Work-out Problems

Study tip: Show all your work!

Exercise 1. Set up, but do NOT solve, a linear programming problem to solve the following:

Pyxie has at most \$20,000 to invest in three different stocks. The TWX company costs \$17.00 per share and pays dividends of \$.20 per share. The GE company costs \$34.00 per share and pays dividends of \$1.00 per share. The WMT company costs \$45.00 per share and pays \$.67 per share in dividends. Pyxie has given her broker the following instructions: Invest at least twice as much money in GE as in WMT. Also, no more than 25% of the total money invested should be in TWX. How should Pyxie invest her money to maximize the dividends?

variables:

$t :=$ number of TWX shares Pyxie purchased

$g :=$ number of GE shares Pyxie purchased

$w :=$ number of WMT shares Pyxie purchased

$D :=$ Pyxie's total dividends, in dollars.

Objective: Maximize $D = 0.20\,t + 1.00\,g + 0.67\,w$

Subject to $17.00\,t + 34.00\,g + 45.00\,w \leq 20\,000$ (total invested)

$$\underbrace{34g}_{\text{amount invested in GE}} \geq 2 \cdot (\underbrace{45w}_{\text{amount invested in WMT}})$$ (ratio of GE to WMT investments)

$$\underbrace{17t}_{\text{amount invested in TWX}} \leq 0.25\left(\underbrace{17t + 34g + 45w}_{\text{total money invested}}\right)$$ (cap on TWX investments)

$t \geq 0,\ g \geq 0,\ w \geq 0$ (nonnegativity constraints)

Exercise 2. Consider the following linear programming problem:

Plan:
① Graph the feasible region.
② Identify all corners of it.
③ Test all corners in objective function, find min.

Objective : Minimize $C = 3x + 5y$

subject to
$$4x + y \geq 36$$
$$2x + y \geq 30$$
$$x + 3y \geq 30$$
$$x \geq 0,\ y \geq 0$$

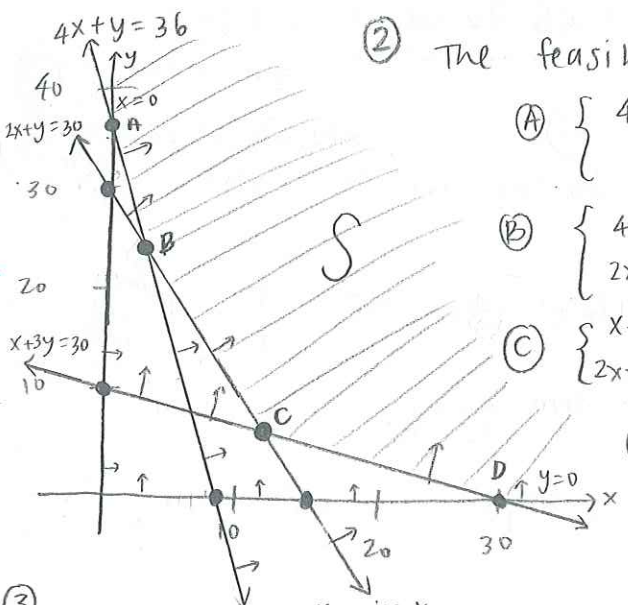
1. Solve the linear programming problem by the method of corners.

Depends: is the feasible region bounded or not?

2. Does this objective function have a maximum value? Explain why or why not.

①

	solid	solid	solid	solid	solid
linear inequality:	$4x+y \geq 36$	$2x+y \geq 30$	$x+3y \geq 30$	$x \geq 0$	$y \geq 0$
boundary line:	$4x+y=36$	$2x+y=30$	$x+3y=30$	$x=0$ vertical line (y-axis)	$y=0$ horizontal line (x-axis)
x-intercept: $(y=0)$	$(9, 0)$	$(15, 0)$	$(30, 0)$		
y-intercept: $(x=0)$	$(0, 36)$	$(0, 30)$	$(0, 10)$		
test point:	$(0,0)$: $4(0)+0 > 36$? $0 > 36$? → FALSE	$(0,0)$: $2(0)+0 > 30$? $0 > 30$? → FALSE	$(0,0)$: $0+3(0) > 30$? $0 > 30$? → FALSE	want x's greater than 0 → to the right of $x=0$	want y's greater than 0 → above $y=0$



② The feasible region is unbounded and has 4 corner points.

