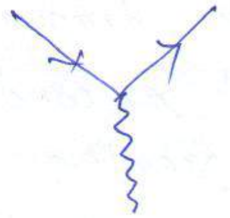
Feynman diagram: continued.



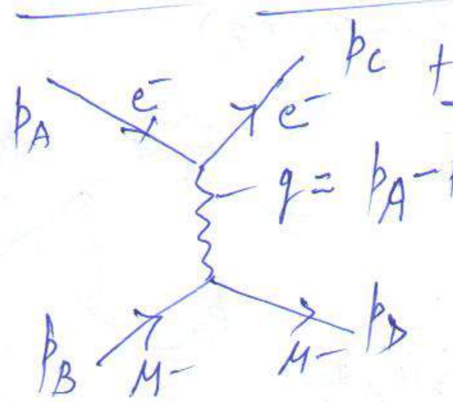
QED primitive vertex

charge does not change at the vertex since photon does not carry any charge.

* charge is conserved at the each vertex.
* Energy and momentum (equivalently four momentum) is conserved for the whole process not at the vertex.

t-channel, s-channel & u-channel Feynman diagram:

e- μ- → e- μ- scattering: (Non-identical particles without any antiparticles)



t-channel

$A + B \longrightarrow C + D$

$p_A + p_B = p_C + p_D$ {Four mom. conservation.}

$s = (p_A + p_B)^2 = (p_C + p_D)^2$

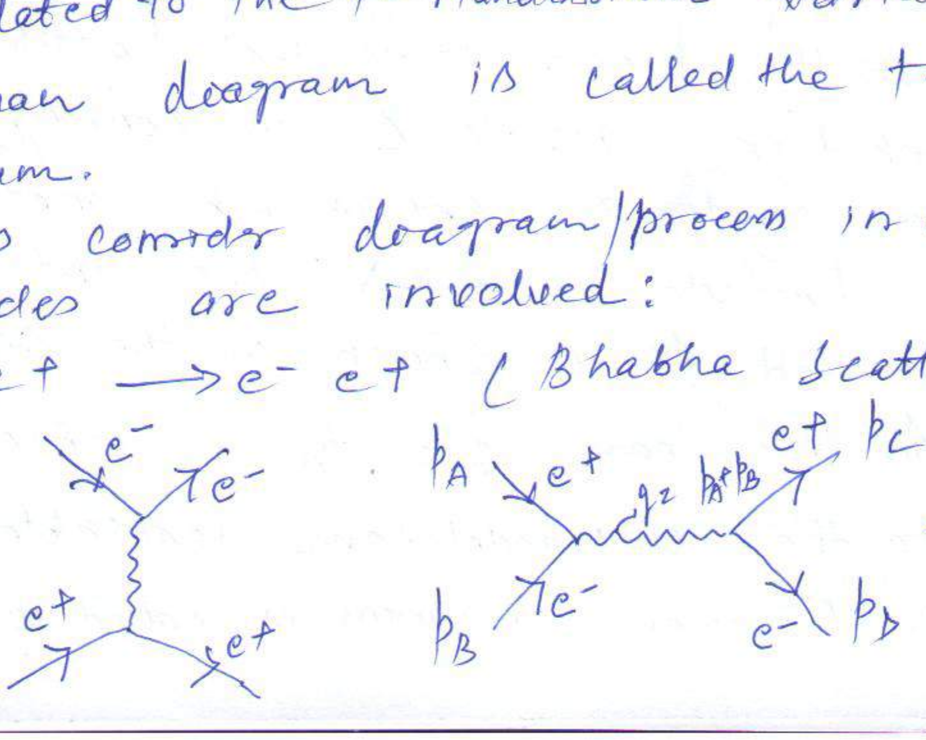
$t = (p_A - p_C)^2 = (p_D - p_B)^2$

$u = (p_A - p_D)^2 = (p_C - p_B)^2$

Since in the above diagram, mom. carried by the photon is related to the t-Mandelstam variable, above Feynman diagram is called the t-channel Feynman diagram.

Let us consider diagram/process in which anti-particles are involved:

$e^- e^+ \longrightarrow e^- e^+$ (Bhabha scattering process)

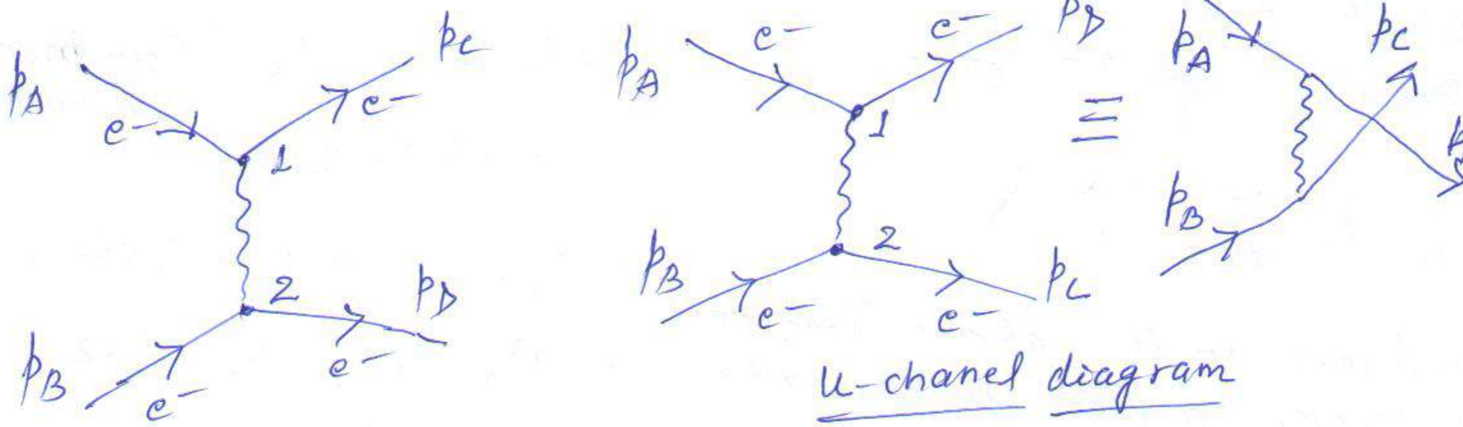


s-channel

In the case of above process, apart from the t-channel, one more Feynman diagram is possible which is shown as second diagram. charge is conserved at the vertex. So no any Issue! In such a case, mom. carried by the photon (virtual particle), $q = p_A + p_B$, which is related to the s-Mandelstam variable. So, above Feynman diagram is called the s-channel Feynman diagram.

