When applying the simplex method, we will use the online website to complete the elementary row operations: `https://www.zweigmedia.com/RealWorld/tutorialsf1/scriptpivot2.html`.

Work-out Problems

Study tip: Show all your work!

Exercise 1. Determine if the specified linear programming problem is a standard maximization problem. If it is, set up the initial simplex tableau.

1. Objective : Maximize $P = 120x + 40y + 60z$
   subject to $x + y + z \leq 100$
   $10x + 4y + 7z \leq 500$
   $x + y + z \geq 60$
   $x \geq 0,\ y \geq 0, z \geq 0$

   constraint not of the form
   $a_1x + a_2y + \dots + a_nz \leq V$
   where a_i is a real #, $V \geq 0$

2. Objective : Minimize $P = 40x + 10y$
   subject to $x + 3y \geq 40$
   $14x + 4y \leq 15$
   $x \geq 0,\ y \geq 0$

   Not "Maximize", so no not a standard maximization problem.

3. Objective : Maximize $P = a + 2b + c + 7d$
   subject to $24 \geq a + 2b + 3c \longleftrightarrow a + 2b + 3c \leq 24$
   $-3a - 6b - c \geq -42 \longleftrightarrow 3a + 6b + c \leq 42$
   $a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0$

   This is a standard maximization problem.

Since this is a std. max. problem, we set up the initial simplex tableau.

Objective: Maximize $P = a + 2b + c + 7d$
Subject to: $a + 2b + 3c \leq 24$ → $a + 2b + 3c + s_1 = 24$
$3a + 6b + c \leq 42$ → $3a + 6b + c + s_2 = 42$
$a \geq 0, b \geq 0, c \geq 0, d \geq 0$
→ $-a - 2b - c - 7d + P = 0$

Initial simplex tableau:

$$\begin{array}{ccccccc|c} a & b & c & d & s_1 & s_2 & P & \\ \end{array}$$
$$\left[\begin{array}{ccccccc|c} 1 & 2 & 3 & 0 & 1 & 0 & 0 & 24 \\ 3 & 6 & 1 & 0 & 0 & 1 & 0 & 42 \\ \hline -1 & -2 & -1 & -7 & 0 & 0 & 1 & 0 \end{array}\right]$$

Exercise 2. Determine whether the given simplex tableau is in final form. If so, find the optimal solution to the associated linear programming problem. If not, find the pivot element to be used in the next iteration of the simplex method.

$$\begin{array}{cccccccc|c}
w & x & y & z & s_1 & s_2 & s_3 & R & \\
\hline
1 & 1/2 & 0 & 1/4 & 0 & 0 & 0 & 0 & 100 \\
0 & 1 & -2 & -1/2 & 1 & 0 & 0 & 0 & 52 \\
0 & 3/2 & 3 & -1/4 & 0 & 1 & 0 & 0 & 100 \\
0 & 1/2 & 2 & -1/4 & 0 & 0 & 1 & 0 & 800 \\
\hline
0 & -1/2 & -1 & 1/4 & 0 & 0 & 0 & 1 & 100
\end{array}$$

There are still negative #s in the bottom row, so no it is not in final form.

To identify the next pivot element, we find the column containing the most negative element in the bottom row (this gives the pivot column) and then the row corresponding to the smallest nonnegative ratio obtained by dividing entries from the far right ("constants") column by the positive entries in the pivot column, excluding the last row (the row corresponding to the smallest nonnegative ratio is the pivot row). The pivot element is the entry in the pivot column and the pivot row.

