Lab 4: Directional Couplers - Additional Information

Objective:

- Understand Coupled Mode Theory and its application in the coupling between waveguides.
- Learn the derivation of the coupling coefficient and understand which design parameters influence coupling.
- Understand the concept of supermodes and learn how to use FDE and supermode propagation for the primary design of directional couplers.

Part 1: Coupled Mode Theory and Inter-Waveguide Coupling

Background:

In class, we briefly went through the Coupled Mode Theory (CMT) and the concept of the coupling coefficient, which determines the periodicity at which light oscillates between two waveguides. In this discussion section, we will go over a detailed derivation of the coupling coefficient using basic CMT and explore the parameters influencing the coupling, aiding the design of directional couplers.

Coupled Mode Theory (CMT) is a perturbative method used to analyze the interaction between modes in various vibrational systems, including mechanical, optical, and electrical systems. The mathematical foundations of CMT were developed for optical waveguides, providing a powerful tool for modeling the mutual coupling between optical modes. This coupling is essential in the design of integrated optical devices. CMT offers valuable insights into how optical energy couples between modes and within waveguides.

<u>Wave Equation</u>

Start from the wave equation for waveguide modes:

$$\nabla^2 E = \epsilon\mu \frac{\partial^2 E}{\partial t^2} \tag{1}$$

The solution to the Helmholtz equation represents the eigenmodes of the waveguide:

$$E(x,t) = \frac{1}{2}\sum_m A_m^+ \, \mathcal{E}_m(x)\, e^{i(\beta_m z - \omega t)} + c.c. \tag{2}$$

Note that:

· $E(x,t)$ is the electric field distribution as a function of both position and time.

· m (and later n) denotes the eigenmodes

· A_m^+ and later A_m^- are the amplitudes of forward and backward traveling waves for each mode.

· $\mathcal{E}_m$ is the time-invariant spatial amplitude distribution for each mode.

· β_m is the propagation coefficient in the z-direction for each eigenmode, and $c.c.$ represents the complex conjugate.

· Assume the mode propagates along the z-direction, with a 1D electric field distribution along the x-direction.

Adding Perturbation

When a perturbation in terms of distributed polarization source is introduced:

$$D = \epsilon E \;\rightarrow\; D = \epsilon E + P_{pert} \tag{3}$$

The modified wave equation becomes:

$$\nabla^2 E = \epsilon\mu \frac{\partial^2 E}{\partial t^2} + \mu \frac{\partial^2 P_{pert}}{\partial t^2} \tag{4}$$

The solution to this modified wave equation can be expressed as a sum of all eigenmodes (guided modes) and radiation modes:

$$\begin{aligned} E(x,t) = {} & \frac{1}{2}\sum_m A_m^+(z)\,\mathcal{E}_m(x)\,e^{i(\beta_m z-\omega t)} + \frac{1}{2}\sum_m A_m^-(z)\,\mathcal{E}_m(x)\,e^{-i(\beta_m z-\omega t)} \\ & + \frac{1}{2}\int_{k_0 n_{start}}^{k_0 n_{final}} A(\beta,z)\mathcal{E}_\beta(x)\,e^{i(\beta_m z-\omega t)} d\beta + c.c. \end{aligned} \tag{5}$$

radiation modes

Here, note that the amplitudes of the forward and backward traveling waves are no longer constants, but vary with propagation distance due to the perturbation.

Substituting the solution back into the wave equation (Eqn. 4):

$$\begin{aligned} & \frac{1}{2}\sum_m \left[\frac{\partial^2 A_m^+}{\partial z^2} + 2i\beta_m \frac{\partial A_m^+}{\partial z}\right] \mathcal{E}_m(x)\,e^{i(\beta_m z-\omega t)} + \frac{1}{2}\sum_m \left[\frac{\partial^2 A_m^-}{\partial z^2} - 2i\beta_m \frac{\partial A_m^-}{\partial z}\right] \mathcal{E}_m(x)\,e^{-i(\beta_m z-\omega t)} + c.c. \\ & \quad = \mu \frac{\partial^2}{\partial t^2} P_{pert} \end{aligned} \tag{6}$$

Using Slow Amplitude Variation Approximation

Assuming the perturbation only causes "slow amplitude variations," we can neglect the second derivative of the amplitude:

$$\frac{\partial^2 A_m}{\partial z^2} \ll \beta_m \frac{\partial A_m}{\partial z} \tag{7}$$

