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# A proposed real options method for assessing investments

Abdollah Arasteh · Alireza Aliahmadi

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**Abstract** Real options analysis is being increasingly used for assessing investments under uncertainty; however, traditional real options methods have some characteristics that restrict their use, such as modeling the value of the underlying asset using geometric Brownian motion and assuming a fixed cost in exercising the options. In this paper, another real options method is expounded that mitigates some of the difficulties posed by traditional methods. Another important aspect that we analyzed in this paper is considering the fuzzy aspects of real options theory. In this section, we are trying to use fuzzy logic concepts integrated with system dynamics to assessing real options in investment projects and we examine dynamic versions of fuzzy logic systems. System dynamics (SD) is an effective method for studying dynamic conditions and changes in complex systems. In this paper, a new dynamic model of real-world systems is designed based on the concepts of system dynamic and fuzzy logic approach. The method is explained with an example from aviation. The analysis offers obvious proof that the integrated fuzzy–SD model could help investors to decide how they should choose an investment program, that managers can use the same results to restructure the program to improve the financial feasibility of the project, and that both investors and managers can define minimum needs to ensure program success.

**Keywords** Investment analysis · Real options (RO) · Automobile industry · System dynamics · Monte Carlo simulation · Fuzzy logic

## 1 Introduction

Real options (RO) analysis has been used for many years to assess investments under uncertainty. The first reference to real options was made in a paper on corporate borrowing written in 1977 by Stewart Myers [1]. Since then, RO has been used in several areas, such as the oil industry and the sales of other marketable good [2–4], business strategy [5–7], transportation regulations [8], real estate [9, 10], the airline industry [11, 12] and functional uses of space in buildings [13].

Traditional real option methods rely on financial option theory. Although RO has been increasingly recognized as an investment assessment tool, traditional methods have defects that restrict their usefulness. Occasionally, when the real option is applied to projects associated with tradable properties such as copper or oil, for example, the leaning is to use the cost of the property as the value of completion and to model it as a stochastic process, such as a geometric Brownian motion [2, 7]. This paper presents a different real options method that addresses some of the problems encountered using traditional methods. First, we have a brief introduction about generalized real options approach and its applications in automobile industry. Then, we have a section about application of system dynamics approach and combining this method with fuzzy logic and discussing about the large potential of this combination in solving complex real-world problems. Finally, we introduce our intuitive method, that is fuzzy system dynamics approach for valuing real options and use this method in a case study about an automobile manufacturer.

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## 2 Generalized real options method

### 2.1 Preliminaries: extending the real options formula

Let  $f_w(w)$  and  $f_s(s)$  be the probability distributions of the certainty similar of the value of completion and the completion cost, respectively, at maturity time  $T$ . In addition, suppose that these distributions can be of any form (see Fig. 1).

With a random completion cost,  $s$ , the value of the option is now contingent upon  $s$ . Therefore,

$$e = E[O(s)] = \int_{s=-\infty}^{\infty} O(s) f_s(s) ds = e^{-m_f T} \left( \int_{s=0}^{\infty} f_s(s) \int_{w=s}^{\infty} w f_w(w) dw ds - \int_{s=0}^{\infty} s f_s(s) \int_{w=s}^{\infty} f_w(w) dw ds \right) \quad (2)$$

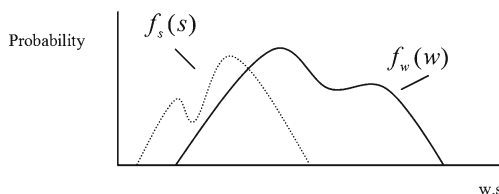
Similarly, the variance of the option value can be explained by applying the explanation of variance for continuous random variables to Eq. 1 as follows:

$$\begin{aligned} V^2(O(s)) &= E[O(s)^2] - (E[O(s)])^2 \\ &= e^{-2m_f T} \left[ \int_{s=-\infty}^{\infty} \left[ \left( \int_{w=s}^{\infty} w f_w(w) dw \right)^2 - 2s \int_{w=s}^{\infty} w f_w(w) dw \int_{w=s}^{\infty} f_w(w) dw + s^2 \left( \int_{w=s}^{\infty} f_w(w) dw \right)^2 \right] f_s(s) ds \right. \\ &\quad - \left. \left( \int_{s=0}^{\infty} f_s(s) \int_{w=s}^{\infty} w f_w(w) dw ds \right)^2 + 2 \left( \int_{s=0}^{\infty} f_s(s) \int_{w=s}^{\infty} w f_w(w) dw ds \right) \left( \int_{s=0}^{\infty} s f_s(s) \int_{w=s}^{\infty} f_w(w) dw ds \right) \right. \\ &\quad - \left. \left( \int_{s=0}^{\infty} s f_s(s) \int_{w=s}^{\infty} f_w(w) dw ds \right)^2 \right] \quad (3) \end{aligned}$$

A significant assumption made in writing Eqs. 2 and 3 is the value of completion and the completion cost is independent. This may be a logical assumption for real projects where, for instance, the relationship between outcomes and costs does not exist or is very weak.

### 2.2 Application of the generalized real options method to a new automobile development program

Transportation is a recurrent industry distinguished by periods of high growth followed by periods of deep decreases in traffic [12, 14]. Advance planning to adjust to this rapidly changing traffic



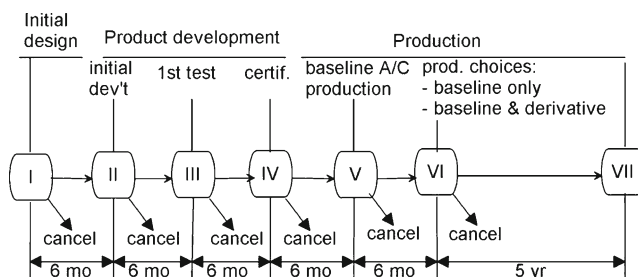
**Fig. 1** The certainty similar of the worth of completion,  $f_w(w)$ , and of the completion cost,  $f_s(s)$ , can be described with general probability distributions

we have the following formula for the value of an option:

$$O(s) = e^{-m_f T} \left( \int_{w=s}^{\infty} w f_w(w) dw - s \int_{w=s}^{\infty} f_w(w) dw \right) \quad (1)$$

The expected value of the option,  $e$ , can be resolved by applying the meaning of the expected value for continuous random variables to Eq. 1 as follows:

demand is a significant problem for all stakeholders. A typical new automobile development program comprises several stages in order. A very simplified drawing of a typical automobile program based on the data provided by a large automobile manufacturer is presented in Fig. 2. It is necessary to explain that this illustration and other related graphs and data are provided from the author's research about automobile industry and manufacturers in Iran. The case selected for this study is the Iran Khodro automobile manufacturing company. Iran Khodro Company, also known as IKCO, is the leading Iranian vehicle manufacturer, with headquarters in Tehran. IKCO is a public



**Fig. 2** The main steps involved in developing new automobile

joint-stock company with the objective of creation and management of factories to manufacture various types of vehicles and parts as well as selling and exporting them. IKCO produces vehicles under 13 brand names. The company has become the largest vehicle manufacturer in the Middle East, Central Asia, and North Africa. In Iran, it is the largest vehicle manufacturing company, having an average share of 65 % of domestic vehicle production. IKCO has 12 production sites around the globe, 6 sites within the borders of Iran and 6 others in IKCO's main export markets [15]. For our study, due to the ease of access, we select the Babol site in the Mazandaran province and by sharing questionnaires between employees and doing interviews with managers, the data were collected.

A project framework, such as the one showed in Fig. 2, includes several real options. For instance, the project manager has the option of continuing or abandoning the process at the end of each stage. Each stage provides the development team with an opportunity to spend a fairly small incremental amount of resources to collect more information about the product before continuing to follow a more great investment. The real options method developed here tells managers whether the project should be performed and, in this case, how much should be spent at each stage.

### 2.3 Simple example: the value of the real option to set up an imitative automobile

To explore this topic, consider a group of investors at step V in Fig. 2 and suppose that two options are available about production in step VI: the investors can continue to produce only the baseline automobile or they can choose to produce the baseline automobile and set up an imitative automobile. In either case, the investors have the right to cancel the project at step VI if the conditions are not favorable. For the special example presented here, the probability distribution of the value of completion can be estimated by combining system dynamics and Monte Carlo simulation, where the system

dynamics model is run many times with different values for the outer variables as showed by the probability distributions chose for each of them (see Fig. 3).