Let us consider a Feynman diagram associated with an identical particles:

$e^- e^- \longrightarrow e^- e^-$ (Moller scattering)



u-chanel diagram

In such cases, where identical particles are involved in the process, it is not possible to decide whether particle C is coming from vertex 1 or vertex 2. That is why another diagram in which particle C and D are interchanged, is also allowed. It is shown in the second diagram In this case, $q = p_A - p_D$, which is related to the u-Mandelstam variable.

So, above Feynman diagram is called u-channel

Feynman diagram.

<u>QCD</u> <u>Feynman</u> <u>diagram</u>:

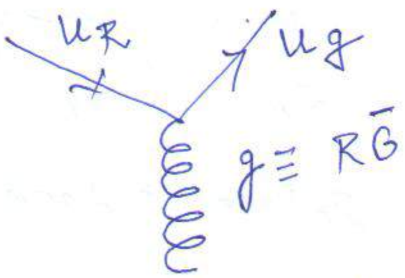
Gluon is the mediator particle in <u>QCD</u>. QCD describes strong force which binds quarks inside <u>hadrons</u>.

<u>Quarks</u> : <u>Colour</u> <u>charge</u>

Gluon (g): <u>Colour</u> <u>charge</u> ; colour charge changes at the QCD <u>vertex</u>.

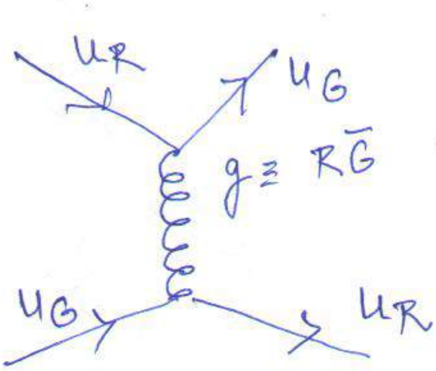
gluon → 

$\begin{pmatrix} u \\ d \end{pmatrix}$ $\begin{pmatrix} c \\ s \end{pmatrix}$ $\begin{pmatrix} t \\ b \end{pmatrix}$ $+\frac{2}{3}$ charge, $-\frac{1}{3}$ charge

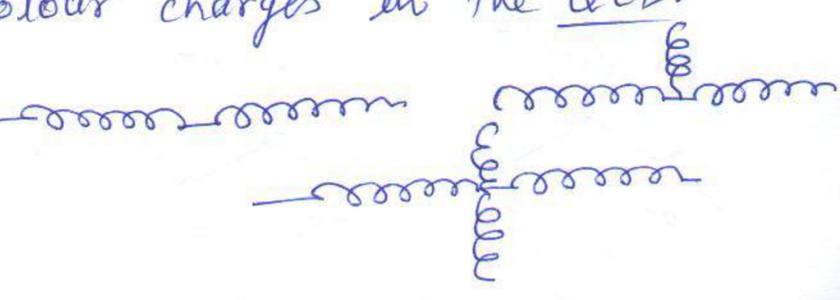


Colour charge → conserve at <u>each vertex</u>

Since gluon has colour charge unlike photons, two gluon, three gluon and four gluon vertices are also possible in <u>QCD</u>. This is called the self interaction between gauge particles. It is related to the non-abelian nature of the <u>QCD</u>. A bigger group of non-abelian gauge theory is called the Yang-Mills theory. <u>QCD</u> is just one member of this big group.



Red, blue, green are three colour charges in the <u>QCD</u>.



Weak Interaction Feynman diagram:

Gauge particles (bosons) are W^+, W^-, Z^0.
They are called Intermediate vector Bosons.

$m_{W^+} \simeq m_{W^-} \simeq 80$ GeV

$m_{Z^0} \simeq 90$ GeV

Range of weak Int. $\propto \dfrac{1}{\text{Mass of exchange particle}}$

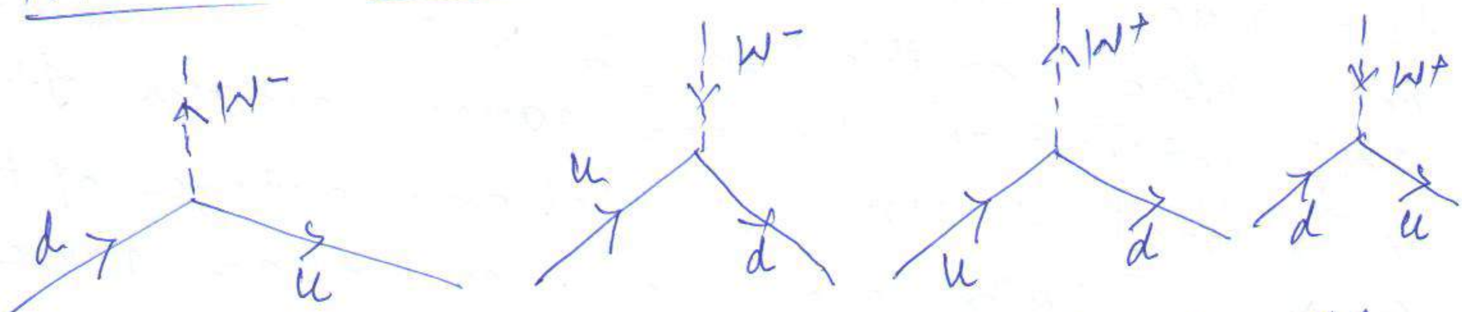
$\Rightarrow$ Short range nature

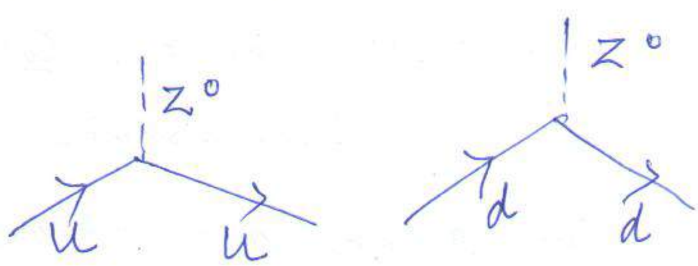
Quarks & Leptons are the particles involved in the weak interaction process.