Thus, we can ignore the second partial term, and Eqn. 6 simplifies to:

$$\frac{1}{2}\sum_m \left[2i\beta_m \frac{\partial A_m^+}{\partial z}\right] \mathcal{E}_m(x)\,e^{i(\beta_m z-\omega t)} + \frac{1}{2}\sum_m \left[-2i\beta_m \frac{\partial A_m^-}{\partial z}\right] \mathcal{E}_m(x)\,e^{-i(\beta_m z-\omega t)} + c.c. = \mu \frac{\partial^2}{\partial t^2} P_{pert} \tag{8}$$

Applying Mode Orthogonality

To simplify the expression further, we need to apply mode orthogonality, assuming no coupling between the x- and z-fields:

$$\int_{-\infty}^{\infty} \mathcal{E}_m(x) \cdot \mathcal{E}_n(x)\, dA = \frac{2\omega\mu}{\beta_m}\delta_{mn} \tag{9}$$

Multiplying both sides of the perturbation equations (Eqn. 8) by $\mathcal{E}_n(x)$ and integrating over x:

$$\frac{1}{2}\sum_m 2i\beta_m \frac{\partial A_m^+}{\partial z}\, e^{i(\beta_m z - \omega t)} \int_{-\infty}^{\infty} \mathcal{E}_m(x) \cdot \mathcal{E}_n(x)\, dx + \frac{1}{2}\sum_m -2i\beta_m \frac{\partial A_m^-}{\partial z}\, e^{-i(\beta_m z - \omega t)} \int_{-\infty}^{\infty} \mathcal{E}_m(x) \cdot \mathcal{E}_n(x)\, dx + c.c.$$
$$= \mu \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} P_{pert}\, \mathcal{E}_n(x) \tag{10}$$

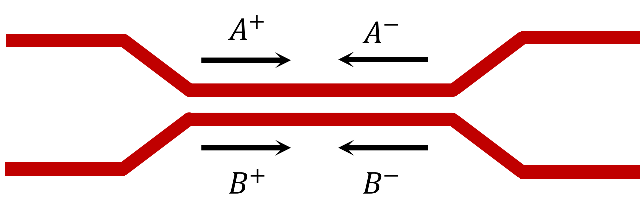
Leverage the orthogonality property:

$$\boxed{\frac{\partial A_n^+}{\partial z}\, e^{i(\beta_n z - \omega t)} - \frac{\partial A_n^-}{\partial z}\, e^{-i(\beta_n z - \omega t)} = \frac{-i}{2\omega}\frac{\partial^2}{\partial t^2}\int P_{pert} \cdot \mathcal{E}_n(x)\, dx} \tag{11}$$

This gives us the primary equation for determining mode coupling.

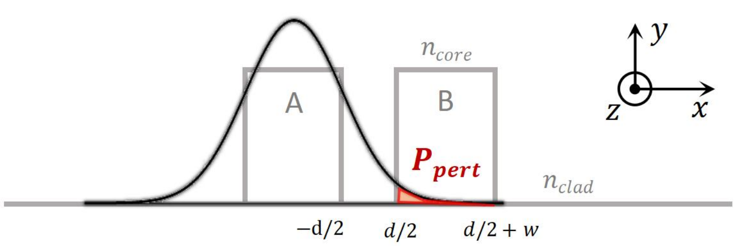
Case for Directional Couplers

Now let's consider a 2x2 optical directional coupler.



Assume that light is initially injected from waveguide A from the left. We can further assume that there is negligible backward light during propagation, so we set both the backward propagating field amplitudes A^- and B^- to 0.

Next, assuming that light propagates along the z-direction, and the electric field distribution is along the x-direction, we can examine the cross-section as shown below and identify the perturbation in waveguide B caused by light propagating in waveguide A.



The perturbation can be expressed as:

$$P_{pert} = \epsilon_0\,(n_{core}^2 - n_{clad}^2)\, E_A(x,t) \tag{12}$$

Deriving Coupling Coefficients

Using the simplified mode coupling equation above, note that the perturbation is only inside waveguide B. Therefore, the integral from $-\infty$ to ∞ reduces to the integral over the region of waveguide B.