Ⓐ $\begin{cases} 4x+y=36 \\ x=0 \end{cases} \rightarrow \left[\begin{array}{cc|c} 4 & 1 & 36 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 36 \end{array}\right] \Rightarrow \begin{cases} x=0 \\ y=36 \end{cases}$
$\Rightarrow A(0,36)$

Ⓑ $\begin{cases} 4x+y=36 \\ 2x+y=30 \end{cases} \rightarrow \left[\begin{array}{cc|c} 4 & 1 & 36 \\ 2 & 1 & 30 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 24 \end{array}\right] \Rightarrow \begin{cases} x=3 \\ y=24 \end{cases}$
$\Rightarrow B(3,24)$

Ⓒ $\begin{cases} x+3y=30 \\ 2x+y=30 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 30 \\ 2 & 1 & 30 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & 6 \end{array}\right] \rightarrow \begin{cases} x=12 \\ y=6 \end{cases}$
$\rightarrow C(12,6)$

Ⓓ $\begin{cases} x+3y=30 \\ y=0 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 30 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 30 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \begin{cases} x=30 \\ y=0 \end{cases} \rightarrow D(30,0)$.

③

Corner point	* min * $C = 3x+5y$
Ⓐ $(0,36)$	$C = 3(0)+5(36) = 180$
Ⓑ $(3,24)$	$C = 3(3)+5(24) = 129$
Ⓒ $(12,6)$	$C = 3(12)+5(6) = 66$ * (min)
Ⓓ $(30,0)$	$C = 3(30)+5(0) = 90$

1. The minimum value of C is 66 and it occurs when $x=12$, $y=6$.

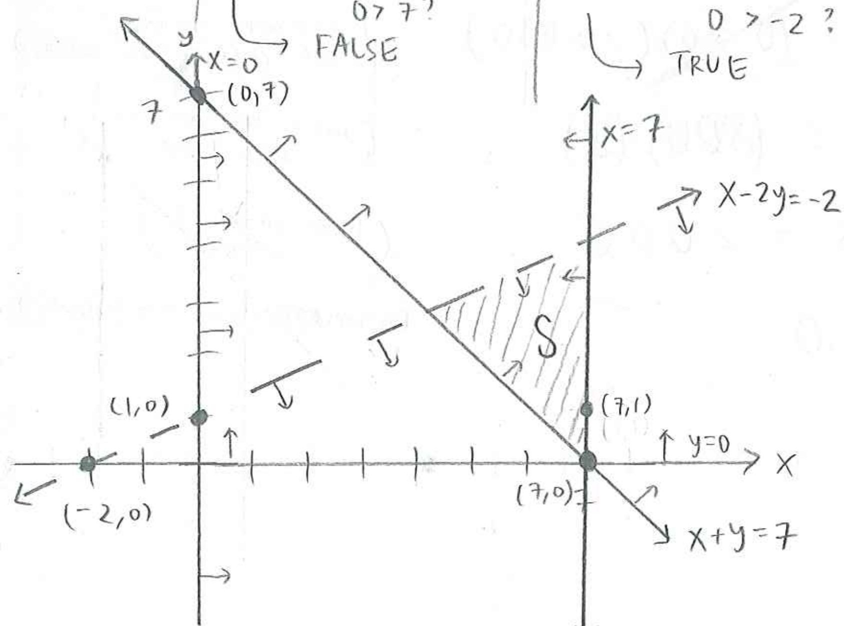
2. The objective function has no maximum value in this feasible region because the feasible region is unbounded.

Exercise 3. Graph the system of linear inequalities.

$$\begin{cases} x + y \geq 7 \\ x - 2y > -2 \\ 0 \leq x < 7, \ y \geq 0 \end{cases}$$

Please label all of your lines, clearly indicate at least two points on each line, distinguish between dashed and solid lines, and clearly indicate the solution set by labeling it with an "S".

	solid	dashed	solid / dashed	solid
linear inequality :	$x+y \geq 7$	$x-2y > -2$	$0 \leq x < 7$	$y \geq 0$
boundary line :	$x+y=7$	$x-2y=-2$	$x=0$ (y-axis) and $x=7$ vertical lines	$y=0$ (x-axis) horizontal line
x-intercept: $(y=0)$	$(7, 0)$	$(-2, 0)$		
y-intercept: $(x=0)$	$(0, 7)$	$(0, 1)$		
test point :	$(0,0)$: $0+0 > 7$? $0 > 7$? → FALSE	$(0,0)$: $0-2(0) > -2$? $0 > -2$? → TRUE	want x's greater than 0 → to the right of $x=0$ and x's less than 7 → to the left of $x=7$	want y's greater than 0 → above $y=0$.