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} ; \begin{pmatrix} e^- \\ \nu_e \end{pmatrix} \begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix} \begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix}$$

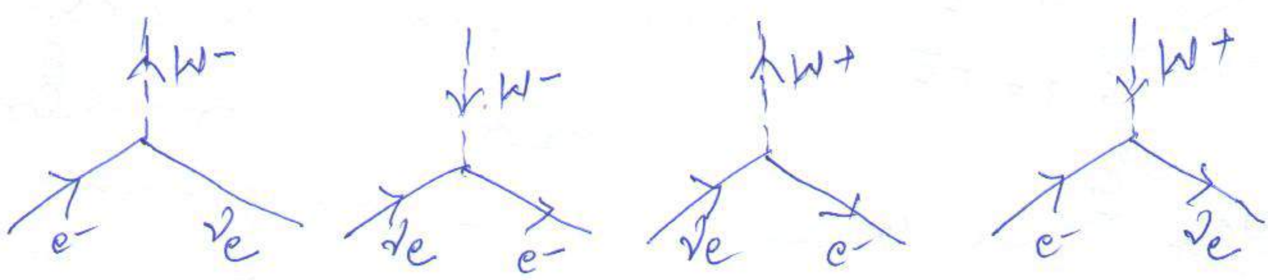
1st gen. 2nd gen. 3rd gen 1st gen. 2nd gen. 3rd gen.

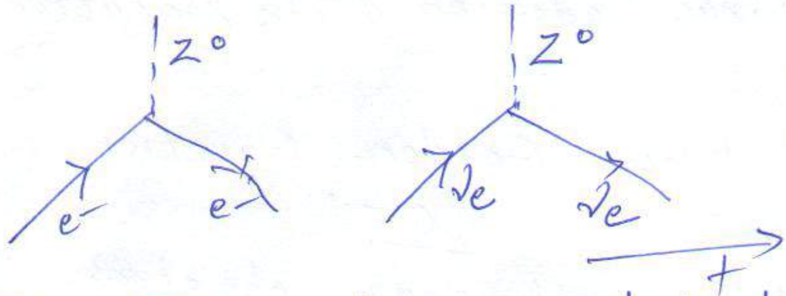
Primitive vertices:





Similarly for other flavours of quarks





Similarly for other flavours of leptons.

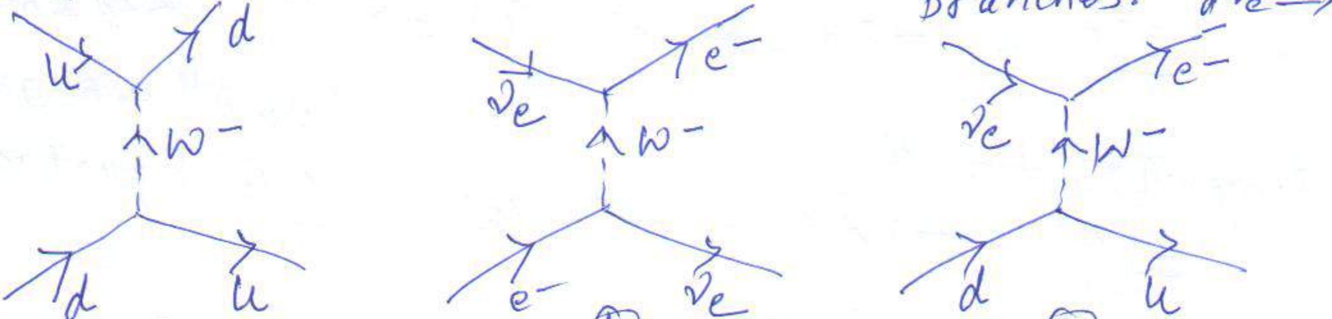
Two Types of Weak Int. process: Based on mediators.

1 Charged Weak Int.; mediated by W^+/W^-

2 Neutral Weak Int. : mediated by Z^0

Three Types of Weak Int. process: Based on partrcles involved in diagram branches.

$ud \rightarrow ud$ $\quad$ $e^-\nu_e \rightarrow e^-\nu_e$ $\quad$ $d\nu_e \rightarrow ue^-$



(a) $\quad$ (b) $\quad$ (c)

1 Quark Weak Int. process : (a)

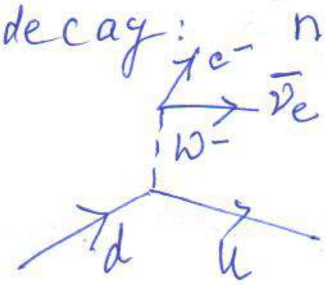
2 Leptonic Weak Int. process : (b)

3 Semi-leptonic or semi-quark or mixed Weak Int. process : (c)

Feynman diagram for beta decay process:

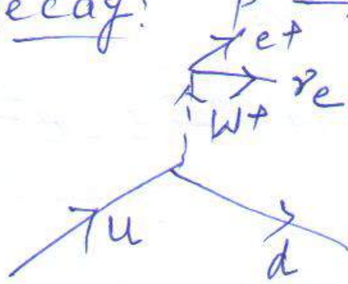
β^- decay: $n \rightarrow p + e^- + \bar{\nu}_e$; $ddu \rightarrow uud + e^- + \bar{\nu}_e$

$\Rightarrow d \rightarrow u + e^- + \bar{\nu}_e$



β^+ decay: $p \rightarrow n + e^+ + \nu_e$; $uud \rightarrow ddu + e^+ + \nu_e$

$\Rightarrow u \rightarrow d + e^+ + \nu_e$



$e^- \rightarrow \beta^-$

$e^+ \rightarrow \beta^+$

Dirac Algebra: Dirac Equation & its solution & related algebra

Shortcomings of the Klein-Gordon Equation:

$$E^2 = p^2 + m^2 \qquad \underline{c=1}$$

Replacing E and p by corresponding operators

$$E = i\frac{\partial}{\partial t} \; ; \quad p = \hat{p} = -i\nabla \qquad \underline{\hbar = 1}$$

$$\Rightarrow \quad -\frac{\partial^2}{\partial t^2} = -\nabla^2 + m^2$$

$$-\frac{\partial^2\phi}{\partial t^2} = -\nabla^2\phi + m^2\phi$$

$$\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi - m^2\phi \quad (1) \Rightarrow \frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = 0$$