$$\begin{aligned}\frac{\partial B^+}{\partial z} e^{i(\beta_B z-\omega t)} + c.c. &= \frac{-i}{2\omega}\frac{\partial^2}{\partial t^2}\int_{d/2}^{d/2+w} P_{pert}\,\mathcal{E}_B(x)\,dx \\ &= i\frac{\omega}{2}\int_{d/2}^{d/2+w} \epsilon_0\,(n_{core}^2 - n_{clad}^2)\,\mathcal{E}_B(x)\left[\frac{A}{2}\mathcal{E}_A(x)\,e^{i(\beta_A z-\omega t)}\right] dx \\ &= i\,\kappa_{BA}\,A\,e^{i(\beta_A z-\omega t)}\end{aligned} \tag{13}$$

in which the coupling coeeficient κ is given by:

$$\kappa_{BA} = \frac{\omega}{4}\epsilon_0\,(n_{core}^2 - n_{clad}^2)\int_{d/2}^{d/2+w} \mathcal{E}_A(x)\,\mathcal{E}_B(x)\,dx \tag{14}$$

Note that this expression is derived for a 1D slab waveguide. If we consider the more general 2D case (with z still being the propagation direction and the modes distributed in the xy-plane), we can modify the expression accordingly:

$$\boxed{\kappa_{BA} = \frac{\omega}{4}\epsilon_0\,(n_{core}^2 - n_{clad}^2)\iint \mathcal{E}_A(x,y)\,\mathcal{E}_B(x,y)\,dxdy} \tag{15}$$

Visualizing Power Transfer Between Waveguides

For two identical waveguides, where their propagation coefficients are equal ($\beta_A = \beta_B$), we can cancel the exponential terms. Thus, the coupling equation (Eqn. 13) simplifies further as:

$$\frac{\partial B}{\partial z} = i\kappa_{BA}A \tag{16}$$

By symmetry, we can write the equation for light in waveguide B as:

$$\frac{\partial A}{\partial z} = i\kappa_{AB}B \tag{17}$$

Combining the two waveguides, we arrive at the second derivative equations for mode coupling. Note that for two identical waveguides, the coupling coefficients are equal ($\kappa_{AB} = \kappa_{BA} = \kappa$).

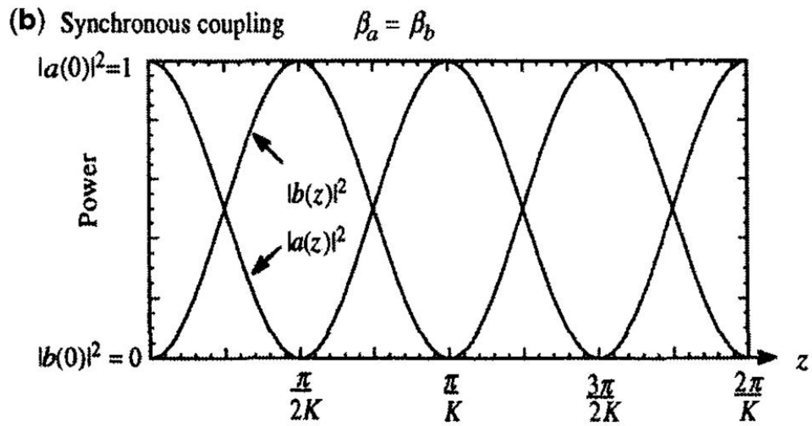
$$\frac{\partial^2 A}{\partial z^2} = -\kappa^2 A, \quad \frac{\partial^2 B}{\partial z^2} = -\kappa^2 B \tag{18}$$

The eigen solutions to the differential equations are sinusoidal functions. Assuming light is injected from waveguide A ($A(0) = 1$, $B(0) = 0$), we can get the following solutions as we saw in the class:

$$\boxed{\begin{cases} A(z) = \cos(\kappa z) \\ B(z) = i\sin(\kappa z)\end{cases}} \tag{19}$$

To calculate the power transmission for each port, we square the amplitude of the respective fields. This gives us the power distribution as shown in Figure 8.9(b) of the textbook.

Textbook, Page 307, Figure 8.9



Further Analysis

From the final expression for the coupling coefficient κ, we observe that it mostly depends on the **overlap** of the evanescent tails of the waveguide modes. A larger overlap increases the coupling coefficient.

Therefore, the simplest way to increase coupling is to reduce the **gap** between the waveguides. We can also see an approximately inverse exponential relationship between the coupling coefficient and the gap.

The coupling coefficient also has **wavelength dependence** due to the frequency term ω, which makes a directional coupler inherently difficult to make broadband. This was observed in the previous part of EME simulation, where **shorter coupler lengths** resulted in a **more broadband** coupling. This is because for the same range of wavelengths or frequencies, a larger coupling length will cause a larger phase mismatch.