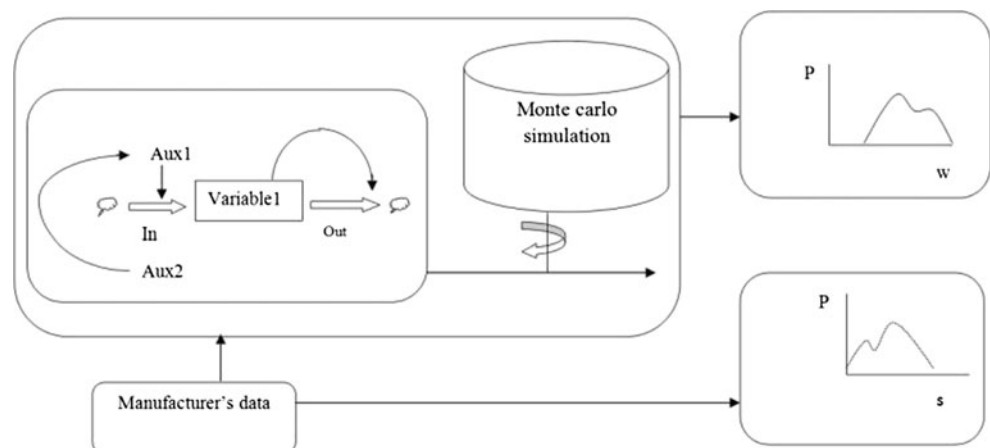
A system dynamics model was developed with input from the automobile manufacturer to choose the value of completion for the real option to set up an imitative automobile (see Fig. 4).

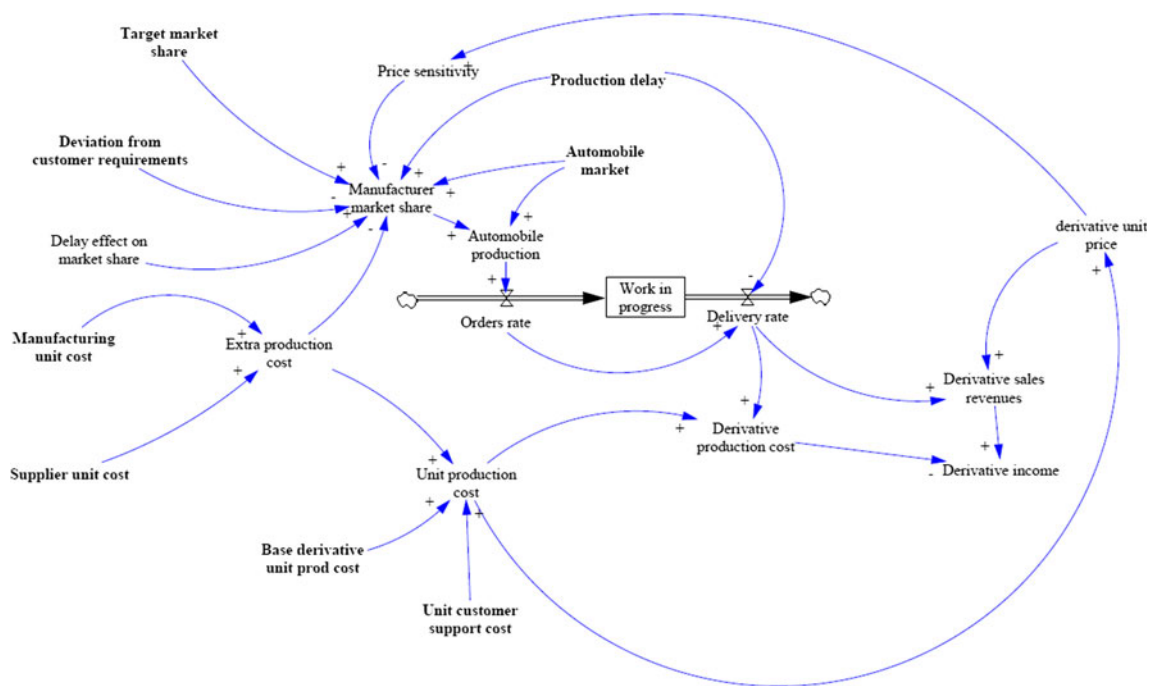
In a Monte Carlo simulation, the system dynamics model is run frequently using different values for the outer variables, which showed from probability distributions designated for each of the variables. The outer variables chosen for this study and their related probability distributions are showed in Table 1.

The probability distributions provided by the automobile manufacturer and shown in Table 1 are not very flat. This finding may be a sign of the scant historical data available about automobile programs at this manufacturer, which is expected because automobile manufacturers typically need several years to begin new or imitative automobile programs. Based on the data provided by the automobile manufacturer, it is assumed that the completion cost of the real option to develop the imitative automobile is 20 % of the imitative development cost. This spending covers building the production facilities for the imitative automobile. The value of the imitative development cost is given by a probability distribution provided by the manufacturer (see Table 2). As with the value of completion, the certainty similar of this quantity is estimated assuming a risk-adjusted discount rate of 18 % and a risk-free discount rate of 5 %.

The probability distributions for the value of completion and the completion cost for the real option to set up the imitative automobile estimated using the system dynamics model and the Monte Carlo simulation are shown in Fig. 5. The graph shows the certainty similar of the value of completion and the completion cost at the maturity of the real option (step VI in Fig. 2).

**Fig. 3** Schematic of the process to get the probability distributions of the value of completion,  $W_{set up imitative}$ , and of the completion cost,  $S_{set up imitative}$ , for the real option to set up an imitative automobile





**Fig. 4** A system dynamics model used to determine the value of completion of the real option to produce an imitative automobile

A series of financial performance parameters can be estimated using these data. These quantities are shown in Table 3.

### 3 Using fuzzy system dynamics as a new approach in real options theory

One of the approaches that have been discussed in recent years as a new approach in the field of real options theory is the use of fuzzy numbers in this analysis. Since using real options valuation mainly raised at uncertain conditions and fuzzy numbers is raised from uncertainty and ambiguity of phenomenon, so it looks like there is a same origin for them. So, in this section of the paper, we consider the compilation the fuzzy logic viewpoint and real options theory and collect them in a dynamic set (or fuzzy dynamic system).

One of the concepts that can be precisely analyzing the time-varying behavior of market is SD. But for using fuzzy concepts in the domain, we should find out whether any SD can be combined with fuzzy logic. If SD has at least one of the following conditions, it can be modeled by fuzzy logic.

- Some levels or rates or other variables are fuzzy
- Time parameter is important
- Some relationships between variables can be replaced with conditional statements that include fuzzy variables. These conditional statements are usually at if-then form;
- Degree of uncertainty of the variables can be expressed as fuzzy probability, when available information may be inaccurate or incomplete.
- Some functions may be fuzzy.

**Table 1** The variables chosen for the Monte Carlo simulation to estimate the probability distribution of the value of completion of the imitative automobile and their related probability distributions

Variable	Probability distribution						
	Unit	Value	<i>P</i> (value)	Value	<i>P</i> (value)	Value	<i>P</i> (value)
Automobile market	Automobile/year	10,000	0.6	14,000	0.2	18,000	0.2
Divergence from customer needs	%	5	0.6	4	0.2	1	0.2
Production postponement	Year	1	0.6	0.75	0.2	0.5	0.2
Market share goal	%	30	0.5	40	0.3	50	0.2
Unit customer support cost	% of unit cost	5	0.5	4	0.4	3	0.1
Unit manufacturing cost	MU million/year	6	0.5	3	0.3	0	0.2
Unit supplier cost	MU million/year	6	0.5	3	0.3	0	0.2

**Table 2** The probability distribution of imitative development cost based on the data provided by the automobile manufacturer

Variable	Probability distribution						
	Unit	Value	<i>P</i> (value)	Value	<i>P</i> (value)	Value	<i>P</i> (value)
Imitative development cost	MU million	1500	0.6	1200	0.2	1800	0.2