Here,

$$\begin{array}{cccccccc|c}
w & x & y & z & s_1 & s_2 & s_3 & R & \\
\hline
1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 100 \\
0 & 1 & -2 & -\frac{1}{2} & 1 & 0 & 0 & 0 & 52 \\
0 & \frac{3}{2} & 3 & -\frac{1}{4} & 0 & 1 & 0 & 0 & 100 \\
0 & \frac{1}{2} & 2 & -\frac{1}{4} & 0 & 0 & 1 & 0 & 800 \\
\hline
0 & -\frac{1}{2} & -1 & \frac{1}{4} & 0 & 0 & 0 & 1 & 100
\end{array}$$

most neg $\Rightarrow$ pivot column

ratios

× (don't divide by 0)

× (don't want neg. ratios)

$\frac{100}{3} \approx 33.3$ (smallest nonneg) $\Rightarrow$ pivot row

$\frac{800}{2} = 40$

So, the next pivot element is the 3 in row 3, column 3.

Exercise 3. A furniture manufacturer produces chairs, sofas, and love seats. The chairs require 5 feet of wood, 1 pound of foam rubber, and 10 square yards of material. The sofas require 35 feet of wood, 2 pounds of foam rubber, and 20 square yards of material. The love seats require 9 feet of wood, 0.2 pounds of foam rubber, and 10 square yards of material. The manufacturer has in stock 405 feet of wood, 25 pounds of foam rubber, and 410 square yards material. If the chairs yield a profit of \$300, the sofas \$200, and the love seats \$220 each, how many of each should be produced to maximize the profit? Find the maximum profit.

1. Solve the linear programming problem.

2. Give an economic interpretation of the slack variables associated with the optimal solution found in Part 1, and determine which resources are in excess (if any).

Variables: $c :=$ number of chairs produced

$s :=$ number of sofas produced

$l :=$ number of love seats produced

$P :=$ profit from selling chairs, sofas, love seats (in dollars)

Objective: Maximize $P = 300c + 200s + 220l$

Subject to:

$5c + 35s + 9l \le 405$ (feet of wood available)

$1c + 2s + 0.2l \le 25$ (pounds of foam rubber available)

$10c + 20s + 10l \le 410$ (square yards of material available)

$c \ge 0,\ s \ge 0,\ l \ge 0$

modified constraints

$5c + 35s + 9l + s_1 = 405$ (feet of wood)

$1c + 2s + 0.2l + s_2 = 25$ (pounds of foam rubber)

$10c + 20s + 10l + s_3 = 410$ (square yards of material)

$-300c - 200s - 220l + P = 0$

c	s	l	s_1	s_2	s_3	P	
5	35	9	1	0	0	0	405
1	2	0.2	0	1	0	0	25
10	20	10	0	0	1	0	410
-300	-200	-220	0	0	0	1	0

ratios

$\frac{405}{5} = 81$

$\frac{25}{1} = 25 \leftarrow$ smallest nonneg $\Rightarrow$ pivot row

$\frac{410}{10} = 41$

most neg $\Rightarrow$ pivot col

8 in R3C3, see back of Bernshtein p3

After pivoting on the 1 in R2, C1, we get the following tableau:

$$\begin{array}{c}
\begin{array}{ccccccc|c} c & s & \ell & s_1 & s_2 & s_3 & P & \end{array}\\
\left[\begin{array}{ccccccc|c}
0 & 25 & 8 & 1 & -5 & 0 & 0 & 280 \\
1 & 2 & \frac{1}{5} & 0 & 1 & 0 & 0 & 25 \\
0 & 0 & 8 & 0 & -10 & 1 & 0 & 160 \\
\hline
0 & 400 & -160 & 0 & 300 & 0 & 1 & 7500
\end{array}\right]
\end{array}$$

most neg $\Rightarrow$ pivot col

ratios:

$\frac{280}{8} = 35$

$\frac{25}{\frac{1}{5}} = 125$

$160/8 = 20 \leftarrow$ smallest nonneg $\Rightarrow$ pivot row

So, the next pivot element is the 8 in R3, C3. After pivoting, we have:

$$\begin{array}{c}
\begin{array}{ccccccc|c} c & s & \ell & s_1 & s_2 & s_3 & P & \end{array}\\
\left[\begin{array}{ccccccc|c}
0 & 25 & 0 & 1 & 5 & -1 & 0 & 120 \\
1 & 2 & 0 & 0 & \frac{5}{4} & -\frac{1}{40} & 0 & 21 \\
0 & 0 & \boxed{1} & 0 & -\frac{5}{4} & \frac{1}{8} & 0 & \boxed{20} \\
\hline
0 & 400 & 0 & 0 & 100 & 20 & 1 & 10700
\end{array}\right]
\end{array}$$