Follow-up Questions

1. What will happen when the two waveguides are not identical. Please provide the results for the revised formula.

Hint:

Starting From Egn. 13, since now β_A and β_B are not equal, the exponential terms can no longer cancel each other, and we need to account for this difference in the coupling equation:

$$\frac{\partial B}{\partial z} = i\kappa_{BA} A\, e^{i(\beta_A - \beta_B)\, z} \tag{20}$$

This modified equation is equivalent to Equation 8.2.8 in the textbook, as shown below. In the textbook's notation, $a = Ae^{i\beta_A z}$ and $b = Be^{i\beta_B z}$

$$\frac{\partial b}{\partial z} = i\kappa_{BA} a + i\beta_B b \tag{21}$$

Assuming a propagation distance of L, integrate over z on both sides from $-\mathrm{L}$ to L. There are two tricks applied in the integral. First, we assume that $\mathrm{B}(-\mathrm{L}) = 0$ since parallel part start to exist and light start to couple at $L = 0$. Second, we also assume that only a small portion of A couples to B, allowing us to treat A as a constant during the integration and focus on the exponential term.

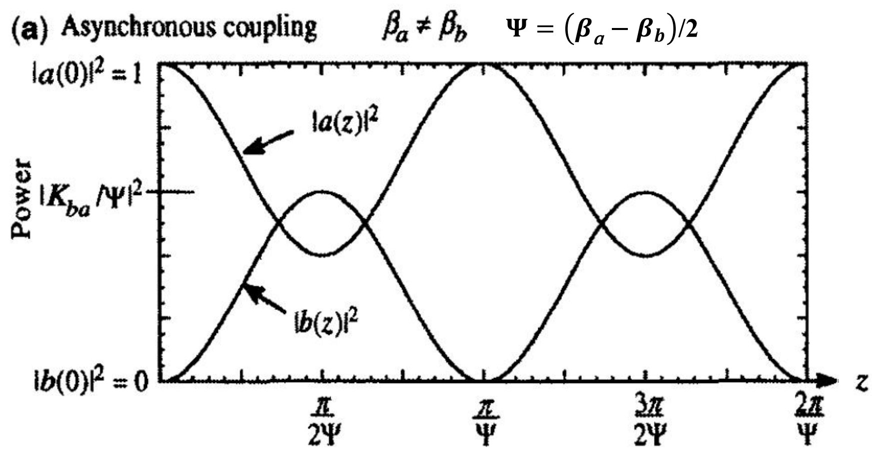
$$\int_{-L}^{L} \frac{\partial B}{\partial z}\, dz \;=\; i\kappa_{BA} A \int_{-L}^{L} e^{i(\beta_A-\beta_B)\,z} dz \;\rightarrow\; B(L) - B(-L) \;=\; \frac{i\kappa_{BA} A}{i(\beta_A-\beta_B)}\left[\, e^{i(\beta_A-\beta_B)\,z} - \, e^{-i(\beta_A-\beta_B)\,z}\right]$$

$$\rightarrow \boxed{B(L) \;=\; 2i\kappa_{BA} A \frac{\sin\left[(\beta_A-\beta_B)L\right]}{\beta_A-\beta_B}} \tag{20}$$

In terms of power, the power transmission equation can be written as below. Figure 8.9 (a) of the textbook can be generated from this equation.

$$P_b(L) \;=\; \frac{|B(L)|^2}{|A(0)|^2} \;=\; \frac{4|\kappa_{BA}|^2}{(\beta_A-\beta_B)^2}\,\sin^2\left[(\beta_A-\beta_B)L\right] \tag{21}$$

Textbook,
Page 307,
Figure 8.9



2. Try an example using two parallel 1D slab waveguides.

First, in *Matlab*, calculate the coupling coefficient and plot the power at the through and drop ports as a function of the coupling length.

Then, perform a 2D simulation of the same setup using *Lumerical FDTD*.

Compare the simulation results with the analytical solution—do they agree? Consider the approximations made in the analytical solution and discuss what factors might have caused any discrepancies.

Part 2: Primary Design of a Directional Coupler using Supermodes Propagation

Background:

In class, we learned about the concept of **supermodes** in a structure with two parallel waveguides. When light starts to crossover, as a result of the evanescent tail interaction, a condition known as "supermodes" occurs, where light is present in both waveguides.