Exercise 4. Set up, but do NOT solve, a linear programming problem to solve the following:

The processing division of the Sunrise Breakfast Company must produce two tons (2000 pounds) of breakfast flakes per day to meet the demand for its Sugar Sweets cereal. Cost per pound of the three ingredients is as follows: Bran flakes cost \$4 per pound, cane sugar costs \$3 per pound, and salt costs \$2 per pound. Government regulations require that the mix contain at least 15% bran flakes and 20% sugar. Use of more than 800 pounds per ton of salt produces an unacceptable taste. How many pounds of each ingredient should be used to minimize the cost of the Sugar Sweets cereal mixture?

variables:

$b :=$ number of pounds of bran flakes used

$c :=$ number of pounds of cane sugar used

$s :=$ number of pounds of salt used

$C :=$ cost of Sugar Sweets cereal mixture, in dollars

Objective: Minimize $C = 4b + 3c + 2s$

subject to

$b \geq (0.15)(20\,000)$ (minimum amount of bran needed)

$c \geq (0.20)(20\,000)$ (minimum amount of cane sugar needed)

$s \leq (800)\cdot(2)$ (max amount of salt used)

$b + c + s = 2000$ (total mixture weight)

$s \geq 0$ (nonnegativity constraints)

Exercise 5. Farmer Rev has 10 acres available to plant maroon and orange carrots. Each acre of maroon carrots will yield 2 tons of carrots and each acre of orange carrots will yield 4 tons of carrots. He wants to have at least three times as many tons of maroon carrots than he does of orange carrots. How many acres of each type of carrots should Farmer Rev plant to maximize his ~~profit~~? yield

1. Set up a linear programming problem that can be used to answer the question.

2. Solve the linear programming problem using the method of corners.

3. Are there any leftover resources when the optimal solution is reached?

1.
Variables:

x := number of acres of maroon carrots planted

y := number of acres of orange carrots planted

P := Farmer Rev's yield (in tons of carrots)

Objective: Maximize: $P = 2x + 4y$

subject to

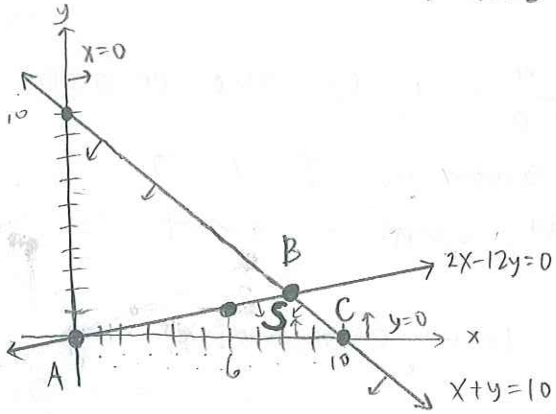
$x + y \leq 10$ (total # acres available)

$(2x) \geq 3(4y) \rightarrow 2x - 12y \geq 0$ (ratio of maroon to orange carrots in tons)

$x \geq 0,\ y \geq 0$ (non negativity constraints)

2.

linear inequality:	$x + y \overset{\text{solid}}{\leq} 10$	$2x - 12y \overset{\text{solid}}{\geq} 0$	$x \overset{\text{solid}}{\geq} 0$	$y \overset{\text{solid}}{\geq} 0$
boundary line:	$x + y = 10$	$2x - 12y = 0$	$x = 0$	$y = 0$
x-intercept $(y=0)$:	$(10, 0)$	$(0, 0)$ → another pt on line: $y = 1 \Rightarrow x = 6 \Rightarrow (6, 1)$	(y-axis)	(x-axis)
y-intercept $(x=0)$:	$(0, 10)$	$(0, 0)$	vertical line	horizontal line
Test point:	$(0,0)$: $0 + 0 < 10$? $0 < 10$? → TRUE	$(3,0)$: $2(3) - 12(0) > 0$, $6 > 0$ → TRUE	want x's greater than 0 → right of $x=0$	want y's greater than 0 → above $y=0$



Identify corner points:

Ⓐ $(0, 0)$, Ⓒ: $(10, 0)$ (already found these)