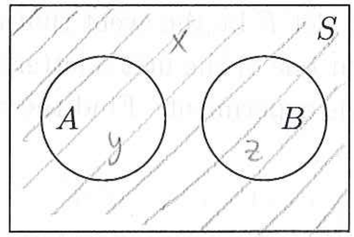
There are no more negatives in the final column, so we're done pivoting, and this is the final simplex tableau. To identify the optimal solution, we identify the basic variables (those heading the unit columns) and the nonbasic variables (those heading the nonunit columns). Use the final simplex tableau to read off the values of basic variables, and set all non-basic variables equal to 0. I've circled the unit columns in the final simplex tableau above.

Basic variables		Nonbasic variables	
$c = 21$	(# chairs)	$s = 0$	(# sofas)
$\ell = 20$	(# love seats)	$s_2 = 0$	(pounds of foam rubber)
$s_1 = 120$	(sq. feet of wood)	$s_3 = 0$	(sq. yards of material)
$P = 10,700$	(profit, in \$)		

1. The manufacturer should produce 21 chairs, 20 love seats, and no sofas in order to maximize their profit at \$10,700.
2. $s_1 = 120 \Rightarrow$ They will have 120 square feet of wood leftover.
   $s_2 = 0 \Rightarrow$ They will have used up all the pounds of foam rubber allotted.
   $s_3 = 0 \Rightarrow$ They will use up every square yards of material allotted.

Exercise 4. Shade each of the following in the given diagram.

1. $A^C \cup B^C$



Write $S = \{x, y, z\}$.

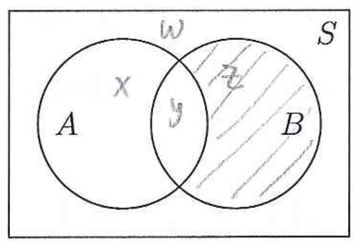
Then $A = \{y\}$ and $B = \{z\}$, so

$$A^c = \{x, z\}$$
$$\cup \quad B^c = \{x, y\}$$

$$\boxed{A^c \cup B^c = \{x, y, z\}}$$

remember, union combines, and we don't list elements more than once.

2. $(A \cup B^C)^C$



Write $S = \{w, x, y, z\}$.

Then $A = \{x, y\}$

$$\cup \quad B^c = \{x, w\}$$

$$A \cup B^c = \{w, x, y\}$$

$$\Rightarrow \boxed{(A \cup B^c)^c = \{z\}}$$

all things in S that are not in $(A \cup B^c)$.

Alternate solution:
By DeMorgan's Law,

$$(A \cup B^c)^c = A^c \cap (B^c)^c$$
$$= A^c \cap B$$
$$= \{z\}$$

since

$$A^c = \{w, z\}$$
$$\cap \quad B = \{y, z\}$$

$$A^c \cap B = \{z\}$$

Exercise 5. Consider the experiment of randomly choosing one of the 26 lower-case letters from the English alphabet. Let S denote the sample space, let $E = \{a, e, i, o, u\}$ be the event that a vowel is chosen, let F be the event that one of the remaining 21 letters is chosen, and let G be the event that one of the first 5 letters of the alphabet is chosen. How many simple events are there in this experiment? Find the events $E \cup F \cup G$, $E^C \cup F^C \cup G^C$, $E \cap F \cap G$, and $E \cup F^C \cup G$.