According to Coupled Mode Theory, a directional coupler composed of two parallel waveguides supports two supermodes: an **even** (symmetric) supermode, where the fields in both waveguides are **in phase**, and an **odd** (antisymmetric) supermode, where the fields are **180° out of phase**.

Textbook Page 304

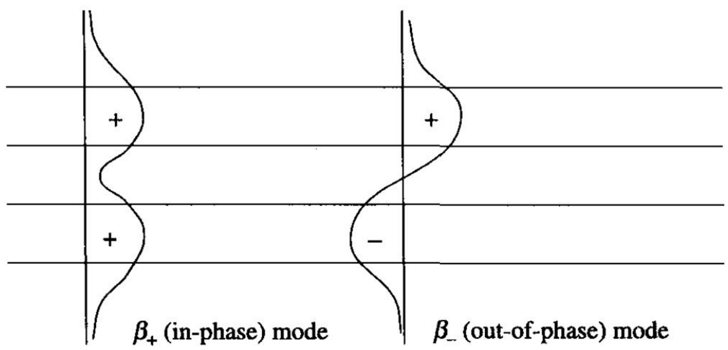


Figure 8.8 A simple sketch of the two system modes: the β_+ (in-phase) mode and the β_- (out-of-phase) mode.

The two supermodes have different propagation constants and effective indices. The combination of their interactions with each other determines how light propagates through the paralleled waveguides system.

Next, we will explore an example using **Lumerical FDE** to calculate the supermodes of two parallel waveguides. This will help us understand the primary design principles for a directional coupler.

To provide some background, the **coupling coefficient κ_{BA}** can be expressed as a function of the **difference between the effective indices of the odd and even supermodes (Δn)** and the free space wavelength (λ_0):

$$\kappa_{BA} = \frac{\pi \, \Delta n}{\lambda_0} \qquad (22)$$

From this, we can calculate the **crossover length L_c**, which is the distance over which the light intensity oscillates between the two waveguides. Once the crossover length is reached, maximum power is transferred to the second waveguide (100% transfer for identical waveguides). The crossover length is given by:

$$L_c = \frac{\lambda_0}{2 \, \Delta n} \qquad (23)$$

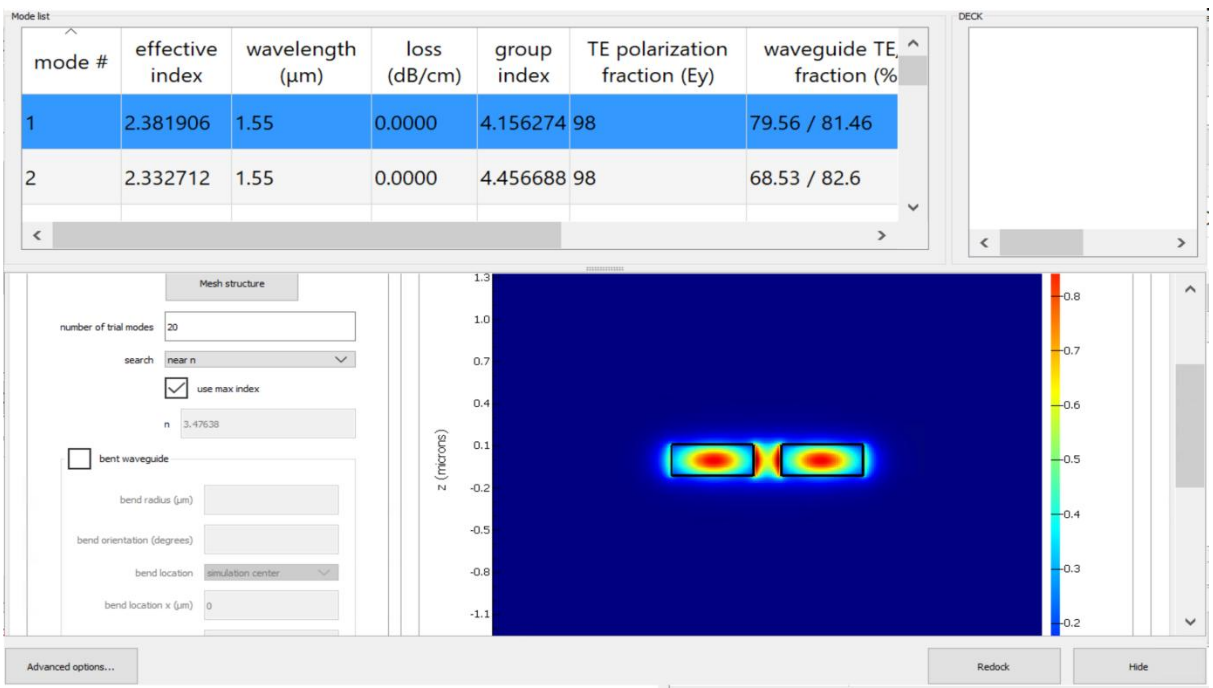
Procedure:

1. <u>Supermodes Calculation</u>

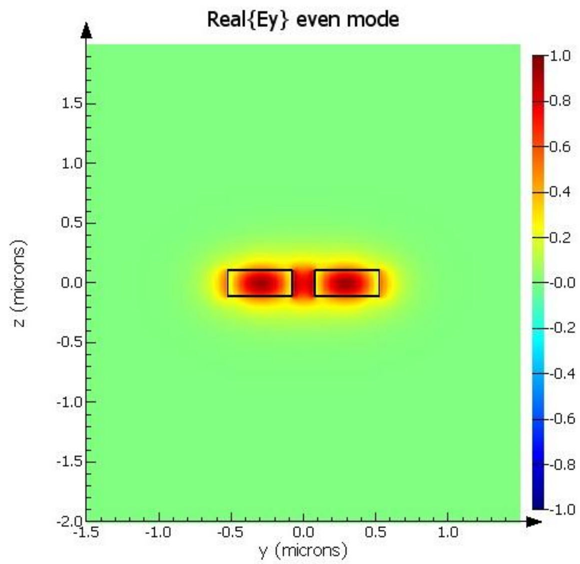
<u>FDE Setup</u>

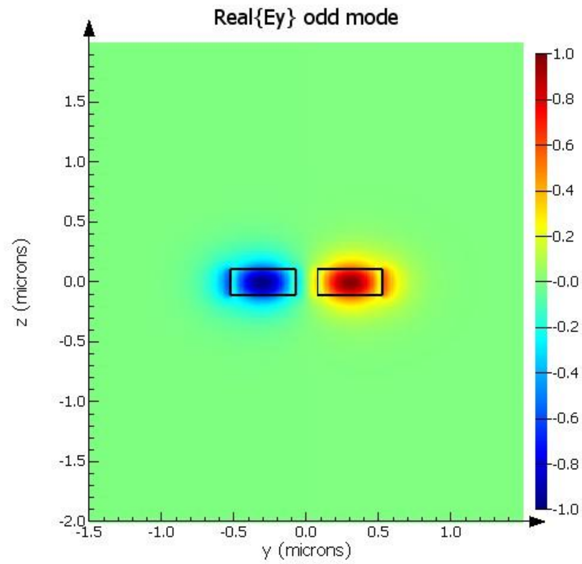
In Lumerical FDE, build the parallel waveguide structure similar to the parallel section of the directional coupler in the previous simulation, using two identical waveguides (named “left” and “right” from the output side) with dimensions 450 nm x 220 nm, and a gap of 150 nm. The corresponding file can be found in ***directional_coupler_fde.lms***.

Add an *FDE* solver to calculate the modes for the cross-sectional structure. Based on the structure setup, select *2D X normal* as the *solver type*. After the modes are calculated, you will observe the two supermodes displayed at the top of the results.



By visualizing the ***real part*** of ***Ey*** as shown below (changing the *plot* from *amplitude* to the *real part* and *component* from *E intensity* to *Ey*), we can distinguish the even and odd modes. The mode with an index of 2.3819 is the odd mode, while the mode with an index of 2.3327 is to the even mode.





Note that for the two Ey plots, I adjusted the color bar limits to be the same ([-1, 1]) so that both plots have the same color bar range. The first plot of even mode should show all positive values.

