Additional Project 2.1 - The Restricted Three-Body Problem

May 3, 2022

Contents

1 Question 1

$J = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Omega(x, y)$

$$\begin{aligned}\frac{dJ}{dt} &= \dot{x}\ddot{x} + \dot{y}\ddot{y} + \frac{d\Omega}{dt} \\ &= \dot{x}\ddot{x} + \dot{y}\ddot{y} + \frac{\partial\Omega}{\partial x}\dot{x} + \frac{\partial\Omega}{\partial y}\dot{y} \\ &= \dot{x}\ddot{x} + \dot{y}\ddot{y} + (2\dot{y} - \ddot{x})\dot{x} - (2\dot{x} + \ddot{y})\dot{y} \\ &= 0\end{aligned}$$

Where we have used (1a) and (1b) in the third equality. This means that J is constant in time, and so J can be determined by initial values: $J = \frac{1}{2}(u_0^2 + v_0^2 + \Omega(x_0, y_0)$.

The first term in J is always positive, so that means that $\Omega(x, y) \leq J, \forall t$, in other words, trajectories are confined to the region

$$\Omega(x, y) \leq \Omega(x_0, y_0) + \frac{1}{2}(u_0^2 + v_0^2)$$

2 Question 2

In polar coordinates, we say $\mathbf{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

So then we can describe derivatives of position like so:

$$\begin{aligned} \mathbf{x} &= r\mathbf{e}_r \\ \dot{\mathbf{x}} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \\ \ddot{\mathbf{x}} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \end{aligned}$$

The equations (1a)-(1b) can be described like so:

$$\ddot{\mathbf{x}} + 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\dot{\mathbf{x}} = -\nabla\Omega$$

Now we write $\begin{pmatrix} a \\ b \end{pmatrix} := a\mathbf{e}_r + b\mathbf{e}_\theta$

$$\begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{pmatrix} + 2\begin{pmatrix} -r\dot{\theta} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} -\frac{0.5}{r^2} \\ 0 \end{pmatrix}$$

After multiplying the $\mathbf{e}_\theta$ component by r, we have

$$\frac{d}{dt}(r^2\dot{\theta} + r^2) = 0 \implies k = r^2\dot{\theta} + r^2$$

as required

Also, we can see now that the $\mathbf{e}_r$ component gives us

$$\ddot{r} = \frac{k^2}{r^3} - r - \frac{0.5}{r^2} = -V'(r)$$

where $V(r) = -\frac{0.5}{r} + \frac{1}{2}r^2 + \frac{1}{2}\frac{k^2}{r^2}$

So (1a)-(1b) certainly imply (7). Conversely, we recover (1a)-(1b) by rewriting the above equations in Cartesian coordinates.

If the initial conditions are such that $\dot{r}(0) = 0$ and $\dot{\theta}(0) = -1 + ka^{-2}$ where $k^2 = a(0.5 + a^3)$ then both equations are satisfied, and so an orbit where $\dot{r} = 0$ for all time is consistent with equations of motion.

Therefore the only possible trajectories are where the starting position is a distance a away from P_h with a constant speed $a(-1 + ka^{-2})$, and velocity directed $\frac{\pi}{2}$ anti-clockwise from the position $\mathbf{x} - \begin{pmatrix} \mu \\ 0 \end{pmatrix}$, with k either positive or negative.

The modified program plot at $a = 0.3$ can be seen in Figure (1). We can check this with the expected analytic solution - After integrating up till $t = 30$ for both k positive and k negative, starting at $(0.5+0.3, 0)$, the maximum deviation over all time in x is 7.87×10^{-4}, and the maximum deviation from y is 7.72×10^{-4}, and the maximum deviation of the radius from 0.3 is 8.31×10^{-6}. All of this evidence points to an accurate numerical solution.

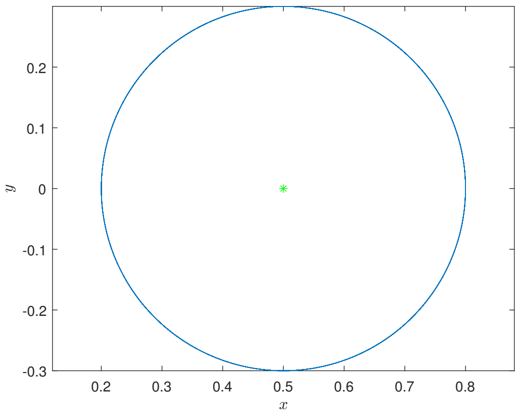
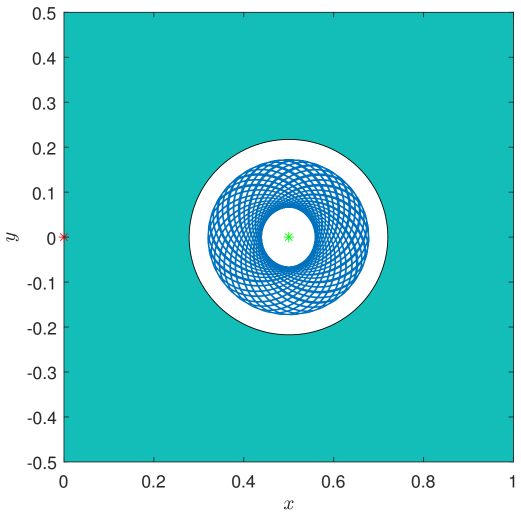


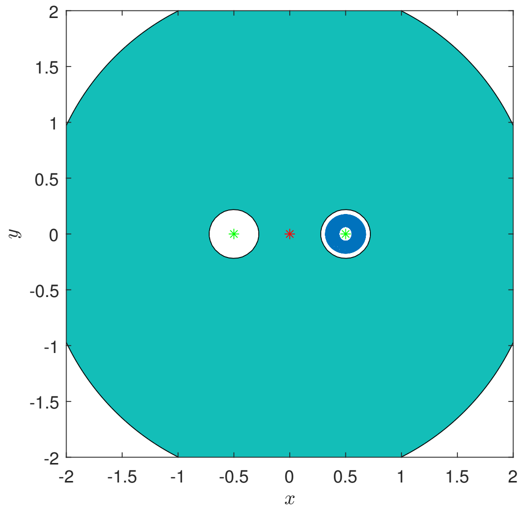
Figure 1: The trajectory as calculated by the modified program for $a = 0.3$, and k positive, with magnitude determined by $\sqrt{a(0.5 + a^3)}$, starting at $(0.5 + 0.3, 0)$. Different starting positions and negative k look identical.