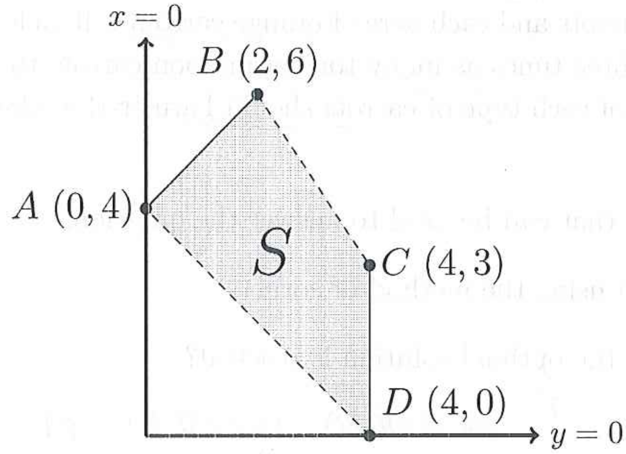
Ⓑ $\begin{cases} x + y = 10 \\ 2x - 12y = 0 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 10 \\ 2 & -12 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \frac{60}{7} \\ 0 & 1 & \frac{10}{7} \end{array}\right] \Rightarrow \left\{ x = \frac{60}{7},\ y = \frac{10}{7} \right. \Rightarrow B\left(\frac{60}{7}, \frac{10}{7}\right)$

corner point	$P = 2x + 4y$ (* max *)
Ⓐ $(0, 0)$	$P = 2(0) + 4(0) = 0$
Ⓑ $\left(\frac{60}{7}, \frac{10}{7}\right)$	$P = 2\left(\frac{60}{7}\right) + 4\left(\frac{10}{7}\right) = \frac{160}{7} \approx 22.8571\ldots$ (* max)
Ⓒ $(10, 0)$	$P = 2(10) + 4(0) = 20$

Solution: Farmer Rev should plant $\frac{60}{7} \approx$ acres of maroon carrots and $\frac{10}{7} \approx$ acres of orange carrots to have a maximum yield of $\frac{160}{7} \approx 22.857$ tons of carrots

3. $x = \frac{60}{7},\ y = \frac{10}{7}$: $x + y = \frac{60}{7} + \frac{10}{7} = 10$ so $x + y = 10$ and $2x - 12y = 2\left(\frac{60}{7}\right) - 12\left(\frac{10}{7}\right) = 0$ so $2x - 12y = 0$. Since all problem constraints are equal at the optimal solution, there are no leftovers.

Exercise 6. Write a system of inequalities describing the solution set S in the figure.



Solid line $\overline{AB}$ has y-intercept $(0, 4)$ and slope $m = \frac{6-4}{2-0} = 1$, so the equation of the line is $y = 1 \cdot x + 4$, which in standard form is $-x + y = 4$. Since $(0, 0)$ is included in the shaded region and $-0 + 0 \leq 4$, the constraint is $\boxed{-x + y \leq 4}$.

Dashed line $\overline{BC}$ has point $(2, 6)$ and slope $\frac{3-6}{4-2} = \frac{-3}{2}$. So the equation of the boundary line is $y - 6 = -\frac{3}{2}(x - 2)$, which in the standard form is $\frac{3}{2}x + y = 9$. Since $(0, 0)$ is included in the shaded region and $\frac{3}{2}(0) + 0 < 9$, the constraint is $\boxed{\frac{3}{2}x + y < 9}$.

Solid line $\overline{CD}$ is a vertical line through $(4, 0)$ and $(4, 3)$, so the equation of the line is $x = 4$. Since we want x's less than 4, the constraint is $\boxed{x \leq 4}$.

Dashed line $\overline{AD}$ has y-intercept $(0, 4)$ and slope $\frac{0-4}{4-0} = -1$. So, the equation of the line is $y = -x + 4$, which in standard form is $x + y = 4$. Since $(0, 0)$ is not included in the shaded region, want $0 + 0 > 4$, so the constraint is $\boxed{x + y > 4}$.

The entire shaded region is restricted to the first quadrant, so we have $\boxed{x \geq 0}$ and $\boxed{y \geq 0}$.

So, the solution set S is defined by:

$$\begin{cases} -x + y \leq 4 \\ \frac{3}{2}x + y < 9 \\ x + y > 4 \\ 0 \leq x \leq 4 \\ y \geq 0 \end{cases}$$

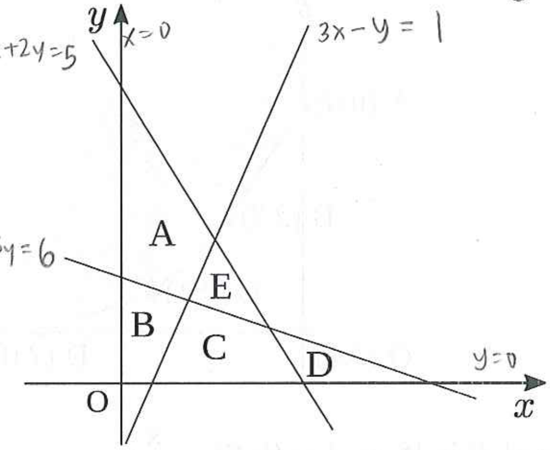
B

Multiple Choice Problems

Study tip: Write out all your work when you complete the multiple-choice problems.