Analytical Formula

The index difference of the even and odd supermodes is:

$$\Delta n = n_1 - n_2 = 2.381906 - 2.332712 = 0.049194 \tag{24}$$

If we start with 100% of the power in waveguide a, the power in waveguide b is given by:

$$P_b(L) = P_0 \sin^2(\kappa_{BA} L) = P_0 \sin^2\left(\frac{\pi \, \Delta n \, L}{\lambda_0}\right) \tag{25}$$

where L is the coupling region length, P_0 is the input power, and λ_0 is the free space wavelength. In this way, we can estimate the cross-over length as:

$$L_c = \frac{\lambda_0}{2 \, \Delta n} = \frac{1.55 \text{ µm}}{2 \cdot 0.049194} = 15.754 \text{ µm} \tag{26}$$

2. Propagating Field Reconstruction

Script File

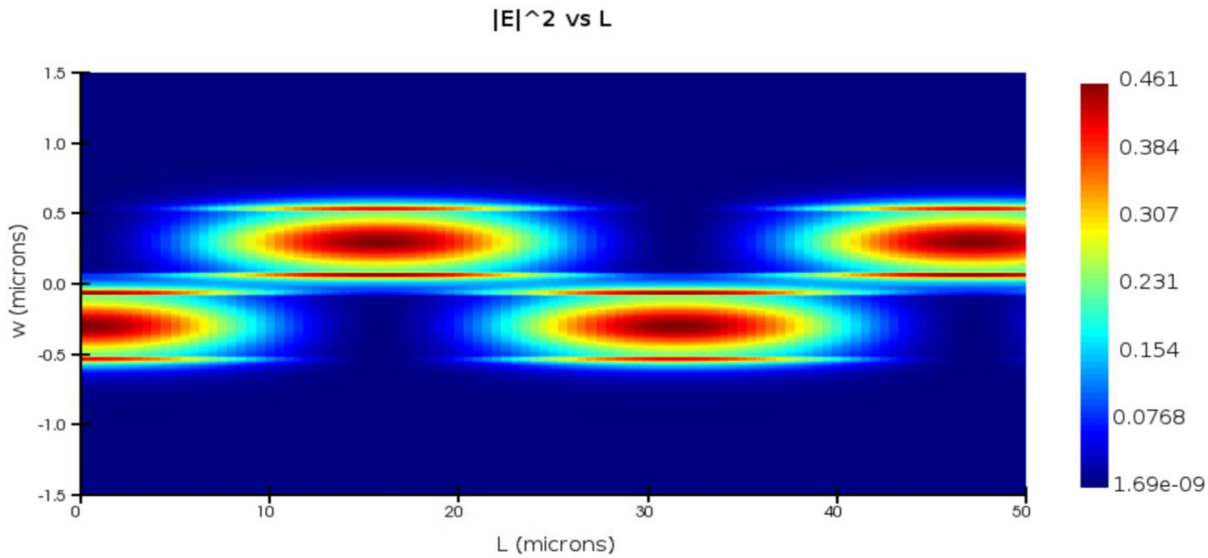
Enter and run the following code in the *Script File Editor*. Alternatively, you can load the code from the file ***directional_coupler_fde.lsf***, which contains the script.

```
# first, disable the right waveguide
switchtolayout;
setactivesolver("MODE");
select("right");
set("enabled",0);

# find modes of the single waveguide
findmodes;
cleardcard("E0");
copydcard("mode1","E0");

# enable the second waveguide back
switchtolayout;
select("right");
set("enabled",1);

# find the modes of the couple device
findmodes;

# choose vector of L
L = linspace(0,50e-6,101);

y = getdata("mode1","y");
z = getdata("mode1","z");
nmin = 2;
nmax = 4;

E2 = matrix(length(y),length(L));
# loop over all lengths and use propagate command
for(i=1:length(L)) {
    outmode = propagate("E0",L(i),nmin,nmax);
    E2_temp = getelectric(outmode);
    cleardcard(outmode);
    E2(1:length(y),i) = E2_temp(1,1:length(y),find(z,0.1e-6));
}

# generate picture
image(L*1e6,y*1e6,transpose(E2),"L (microns)","w (microns)","|E|^2 vs L");
```

The code provided here reconstructs the field image as it propagates over long distances. Initially, the right waveguide is disabled, and the field from the input left waveguide is calculated and stored in a global variable, ***E0***.

Next, the right waveguide is reintroduced by the script, and the supermodes of the coupled waveguide system are computed. The mode E0 is then projected onto the two supermodes of the coupled system. It is subsequently propagated over an arbitrary distance using the ***propagate command***. This process is repeated for 51 different lengths, ranging from 0 to 50 microns.

Finally, a figure is generated showing the electric field intensity as a function of position along the width direction *w* and propagation length *L*, measured at the cross-section located at half of the waveguide height.



From the figure, it can be seen that the crossover length is ~16 µm, which agrees with the previous calculation.

Interpretation

When we disable the right waveguide, we obtain the mode for only the left waveguide, which represents the situation where light is propagating solely in the left waveguide.