3 Question 3

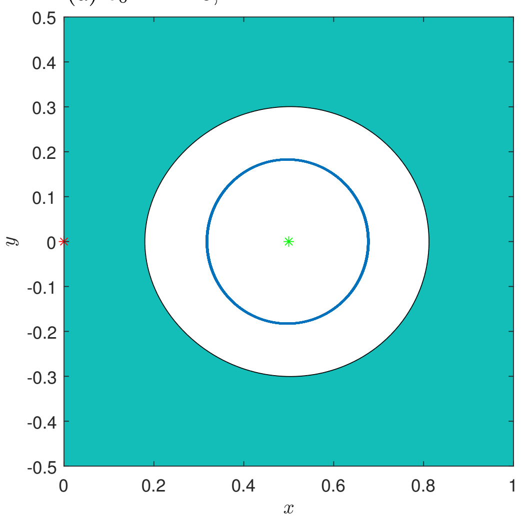
3.1 Trajectory and forbidden region plots



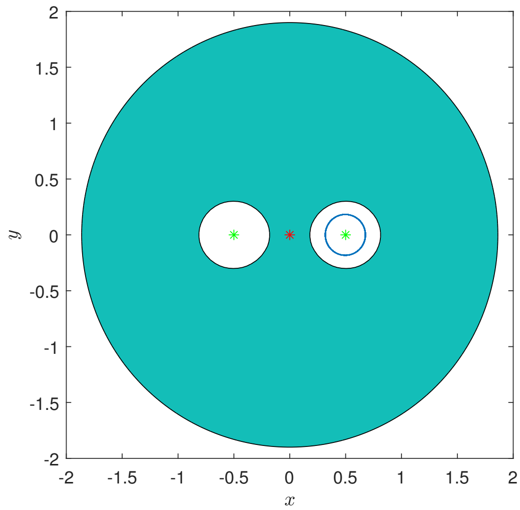
(a) $v_0 = -1.0$, zoomed in for details



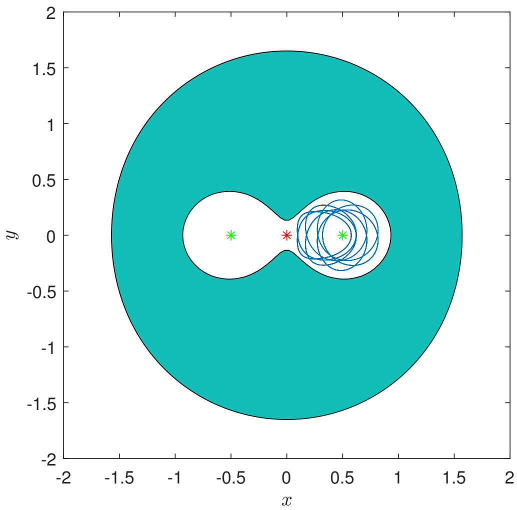
(b) $v_0 = -1.0$, zoomed out



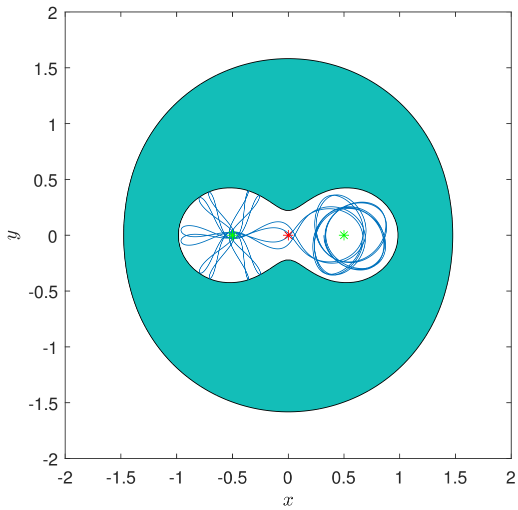
(c) $v_0 = -1.5$, zoomed in for details



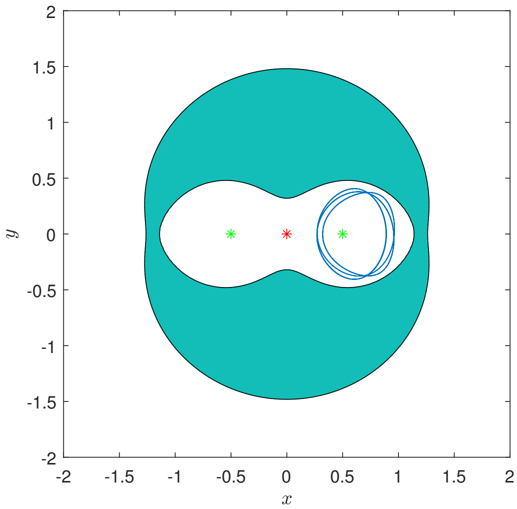
(d) $v_0 = -1.5$, zoomed out



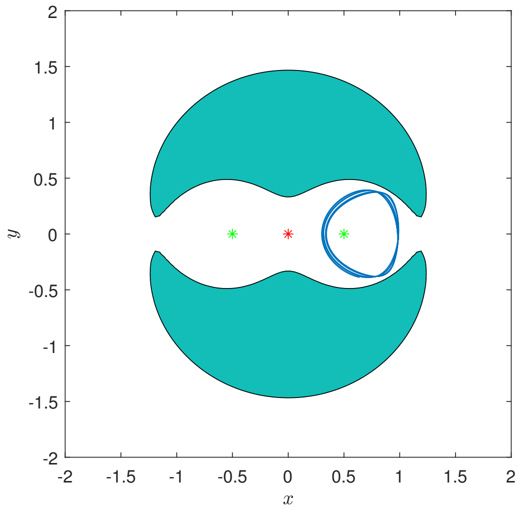
(a) $v_0 = -1.73$



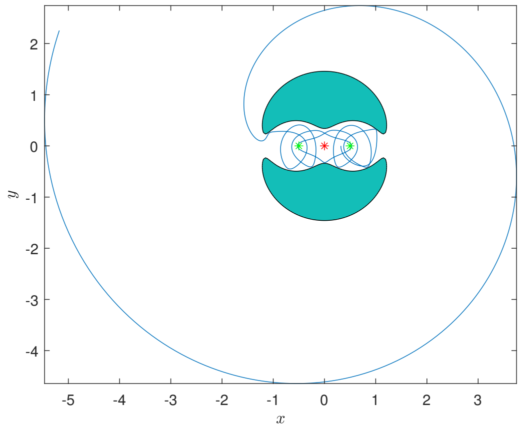
(b) $v_0 = -1.78$



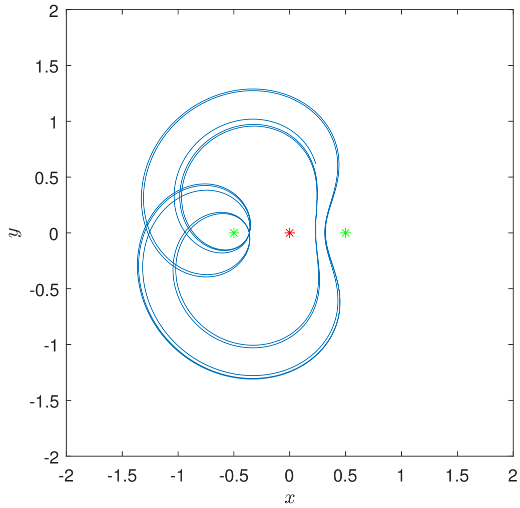
(c) $v_0 = -1.845$, integrated up to $t = 1000$