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

K G Equation

Writing (1) for ϕ^*

$$\frac{\partial^2\phi^*}{\partial t^2} = \nabla^2\phi^* - m^2\phi^* \quad (2)$$

$\phi^* \times (1) - \phi \times (2)$, we get,

$$\phi^*\frac{\partial^2\phi}{\partial t^2} = \phi^*\nabla^2\phi - m^2\phi^*\phi \quad (3)$$

$$\phi\frac{\partial^2\phi^*}{\partial t^2} = \phi\nabla^2\phi^* - m^2\phi\phi^* \quad (4)$$

$$\phi^*\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^*}{\partial t^2} = \phi^*\nabla^2\phi - \phi\nabla^2\phi^*$$

$$\frac{\partial}{\partial t}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) = \vec{\nabla}\cdot\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)$$

$$\frac{\partial}{\partial t}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) - \vec{\nabla}\cdot\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right) = 0$$

Multiplying by i on both the sides

$$i\frac{\partial}{\partial t}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) - i\vec{\nabla}\cdot\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right) = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot J = 0 \quad (5)$$

where $\rho = i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)$; $J = -i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)$ — (6)

Solution of K.G equation

$$\phi(x,t) = N e^{-ip\cdot x} = N e^{-i(Et - \vec{p}\cdot\vec{x})} = N e^{i(\vec{p}\cdot\vec{x} - Et)}$$

$$\phi^*(x,t) = N e^{+ip\cdot x}$$

By using ϕ, ϕ^*, $\partial\phi/\partial t$, $\partial\phi^*/\partial t$, $\vec{\nabla}\phi$, $\vec{\nabla}\phi^*$ one can finally write ρ and J as:

$$\rho = 2|N|^2 E \quad ; \quad \vec{J} = 2|N|^2 \vec{p} \quad - (7)$$

E and $\vec{p}$ form 4-vector, so ρ and J too

$$J^\mu = (\rho, \vec{J})$$

Eq. (7) can be written as:

$$J^\mu = 2|N|^2 p^\mu$$

Since $E^2 = p^2 + m^2 \Rightarrow E = \pm\sqrt{p^2+m^2}$

$E = +\sqrt{p^2+m^2}$ OK No issue

$E = -\sqrt{p^2+m^2}$ -ve ; What is the meaning of -ve E ?

Shortcoming 1

Since $\rho = 2|N|^2 E$

for $E = +$ve $\quad \rho = +$ve No issue

But for $E = -$ve $\quad \rho = -$ve ; What is the meaning of -ve ρ?

Shortcoming 2

Shortcomings 1 & 2 motivated or compelled Dirac to think/derive new equation called the Dirac equation. Derivation and solution/prediction of the Dirac equation is an important milestone in the history of the particle Physics

After a lot of effort/thinking the Dirac was able to identify the main reason behind the -ve energy solutions of the K.G equation. Dirac pointed out that negative energy was due to second order time and second order space derivative in the K.G equation. Schrodinger eqn. has 2nd order space derivative but first order time derivative. It does not give -ve energy issue. In view of the above, Dirac had reached to the conclusion that either one 2nd order space or time derivative or no second order derivative should be the solution of the K.G eqn. shortcoming.

$$E^2 = p^2 + m^2$$

Tried to write above as $E = p + m$, which is not correct. Then again he tried to write $E = p + m$ by introducing constants α and β. In fact α & β were quantities whose properties were not known prior to the derivation.

$E = p + m$ (not correct) was corrected using α & β as

$$E = \alpha \cdot \hat{p} + \beta m$$

$$\hat{H}_D = \alpha \cdot \hat{p} + \beta m = \alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z + \beta m$$

$\hat{H}_D = \alpha \cdot \hat{p} + \beta m$ is called the Dirac Hamiltonian.

using $\hat{H}_D$, one can write Dirac equation as

$$H_D \psi = E \psi$$

$$(\alpha \cdot \hat{p} + \beta m) \psi = E \psi \quad \text{— (1)}$$

Dirac Equation Continued:

$$\hat{H}_D = \alpha \cdot \hat{p} + \beta m$$

or $\hat{H}_D = \alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z + \beta m$

or $\hat{H}_D = \alpha_j p_j + \beta m$

$H_D \psi = E\psi$: Dirac equation.

Properties of α_s and β:

$$\hat{H}_D \psi = E\psi$$

$$(\alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z + \beta m)\psi = E\psi$$

Writing $\hat{p}_x = -i\frac{\partial}{\partial x}$; $\hat{p}_y = -i\frac{\partial}{\partial y}$; $\hat{p}_z = -i\frac{\partial}{\partial z}$; $E = i\frac{\partial}{\partial t}$

with $\hbar = 1$ in NUs.

$$\left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi = i\frac{\partial \psi}{\partial t}$$

Squaring and then operating on ψ, we get

$$\left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)^2 \psi = \left(i\frac{\partial}{\partial t}\right)^2 \psi$$

$$\left(i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} - \beta m\right)^2 \psi = -\frac{\partial^2 \psi}{\partial t^2}$$

$$\Big(-\alpha_x^2 \frac{\partial^2}{\partial x^2} - \alpha_y^2 \frac{\partial^2}{\partial y^2} - \alpha_z^2 \frac{\partial^2}{\partial z^2} - \alpha_x \alpha_y \frac{\partial}{\partial x}\frac{\partial}{\partial y} - \alpha_y \alpha_x \frac{\partial}{\partial y}\frac{\partial}{\partial x} - \alpha_y \alpha_z \frac{\partial}{\partial y}\frac{\partial}{\partial z} - \alpha_z \alpha_y \frac{\partial}{\partial z}\frac{\partial}{\partial y} - \alpha_x \alpha_z \frac{\partial}{\partial x}\frac{\partial}{\partial z} - \alpha_z \alpha_x \frac{\partial}{\partial z}\frac{\partial}{\partial x} + \beta^2 m^2\Big)\psi = -\frac{\partial^2 \psi}{\partial t^2}$$