Once the second waveguide is introduced, we now have two supermodes, as calculated earlier. If we consider the sum of them, we observe that the original left waveguide mode matches well with the sum of the two supermodes that exist when both waveguides are present. This indicates that the original waveguide mode excites both of these modes equally, but they propagate with different phase velocities.

The electric field at a distance L from the point where the second waveguide is introduced is given by:

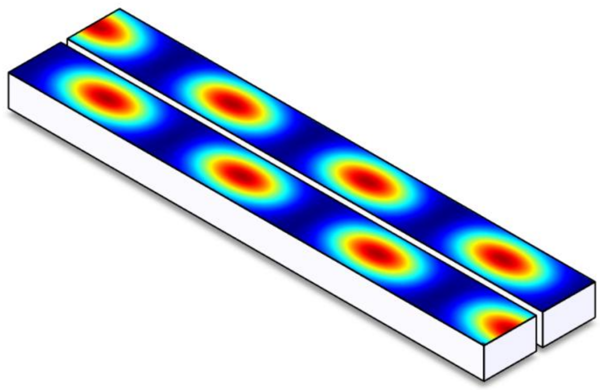
$$\begin{aligned} \vec{E}(L) &= \overrightarrow{E_1}\, e^{i\left(\frac{2\pi n_1}{\lambda_0} L\right)} + \overrightarrow{E_2}\, e^{i\left(\frac{2\pi n_2}{\lambda_0} L\right)} \approx \left(\overrightarrow{E_a} + \overrightarrow{E_b}\right) e^{i\left(\frac{2\pi n_1}{\lambda_0} L\right)} + \left(\overrightarrow{E_a} - \overrightarrow{E_b}\right) e^{i\left(\frac{2\pi n_2}{\lambda_0} L\right)} \\ &= 2\,\overrightarrow{E_a}\, e^{i\left(\frac{2\pi n_0}{\lambda_0} L\right)} \cos\left(\frac{\pi\,\Delta n}{\lambda_0} L\right) + 2i\,\overrightarrow{E_b}\, e^{i\left(\frac{2\pi n_0}{\lambda_0} L\right)} \sin\left(\frac{\pi\,\Delta n}{\lambda_0} L\right) \end{aligned} \qquad (27)$$

where $n_0 = (n_1 + n_2)/2$, $\overrightarrow{E_1}$ and $\overrightarrow{E_2}$ are the supermodes of the coupled waveguide system, and $\overrightarrow{E_a}$ and $\overrightarrow{E_b}$ are the unperturbed modes of the single waveguide positioned on the left and right, respectively. The sine and cosine terms indicate that the light oscillates periodically between the two waveguides during propagation.

Conclusion

From the previous simulation, we can see that this method provides a good estimate of the crossover length. Both the initial calculation and the results obtained using the propagate command closely match the actual crossover length from our earlier EME simulation.

Therefore, this simple code can serve as a primary design tool for directional couplers. By adjusting the crossover length (while keeping the gap and wavelength fixed), we can control the percentage of power that couples from one waveguide to another. After this, we can add the bending section and proceed with further design using EME or FDTD simulations.



Follow-up Questions

1. Try using this method with asymmetric waveguides. What does the propagation pattern look like? How does it compare with the *EME* and *FDTD* simulations?

2. What are the approximations in this method? Why is it less accurate than the *EME* or *FDTD* simulations (ignoring the bending section).

References

[1] Chuang, Shun L. *Physics of Photonic Devices*. 2nd ed., John Wiley & Sons, 2009.

[2] Pollock, Clifford R., and Michal Lipson. *Integrated Photonics*. Springer, 2003. https://doi.org/10.1007/978-1-4757-5522-0.

[3] "Evanescent Waveguide Couplers." *Ansys Optics*, Ansys, https://optics.ansys.com/hc/en-us/articles/360042304694-Evanescent-waveguide-couplers. Accessed 15 May 2025.

[4] Khan, Umar, et al. "Extracting Coupling Coefficients of Directional Couplers." *Optics Express*, vol. 26, no. 26, 2018, pp. 33873–33883. https://doi.org/10.1364/OE.26.033873.

[5] "Directional Couplers." *CamachoLab Photonics Bootcamp*, CamachoLab, 2023, https://byucamacholab.github.io/Photonics-Bootcamp/pages/directional_couplers.html. Accessed 15 May 2025.