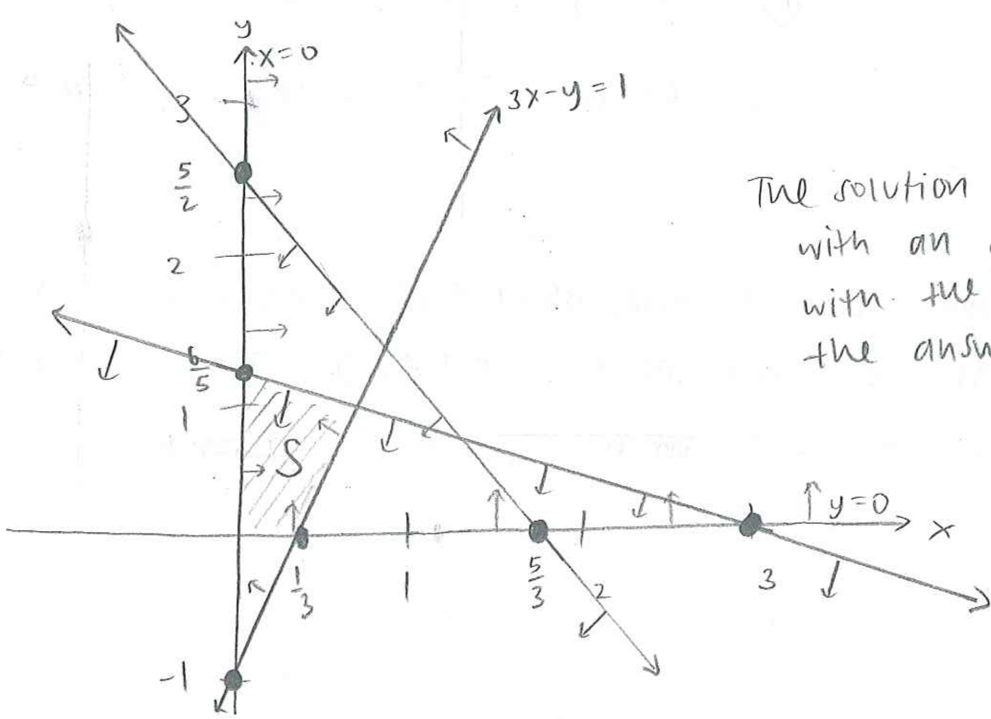
Multiple Choice 1. Find the solution set for the constraints in the figure on the right.

$$\begin{cases} 3x + 2y \le 5 \\ 2x + 5y \le 6 \\ 3x - y \le 1, \\ x \ge 0,\ y \ge 0 \end{cases}$$



(a) A (b) B (c) C (d) D (e) E

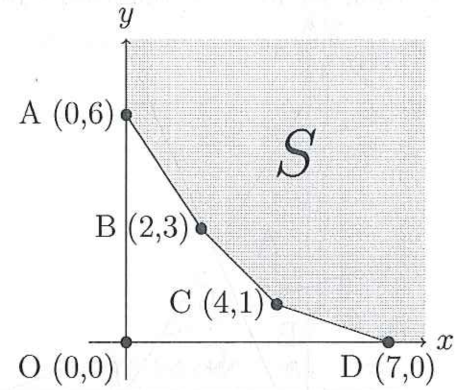
linear inequality:	solid ↑ $3x + 2y \le 5$	solid ↑ $2x + 5y \le 6$	solid ↑ $3x - y \le 1$	solid ↑ $x \ge 0$	solid ↑ $y \ge 0$
boundary line:	$3x + 2y = 5$	$2x + 5y = 6$	$3x - y = 1$	$x = 0$ (y-axis) vertical line	$y = 0$ (x-axis) horizontal line
x-intercept: $(y=0)$	$(\frac{5}{3}, 0)$	$(3, 0)$	$(\frac{1}{3}, 0)$	want x's greater than 0 → right of $x=0$	want y's greater than 0 → above $y=0$
y-intercept: $(x=0)$	$(0, \frac{5}{2})$	$(0, \frac{6}{5})$	$(0, -1)$		
Test point:	$(0,0)$: $3(0)+2(0) < 5$? $0 < 5$? ↳ TRUE	$(0,0)$: $2(0)+5(0) < 6$? $0 < 6$? ↳ TRUE	$(0,0)$: $3(0) - 0 < 1$? $0 < 1$? ↳ TRUE		



The solution set is labelled with an S. comparing with the picture above, the answer is region B.