### 3.1 Why use fuzzy numbers in simulations?

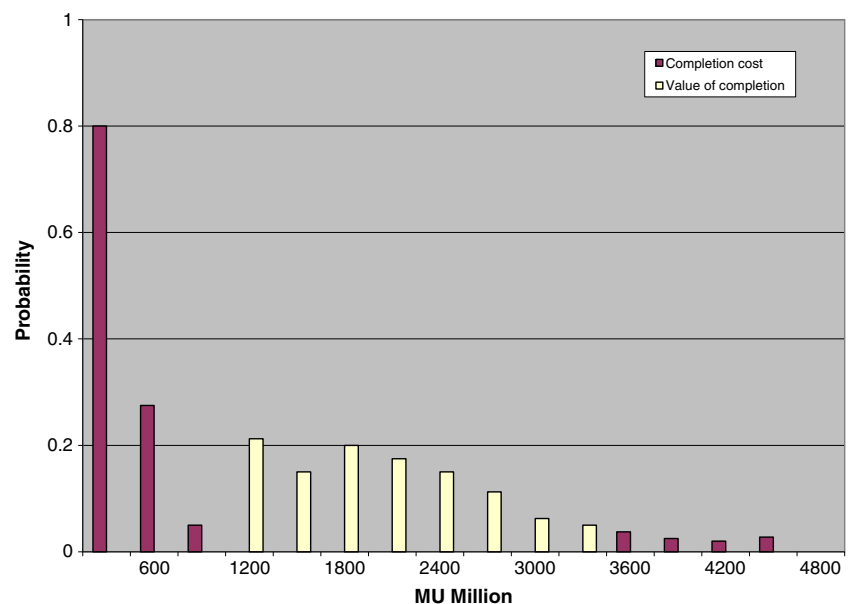
In real-world applications, we need qualitative approach and fuzzy concepts because we are often unable to designate system structures with precise numerical accuracy. Our need to avoid numerical model arises from two origins: our comprehension are often uncertain and usually vague. In addition, numerical items often have vague behavior forms and interferes our recognition and comprehension of the influencing structural parts underlying these forms. As a result, we need qualitative methods to remove from unimportant items and help us concentrate on structural parts underlying the important dynamic features, appearing in the time and state space of these systems.

In system dynamics, qualitative considerations, relied on stock and flow, feedback loop, and state space diagrams are completely popular. The dynamic qualities of several nonlinear systems have been described more strictly in the literature. Relied on several combinations of graphical analysis [16, 17], empirical experiments [18, 19], deterministic analysis [16, 20–29], simulations experiments where initial conditions are being concerned [19, 30–33], statistical research in the parameter space [17], and piecewise linear analysis [34], the dynamics of equilibria, i.e., stability criteria, dominant feedback loop polarities, bifurcations, and features of chaos and self-organization have been studied greatly.

Fuzzy sets theory and fuzzy numbers allow us to represent vague conditions symbolically. In fact, by using fuzzy numbers, we evade the normalization needed when implementing probabilities. Also, it enables us to represent several levels of uncertainty. Fuzzy concept allows us to analyze a linguistic sign as a collect representation of a reality that can be described in larger feature using precise numerical representation [35]. To confirm model strength, the aggregation must create a uniform mapping to keep the relation between the input and the output route of the model. Fuzzy dynamic models share some features of real models—specifically, stochastic models that are conditional on the Monte Carlo simulation. Their dynamic feature is specified in often the same way by the basic systems structure, i.e. by feedback, nonlinearities, etc. that contribute to duplicate uncertainty and vagueness all through each model.

### 3.2 How to use fuzzy numbers in simulations?

A fuzzy arithmetic has been expanded for fuzzy numbers; see e.g., [36]. Fuzzy arithmetic can be considered as a generalization of Moore's interval arithmetic. Fuzzy numbers generally represent unawareness about the precise value of some variable. If there are some relations between the values of variables, this may lead to too broad intervals after the application of interval arithmetic. System dynamic simulation models

**Fig. 5** Probability distribution of the value of completion and the completion cost for the real option of setting up an imitative automobile as determined using the system dynamics model and the Monte Carlo simulation



**Table 3** The expected value (in MU million) of various quantities of interest for the real option to set up an imitative automobile

	Value flexible (RO)	Value inflexible	Engineering cost	Value project		Value of flexibility
				Flexible	Inflexible	
Expected value	1,582.2	1,582.2	1,200.0	382.2	382.2	0.0
Standard deviation	62.4	62.4	189.6	199.2	199.2	0.0

usually include many functions and operations on variables that are not mutually independent. If we ignore these interrelations, the following values of such functions and operations become too broad. Another feature of dynamic models is that functions generally are nonmonotonic. This also bans the use of interval arithmetic, and look for other methods are needed. Integrating fuzzy arithmetic and system dynamics is a method for this problem that we used in our proposed method.

### 3.3 Fuzzy framework for our example

As we noted earlier, models of the dynamics of complex systems give potential advantages to the extensive understanding of the available system besides performing particular tasks. The dynamics of such systems result from complex interacting mechanisms, to comprehend which is many times qualitative and very incomplete. This makes the traditional differential modeling problem entirely hard to solve. As an alternative, data-driven fuzzy recognition methods [37–43] can successfully be appealed when an enough set of samples is accessible. But unfortunately, for complex systems, it can often occur that such a data set is inadequate, and the identified model may result to be incapable of generalization. Moreover, the model may result to be numerically unstable and unintelligible due to the way the fuzzy partitions and the rule base are built.

Sine our proposed model is relied on fuzzy and qualitative modeling frameworks of system dynamics, for better understanding and interpretation of our model and its features, this section outlines the basics of both fuzzy and qualitative modeling frameworks.

#### A. A fuzzy framework for system dynamics modeling

Determining nonlinear system dynamics from data could be considered as a problem of modeling nonlinear discrete time dynamic systems, and concern the reformulating of the unknown function  $f: X \subseteq R^n \rightarrow R$  that suggests the practical relation between the input ( $\underline{x}$ ) and output ( $y$ ). Among the possible forms to designate the dynamics of  $y$  [44], let us consider:

$$y_k = f(\underline{x}_{k-1}, \underline{\theta}) + \varepsilon_k \quad (4)$$

In fuzzy conditions, the problem includes finding a continuous function approximator  $\tilde{f}$  of  $f$  within a class of

fuzzy systems known to possess the universal approximation property. To encoding the qualitative illustration of relations revealed through if-then rules, we can use Mamdani's or Sugeno's approaches that we express elaborately later. Generally, the problem of building  $\tilde{f}$  could be divided in two subproblems:

- (1) structural determination
- (2) parameter estimation

To make the modeled system behavior easily interpretable and clear, these two subproblems should be individually solved.

#### B. A qualitative framework for system dynamics modeling

An approach that qualitatively summarizes ordinary differential equations (ODE) is given by qualitative differential equations (QDE) [45]. A QDE designates a system dynamics in the similar expressions as an ODE does, except that (1) the values of variables are qualitatively suggested about their ordinal relations with indicator values, and (2) functional relationships between variables are designated about districts of monotonicity.

The outline of our proposed method is showed in Fig. 6.

According to what was said, we could simply consider the earlier SD that designed for automobile manufacturer as fuzzy. This means the existing variables in the SD are fuzzy in calculations that might be a more accurate view of the real world. Fuzzy-SD diagram is similar to the diagram that discussed in the previous section. The only difference is the look of the variables that in the new state, the input variables are assumed to be fuzzy.

In working with fuzzy variables, one of the important things that need to be considered is the fuzzy inference system (FIS). FIS is the process of devising the mapping from inputs to outputs using fuzzy logic, the mapping then gives a base from which a decision can be made. It is used for assessing fuzzy semantic statements using ideas like membership functions and if-then rules [46]. Because of its multidisciplinary character, FIS are connected to a few names, like fuzzy expert systems, fuzzy associative memory, fuzzy logic controllers, and simply, fuzzy systems. Totally, there are two common types of FIS: Mamdani and Sugeno types [47]. These two types of FIS alter slightly impeding outputs are decided. For these two types, there are complete explanations in the

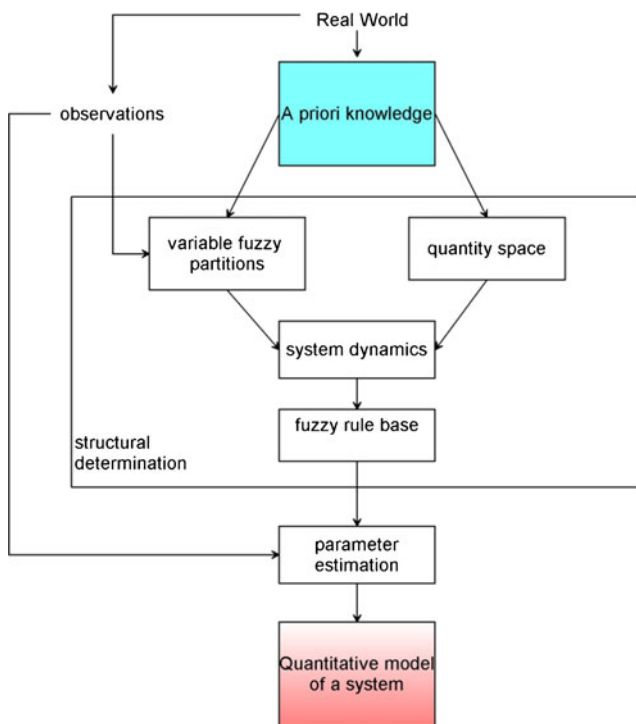


Fig. 6 Outline of our proposed method

literature [41, 48, 49]. The FIS diagram illustrated in Fig. 7 [50] is the composite of all the smaller diagrams. It concurrently shows all parts of the fuzzy inference process.