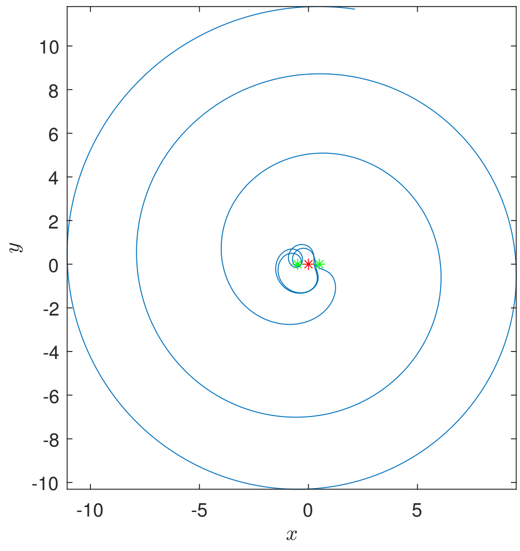
(d) $v_0 = -1.853$



(a) $v_0 = -1.858$



(b) $v_0 = -2.3$



(c) $v_0 = -2.31$

3.2 Question 3(ii)

Table 1: Final positions of (x, y) at $t = 30$

$-v_0$	x	y	Absolute deviation in J
-1	0.428	-0.067	6.176e-06
-1.5	0.322	0.042	2.000e-06
-1.73	0.182	0.170	2.232e-07
-1.78	0.660	-0.068	3.438e-06
-1.853	0.640	-0.389	2.409e-08
-1.858	-5.177	2.255	2.933e-07
-2.3	0.233	0.620	2.898e-07
-2.31	2.133	11.687	2.215e-07

We can judge accuracy of these final positions by looking at the deviation of J, at $t = 30$, which does not exceed 6.1768×10^{-6}. Given that $|J|$ is always greater than 0.7, this means that the relative deviation is very small (on the order of 10^{-5}), so the program is accurate.

In addition to this, varying the tolerances on the black box ODE solver varies the timestep accordingly, and gives similar trajectories, so this also points to an accurate solution.

3.3 Comment on Trajectories and Forbidden regions

The trajectories are very unpredictable, and even very small variations in v_0 result in vastly different trajectories. For example, Figure (3d) and Figure (4a) only differ in v_0 by 0.005, but the latter exits the forbidden region and goes off to infinity, whereas the former does not. Similarly, despite v_0 only differing by 0.01 between Figures (4b) and (4c), the latter goes off to infinity and the former does not.

As $-v_0$ increases, the forbidden region decreases in size, until it entirely disappears above around -2.031. This decrease can be explained by looking at equation (4) in Question 1. As $\sqrt{u_0^2 + v_0^2}$ increases, the upper bound for the allowed region increases, so there are more points which are allowed. The allowed region begins as two small circles around P_h and P_l, which increase in size until they join in the middle, separating the forbidden region into two regions, which decrease in size.

The allowed region is a somewhat useful guide to the trajectory, especially while it is very small, as in Figures (2b) and (2d), because the trajectory really is restricted to this small circle. However as soon as the circles join, it is no longer a very useful guide, because while the trajectory may not go to infinity, the existence of a possible path from P_h to P_l does not imply that such a path actually happens, as seen in Figure (3a) (although it can, as seen in Figure (3b)). In fact, even when the forbidden region splits, the trajectory can remain orbiting around P_h, as in (3d), although figure (4a) demonstrates that it is possible for the trajectory to go to infinity with a similarly shaped forbidden region. Figure (4b) shows that a lack of forbidden region does not imply that the trajectory must go to infinity, although it can do, as demonstrated by Figure (4c).

(All claims about trajectories remaining bounded, or only orbiting P_h were checked by integrating up to $t = 1000$, as demonstrated in Figure (3c))

In order to get from P_h to P_l, we would want $-v_0$ small enough that the forbidden region bounds the trajectory, but also large enough that the allowed regions are connected. However even once these conditions are met, there are cases for small $-v_0$ (Figure (3a)) and large $-v_0$ (Figure (3c)), where the trajectory stays orbiting P_h, so the ideal value of $-v_0$ is in the approximate range $[1.75, 1.83]$. An ideal value is therefore $-v_0 = 1.8$.

4 Question 4

4.1 Contour plots of Omega

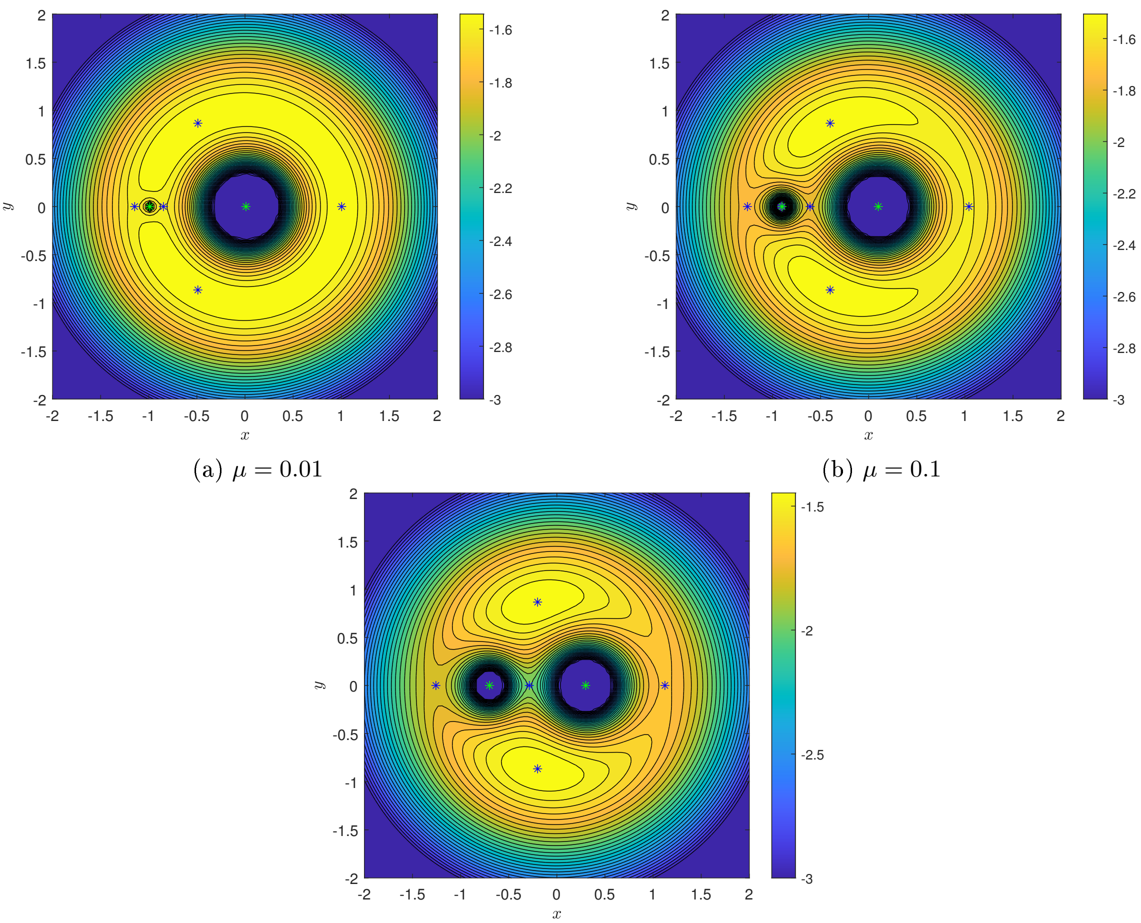
Writing $x_1 = x, x_2 = y, x_3 = \dot{x}, x_4 = \dot{y}$, we can write our system of second order DEs as 4 first order DEs:

$$
\begin{aligned}
\dot{x_1} &= x_3 \\
\dot{x_2} &= x_4 \\
\dot{x_3} &= 2x_4 - \frac{\partial \Omega}{\partial x_1}(x_1, x_2) \\
\dot{x_4} &= -2x_3 - \frac{\partial \Omega}{\partial x_2}(x_1, x_2)
\end{aligned}
$$

Which we may rewrite as

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$$

after defining $\mathbf{X}$ appropriately. We obtain the equilibrium points by setting $\mathbf{X}(\mathbf{x}) = 0$, which implies that $\nabla\Omega = 0$, from which we can find all five of the Lagrange points. In fact, by examining contours plots of Ω, as in Figure (5), we can see the saddle points, minima and maxima, which correspond to equilibrium points.