E

Multiple Choice 2. The shaded region in following figure illustrates the unbounded feasible region of a linear programming problem. Given the objective function $P = 2x + 3y$, which of the following is **TRUE**?



The feasible region is unbounded so the objective function has no maximum, but it does have a minimum at one of the four corner points A, B, C, or D. (Notice that (0,0) is not a corner point of the feasible region.)

I. The maximum of P is 18 at $A = (0, 6)$. ×
II. The minimum of P is 0 at $O = (0, 0)$. ×
III. The maximum of P is 21 at $D = (7, 0)$. ×
IV. The minimum of P is 11 at $C = (4, 1)$. ✓

(a) I only

(b) I and II.

(c) II and III.

(d) II only.

(e) IV only.

	Corner point	* min * $P = 2x + 3y$	
A	(0, 6)	$P = 2(0) + 3(6) = 18$	
B	(2, 3)	$P = 2(2) + 3(3) = 13$	
C	(4, 1)	$P = 2(4) + 3(1) = 11$	* (min)
D	(7, 0)	$P = 2(7) + 3(0) = 14$	

Solution: The minimum of the objective function is $P = 11$ which happens at the point C (4, 1). The objective function has no maximum in this region.

E

Multiple Choice 3. Consider the system of linear inequalities:

$$\begin{cases} 3x + y \geq 7 \\ 2x + 3y \leq 14 \\ x + 3y \leq 10 \\ x \geq 0,\ y \geq 0 \end{cases}$$

What are the corners of the feasible region?

(a) $(1,4), \left(\frac{11}{8}, \frac{23}{8}\right), (4,2)$

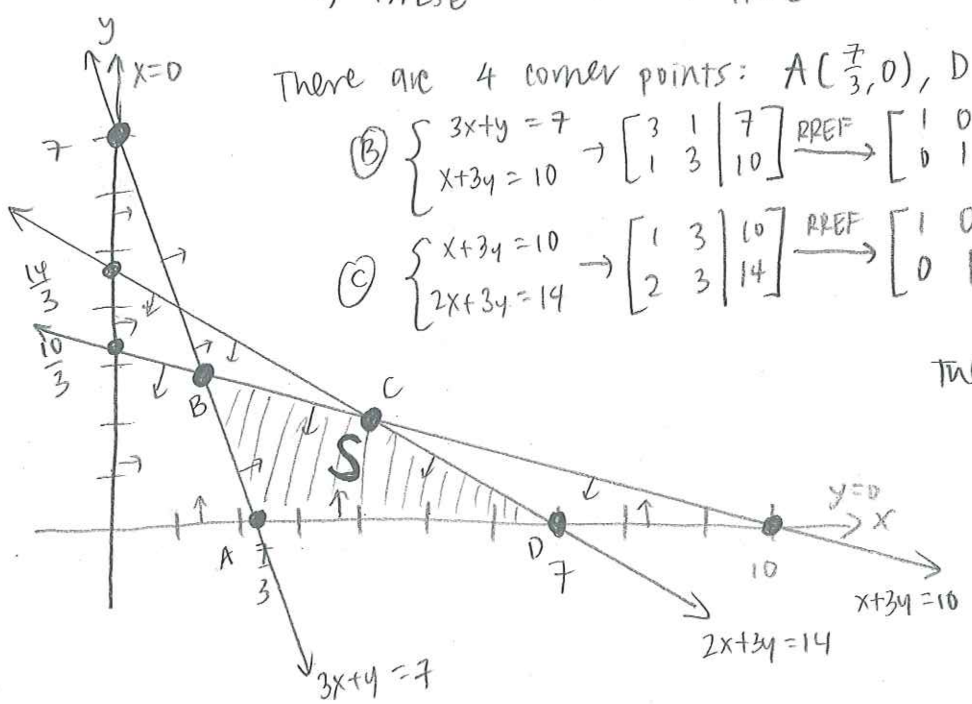
(b) $(0,7), (1,4), (4,2), (10,0)$

(c) $(0,0), \left(0, \frac{10}{3}\right), \left(\frac{11}{8}, \frac{23}{8}\right), \left(\frac{7}{3}, 0\right)$