### 3.4 Qualitative and quantitative modeling of the project

A flowchart representing different steps of determining the real options using the proposed fuzzy–SD approach is shown in Fig. 8. As it can be seen in this figure, the proposed fuzzy–SD approach can determine the real options considering all the impacting factors besides the existing risks and uncertainties. Therefore, first, the qualitative model of project is created using cause and effect feedback loops. Then, the interrelationship occurred between different factors are explained by mathematical equations and the quantitative model of project

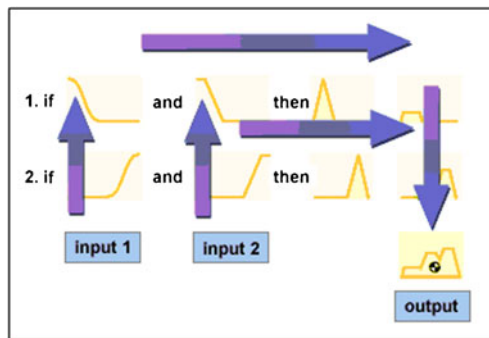


Fig. 7 Fuzzy inference diagram [50]

is created. The importance of input factors is mentioned by fuzzy numbers. Having created the quantitative model of project, the net present value (NPV) and thereafter real options valuation (ROV) is simulated at different  $\alpha$ -cut levels and is presented as a fuzzy number. To be more precise, after computing NPV value, ROV value can be solved by any of the usual methods such as fuzzy payoff method proposed by Collan et al. [51] or Black–Scholes method. The ROV is now determined as a fuzzy number using the calculated value of the project's NPV. Finally, the achieved fuzzy number of ROV is defuzzified and a crisp value for ROV is presented.

#### 3.4.1 Qualitative modeling of project

There are various parameters influencing a project. These parameters have complex interactions with each others. To determine the ROV properly, it is necessary to account for these influencing factors. As we noted earlier in this paper, system dynamics (SD) introduced by Forrester is an object-oriented simulation methodology allowing us to model the complex interrelated structure of different factors influencing a project [52]. System dynamics modeling is intrinsically creative and individual modelers have different approaches. In Fig. 4, we show the SD diagram of an automobile production and try to show all affecting parameters in the project. In Fig. 9, a high-level diagram of different factors affecting a project's ROV is shown.

As it can be seen in this figure, the project's ROV is affected by several parameters. These parameters involve annual capital investment, manufacturing duration, annual supporting cost, and annual discount rate [53–59]. The complex interactions existed between these factors based on Fig. 4 are depicted in Fig. 9.

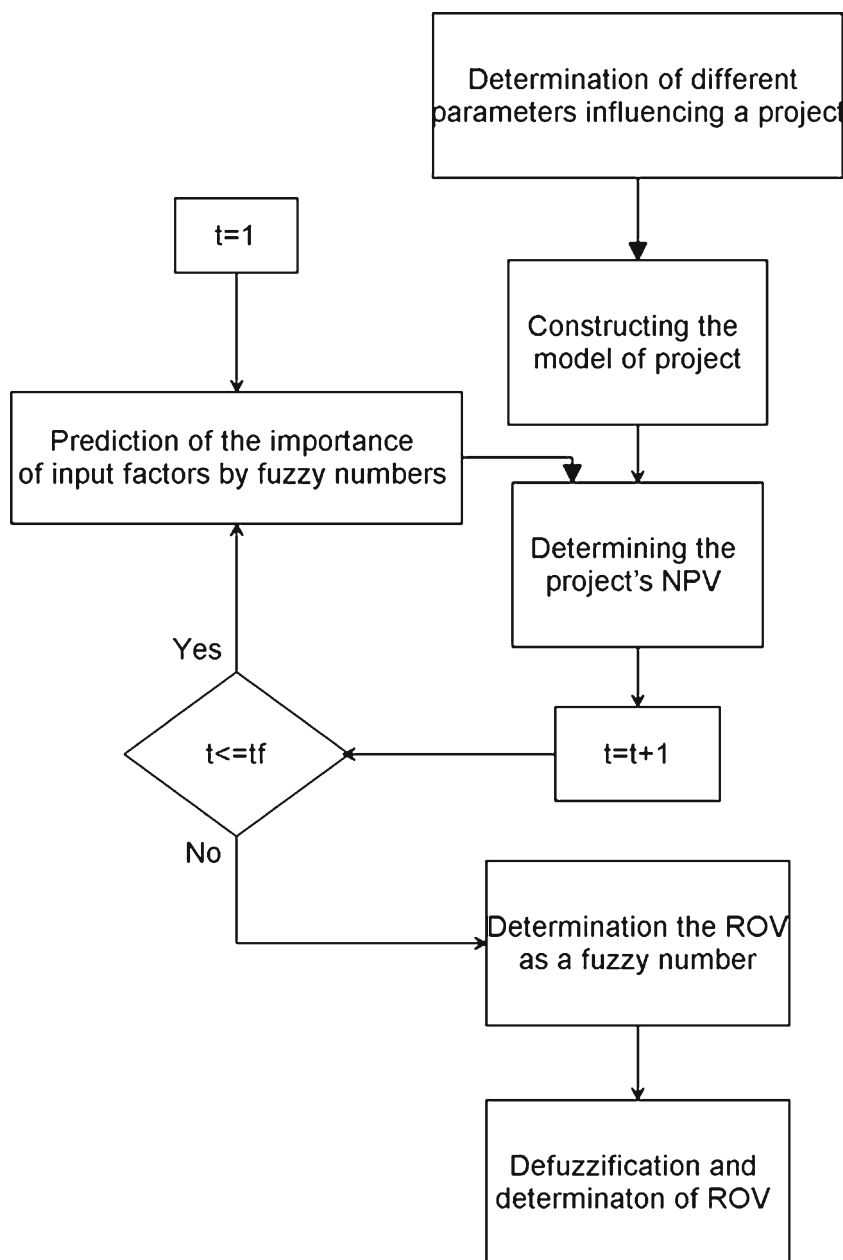
#### 3.4.2 Quantitative modeling of project

Having influenced the qualitative model of project using feedback loops, the relation existing between different parameters are determined and the quantitative model of the real options is built. The relationships between some of the parameters are clear and can directly be determined by mathematical functions. Although, there exist some other parameters for which these relationships can't be simply determined. In these cases, the relationships have been evaluated either by extrapolation or expert judgment. If historical data were available, then the relations were determined by extrapolation and if historical data were not available, the relations were determined by expert judgment using fuzzy inference mechanism.