$$\Big(\alpha_x^2 \frac{\partial^2}{\partial x^2} + \alpha_y^2 \frac{\partial^2}{\partial y^2} + \alpha_z^2 \frac{\partial^2}{\partial z^2} + \alpha_x \alpha_y \frac{\partial}{\partial x}\frac{\partial}{\partial y} + \alpha_y \alpha_x \frac{\partial}{\partial y}\frac{\partial}{\partial x} + \alpha_y \alpha_z \frac{\partial}{\partial y}\frac{\partial}{\partial z} + \alpha_z \alpha_x \frac{\partial}{\partial z}\frac{\partial}{\partial x} + \alpha_z \alpha_y \frac{\partial}{\partial z}\frac{\partial}{\partial y} + \alpha_x \alpha_z \frac{\partial}{\partial x}\frac{\partial}{\partial z} - \beta^2 m^2\Big)\psi = \frac{\partial^2 \psi}{\partial t^2} \quad (1)$$

K.G equation:

$$E^2 = p^2 + m^2$$

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

$$\left(\frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} + m^2\right)\phi = 0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-m^2\right)\phi = \frac{\partial^2\phi}{\partial t^2} \quad \text{--- (2)}$$

Claiming that Eqn. (1) and (2) are the same.
Comparing we get.

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I$$

$$\alpha_x\alpha_y + \alpha_y\alpha_x = \alpha_y\alpha_z + \alpha_z\alpha_y = \alpha_x\alpha_z + \alpha_z\alpha_x = 0$$

Furthermore.

$$Tr(\alpha_i) = Tr(\alpha_i\beta\beta) = Tr(\beta\alpha_i\beta) = -Tr(\beta\beta\alpha_i)$$

$$Tr(\alpha_i) = -Tr(\alpha_i) \Rightarrow 2Tr(\alpha_i) = 0$$

$$\Rightarrow Tr(\alpha_i) = 0$$

Similarly, $Tr(\beta) = Tr(\beta\alpha_i\alpha_i) = Tr(\alpha_i\beta\alpha_i)$

$$= -Tr(\alpha_i\alpha_i\beta)$$

$$Tr(\beta) = -Tr(\beta)$$

$$2Tr(\beta) = 0 \Rightarrow Tr(\beta) = 0$$

$\Rightarrow \alpha_x, \alpha_y, \alpha_z$ and β are Traceless matrices.

Let they have eigen values, λ

$$\alpha_i X = \lambda X$$

$$\alpha_i\alpha_i X = \alpha_i\lambda X$$

$$\alpha_i^2 X = \lambda\alpha_i X = \lambda\lambda X$$

$$X = \lambda^2 X \Rightarrow (\lambda^2 - 1)X = 0$$

$$\lambda = \pm 1$$

Similarly for β

$\Rightarrow \alpha_x, \alpha_y, \alpha_z$ and β are matrices having eigen values ± 1.

In view of the above properties, α_is β cannot be real numbers, complex numbers or even vectors. But these are anticommutating matrices with squares unity, traceless and having eigen values ± 1.

Also $\hat{H}_D$ should have real eigen values, and therefore $\hat{H}_D^\dagger = H_D \Rightarrow \hat{H}_D$ should be a Hermitian matrix.

$\Rightarrow \alpha_i, \beta$ should be Hermitian matrices.

$$\alpha_x^\dagger = \alpha_x; \quad \alpha_y^\dagger = \alpha_y; \quad \alpha_z^\dagger = \alpha_z. \quad \beta^\dagger = \beta$$

Matrix Representation of α_i and β:

α_i, β are a set of four matrices.

We have three Pauli matrices which satisfy all the above discussed properties.

Also since Trace is equal to the sum of eigen values, for trace to be zero, matrices should be of even order. These cannot be of order 2, since we have only three Pauli matrices of order 2. So, let us think about matrices of order 4.

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}; \quad \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

$$\alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

or
$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \qquad \text{--- } (3)$$

where
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Set of matrices (3) ~~is~~ of the order 4×4. They satisfy all the above mentioned properties. These are called the Dirac-Pauli representation of the Dirac α_is and β matrices. In fact, they are not unique. One can write several α_is and βs using an unitary matrix say U.

$$\alpha_i' = U\alpha_i U^{-1} \; ; \; \beta' = U\beta U^{-1}$$

Prediction of the Dirac equation does not depend on the particular representation, but it depend on the algebra satisfied by these matrices. In this way, non-uniqueness does not create any issue.

Covariant Form of the Dirac Equation: Dirac gamma Matrices:

$$\hat{H}_D = (\alpha \cdot \hat{p} + \beta m)$$

$$\hat{H}_D \psi = \hat{E} \psi$$

$$(\alpha \cdot \hat{p} + \beta m) = \hat{E} \psi$$

$$(\alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z + \beta m) \psi = \hat{E} \psi$$

$$\left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right) \psi = i\frac{\partial \psi}{\partial t}$$

$$\left(i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} + i\frac{\partial}{\partial t} - \beta m\right) \psi = 0$$

Multiplying by β on both the sides.

$$\left(i\beta\alpha_x \frac{\partial}{\partial x} + i\beta\alpha_y \frac{\partial}{\partial y} + i\beta\alpha_z \frac{\partial}{\partial z} + i\beta\frac{\partial}{\partial t} - \beta^2 m\right) \psi = 0$$

Let us make a substitution.

$$\gamma^0 = \beta \; ; \; \gamma^1 = \beta\alpha_x, \; \gamma^2 = \beta\alpha_y \; ; \; \gamma^3 = \beta\alpha_z$$

$$\left(i\gamma^1 \frac{\partial}{\partial x} + i\gamma^2 \frac{\partial}{\partial y} + i\gamma^3 \frac{\partial}{\partial z} + i\gamma^0 \frac{\partial}{\partial t} - m\right)\psi = 0 \quad -(1)$$

$$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$$

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\gamma^\mu \partial_\mu = \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \quad -(2)$$

Substituting (2) into (1), we get

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad -(3)$$

Equation (3) is called the Dirac equation in the covariant form. $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are called the Dirac gamma matrices.

