

$$\equiv \neg P \vee (Q \rightarrow ((P \vee Q) \rightarrow (P \wedge Q)))$$

Apply implication elimination to the next implication:

$$\equiv \neg P \vee (\neg Q \vee ((P \vee Q) \rightarrow (P \wedge Q)))$$

Apply implication elimination to the innermost implication:

$$\equiv \neg P \vee (\neg Q \vee (\neg(P \vee Q) \vee (P \wedge Q)))$$

By De Morgan's law:  $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

$$\equiv \neg P \vee (\neg Q \vee ((\neg P \wedge \neg Q) \vee (P \wedge Q)))$$

Applying associativity of disjunction:

$$\equiv \neg P \vee \neg Q \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

By absorption law:  $A \vee (A \wedge B) \equiv A$ , we observe that  $\neg P \vee (\neg P \wedge \neg Q) \equiv \neg P$ :

$$\equiv \neg P \vee \neg Q \vee (P \wedge Q)$$

Now we can factor this expression. Notice that:

$$\neg P \vee \neg Q \vee (P \wedge Q) \equiv (\neg P \vee \neg Q) \vee (P \wedge Q)$$

By De Morgan's law in reverse:  $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

$$\equiv \neg(P \wedge Q) \vee (P \wedge Q)$$

This is the law of excluded middle:  $\neg A \vee A \equiv \top$  (always true)

$$\equiv \top$$

Therefore, the formula is a **tautology**.

$$\boxed{P \rightarrow (Q \rightarrow ((P \vee Q) \rightarrow (P \wedge Q))) \equiv \top}$$

## 2 Finding the ANF (Zhegalkin Polynomial)

Find the ANF (Zhegalkin polynomial) of the following Boolean function and determine whether it can be expressed as a linear Zhegalkin polynomial:

$$f(x, y, z) = xyz \vee xy'z \vee x'yz' \vee x'y'z'$$

### 2.1 Constructing the Truth Table

First, we construct the truth table for the given function:

$x$	$y$	$z$	$f(x, y, z)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

### 2.2 Computing the ANF using Pascal's Triangle Method

To find the ANF, we use the XOR-based method. We create an auxiliary table where each row is computed by XORing adjacent values:

Row	000	001	010	011	100	101	110	111
0	1	0	1	0	0	1	0	1
1	1	1	1	0	1	1	1	
2	0	0	1	1	0	0		
3	0	1	0	1				
4	1	1	1					
5	0	0						
6	0							

### 2.3 Reading the ANF Coefficients

The ANF coefficients correspond to the first column of each row. Reading from top to bottom, we get the coefficients for terms:  $1, x, y, xy, z, xz, yz, xyz$

From our table: coefficients are  $1, 1, 0, 0, 1, 0, 0, 0$

Therefore, the ANF is:

$$f(x, y, z) = 1 \oplus x \oplus z$$

### 2.4 Determining Linearity

A Zhegalkin polynomial is linear if it contains only terms of degree 0 (constant) and degree 1 (single variables), with no products of variables.

Our ANF contains only the constant term 1 and the linear terms  $x$  and  $z$ .

Therefore:

The function IS a linear Zhegalkin polynomial

### 3 Constructing Truth Table of Boolean Function

Construct the truth table of the following boolean function:

$$f(x, y, z) = x' \rightarrow (x \leftrightarrow (y \oplus (xz)))$$

#### 3.1 Understanding the Operators

- $\oplus$  is XOR (exclusive or):  $A \oplus B$  is true when exactly one of  $A$  or  $B$  is true
- $\leftrightarrow$  is biconditional (XNOR):  $A \leftrightarrow B$  is true when  $A$  and  $B$  have the same truth value
- $\rightarrow$  is implication:  $A \rightarrow B$  is false only when  $A$  is true and  $B$  is false

#### 3.2 Building the Truth Table

We evaluate the function step by step for each combination:

$x$	$y$	$z$	$x'$	$xz$	$y \oplus (xz)$	$x \leftrightarrow (y \oplus (xz))$	$f(x, y, z)$
0	0	0	1	0	0	1	1
0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	0	0	0	1
1	0	1	0	1	1	1	1
1	1	0	0	0	1	1	1
1	1	1	0	1	0	0	1

#### 3.3 Analysis

The function evaluates to 0 when  $x = 0$  and  $y = 1$  (rows 3 and 4), and evaluates to 1 otherwise. Therefore:

$$f(x, y, z) \neq 1 \text{ (not a tautology)}$$

The function can be expressed in DNF (Disjunctive Normal Form) by taking the minterms where  $f = 1$ :

$$f(x, y, z) = \bar{x}\bar{y}\bar{z} \vee \bar{x}\bar{y}z \vee x\bar{y}\bar{z} \vee x\bar{y}z \vee xy\bar{z} \vee xyz$$