Here we discussed about fuzzy conditions and relations between variables. As we said, when historical data are not



**Fig. 8** Diagram of our proposed method



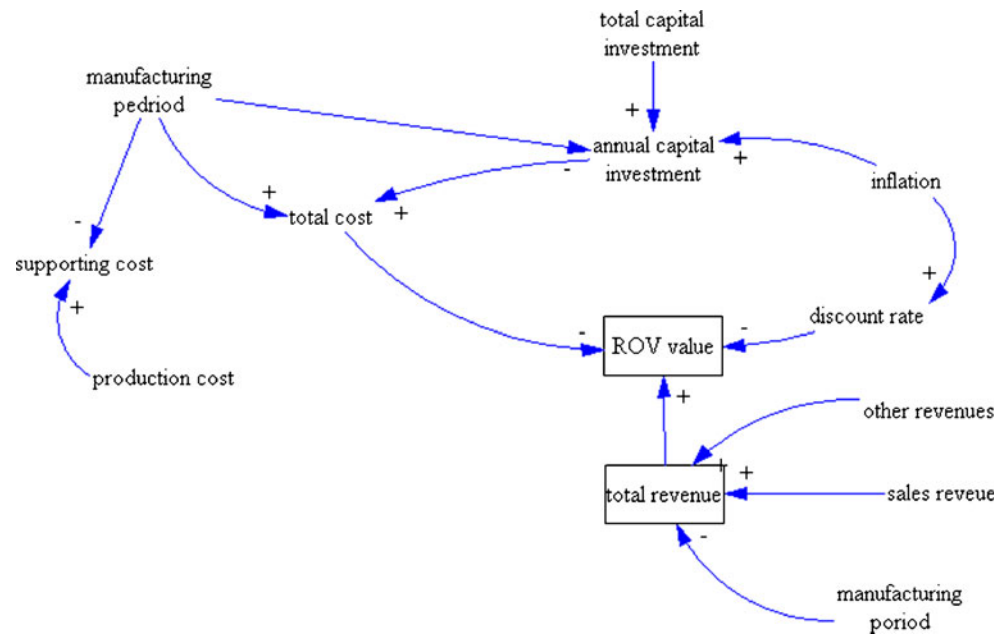
available, the relations between parameters can be determined by inference mechanism. The fuzzy logic if-then rule performs approximate reasoning with vague dependencies. The Mamdani inference mechanism, as one of the most popular types of fuzzy controllers, is used in this research [60]. The main idea is to designate process state by using linguistic variables and to use these variables as inputs to control rules [61, 62]

As explained above, there are various parameters influencing the project's ROV. The value of these influencing factors can't be evaluated with certainty because of the existing uncertainties. The values of these parameters are supposed as a fuzzy number relied

on the opinions of different experts involved in the project. Using the proposed fuzzy-SD approach, the ROV value is calculated as a fuzzy number. The proposed method is explained below briefly:

- (1) A specific  $\alpha$ -cut value is selected, where  $0 \leq \alpha \leq 1$ .
- (2) The related crisp values of the fuzzy number of the value of different input factors correspond to  $\alpha$  are specified as  $[a_\alpha, b_\alpha]$ .
- (3) The ROV value is calculated with these crisp values as input to the simulation model.
- (4) Steps 1–3 are repeated for as many values of  $\alpha$  needed to get the solution. Using proposed fuzzy-SD approach, the ROV value could be specified throughout the project life cycle.

**Fig. 9** A high-level diagram of different factors affecting the project's ROV



### 3.5 Building FIS for SD modeling of automobile manufacturer company

As mentioned, one of the important applications of fuzzy logic and SD combination of it is building the FIS. In this section, we present the steps of proposed approach and Mamdani type inference process will adopt.

#### Step 1 Determination of variables

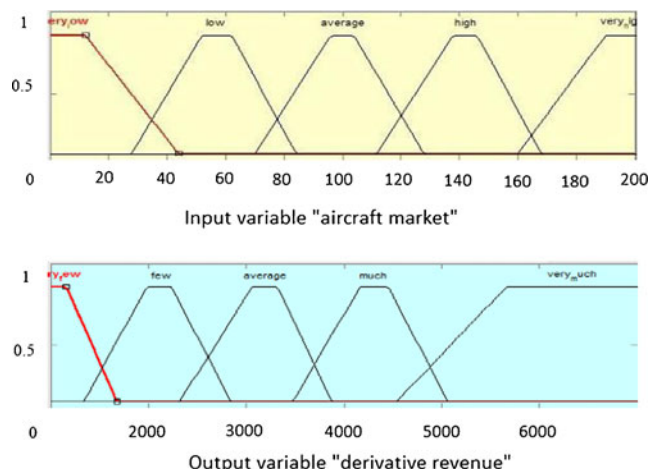
The first step in planning the FIS is to determine the fuzzy input and output variables of the controller. An ordinary problem in most of the research is selecting a restricted number of variables that are applicable for the application, and for which dependable data can be acquired [63]. The input variable selection can be executed by using data-driven methods like artificial neural networks or genetic algorithms [64–66], but has not yet been appealed for rule-based fuzzy model because these methods often need a large dataset, and then the specialist knowledge has mainly been used [67].

As mentioned previously, in this study, the number of variables is acquired from the market and environment of automobile manufacturer companies' analyses and exact consideration of flow diagram of the automobile manufacturing process in these companies. These variables are exactly as in the previous case by definition, interrelationships and operation in the considered SD. The only difference is that in the previous case, we consider the variables as possibilistic variables and by using Monte Carlo simulation, we get quantities for intermediate variables and finally, output of the system, but here, by using

fuzzy logic and if-then rules to define the relationships between these variables and finally calculate the system's output.

#### Step 2 Choosing fuzzy set and membership functions

After determination of fuzzy variables, the next step is determining the fuzzy set for each variable and choosing the suitable membership functions for variables. In this paper, the trapezoidal membership function is engaged to show linguistic variables. Figure 10 depicts some of these membership functions with linguistic variables. Of course, these membership functions are for first example, therefore "Monte Carlo simulation for value of the real option to launch a derivative automobile."



**Fig. 10** Membership functions of input and output variables

### Step 3 Create fuzzy rule base and set the inference process

Once the membership functions for each variable are built, the fuzzy rule base is expanded to connect the input variables to the output variables by if-then rules. The fuzzy operators such as “and” or “or” are appealed across the rules that are built relied on information about SD diagram in Fig. 4. In this study, fuzzy rule bases to modeling automobile production market are written in the 74 rules to relate fuzzy input variables with the fuzzy output variables. Figure 11 shows some of these rules which are written by MATLAB Fuzzy Logic Toolbox.

We use Mamdani FIS. Mamdani suggested a fuzzy inference. His fuzzy logic operator is given as:

$$f(\mu_A(x), \mu_B(y)) = \mu_A(x) \wedge \mu_B(y) \quad (5)$$

where  $\mu_A(x)$  indicates the membership function of  $x$ .

In the Mamdani inference rule, the “or” link is substituted with the “max” operator and the “and” link is substituted with the “min” operator. Besides the Mamdani inference operator (i.e., min operator) is appealed between the result of preceding and the resulting membership function. Finally, all the outputs are collected using collection methods, e.g., maximum (max), probabilistic OR (probor) and customized methods

### Step 4 Choosing the defuzzification methods

The final step is defuzzifying the fuzzy output into numeric values. The input of this process is a fuzzy set and the output is a crisp number. There are

distinctive defuzzification methods like centroid, bi-sector, middle of the max, etc. The centroid method is appealed in this study which yields the center of area under the curve.

After building the FIS to modeling the automobile manufacturing SD, the results of the FIS are considered. The following section represents the results of simulation.

## 4 Model simulation result

The MATLAB software is used to design the FIS which computes the value of the output relied on different input values. Figure 12 shows the curves which describe the mapping input to output variables. These figures show there is a positive correlation between the input variable and the output variable.

The MATLAB fuzzy toolbox is used for designing FIS that calculate the output variable based on different input variables. As considered with the SD and already mentioned previous, some variables have a positive effect on the output and others have a negative impact. For instance, automobile market variable has a positive impact on the output variable and manufacturing unit cost variable has negative impact. By exact description of fuzzy rules, the relationship between input and output variables could be showed in distinct diagrams and by changing the values of input variables, get different values