First, we explicitly list out the outcomes in the events S, E, F, and G:

sample space

$S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

$E = \{a, e, i, o, u\}$

$F = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$

$G = \{a, b, c, d, e\}$

Counting the outcomes in S, we see that this experiment has <u>26 simple events</u>.

Then

$E \cup F \cup G = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\} = S$

{vowels} or {consonants} or {first 5 letters}

$E^C \cup F^C \cup G^C = F \cup E \cup \{f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

{consonants} or {vowels} or {last 21 letters}

$= \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$
$= S$

$E \cap F \cap G = \emptyset$

{vowels} and {consonants} and {first five letters}

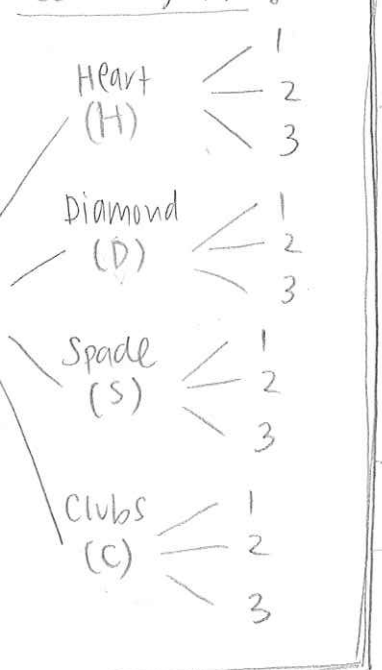
a letter cannot be a vowel and a consonant at the same time; in other words, E and F are <u>mutually exclusive</u>.

$E \cup F^C \cup G = E \cup E \cup G = E \cup G = \{a, e, i, o, u\} \cup \{a, b, c, d, e\}$
$= \underline{\{a, b, c, d, e, i, o, u\}}.$

Exercise 6. Consider the following experiment: First, a card is drawn from an standard 52-card deck and the suit is recorded. Next, a fair 3-sided die is rolled and the number showing uppermost is recorded.

1. Write the sample space for this experiment. (Use a tree diagram to help.)
2. State the total number of possible events of the sample space.
3. Write the outcomes in the event "A number greater than 3 is rolled."
4. Write the outcomes in the event "A red card is drawn."
5. Write the outcomes in the event "A 2 is rolled **and** a clubs is drawn."
6. Write the outcomes in the event "A 2 is rolled **or** a clubs card is drawn."
7. Which of the events from Parts 3 – 6 are simple events, if any?

Tree diagram:



1. $S = \{H1, H2, H3, D1, D2, D3, S1, S2, S3, C1, C2, C3\}$

2. There are $n = \underbrace{4}_{\text{suits}} \cdot \underbrace{3}_{\#\text{s on die}} = 12$ simple events in S, so in total there are $2^{12} = \boxed{4096 \text{ possible events}}$ for this sample space.

3. "A number greater than 3 is rolled." This is impossible, so the event is $\emptyset$.

4. "A red card is drawn." : $\{H1, H2, H3, D1, D2, D3\}$.

Let E be the event "A 2 is rolled." Then $E = \{H2, D2, S2, C2\}$.

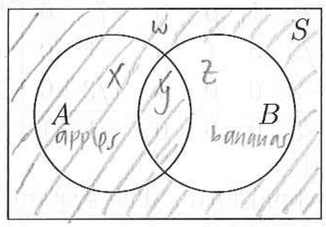
Let F be the event "A clubs is drawn." Then $F = \{C1, C2, C3\}$.

5. "A 2 is rolled and a clubs is drawn" $= E \cap F = \{C2\}$.

6. "A 2 is rolled or a clubs is drawn" $= E \cup F = \{H2, D2, S2, C2, C1, C3\}$.

7. The only simple event from parts 3-6 is the one from part 5 ("A 2 is rolled and a clubs is drawn") since it consists of a single outcome.