## 4 Using K-maps to Simplify Boolean Expression

Using K-maps simplify the following boolean expression:

$$f(w, x, y, z) = \bar{w}xy\bar{z} + w\bar{x}yz + wx\bar{y}\bar{z} + \bar{w}xy\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}x\bar{y}z + w\bar{x}\bar{y}\bar{z} + \bar{w}xyz + w\bar{x}yz + wx\bar{y}z + \bar{w}xy\bar{z} + \bar{w}x\bar{y}\bar{z}$$

### 4.1 Identifying the Minterms

First, we identify which minterms are present (removing duplicates):

- $\bar{w}xy\bar{z} \rightarrow 0110 \rightarrow \text{minterm } 6$
- $w\bar{x}yz \rightarrow 1011 \rightarrow \text{minterm } 11$
- $wx\bar{y}\bar{z} \rightarrow 1100 \rightarrow \text{minterm } 12$
- $\bar{w}\bar{x}y\bar{z} \rightarrow 0010 \rightarrow \text{minterm } 2$
- $\bar{w}x\bar{y}z \rightarrow 0101 \rightarrow \text{minterm } 5$
- $w\bar{x}\bar{y}\bar{z} \rightarrow 1000 \rightarrow \text{minterm } 8$
- $\bar{w}xyz \rightarrow 0111 \rightarrow \text{minterm } 7$
- $wx\bar{y}z \rightarrow 1101 \rightarrow \text{minterm } 13$
- $\bar{w}x\bar{y}\bar{z} \rightarrow 0100 \rightarrow \text{minterm } 4$

Minterms: 2, 4, 5, 6, 7, 8, 11, 12, 13

### 4.2 Constructing the K-map

We fill in a 4-variable Karnaugh map:

$wx \backslash yz$	00	01	11	10
00	0	0	0	1
01	1	1	1	1
11	1	1	0	0
10	1	0	1	0

Where rows represent  $wx$ : 00, 01, 11, 10 and columns represent  $yz$ : 00, 01, 11, 10.



### 4.3 Grouping the Ones

We identify the prime implicants by grouping adjacent 1s in the K-map. The goal is to form the largest possible groups (powers of 2) to minimize the final expression.

**Group 1** (size 4): Covers minterms 4, 5, 6, 7

- Row 01, all columns
- Variables that remain constant:  $w = 0, x = 1$
- Prime implicant:  $\bar{w}x$

**Group 2** (size 2): Covers minterms 12, 13

- Row 11, columns 00 and 01
- Variables that remain constant:  $w = 1, x = 1, y = 0$
- Prime implicant:  $wx\bar{y}$

**Group 3** (size 2): Covers minterms 2, 6

- Column 10, rows 00 and 01
- Variables that remain constant:  $w = 0, y = 1, z = 0$
- Prime implicant:  $\bar{w}y\bar{z}$

**Group 4** (size 2): Covers minterms 8, 12

- Column 00, rows 10 and 11
- Variables that remain constant:  $w = 1, y = 0, z = 0$
- Prime implicant:  $w\bar{y}\bar{z}$

**Group 5** (size 1): Covers minterm 11

- This minterm cannot be grouped with others
- Prime implicant:  $w\bar{x}yz$

### 4.4 Simplified Expression

Combining all prime implicants that are necessary to cover all minterms:

$$f(w, x, y, z) = \bar{w}x + wx\bar{y} + \bar{w}y\bar{z} + w\bar{y}\bar{z} + w\bar{x}yz$$

Note:  $\bar{w}y\bar{z}$  is essential for minterm 2,  $w\bar{y}\bar{z}$  is essential for minterm 8, and  $w\bar{x}yz$  is essential for minterm 11.

## ՈՒՍՈՒՄՆԱՄԵԹՈՂԱԿԱՆ ԳՐԱԿԱՆՈՒԹՅՈՒՆ

1. Mendelson Elliott, Introduction to Mathematical Logic, Van Nostrand company inc., 1964
2. Игошин В. И., Математическая логика и теория алгоритмов, М., "Академия", 2008
3. Игошин В. И., Задачи и упражнения по математической логике и теории алгоритмов, М., 2007
4. Տոնոյան Ռ. Ն., Դիսկրետ մաթեմատիկայի դասընթաց: Երևան, ԵՊՀ, 1999, 130 էջ:
5. Rosen, Kenneth H, Discrete mathematics and its applications-8th edition, 2021, 1118 pages.