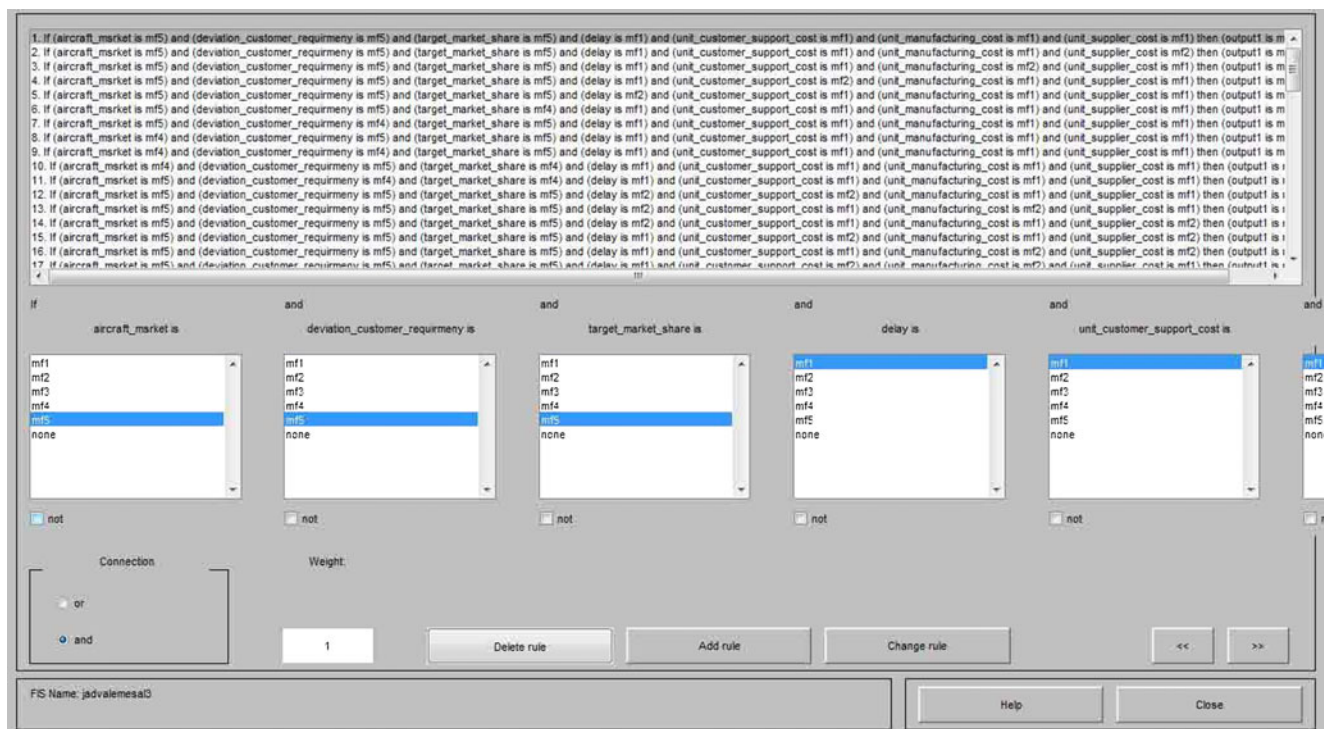
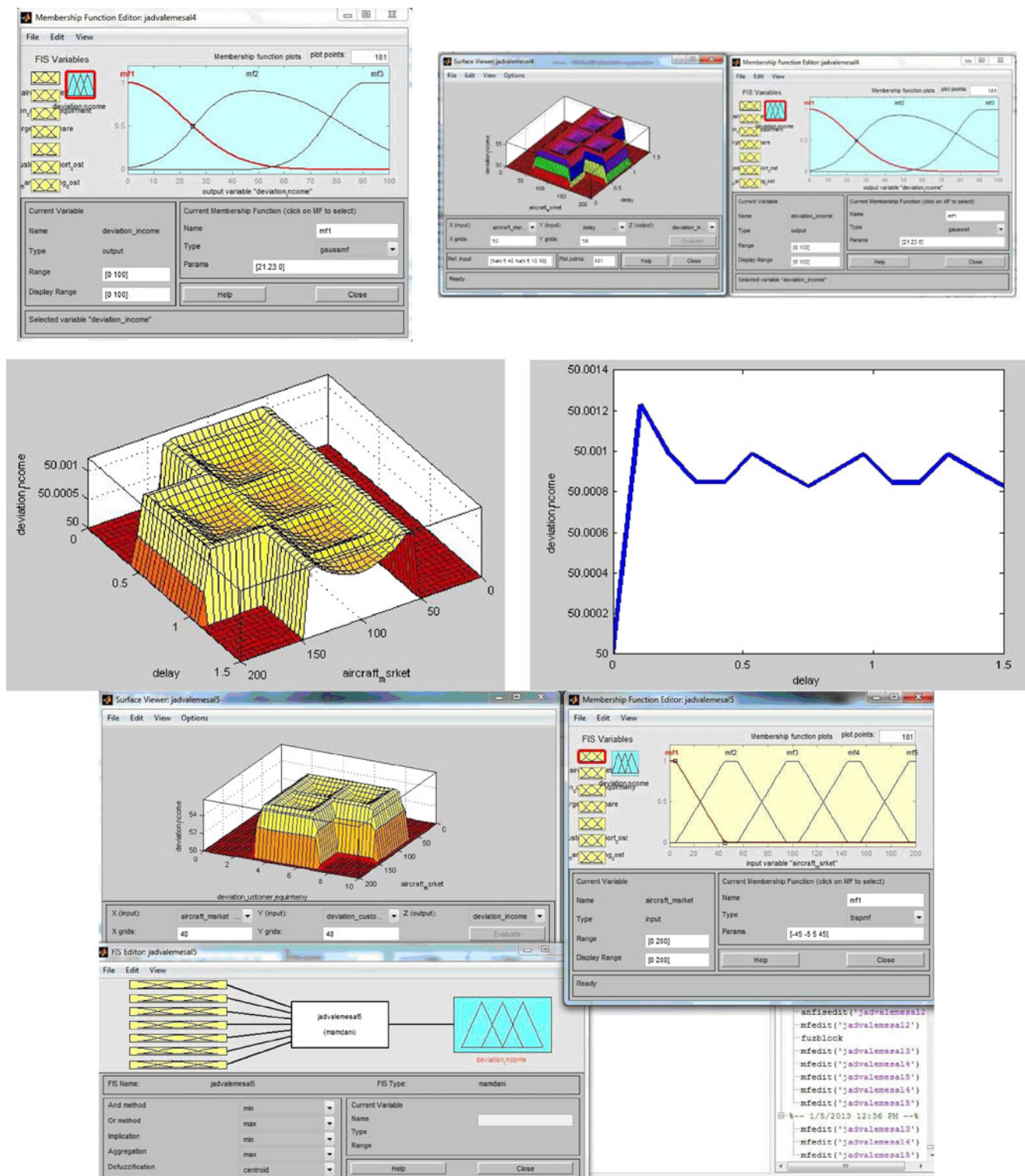


Fig. 11 An example of the rules used in the FIS



**Fig. 12** FIS input and state of interactions between and output variables

for output variables. The following diagrams show the manner of interaction and impact of input variables on the output variables.

After getting the output value that could be a triangular or trapezoidal fuzzy number—that distinct the project's value of

completion—the mean and variance of real option could be achieved. In other words, the output of this fuzzy toolbox that can be written with MATLAB explains the inserted real options in different phases of the project and then the numerical values of mean and variance could be formulated.



For simplicity, here, we use a typical trapezoidal membership function for expressing real options inserted in different stages of the project which we call the  $p_f$  function. Price variant function  $p_f$ , as showed in Fig. 13, is used for specifying changes in technology, the value of options in the set of flexibility opportunities (real options) by using trapezoidal membership functions. In other words, the output of this FIS get the value of completion of the project that some indication of the options that is embodied in different phases of the project. So, the output of this FIS can be expressed in two forms that are equivalent. Even we can consider it as the value of completion of the project and even we can consider it as membership function of flexibility and time of exercising option.

If we suppose that  $p_f$  is a trapezoidal fuzzy number, the below equation presents the trapezoidal membership function of  $p_f$

$$\text{trap}\left(t : p_{fa}, p_{fb}, p_{fc}, p_{fd}\right) = \begin{cases} 0 & ; t < p_{fa}, t \geq p_{fd} \\ 1 & ; p_{fb} \leq t < p_{fc} \\ \frac{t-p_{fa}}{p_{fb}-p_{fa}} & ; p_{fa} \leq t < p_{fb} \\ \frac{p_{fd}-t}{p_{fd}-p_{fc}} & ; p_{fb} \leq t < p_{fc} \end{cases} \quad (6)$$

Where ( $t < T$ ) shows a small window of chance attainable to the user. The user can exercise, wait, select an alternative, defer, or abandon policies about options. The individual chances in Fig. 14 like exercise, defer, alternative, wait, and abandon can be mapped to the fuzzy membership function of  $p_f$  in Fig. 13.

This means that a classification of the chances attainable to the user mapped into the fuzzy membership function such that for each class of user chances, and for each of the levels in the similar membership function, there is a similar investment chances also classified about the user chances as very low, low, mid, high, and very high.

Therefore, both explained analysis lead to our considered purpose therefore got the option membership function.