Properties of Dirac gamma matrices and its matrix representation:

Writing $\cancel{\gamma^0}$ $\gamma^1, \gamma^2, \gamma^3$ by γ^k

$$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = (\gamma^0, \gamma^k)$$

$$\gamma^{0^2} = \gamma^0 \gamma^0 = \beta\beta = I$$

$$\gamma^{k^2} = \gamma^k \gamma^k = \beta\alpha^k \beta\alpha^k = -\beta\alpha^{k^2}\beta = -\beta^2 = -I$$

$$\gamma^{k^2} = -I$$

$$\gamma^0 \gamma^k = \beta\beta\alpha^k = -\beta\alpha^k\beta = -\gamma^k\gamma^0$$

$$\gamma^1\gamma^2 = \beta\alpha_x \beta\alpha_y = \beta\alpha^1\beta\alpha^2 = -\beta\alpha^1\alpha^2\beta$$

$$= +\beta\alpha^2\alpha^1\beta$$

$$= -\beta\alpha^2\beta\alpha^1$$

$$\gamma^1\gamma^2 = -\gamma^2\gamma^1$$

Similarly, $\gamma^2\gamma^3 = -\gamma^2\gamma^3$; $\gamma^1\gamma^3 = -\gamma^3\gamma^1$

All the above Dirac gamma matrices properties can be combined into a single equation: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$\gamma^{0\dagger} = \beta^\dagger = \beta = \gamma^0$

$\gamma^{k\dagger} = (\beta\alpha^k)^\dagger = \alpha^{k\dagger}\beta^\dagger = \alpha^k\beta = -\beta\alpha^k = -\gamma^k$

Hermitian conjugate properties of γ- matrices can be expressed as

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

$$\text{or } \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$$

$$\text{or } \gamma^0\gamma^{\mu\dagger} = \gamma^\mu\gamma^0$$

$$\text{or } \gamma^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0$$

Expression for Probability density and Probability Current:

$\psi^* \longrightarrow \psi^\dagger = (\psi^*)^T$ since ψ ~~has~~ is now a four component column vector.

$$H\psi = E\psi$$

$$(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m)\psi = E\psi$$

$$-i\alpha_x\frac{\partial\psi}{\partial x} - i\alpha_y\frac{\partial\psi}{\partial y} - i\alpha_z\frac{\partial\psi}{\partial z} + \beta m\psi = i\frac{\partial\psi}{\partial t} \quad -(1)$$

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial\psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial\psi^\dagger}{\partial z}\alpha_z^\dagger + m\psi^\dagger\beta^\dagger = -i\frac{\partial\psi^\dagger}{\partial t} \quad -(2)$$

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x + i\frac{\partial\psi^\dagger}{\partial y}\alpha_y + i\frac{\partial\psi^\dagger}{\partial z}\alpha_z + m\psi^\dagger\beta = -i\frac{\partial\psi^\dagger}{\partial t} \quad (2)$$

$\psi^\dagger \times (1) - (2)\times\psi$ gives

$$\psi^\dagger\left(-i\alpha_x\frac{\partial\psi}{\partial x} - i\alpha_y\frac{\partial\psi}{\partial y} - i\alpha_z\frac{\partial\psi}{\partial z} + \beta m\psi\right) - \left(i\frac{\partial\psi^\dagger}{\partial x}\alpha_x + i\frac{\partial\psi^\dagger}{\partial y}\alpha_y + i\frac{\partial\psi^\dagger}{\partial z}\alpha_z + m\psi^\dagger\beta\right)\psi$$

$$= i\psi^\dagger\frac{\partial\psi}{\partial t} + i\frac{\partial\psi^\dagger}{\partial t}\psi$$

$$-i\frac{\partial}{\partial x}(\psi^\dagger\alpha_x\psi) - i\frac{\partial}{\partial y}(\psi^\dagger\alpha_y\psi) - i\frac{\partial}{\partial z}(\psi^\dagger\alpha_z\psi)$$

$$= i\frac{\partial(\psi^\dagger\psi)}{\partial t}$$

$$+i\vec{\nabla}\cdot(\psi^\dagger\alpha\psi) + i\frac{\partial(\psi^\dagger\psi)}{\partial t} = 0$$

$$\vec{\nabla}\cdot(\psi^\dagger\alpha\psi) + \frac{\partial(\psi^\dagger\psi)}{\partial t} = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$: <u>Continuity equation</u>

where $\rho = \psi^\dagger\psi$; $\vec{J} = \psi^\dagger\alpha\psi$

$$\psi^\dagger = (\psi_1^*\ \psi_2^*\ \psi_3^*\ \psi_4^*)$$

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \Rightarrow \rho = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

ρ = <u>+ve</u> definite quantity for +ve E as well as −ve E

$$\rho = \psi^\dagger\psi = \psi^\dagger\gamma^{02}\psi = \psi^\dagger\gamma^0\gamma^0\psi$$

$$= \bar{\psi}\gamma^0\psi$$

$$J = \psi^\dagger\alpha\psi = \psi^\dagger\gamma^{02}\alpha\psi = \psi^\dagger\gamma^0\gamma^0\alpha\psi$$

$$= \bar{\psi}\beta\alpha\psi$$

$$= \bar{\psi}\gamma\psi$$

$$J^\mu = (\rho, \vec{J}) = (\bar{\psi}\gamma^0\psi, \bar{\psi}\gamma\psi)$$

$\bar{\psi} = \psi^\dagger\gamma^0$ = Dirac Adjoint spinor

ψ = Dirac spinor

Dirac Equation for Adjoint Spinor and Continuity equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^k \frac{\partial \psi}{\partial x^k} - m\psi = 0$$

Taking Hermitian conjugate

$$-i\frac{\partial \psi^\dagger}{\partial t}\gamma^{0\dagger} - i\frac{\partial \psi^\dagger}{\partial x^k}\gamma^{k\dagger} - m\psi^\dagger = 0$$

$$-i\frac{\partial \psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial \psi^\dagger}{\partial x^k}(-\gamma^k) - m\psi^\dagger = 0$$