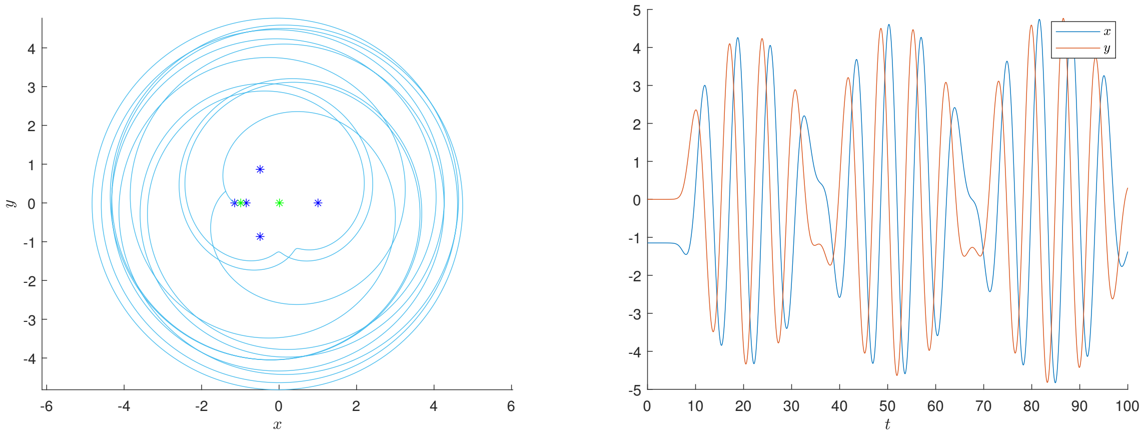
(a) $\mu = 0.01$

(b) $\mu = 0.1$

(c) $\mu = 0.3$

Figure 5: Contour plots for various μ, with Ω capped above -3 for clarity of colour scale around equilibria. Equilibria points are marked in blue, and P_h and P_l are marked in green.

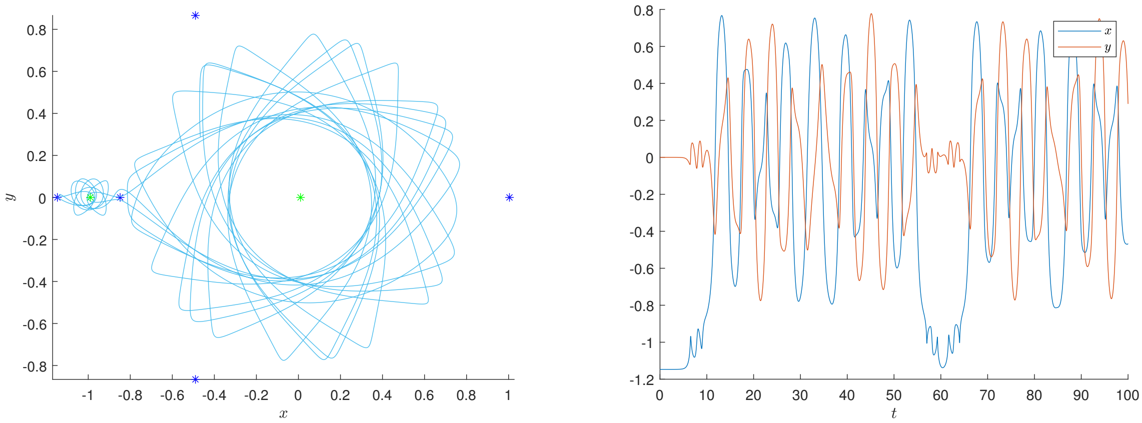
4.2 Numerical investigation of stability of collinear Lagrange points



(a) Trajectory plot

(b) x, y plotted against t

Figure 6: Trajectory when starting at rest, 10^{-7} left of the leftmost collinear equilibrium point when $\mu = 0.01$



(a) Trajectory plot

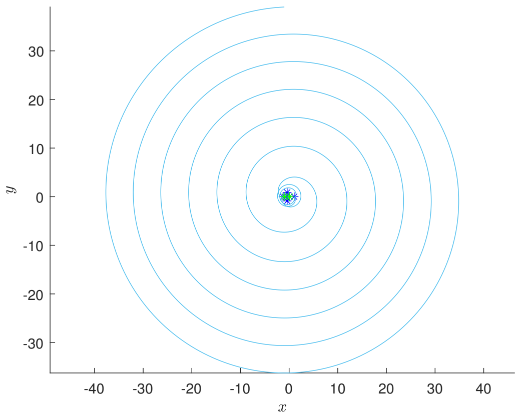
(b) x, y plotted against t

Figure 7: Trajectory when starting at rest, 10^{-7} right of the leftmost collinear equilibrium point when $\mu = 0.01$

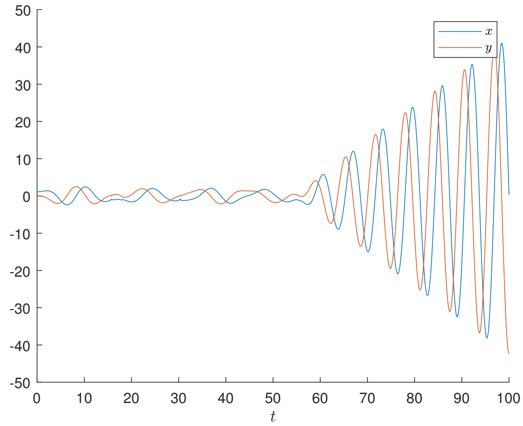
As can be seen in Figure (6) and Figure (7), small perturbations either side of the first collinear point result in unstable orbits when $\mu = 0.01$. While the trajectories may remain in a neighbourhood of the equilibrium point, they do not tend to the point. In some cases, as in Figure (8), the trajectory goes to infinity, and in some cases, as in Figure (9), the trajectory enters a periodic orbit, however there is no case where the the trajectory stays at the equilibrium.