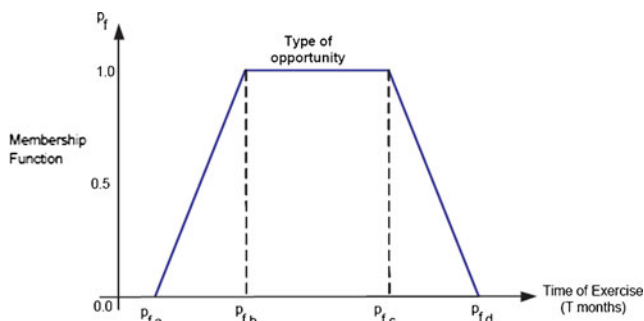


Fig. 13 Price variant function

But in the first case, the FIS output value is set to the fuzzy function of the value of completion and must use this value by 2 and 3 formulas (similar to possibilistic state) and calculate the mean and variance of the real option. However, in this case, to get the mean and variance of the normal amount, we need to know the cost of completion. As mentioned, the value of the cost of completion achieving from the manufacturer's data reduction can be achieved. Here, as in probabilistic state that was previously considered, this cost could be achieved from development cost of derivative automobile. According to data provided from automobile manufacturer, suppose the completion cost of the real option for performing derivative automobile is 20 % of derivative development cost. Therefore, for getting mean and variance of real option, like the previous case, we have Eqs. 2 and 3. But note that in this case, all functions in these equations are fuzzy. Here, we try to discuss this problem in the most general case, and give a solution for it. To calculate these fuzzy integrals, we have to express some of the concepts related to fuzzy integrals. There are various definitions about fuzzy integrals. Sugeno presented the ideas of fuzzy measure and fuzzy integral [68].

According to definition, if  $f(x)$  is a nonnegative real-valued measurable function on  $(X, F, \mu)$  and  $A \in F$ , the fuzzy integral of  $f$  on  $A$  about  $\mu$  is explained by

$$\int_A f d\mu = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \quad (7)$$

Where  $F_\alpha = \{x : f \geq \alpha\}$ ,  $\alpha \in [0, \infty]$ .

In another definition, Ralescu and Adams provide a definition of a fuzzy integral [69]. They proved some useful propositions about fuzzy integrals, such as

$$\int_A f d\mu = \int f \cdot \chi_A d\mu \quad (8)$$

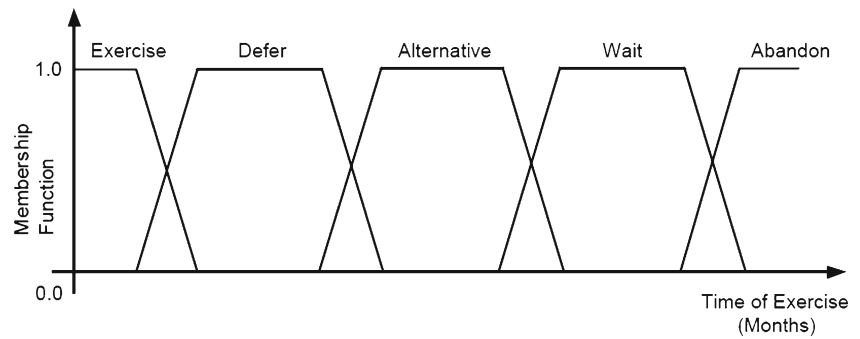
Where

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \quad (9)$$

One of the most common types of equations in the literature is the Fredholm integral equation of the second kind is

$$F(u) = \alpha \int_I V(\beta, u) F(\beta) d\beta + f(u) \quad (10)$$

Besides, if  $f(u)$  be a crisp function then the solution is also crisp. On the other hand, if  $f(u)$  be a fuzzy function we have Fredholm fuzzy integral equation of the second kind which may only process fuzzy solutions. Necessary conditions for the resolvent existence in fuzzy situation are given in [70, 71].

**Fig. 14** Trapezoidal membership function for flexibility chance

In order to give a numerical solution for solving a system of Fredholm equation, we write the parametric form of the given equations:

$$\bar{F}_i(u, \gamma) = \sum_{j=1}^n \left( \alpha_{ij} \int_l^k \bar{W}_{i,j}(\beta, \gamma) d\beta \right) + \bar{f}_i(u, \gamma) \quad (11)$$

Where

$$\bar{W}_{i,j}(\beta, \gamma) = \begin{cases} V_{i,j}(\beta, u) \bar{F}_j(\beta, \gamma) & V_{i,j}(\beta, u) \geq 0 \\ V_{i,j}(\beta, u) \underline{F}_j(\beta, \gamma) & V_{i,j}(\beta, u) < 0 \end{cases} \quad (12)$$

This formula is also expressed exactly for  $\underline{F}_j(u, \gamma)$ . Because of likeness and the symmetrical nature of the formula,

we refuse writing it for  $\underline{F}_j(u, \gamma)$  and it will be enough to write formula related to  $\bar{F}_j(u, \gamma)$ . By using Taylor expansion, we can get the solution to the above equations in the form of

$$\bar{F}_{j,N}(u, \gamma) = \sum_{i=0}^N \left( \frac{1}{i!} \cdot \frac{\partial^{(i)} \bar{F}_j(u, \gamma)}{\partial u^i} \Big|_{u=\zeta} \cdot (u-\zeta)^i \right), l \leq u, \zeta \leq k, 0 \leq \gamma \leq 1 \quad (13)$$

(for  $j=1, \dots, n$ ) which are the Taylor expansions of degree  $N$  at  $u = \zeta$  for the unknown function  $\bar{F}_j(u, \gamma)$ . For this area, we distinguish above formula  $(N+1)$  times (for  $q=0, \dots, N$ ) about  $u$  and get

$$\frac{\partial^{(q)} \bar{F}_i(u, \gamma)}{\partial u^q} = \frac{\partial^{(q)} \bar{f}_i(u, \gamma)}{\partial u^q} + \sum_{j=1}^n \alpha_{ij} \left( \int_l^{c_{i,j}} \frac{\partial^{(q)} V_{i,j}(\beta, u)}{\partial u^q} \cdot \bar{F}_j(\beta, \gamma) d\beta + \int_{c_{i,j}}^k \frac{\partial^{(q)} V_{i,j}(\beta, u)}{\partial u^q} \cdot \underline{F}_j(\beta, \gamma) d\beta \right) \quad (14)$$

Where ( $i=1, \dots, n$ ). For shortness, we determine below character as:

$$\bar{F}_j^{(q)}(\zeta, \gamma) := \frac{\partial^{(q)} \bar{F}_j(u, \gamma)}{\partial u^q} \Big|_{u=\zeta} \quad j = 1, \dots, n \quad (15)$$

The goal of this research is defining of the coefficients  $\bar{F}_j^{(q)}(\zeta, \gamma)$  (and  $\underline{F}_j^{(q)}(\zeta, \gamma)$ ), (for  $q=0, \dots, N; j=1, \dots, n$ ). For this goal, we broaden  $\bar{F}_j(\beta, \gamma)$  in Taylor series and replaced its  $N$ -th slice in (5). Therefore, we can write:

$$\bar{F}_i^{(q)}(l, \gamma) = \frac{\partial^{(q)} \bar{f}_i(u, \gamma)}{\partial u^q} \Big|_{u=l} + \sum_{j=1}^n \left( \sum_{s=0}^N \omega_{q,s}^{(i,j)} \cdot \bar{F}_j^{(s)}(l, \gamma) + \sum_{s=0}^N \omega'_{q,s}^{(i,j)} \cdot \underline{F}_j^{(s)}(l, \gamma) \right) \quad (16)$$

Where

$$\omega_{q,s}^{(i,j)} = \frac{\alpha_{i,j}}{s!} \int_l^{c_{i,j}} \frac{\partial^{(q)} V_{i,j}(\beta, u)}{\partial u^q} \Big|_{u=l} \cdot (\beta-l)^s d\beta, \quad q, s = 0, \dots, N \quad (17)$$

and

$$\omega'_{q,s}^{(i,j)} = \frac{\alpha_{i,j}}{s!} \int_{c_{i,j}}^k \frac{\partial^{(q)} V_{i,j}(\beta, u)}{\partial u^q} \Big|_{u=l} \cdot (\beta-l)^s d\beta, \quad i, j = 1, \dots, n \quad (18)$$

We could write the expression (16) as the matrix form, and then it is easy to prove convergence of this proposed numerical method.

In the second case, the output of the FIS is membership function of flexibility and exercising time of the real option. The following figure shows the controller design. We use fuzzy logic to capture the inherent uncertainty at each stages of the derivative automobile production. Classification follows the specified real option set as a trapezoidal fuzzy number, as



showed in Fig. 15. Certainty in effects related to technical developments could be achieved from modulation of  $p_f$  and supposed algorithm. For simplicity, we supposed the output of FIS is given as a triangular fuzzy number  $(\alpha_t, \beta_t, \gamma_t)$ .

