Numerical Methods

Homework #4 Problem #6

Find the gradient & Hessian for the following functions.

A) $f(x,y) = \ln(x^2 + 3xy + 2y^2)$

$$\nabla f(x,y) = \begin{bmatrix} \dfrac{2x + 3y}{x^2 + 3xy + 2y^2} \\[2ex] \dfrac{3x + 4y}{x^2 + 3xy + 2y^2} \end{bmatrix}$$

$$\text{Hessian } f(x,y) = \begin{vmatrix} \dfrac{-2x^2 + 6xy + 5y^2}{(x^2 + 3xy + 2y^2)^2} & -\dfrac{2x + 3y}{(x^2 + 3yx + 2y^2)^2} \\[2ex] \dfrac{-2x + 3y}{(x^2 + 3yx + 2y^2)^2} & \dfrac{-5x^2 + 12xy + 8y^2}{(x^2 + 3xy + 2y^2)^2} \end{vmatrix}$$

$$= \frac{(-2x^2 + 6xy + 5y^2)(-5x^2 + 12xy + 8y^2)}{(x^2 + 3xy + 2y^2)^4} - \frac{(2x + 3y)^2}{(x^2 + 3yx + 2y^2)^4}$$

B) $f(x,y,z) = x^2 + y^2 + 3z^2$

$$\nabla f(x,y,z) = \begin{bmatrix} 2x \\ 2y \\ 6z \end{bmatrix} \qquad \text{Hessian} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{vmatrix} = \underline{\underline{24}}$$

Homework #4 Problem # 7

A) Start w/ an initial guess of $(x_0, y_0) = (1,1)$ and apply two iterations of the steepest ascent method to maximize $f(x,y)$

$f(x,y) = 2xy + 2y - 1.5x^2 - 2y^2$

$\nabla f(x,y) = \begin{bmatrix} 2y - 3x \\ 2x + 2 - 4y \end{bmatrix}$

$\nabla f(x,y) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$g(h) = f(1-h, 1)$

$= 2(1-h) + 2 - 1.5(1-h)^2 - 2(1)^2$

$= .5h - 1.5h^2$

$g'(h) = -3h + 1 \qquad h^* = 1/3$

$(x_1, y_1) = (x_0, y_0) + h\nabla f(x_0, y_0)$

$= (1,1) + \frac{1}{3}(-1,0)$

Iteration #1 $\rightarrow$ $= (2/3, 1)$

$g(h) = f((\frac{2}{3}, 1) + h(0, 2/3))$

$= f(\frac{2}{3}, 1 - \frac{2}{3}h)$

$= 2(\frac{2}{3})(1-\frac{2}{3}h) + 2(1-\frac{2}{3}h) - 1.5(\frac{2}{3})^2 - 2(1-\frac{2}{3}h)^2$

$= \frac{-4(h+.5)(2 \cdot h - 3)}{9}$

$g'(h) = .\bar{8} - 1.\bar{7}h \qquad h^* = \frac{1}{2}$

$(x_2, y_2) = (x_1, y_1) + h\nabla f(x_1, y_1)$

$= (2/3, 1) + \frac{1}{2}(0, -2/3)$

Iteration #2 $\rightarrow$ $= (2/3, \frac{2}{3})$

B)

$$f(x,y) = 2xy + 2y - 1.5x^2 - 2y^2$$

$$\nabla f(x,y) = \begin{bmatrix} 2y - 3x \\ 2x - 4y + 2 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} = -3$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = -4$$

$$\text{Hessian } f(x,y) = \begin{vmatrix} -3 & -2 \\ 2 & -4 \end{vmatrix} = 8$$

Because $|H|$ is > 0 & $\frac{\partial^2 f}{\partial x^2} < 0$, the value is a <u>max</u>

Solving for max point:

$2y - 3x = 0$ $\quad x = 1/2$

$2x - 4y + 2 = 0$ $\quad y = 3/4$