We can be satisfied that these small perturbations capture the behaviour of infinitesimally small perturbations, because the behaviour for small time, for example $0 \leq t \leq 0.5$, the size of the perturbation from the Lagrange point is still small - in all three cases of μ and all three possible collinear points, the maximal deviation in x or y does not exceed 10^{-5} which means that second (and higher) order terms remain negligibly small - on the order of 10^{-10}. Therefore the behaviour of perturbations of size 10^{-7} mimics that of formally infinitesimal perturbations. In fact, we see an exponential increase (10), as predicted by linear stability analysis (4.3.1).

For these values of μ, every collinear point is linearly unstable. We can prove this analytically by doing a linear stability analysis.

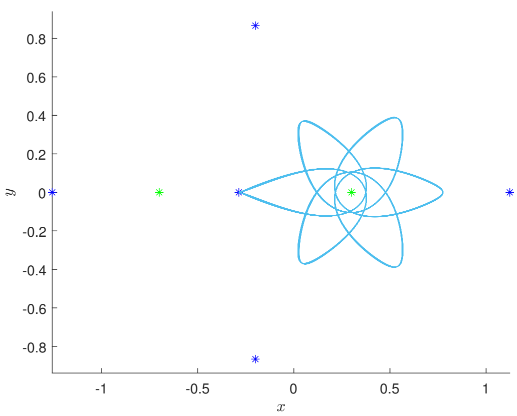


(a) Trajectory plot

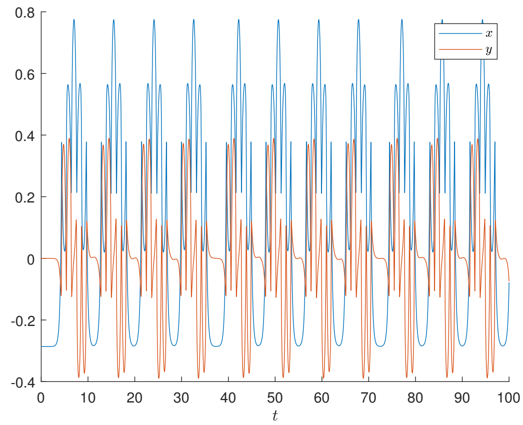


(b) x, y plotted against t

Figure 8: Trajectory when starting at rest, 10^{-7} right of the rightmost collinear equilibrium point when $\mu = 0.1$



(a) Trajectory plot



(b) x, y plotted against t

Figure 9: Trajectory when starting at rest, 10^{-7} right of the middle collinear equilibrium point when $\mu = 0.3$

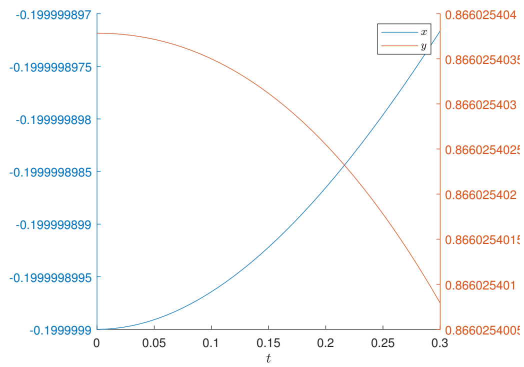


Figure 10: Demonstration of exponential increase of unstable value for small time, with $\mu = 0.3$

4.3 Linear Stability Analysis

If we let x_i^0 be coordinates of an equilibrium point and let $x_i = x_i^0 + \eta_i(t)$, we can use a Taylor series to write $\dot{x}_i = \dot{\eta}_i = \mathbf{X}_i(x^0 + \eta(t)) = \mathbf{X}_i(x^0) + \eta \cdot \nabla\mathbf{X}_i(x^0) + O(|\eta|^2)$

Since these are equilibria, $\mathbf{X}_i(x^0) = 0$

So then we can write:

$$\dot{\eta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Omega_{x_1x_1} & -\Omega_{x_1x_2} & 0 & 2 \\ -\Omega_{x_2x_1} & -\Omega_{x_2x_2} & -2 & 0 \end{pmatrix} \eta \tag{1}$$

The characteristic equation of this is:

$$\lambda^4 + (4 + \Omega_{xx} + \Omega_{yy})\lambda^2 + \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = \Lambda^2 + (4 + \Omega_{xx} + \Omega_{yy})\Lambda + \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2$$

where $\Lambda = \lambda^2$. The complementary solution (which can be verified by substitution) to the first order matrix ODE in equation (1) can be written like

$$Y_c = \sum_{i=1}^{4} a_i \mathbf{v}_i e^{\lambda_i t}$$

where $\mathbf{X}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for every i

If $\exists\lambda$ s.t. $\Re(\lambda) > 0$ then this means that the solution is unstable, because choosing $\eta = \mathbf{v}$ (the corresponding eigenvector), we can see that the complementary solution is $A\mathbf{v}e^{\lambda t}$ which is unbounded as $t \to \infty$, so the Lagrange point is unstable.

Since we have a quadratic in $\Lambda = \lambda^2$, if λ is a solution, then so is $-\lambda$. Therefore the system is only stable if both roots of Λ are real and negative.

4.3.1 Linear Stability of Collinear Points

From the contours at collinear Lagrange points, we can see that $\Omega_{xx} < 0, \Omega_{yy} > 0, \Omega_{xy} = 0$, so we can write the characteristic equation like so:

$$\Lambda^2 + 2\beta_1\Lambda - \beta_2^2$$

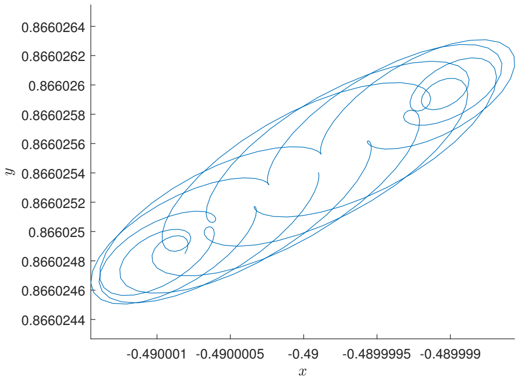
where $\Lambda = \lambda^2$, $\beta_1 = 2 + \frac{\Omega_{xx}+\Omega_{yy}}{2}$, $-\beta_2^2 = \Omega_{xx}\Omega_{yy} > 0$

So then have $\Lambda = -\beta_1 \pm \sqrt{\beta_1^2 + \beta_2^2}$, which implies Λ_+ is real and positive, so then the system is unstable.

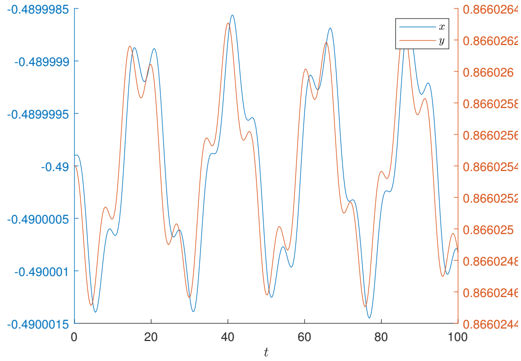
5 Question 5

As μ increases, the behaviour changes significantly for values of μ above 0.025. I have plotted the trajectories until $t = 100$ to differentiate the behaviour of stable points which remain bounded, and unstable ones, which are unbounded and chaotic. For large μ, we initially observe a spiral as in Figure (14), while the trajectory is close to the equilibrium point and linear stability analysis is relevant. Beyond this, chaotic motion ensues. For small values of μ, there is an interesting pattern which is bounded by a small ellipse of size on the order of 10^{-5}, as seen in Figure (17). This implies a stable orbit for sufficiently small μ.