(d) $(0,0), (0,7), (1,4), \left(\frac{11}{8}, \frac{23}{8}\right), (10,0)$

(e) None of these

	solid	solid	solid	solid	solid
linear inequality:	$3x+y \geq 7$	$2x+3y \leq 14$	$x+3y \leq 10$	$x \geq 0$	$y \geq 0$
boundary line:	$3x+y=7$	$2x+3y=14$	$x+3y=10$	$x=0$ (y-axis) vertical line	$y=0$ (x-axis) horizontal line
x-intercept (y=0):	$(\frac{7}{3}, 0)$	$(7,0)$	$(10,0)$		
y-intercept (x=0):	$(0,7)$	$(0, \frac{14}{3})$	$(0, \frac{10}{3})$		
Test point:	$(0,0)$: $3(0)+0>7$? $0>7$? → FALSE	$(0,0)$: $2(0)+3(0)<14$? $0<14$? → TRUE	$(0,0)$: $0+3(0)<10$? $0<10$? → TRUE	want x's greater than 0 → right of $x=0$	want y's greater than 0 → above $y=0$



There are 4 corner points: $A(\frac{7}{3},0)$, $D(7,0)$, and B and C.

(B) $\begin{cases} 3x+y=7 \\ x+3y=10 \end{cases} \to \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 3 & 10 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \frac{11}{8} \\ 0 & 1 & \frac{23}{8} \end{array}\right] \Rightarrow \begin{cases} x = \frac{11}{8} \\ y = \frac{23}{8} \end{cases} \Rightarrow B\left(\frac{11}{8}, \frac{23}{8}\right)$

(C) $\begin{cases} x+3y=10 \\ 2x+3y=14 \end{cases} \to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 2 & 3 & 14 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{cases} x=4 \\ y=2 \end{cases} \Rightarrow C(4,2)$.

The four corner points are:

$A(\frac{7}{3}, 0)$,
$B(\frac{11}{8}, \frac{23}{8})$,
$C(4,2)$, and
$D(7,0)$.

D

Multiple Choice 4. A feasible region is given by

$$\begin{cases} x + y \le 6 \\ 3x + y \le 15 \\ x + 3y \le 15 \\ x \ge 0,\ y \ge 0\,. \end{cases}$$

At how many points is the objective function $P = 0.5x + 1.5y$ maximized over the feasible region?

The feasible region is bounded so a max exists.

(a) There is no maximum. X

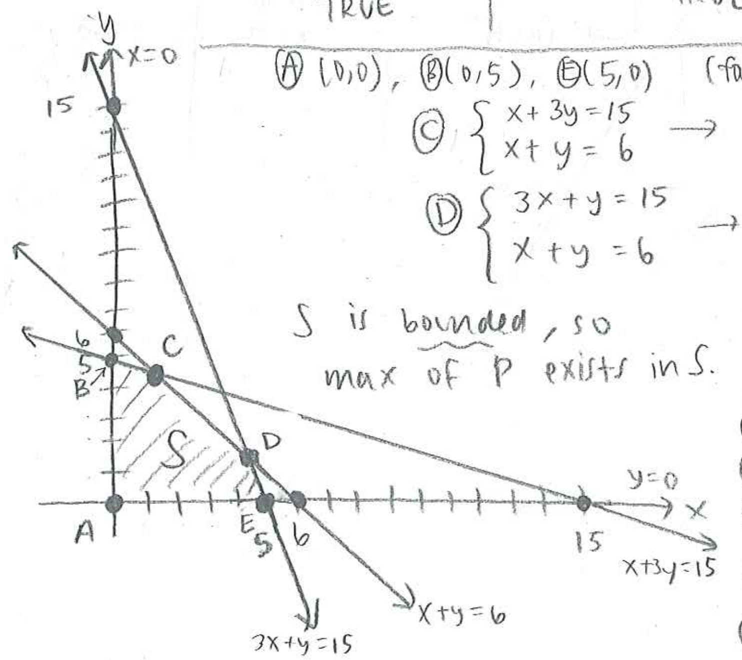
(b) At exactly one point.

(c) At exactly two points.

(d) At infinitely many points.