Exercise 7. A survey of supermarket shoppers is taken to analyze which fruits they buy regularly, in order to determine which ones should be put on sale. The supermarket only sells two types of fruits: apples and bananas. In the survey, some shoppers regularly purchase bananas, and some purchase apples. Some people purchase neither, and some people regularly purchase both apples and bananas. Let A denote the event that a shopper regularly purchases apples and let B denote the event that a shopper regularly purchases bananas. Shade the region that corresponds to the event that "A shopper regularly purchases ~~an apple~~ apples or no fruit at all" in the Venn diagram below.



A : "A shopper regularly purchases apples."

B : "A shopper regularly purchases bananas."

We can express "A shopper regularly purchases apples or no fruit at all." using A and B, namely by:

$$A \cup (A \cup B)^c.$$

Write $S = \{w, x, y, z\}$.

Then
$$\begin{array}{rl} & A = \{x, y\} \\ \cup & B = \{y, z\} \\ \hline & A \cup B = \{x, y, z\} \end{array}$$

$\Rightarrow (A \cup B)^c = \{w\}$.

Now
$$\begin{array}{rl} & A = \{x, y\} \\ \cup & (A \cup B)^c = \{w\} \\ \hline & A \cup (A \cup B)^c = \{w, x, y\} \end{array}$$

Multiple Choice Problems

Study tip: Write out all your work when you complete the multiple-choice problems.

C

Multiple Choice 1. Determine whether the given simplex tableau is in final form. If so, find the optimal solution to the associated linear programming problem. If not, find the pivot element to be used in the next iteration of the simplex method.

x	y	z	s_1	s_2	s_3	P	
1/4	2	1	0	0	7	0	16
1	1/3	0	1	0	−1	0	6
−2	−2	0	0	1	1/3	0	8
−2	−5	0	0	0	−1	1	80

$16/2 = 8$ ← pivot row

$6/(\frac{1}{3}) = 18$

X

pivot col

(a) Yes, the simplex tableau is in final form. The system has a maximum value of 80 at $(x, y, z) = (16, 6, 8)$.

(b) Yes, the simplex tableau is in final form. The system has a maximum value of 80 at $(x, y, z) = (0, 0, 16)$.

(c) No, the simplex tableau is not in final form. The next pivot element is the 2 in the first row, second column.

(d) No, the simplex tableau is not in final form. The next pivot element is the −2 in the third row, second column.

(e) No, the simplex tableau is not in final form. The pivot element is the 7 in the first row, sixth column.

D

Multiple Choice 2. A company makes small and large desks that require wood and finish to make. The number of units of wood and finish required for each small and large desk along with the amount of wood and finish available are given in the table below:

	small	large	available
wood	2	4	100
finish	3	5	300

If the profit from each small desk is \$4.50 and the profit from each large desk is \$6, how many desks of each size should they make to maximize their profit?

In solving the above problem, the initial simplex tableau is:

$x :=$ # small desks produced
$y :=$ # large desks produced
$P :=$ company's profit, in \$

x	y	s_1	s_2	P		
2	4	1	0	0	100	(units of wood)
3	5	0	1	0	300	(units of finish)
−4.5	−6	0	0	1	0	

and the final simplex tableau is:

x	y	s_1	s_2	P	
1	2	0.5	0	0	50
0	−1	−1.5	$\boxed{1}$	0	$\boxed{150}$
0	3	2.25	0	1	225

Basic	Non basic
$x = 50$	$y = 0$
$S_2 = 150$	$S_1 = 0$
$P = 225$	

Which of the following statements is true?

(a) At the optimal solution, there is 50 units of leftover wood and 150 units of leftover finish.

(b) At the optimal solution, there is 150 units of leftover wood and no leftover finish.

(c) At the optimal solution, there is no leftover wood and no leftover finish.

(d) At the optimal solution, there is no leftover wood and 150 units of leftover finish.