Having integrated till large t for $\mu = 0.035, 0.042$, (Figures (17) and 18)), the instability of the latter can be seen clearly. The former remains bounded within 10^{-5} of the coordinates of the Lagrange points implies that it is stable. So numerically, we can conclude that $\mu_c \approx 0.038$.

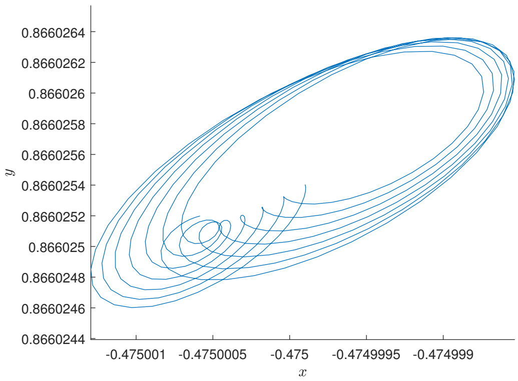


(a) Trajectory plot

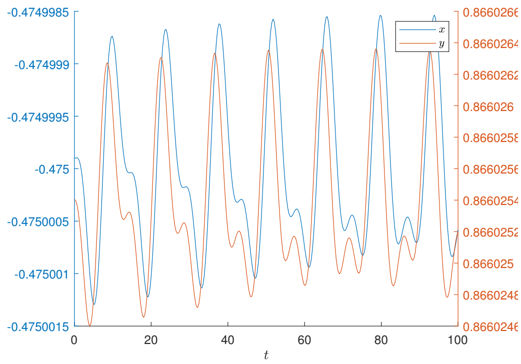


(b) x, y plotted against t

Figure 11: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.01$

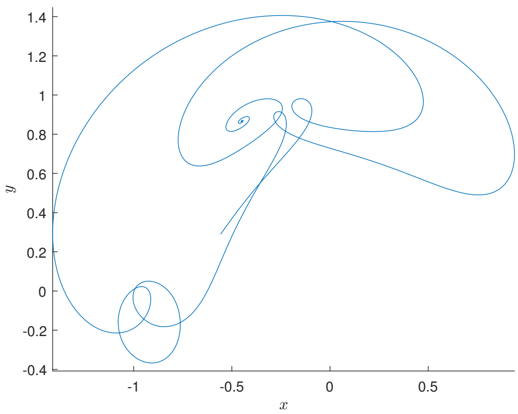


(a) Trajectory plot

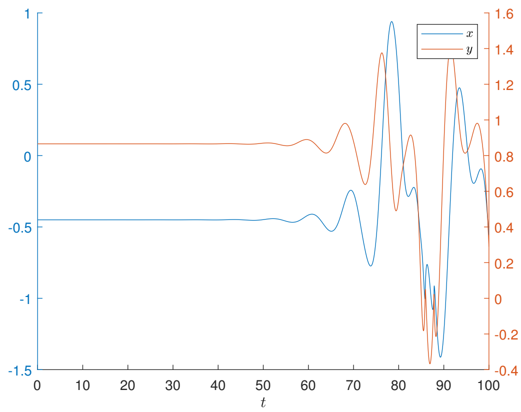


(b) x, y plotted against t

Figure 12: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.025$

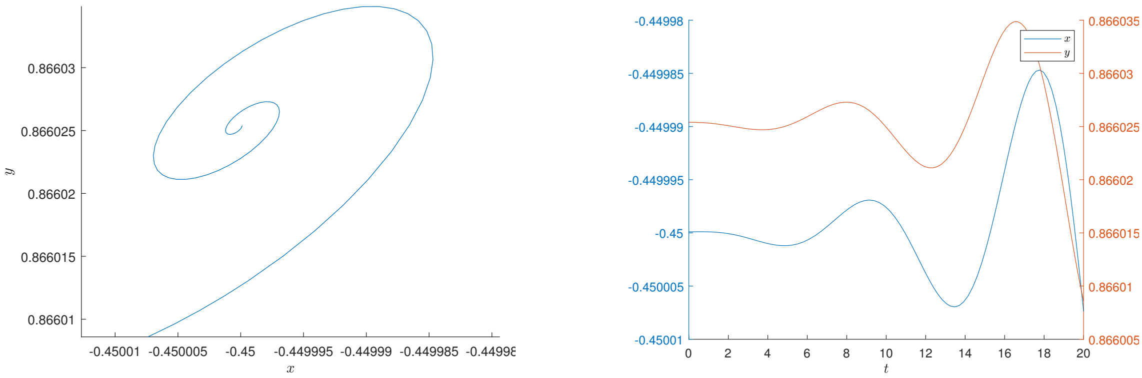


(a) Trajectory plot



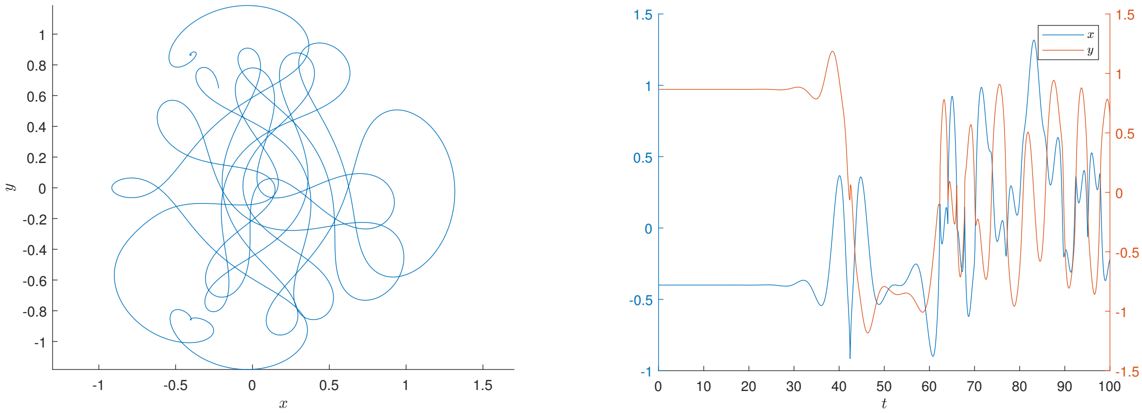
(b) x, y plotted against t

Figure 13: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.05$



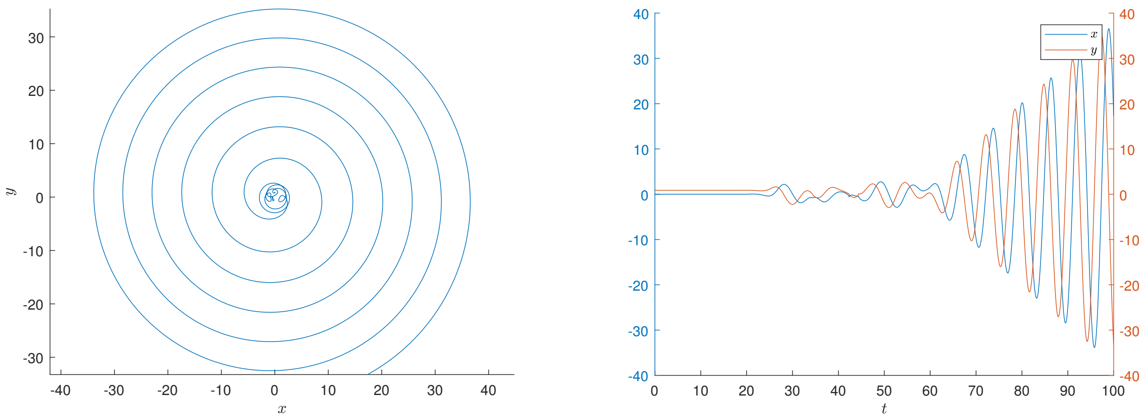
(a) Trajectory plot (b) x, y plotted against t