Multiplying by γ^0 from right we get

$$-i\frac{\partial \psi^\dagger}{\partial t}\gamma^0\gamma^0 - i\frac{\partial \psi^\dagger}{\partial x^k}(-\gamma^k\gamma^0) - m\psi^\dagger\gamma^0 = 0$$

$$-i\frac{\partial \bar\psi}{\partial t}\gamma^0 - i\frac{\partial \psi^\dagger}{\partial x^k}\gamma^0\gamma^k - m\bar\psi = 0$$

$$-i\frac{\partial \bar\psi}{\partial t}\gamma^0 - i\frac{\partial \bar\psi}{\partial x^k}\gamma^k - m\bar\psi = 0$$

$$i\frac{\partial \bar\psi}{\partial t}\gamma^0 + i\frac{\partial \bar\psi}{\partial x^k}\gamma^k + m\bar\psi = 0$$

$$i\,\partial_\mu \bar\psi\,\gamma^\mu + m\bar\psi = 0$$: Dirac equation for adjoint spinor

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$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad -(1)$$

$$i\,\partial_\mu \bar\psi\,\gamma^\mu + m\bar\psi = 0 \quad -(2)$$

$\bar\psi \times (1) + (2) \times \psi$. we get

$$i\bar\psi\gamma^\mu\partial_\mu\psi - \cancel{m\bar\psi\psi} + i\,\partial_\mu\bar\psi\,\gamma^\mu\psi + \cancel{m\bar\psi\psi} = 0$$

$$= i\,\partial_\mu(\bar\psi\gamma^\mu\psi) = 0 \Rightarrow \partial_\mu J^\mu = 0$$

where $J^\mu = \bar{\psi}\gamma^\mu\psi$

$J^0 = \bar{\psi}\gamma^0\psi = \rho$

$J^k = \bar{\psi}\gamma^k\psi \neq$

$J^\mu = (\bar{\psi}\gamma^0\psi, \ \bar{\psi}\vec{\gamma}\psi)$

Commutator H_D with angular momentum:

$$H_D = \alpha\cdot\vec{p} + \beta m = \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m$$

$[H_D, \vec{L}]$:

$$[H_D, L_x] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m, \ y p_z - z p_y]$$

$$= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \ y p_z - z p_y] + [\beta m, \ y p_z - z p_y]$$

$[x, p_x] = [y, p_y] = [z, p_z] = i$

$$[H_D, L_x] = \alpha_y [p_y, y] p_z - \alpha_z [p_z, z] p_y$$

$$= -i\alpha_y p_z + i\alpha_z p_y$$

$$= -i(\alpha \times \vec{p})_x$$

$$[H_D, \vec{L}] = -i(\alpha \times \vec{p}) \quad \text{--- (1)}$$

$\Rightarrow$ L is not a conserved quantity for a Dirac particle.

Let us calculate the commutator of H_D with S.

$$S = \frac{1}{2}\Sigma = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

$$S_x = \frac{1}{2}\Sigma_x = \frac{1}{2}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$$

$$S_y = \frac{1}{2}\Sigma_y = \frac{1}{2}\begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}; \quad S_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

To determine $[H_D, \vec{S}]$. let us determine $[H_D, \Sigma]$

$$[H_D, \Sigma_x] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m, \Sigma_x]$$

$$= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] + [\beta m, \Sigma_x]$$

$$= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x]$$

$$= [\alpha_x \, \Sigma_x] p_x + [\alpha_y \, \Sigma_x] p_y + [\alpha_z, \Sigma_x] p_z$$

$$[\alpha_i \, \Sigma_x] = \alpha_i \Sigma_x - \Sigma_x \alpha_i$$

$$= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_i \sigma_x \\ \sigma_i \sigma_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_x \sigma_i \\ \sigma_x \sigma_i & 0 \end{pmatrix}$$

$$[\alpha_i, \Sigma_x] = \begin{pmatrix} 0 & [\sigma_i, \sigma_x] \\ [\sigma_i, \sigma_x], & 0 \end{pmatrix}$$

$$[\alpha_x, \Sigma_x] = 0$$

$$[\alpha_y \, \Sigma_x] = \begin{pmatrix} 0 & [\sigma_y, \sigma_x] \\ [\sigma_y, \sigma_x] & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z, & 0 \end{pmatrix}$$

$$= -2i \, \alpha_z$$

$$[\alpha_z, \Sigma_x] = \begin{pmatrix} 0 & [\sigma_z, \sigma_x] \\ [\sigma_z, \sigma_x] & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i\sigma_y \\ 2i\sigma_y & 0 \end{pmatrix} = 2i \, \alpha_y$$

$$[H_D, \Sigma_x] = -2i \, \alpha_z p_y + 2i \, \alpha_y p_z = 2i(\alpha \times \vec{p})_x$$

$$[H_D, \Sigma] = \underline{2i(\alpha \times \vec{p})}$$

$[H_D, S] = 2i(\vec{\alpha}\times\vec{p})$

$\Rightarrow$ S is not a conserved quantity for a Dirac particle.

$[H_D, \vec{L}+\vec{S}] = 0 \Rightarrow [H_D, \vec{J}] = 0$

$\Rightarrow$ Total angular momentum $\vec{J}$ is a conserved quantity for a Dirac particle.

$$S = \frac{1}{2}\Sigma \Rightarrow S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$ & similarly S_y & S_z

$$S^2 = S_x^2 + S_y^2 + S_z^2$$
$$= \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Moreover $S^2 = s(s+1)\hbar^2 \Rightarrow \frac{3}{4}\hbar^2 = \frac{3}{4}$ for $s = \frac{1}{2}$

$\Rightarrow$ Dirac equation describes spin ½ particles.