(e) There is not enough information to determine.

linear inequality:	$x + y \le 6$ (solid)	$3x + y \le 15$ (solid)	$x + 3y \le 15$	$x \ge 0$	$y \ge 0$
boundary line:	$x + y = 6$	$3x + y = 15$	$x + 3y = 15$	$x = 0$ (y-axis)	$y = 0$ (x-axis)
x-intercept ($y=0$):	$(6, 0)$	$(5, 0)$	$(15, 0)$	vertical line	horizontal line
y-intercept ($x=0$):	$(0, 6)$	$(0, 15)$	$(0, 5)$		
Test point:	$(0,0)$: $0+0<6$? $0<6$? TRUE	$(0,0)$: $3(0)+0<15$? $0<15$? TRUE	$(0,0)$: $0+3(0)<15$? $0<15$? TRUE	want x's greater than 0 → right of $x=0$	want y's greater than 0 → above $y=0$



Ⓐ $(0,0)$, Ⓑ$(0,5)$, Ⓔ$(5,0)$ (found these already when graphing)

Ⓒ $\begin{cases} x + 3y = 15 \\ x + y = 6 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 15 \\ 1 & 1 & 6 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{9}{2} \end{array}\right] \Rightarrow \begin{cases} x = \frac{3}{2} \\ y = \frac{9}{2} \end{cases} \Rightarrow Ⓒ\left(\frac{3}{2}, \frac{9}{2}\right)$

Ⓓ $\begin{cases} 3x + y = 15 \\ x + y = 6 \end{cases} \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 15 \\ 1 & 1 & 6 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \frac{9}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right] \Rightarrow \begin{cases} x = \frac{9}{2} \\ y = \frac{3}{2} \end{cases} \Rightarrow Ⓓ\left(\frac{9}{2}, \frac{3}{2}\right)$

S is bounded, so max of P exists in S.

Corner point	* max * $P = 0.5x + 1.5y$	
Ⓐ $(0,0)$	$P = 0.5(0) + 1.5(0) = 0$	
Ⓑ $(0,5)$	$P = 0.5(0) + 1.5(5) = \frac{15}{2}$	* max
Ⓒ $\left(\frac{3}{2}, \frac{9}{2}\right)$	$P = 0.5\left(\frac{3}{2}\right) + 1.5\left(\frac{9}{2}\right) = \frac{15}{2}$	* max
Ⓓ $\left(\frac{9}{2}, \frac{3}{2}\right)$	$P = 0.5\left(\frac{9}{2}\right) + 1.5\left(\frac{3}{2}\right) = \frac{9}{2}$	
Ⓔ $(5,0)$	$P = 0.5(5) + 1.5(0) = \frac{5}{2}$	

The max occurs @ two corner points, so at the infinitely many points on the line segment connecting them.

A

Multiple Choice 5. Kane Manufacturing has a division that produces two models of grates, model A and model B. To produce each model A grate requires 3 pounds of cast iron and 6 minutes of labor. To produce each model B grate requires 4 pounds of cast iron and 3 minutes of labor. The profit for each model A grate is \$2, and the profit for each model B grate is \$1.50. There is no more than 100 pounds of cast iron available and at least 20 **hours** of labor must be made available for grate production each day. Because of backlog orders for model B grates, Kane's manager has decided to produce at least 180 model B grates per day. If A denotes the number of model A grates produced each day and B denotes the number of model B grates produced each day, which of the following will help the company maximize, P, Kane Manufacturing's profits? obj fnc

(a)

$$\begin{aligned} \text{Objective : Maximize} \quad & P = 2A + 1.5B \\ \text{subject to} \quad & 3A + 4B \leq 100 \\ & 6A + 3B \geq 1200 \\ & A \geq 0,\ B \geq 180 \end{aligned}$$

(b)

$$\begin{aligned} \text{Objective : Maximize} \quad & P = 2A + 1.5B \\ \text{subject to} \quad & 3A + 4B \geq 100 \\ & 6A + 3B \leq 20 \\ & A \geq 0,\ B \geq 180 \end{aligned}$$

(c)

$$\begin{aligned} \text{Objective : Maximize} \quad & P = 2A + 1.5B \\ \text{subject to} \quad & 3A + 4B \leq 100 \\ & 6A + 3B \geq 1200 \\ & A \geq 0,\ B \geq 0 \end{aligned}$$

(d)

$$\begin{aligned} \text{Objective : Maximize} \quad & P = 2A + 1.5B \\ \text{subject to} \quad & 3A + 4B \leq 100 \\ & 6A + 3B \geq 20 \\ & A \geq 0,\ B \geq 0 \end{aligned}$$

(e) None of these.