Figure 14: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.05$, only integrated to $t = 20$ to demonstrate initial spiral



(a) Trajectory plot (b) x, y plotted against t

Figure 15: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.1$



(a) Trajectory plot (b) x, y plotted against t

Figure 16: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.5$

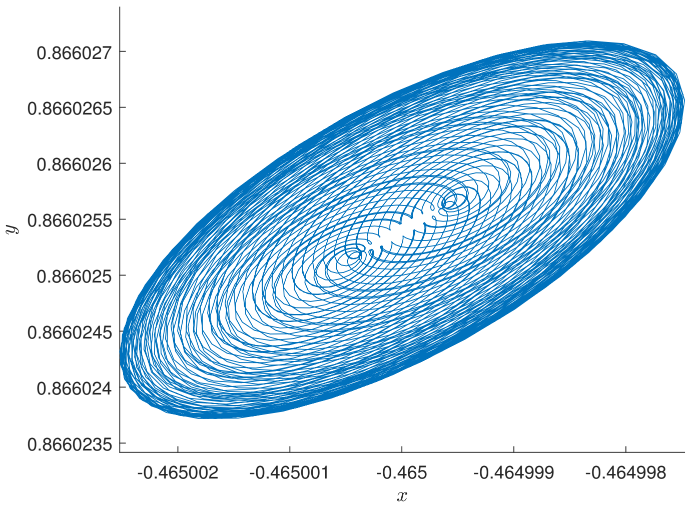


Figure 17: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.035$, integrated up till $t = 1000$

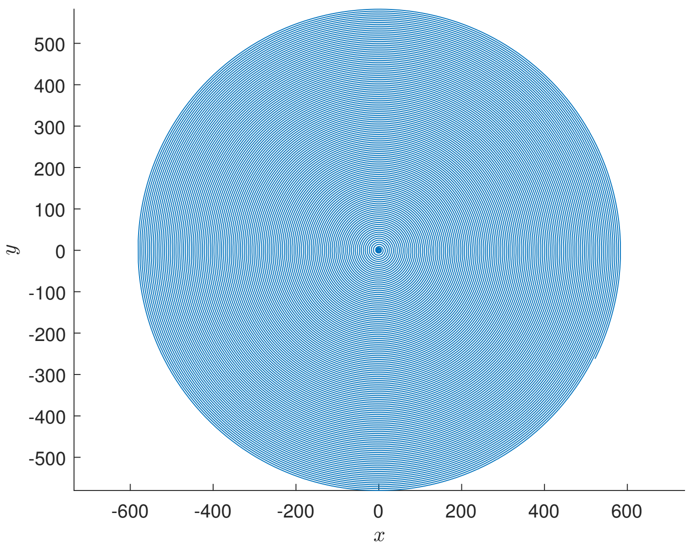


Figure 18: Trajectory when starting at rest, 10^{-7} right of the upper equilateral equilibrium point when $\mu = 0.042$, integrated up till $t = 1000$

5.1 Linear Stability of Equilateral Lagrange Points

Letting $r_l = \sqrt{(x-\mu+1)^2+y^2}, r_h = \sqrt{(x-\mu)^2+y^2}$

$$\begin{aligned}
\Omega_{xx} &= -\frac{3(1-\mu)(x-\mu)^2}{r_h^5} + \frac{1-\mu}{r_h^3} + \frac{\mu}{r_l^3} - \frac{3\mu(x-\mu+1)^2}{r_l^5} - 1 \\
\Omega_{xy} &= -\frac{3(1-\mu)(x-\mu)y}{r_h^5} - \frac{3\mu(x-\mu+1)y}{r_l^5} \\
\Omega_{yy} &= -\frac{3(1-\mu)y^2}{r_h^5} + \frac{1-\mu}{r_h^3} + \frac{\mu}{r_l^3} - \frac{3\mu y^2}{r_l^5} - 1
\end{aligned}$$

As we are looking at equilateral Lagrange points, $r_h = r_l = 1$, so then $\Omega_{xx} = -\frac{3}{4}, \Omega_{yy} = -\frac{9}{4}, \Omega_{xy} = \frac{3\sqrt{3}}{4}(1-2\mu)$

Substituting these values into the characteristic equation yields the equation

$$\Lambda^2 + \Lambda + \frac{27}{4}\mu(1-\mu) = 0 \implies \Lambda = \frac{-1 \pm \sqrt{1-27\mu(1-\mu)}}{2}$$

If the discriminant is negative, then we have imaginary Λ, and the system is unstable. Solving the inequality $1-27\mu(1-\mu) < 0$, we see that this happens when $\mu > \frac{1}{2}(1-\frac{\sqrt{69}}{9}) = \mu_c$

When $\mu \leq \mu_c$, we get pure imaginary eigenvalues, so we get a centre, which is stable. While centres take the shape of an ellipse in a system of two ODEs, we do not quite observe an ellipse in this case. This is expected, however, because we have a system of four ODEs, so we can only predict that the state space $(x, y, \dot{x}, \dot{y})$ remains in a four dimensional ellipsoid, and we are simply observing the projection of this trajectory onto the (x, y) plane, so all we can say is that the motion is bounded by an ellipse. Indeed, this is seen in Figure (17).

6 Question 6

The persistence of Jupiter Trojans is consistent with the above findings, as $\mu = 9.54 \times 10^{-4} \leq \mu_c$, so the equilateral Lagrange points are stable.

While the Earth-Moon system has $\mu < \mu_c$, this μ is closer to the critical value, so there is a smaller region around it in which there are stable orbits. In addition to this, the existence of the Sun's gravitation field could disturb the stable equilibrium - the system could be construed as a restricted four body problem. This means that we would not expect our current results to hold.