The value of the project's NPV and ROV throughout the project life cycle for our case study is shown in Fig. 16 graphically.

Applying the proposed fuzzy–SD method in this project case example showed the effectiveness of the method in specifying the ROV. The reached results by the proposed method are more reliable in comparison to the other existing methods since the interrelated structure of affecting factors besides the existing risk and uncertainties are considered.

## 5 Discussions

An application of the generalized real options methodology developed in this paper was showed with the analysis of strategies in a new automobile development program. The following results were made:

- (1) Basic features of projects in transportation, such as large capital costs and multiple technical and market uncertainties, specify that a flexible investment strategy can notably improve the financial performance of investments in this area.
- (2) For case study, a typical new automobile development program is built as a series of sequential stages. Each stage can be viewed as a real option, as managers have the flexibility to continue or stop the process after each stage, according to the most currently obtainable information. The analysis presented here differs from previous because the value of completion and the completion cost have been analyzed with a bottom-up approach in contrast with top-down models.
- (3) Combining system dynamics and Monte Carlo simulations were used to specify the probability distributions of the value of completion and the completion cost used in

the real options valuation. The model was adjusted with information prepared by a major automobile manufacturer.

- (4) In many other examples of real options, the value of flexibility is inclined to increase all over the life of the option. In the particular case of automobile manufacturing, although, this does not seem to be the case. Indeed, in typical automobile development programs, much of the costs happen in the early stages of the process and they decrease as the project continues. So, the value of the ability of waiting to invest decreases because fewer costs are left undone.
- (5) Numerical results suggests that if there are reasons other than profit maximization for having such an automobile program, i.e., national security and job creation outside involvement in the early stages of the project may be adjusted to guarantee its feasibility until it reaches a point of self-sufficiency.
- (6) A heuristic method for the combination of fuzzy system dynamic and real options theory is proposed and the applicability of this method is considered with MATLAB software. Also, a few discussions about interpreting results are made and we express the two seemingly different, but the same analysis about the results and some mathematical analyses are performed about them.

## 6 Conclusions

In this paper, a generalized real options method is proposed that can be used to assess uncertain investments. If the probability distributions of the value of completion and the completion cost can be provided analytically, then the generalized method can be used to discover a precise solution to the problem of options valuation. Otherwise, if the essential distributions are not familiar, then numerical simulation, such as a combination of system dynamics and Monte Carlo modeling, may be used to determine the probability distributions. This flexibility enables the user to designate the distributions

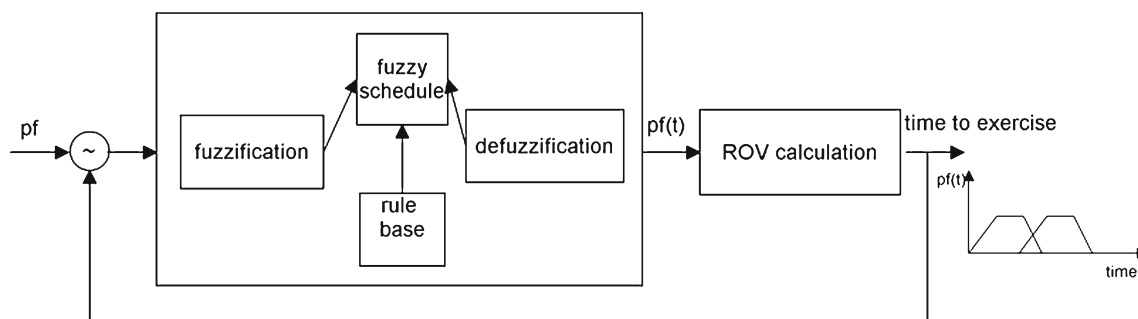
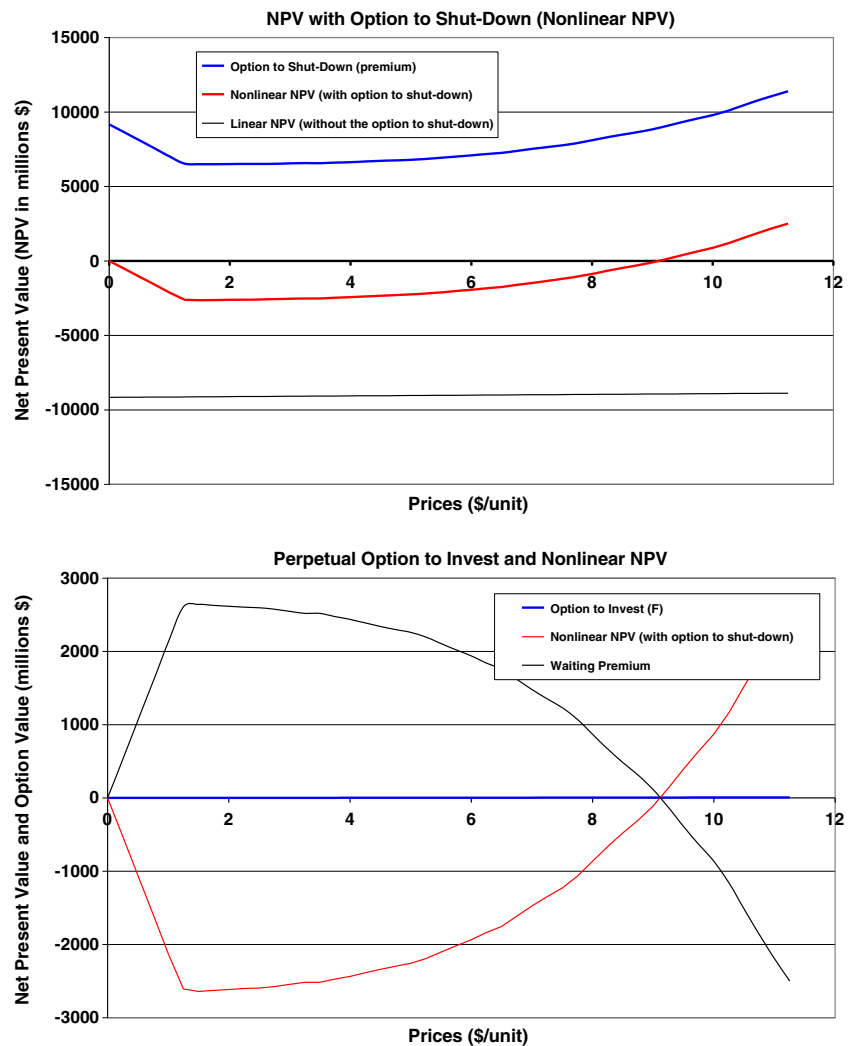


Fig. 15 The structure of fuzzy controller design

**Fig. 16** Values of NPV and ROV of the project

that best depict the value and the costs of the project under examination. As with any numerical model, many simplifying suppositions were used in developing the system dynamics and Monte Carlo simulation to calculate the value of real options in the new automobile development program.

Also, we considered a fuzzy based system dynamics approach to specify the real option value representing for the complex interrelated structure of affecting factors besides the existing uncertainties. The complex interrelated structure of different factors affecting a project was explained by the fulfilled SD-based approach. So, the qualitative model of project was first created using cause and effect feedback loops. A high-level diagram of ROV was presented using cause and effect feedback loops. The relations existed between different factors were then specified and the quantitative model of the project was created. The relations existed between different factors were clearly specified by mathematical functions in most of the cases. Although, there existed some other factors for which these relations could not be clearly specified.

In these cases, the relations were evaluated either by extrapolation or expert judgment (using fuzzy inference mechanism). It was shown that the existing uncertainties could be considered by combining fuzzy logic into the proposed SD-based model. Performing the proposed method in specification of the ROV was assessed by its implementation in an automobile project. The values of different input factors influencing the ROV were specified by fuzzy numbers. Using the proposed integrated fuzzy–SD approach, the value of the ROV was specified as a fuzzy number. Finally, they reached fuzzy number of ROV was defuzzified and a crisp value for the ROV was presented.

It is believed that the proposed fuzzy–SD method may offer a pliable and powerful method for specifying the ROV since both the complex interrelated structure of effecting factors besides the existing uncertainties is considered. The proposed combined fuzzy–SD model can be appealed for all projects for specify the ROV.

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