(e) None of these

$S_1 = 0$, S_1 represents excess units of wood
$\Rightarrow$ there are 0 units of leftover wood.
$S_2 = 150$, S_2 represents excess units of finish
$\Rightarrow$ there are 150 units of leftover finish

E

Multiple Choice 3. If $S = \{1, 2, 3, a, b, c\}$ is a uniform sample space with events $E = \{1, a, c\}$ and $F = \{3, a, b\}$, which of the following is FALSE? (Note: There is only one false statement.)

(a) $(E \cup F) = \{1, 3, a, b, c\}$ ✓

(b) There are exactly 2 outcomes in $(E \cap F^c)$. ✓

(c) $E \cap F$ is a simple event. ✓

(d) The two events F and $(E \cap F^C)$ are mutually exclusive. ✓

(e) $E \cup F^c = \{1, c\}$ ✗

check (a)

$E \cup F = \{1, a, c\} \cup \{3, a, b\} \Rightarrow \{1, 3, a, b, c\}$ ✓

check (b)

$E \cap F^c$: $E = \{1, a, c\}$

$\cap \; F^c = \{1, 2, c\}$

$\Rightarrow E \cap F^c = \{1, c\}$ has 2 outcomes ✓

check (c)

$E \cap F$: $E = \{1, a, c\}$

$\cap \; F = \{3, a, b\}$

$\Rightarrow E \cap F = \{a\}$, has exactly one outcome, so is a simple event. ✓

check (d) Is $F \cap (E \cap F^c) = \emptyset$?

$F = \{3, a, b\}$

$\cap \; (E \cap F^c) = \{1, c\}$

$F \cap (E \cap F^c) = \emptyset \Rightarrow$ F and $(E \cap F^c)$ are mutually exclusive. ✓

check (e):

$E = \{1, a, c\}$

$\cup \; F^c = \{1, 2, c\}$

$E \cup F^c = \{1, 2, a, c\}$ So, (e) is false.

C

Multiple Choice 4. Find the initial simplex tableau used to find the maximum of the objective function $P = 2x + 3y$ subject to the constraints

$$\begin{cases} 2x + 3y \le 90 \\ 3x + y \le 30 \\ 4x + 2y \le 40 \\ x \ge 0,\ y \ge 0 \end{cases}$$

$2x + 3y + s_1 = 90$
$3x + y + s_2 = 30$
$4x + 2y + s_3 = 40$
$P = 2x + 3y \rightarrow -2x - 3y + P = 0$

Have one slack variable for each of the three problem constraints.

(a)

x	y	s_1	s_2	P	
2	3	1	0	0	90
3	1	0	1	0	30
4	2	0	0	1	40
−2	−3	0	0	0	0

(b)

x	y	s_1	s_2	P	
2	3	1	0	0	90
3	1	0	1	0	30
4	2	0	0	0	40
−2	−3	0	0	1	0

(c)

x	y	s_1	s_2	s_3	P	
2	3	1	0	0	0	90
3	1	0	1	0	0	30
4	2	0	0	1	0	40
−2	−3	0	0	0	1	0

(d)

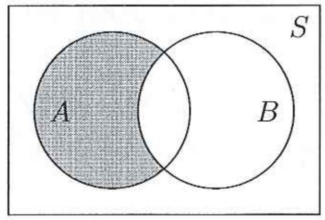
x	y	s_1	s_2	s_3	P	
2	3	1	0	0	0	90
3	1	0	1	0	0	30
4	2	0	0	1	0	40
2	3	0	0	0	1	0

To get the modified constraints move x and y to the left hand side of the eqn.

(e) None of these

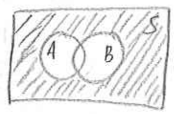
A

Multiple Choice 5. Which of the following represents the shaded region?

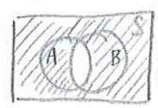


(a) $A \cap B^{C}$

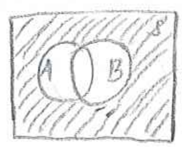
(b) $(A \cup B)^{C}$



(c) $(A \cap B)^{C}$



(d) $A^{C} \cap B^{C}$



(e) None of these