7 Code

7.1 main1.m

```
% Have Y be a 4 dimensional vector which stores x,x',y,y'
%% Initial conditions
mu=0.5;
x0=0.32;
v0List=[-1 -1.5 -1.73 -1.78 -1.845 -1.853 -1.858 -2.3 -2.31];
u0=0;
y0=0;
EndCoords=zeros(8,2);
JDiff=zeros(length(v0List),1);
for i=1:length(v0List)
    v0=v0List(i);
    Y0=[x0; u0; y0; v0];
    tspan=[0 30];

```

```

    %% Solving the ODE
    opts=odeset('RelTol',1e-7,'AbsTol',1e-9);
    [t,Y]=ode45(@(t,Y) ODE(Y,mu),tspan,Y0,opts);
    sizeT=length(t);

    x=Y(:,1);
    u=Y(:,2);
    y=Y(:,3);
    v=Y(:,4);

    %% Defining J and Omega
    r1=@(x,y)((x+1-mu).^2+y.^2).^(1/2);
    r2=@(x,y) ((x-mu).^2+y.^2).^(1/2);
    Omega=@(x,y) -(x.^2+y.^2)/2 -mu./r1(x,y)-(1-mu)./r2(x,y);
    Omega0=Omega(x0,0);
    J0=(u0^2+v0^2)/2+Omega0;

    %% Creating contours
    size=2;
    xList = linspace(-size,size);
    yList = linspace(-size,size);

    [XList,YList] = meshgrid(xList,yList);
    Z = Omega(XList,YList);
    contourf(XList,YList,Z,[J0 J0])

    %% Plotting
    hold on
    plot(Y(:,1),Y(:,3))
    plot(mu-1,0,'g*')
    plot(mu,0,'g*')
    plot(0,0,'r*')
    axis equal
    xlabel('$x$','interpreter','latex')
    ylabel('$y$','interpreter','latex')
    hold off

    %% End Coordinates
    EndCoords(i,1)=x(end);
    EndCoords(i,2)=y(end);

    %% Deviation from J
    JDiff(i)=abs((u(end)^2+v(end)^2)/2+Omega(x(end),y(end))-J0); % This measure the deviation of J from the original value of J.

end
```

7.2 ODE.m

```
function dYdt = ODE(Y,mu)
x=Y(1);
u=Y(2);
y=Y(3);
v=Y(4);
```

```
r1=((x+1-mu)^2+y^2)^(1/2);
r2=((x-mu)^2+y^2)^(1/2);
dYdt=zeros(4,1);
dYdt(1)=u;
dYdt(2)=2*v+x-mu*(x+1-mu)/r1^3-(1-mu)*(x-mu)/r2^3;
dYdt(3)=v;
dYdt(4)=-2*u+y-mu*y/r1^3-(1-mu)*y/r2^3;
end
```

7.3 main2.m

```
%% Initial conditions
mu=0.5;
a=0.3;
x0=0.5+a;
k=(a*(0.5+a^3))^0.5;
u0=0;
y0=0;
v0=a*(k/a^2-1);
Y0=[x0; u0; y0; v0];
tspan=[0 30];

%% Solving ODE
opts=odeset('RelTol',1e-6,'AbsTol',1e-9);
[t,Y]=ode45(@(t,Y) newOmegaODE(Y),tspan,Y0,opts);
sizeT=length(t);
x=Y(:,1);
u=Y(:,2);
y=Y(:,3);
v=Y(:,4);

%% Deviation from analytic solution
xExact=0.5+0.3*cos((-1+k/a^2)*t);
yExact=0.3*sin((-1+k/a^2)*t);
worstXError=max(abs(x-xExact));
worstYError=max(abs(y-yExact));

%% Plotting
plot(Y(:,1),Y(:,3))
hold on
plot(mu,0,'g*')
hold off
axis equal
xlabel('$x$','interpreter','latex')
ylabel('$y$','interpreter','latex')
```

7.4 newOmegaODE.m

```
function dYdt = ODE(t,Y)
x=Y(1);
u=Y(2);
y=Y(3);
v=Y(4);
r=((x-0.5)^2+y^2)^(1/2);
```

```
dYdt=zeros(4,1);
dYdt(1)=u;
dYdt(2)=2*v-0.5*(x-0.5)/r^3;
dYdt(3)=v;
dYdt(4)=-2*u-0.5*y/r^3;
end
```

7.5 q4contour.m

```
%% Define Omega
r1=@(x,y)((x+1-mu).^2+y.^2).^(1/2);
r2=@(x,y) ((x-mu).^2+y.^2).^(1/2);
Omega=@(x,y) -(x.^2+y.^2)/2 -mu./r1(x,y)-(1-mu)./r2(x,y);

%% Find collinear points
r1=@(x)((x+1-mu)^2)^(1/2);
r2=@(x)((x-mu)^2)^(1/2);
OmegaX=@(x) x-mu*(x+1-mu)/r1(x)^3-(1-mu)*(x-mu)/r2(x)^3;
collinear1=fzero(OmegaX,-1.25);
collinear2=fzero(OmegaX,-0.5);
collinear3=fzero(OmegaX,1.25);

%% Contours
xList = linspace(-2,2);
yList = linspace(-2,2);
[XList,YList] = meshgrid(xList,yList);
OmegaCap=@(x,y) max(-3,Omega(x,y));
Z = OmegaCap(XList,YList);
contourf(XList,YList,Z,30)

%% Plotting
hold on
plot(mu-0.5,sqrt(3)/2,'b*')
plot(mu-0.5,-sqrt(3)/2,'b*')
plot(collinear1,0,'b*')
plot(collinear2,0,'b*')
plot(collinear3,0,'b*')
plot(mu-1,0,'g*')
plot(mu,0,'g*')
axis equal
hold off
colorbar
xlabel('$x$','interpreter','latex')
ylabel('$y$','interpreter','latex')
```

7.6 StabilityTrajectory.m

```
%% Find collinear points
r1=@(x)((x+1-mu)^2)^(1/2);
r2=@(x)((x-mu)^2)^(1/2);
OmegaX=@(x) x-mu*(x+1-mu)/r1(x)^3-(1-mu)*(x-mu)/r2(x)^3;
collinear1=fzero(OmegaX,-1.25);
collinear2=fzero(OmegaX,-0.2);
collinear3=fzero(OmegaX,1.25);
```

```
muList=[0.01 0.025 0.05 0.1 0.5];
for i=1:5
    %% Initial Condition
    mu=muList(i);
    x0=mu-0.5+1e-7;
    u0=0;
    y0=sqrt(3)/2;
    v0=0;
    Y0=[x0; u0; y0; v0];
    tspan=[0 20];

    %% Solving ODE
    opts=odeset('RelTol',1e-7,'AbsTol',1e-9);
    [t,Y]=ode45(@(t,Y) ODE(Y,mu),tspan,Y0,opts);
    sizeT=length(t);
    x=Y(:,1);
    u=Y(:,2);
    y=Y(:,3);
    v=Y(:,4);

    %% Plotting trajectory
    hold on
    plot(Y(:,1),Y(:,3))
    axis equal
    xlabel('$x$','interpreter','latex')
    ylabel('$y$','interpreter','latex')
    hold off
    clf

    %% Plotting x,y against t
    hold on
    yyaxis left
    plot(t,Y(:,1))
    yyaxis right
    plot(t,Y(:,3))
    hold off
    xlabel('$t$','interpreter','latex')
    legend('$x$','$y$','interpreter','latex')
    clf
end

%% Investigating deviation from starting position, to validate linear stability
    analysis
xDiff=abs(x0-Y(:,1));
max(xDiff);
xDiff=abs(y0-Y(:,3));
max(yDiff);
```

References

[1] Szebehely, V. (1967) *Theory of Orbits: The Restricted Problem of Three Bodies*, Academic Press Inc.