Solution of the Dirac Equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$i\gamma^0 \frac{\partial\psi}{\partial t} + i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = 0 \quad (1)$$

Let solution of Eqn. (1) is given by

$$\psi = u(E,\vec{p})e^{-ip\cdot x} = u(E,\vec{p})e^{-i(Et-\vec{p}\cdot\vec{x})}$$

$$\psi = u(E,\vec{p})e^{i(\vec{p}\cdot\vec{x}-Et)} \quad (2)$$

$$\frac{\partial\psi}{\partial t} = -iE\psi;\ \frac{\partial\psi}{\partial x} = ip_x\psi;\ \frac{\partial\psi}{\partial y} = ip_y\psi$$

$$\frac{\partial\psi}{\partial z} = ip_z\psi$$

Substituting values in Eqn. (I), we get

$$i\gamma^0(-iE)\psi + i\gamma^1(ip_x)\psi + i\gamma^2(ip_y)\psi + i\gamma^3(ip_z)\psi - m\psi = 0$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)\psi = 0$$

$$(\gamma^\mu p_\mu - m)\, u(E,\vec{p})\, e^{i(\vec{p}\cdot\vec{x} - Et)} = 0$$

$$\Rightarrow (\gamma^\mu p_\mu - m)\, u(E,\vec{p}) = 0$$

$u(E,\vec{p})$ = Dirac energy momentum spinor

<u>Case I</u>: Solution when particle is at rest:

$$(\gamma^\mu p_\mu - m)\, u = 0$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)\, u = 0$$

$$\vec{p} = 0 \Rightarrow p_x = p_y = p_z = 0$$

$$\Rightarrow \gamma^0 E u = m u$$

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} u = m u$$

<u>Diagonal</u> <u>matrix</u> $\Rightarrow$ u would have orthogonal solutions.

$$u_1 = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = N_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_3 = N_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad u_4 = N_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

u_1 & u_2 $\Rightarrow$ $E = +m$

Particle with mass m

u_3 & u_4 $\Rightarrow$ $E = -m$

Particle with mass $-m$

Dirac interpreted this as antiparticle with mass m. Overall spinor ψ can be written as

$$\psi_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} ; \quad \psi_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}$$

$$\psi_3 = N_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt} ; \quad \psi_4 = N_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

u_1 & u_2 are eigen states of $\hat{S}_z$ with $+1/2$ and $-1/2$ eigen values, respectively.

$\Rightarrow$ u_1 & u_2 describes particles with spin up and spin down, respectively.

<u>Case II</u>: Particle with $\vec{p} \neq 0$.

$$u_3 = N_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad ; \quad u_4 = N_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

u_1 & $u_2 \Rightarrow E = +m$

Particle with mass m

u_3 & $u_4 \Rightarrow E = -m$

Particle with mass $-m$

Dirac interpreted this as antiparticle with mass m.

Overall spinor ψ can be written as

$$\psi_1 = N_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imt} \quad ; \quad \psi_2 = N_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-imt}$$

$$\psi_3 = N_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{+imt} \quad ; \quad \psi_4 = N_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{+imt}$$

u_1 & u_2 are eigen states of $\hat{S}_z$ with $+\frac{1}{2}$ and $-\frac{1}{2}$ eigen values, respectively.

$\Rightarrow$ u_1 & u_2 describes particles with spin up and spin down, respectively.

Case II: Particle with $\vec{p} \neq 0$.

$$(\gamma^\mu p_\mu - m)\, u = 0$$

$$(E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m)\, u = 0$$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma\cdot p \\ -\sigma\cdot p & 0 \end{pmatrix} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0$$

$$\sigma\cdot p = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$\begin{pmatrix} (E-m)I & -\sigma\cdot p \\ \sigma\cdot p & -(E+m)I \end{pmatrix} u = 0$$

Let $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

$$\begin{pmatrix} (E-m)I & -\sigma\cdot p \\ \sigma\cdot p & -(E+m)I \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$u_A = \frac{\sigma\cdot p}{E-m}\, u_B \quad \text{— (1)}$$

$$u_B = \frac{\sigma\cdot p}{E+m}\, u_A \quad \text{— (2)}$$

Two simplest orthogonal choices of u_A

$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow u_B = \frac{1}{(E+m)}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_B = \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix}$$

Similarly $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$

$$u_B = \frac{1}{(E+m)} \begin{pmatrix} p_z & p_x - ip_y \\ p_x+ip_y & -p_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

$$\Rightarrow u_1 = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = N_1 \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \quad — (3)$$

$$u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix} \quad — (4)$$

Similarly two orthogonal choices of the u_B

$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

would give two more solutions, namely, u_3, u_4.

$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives

$$u_A = \frac{1}{E-m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} p_z/(E-m) \\ (p_x+ip_y)/(E-m) \end{pmatrix}$$

$$u_3 = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = N_3 \begin{pmatrix} p_z/(E-m) \\ (p_x+ip_y)/(E-m) \\ 1 \\ 0 \end{pmatrix} \quad \text{— (5)}$$

$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives

$$u_A = \frac{1}{E-m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} (p_x-ip_y)/(E-m) \\ -p_z/(E-m) \end{pmatrix}$$

$$u_4 = N_4 \begin{pmatrix} (p_x-ip_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix} \quad \text{— (6)}$$

u_1, u_2, u_3 & u_4 given by Eqs. (3), (4), (5) & (6) are four solutions of the Dirac equation.
Back substitution of u_1 & u_2 into $(\gamma^{\mu}p_{\mu} - m)u = 0$

gives $E = +\sqrt{p^2 + m^2}$

$\Rightarrow$ u_1, u_2 are +ve energy solutions.

Similarly, u_3, u_4 gives
$$E = -\sqrt{p^2 + m^2}$$
$\Rightarrow$ u_3 & u_4 are -ve energy solutions.

Dirac had explained the -ve energy solutions by postulating the existance of the antiparticles.
See Dirac sea interpretation.

Feynman-Stuckelberg Interpretation of antiparticles.:
Antiparticle with +ve energy moving forward in time is equivalent to particle with -ve energy moving backward in time.

u_3 and u_4 are never used for antiparticles solutions. Instead, antiparticle solutions are determined. They are written

$$\psi = u(-E, -\vec{p})\, e^{i(-\vec{p}\cdot\vec{x} - (-E)t)}$$
$$\psi = v(E, \vec{p})\, e^{-i(\vec{p}\cdot\vec{x} - Et)} \quad — 1$$

$v(E, \vec{p}) \equiv$ Antiparticle spinor

Antiparticle solution of the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad — (1)$$

$$\psi = v(E, \vec{p})\, e^{-i(\vec{p}\cdot\vec{x} - Et)} \quad — (2)$$

$$\frac{\partial\psi}{\partial t} = iE\psi \;;\; \frac{\partial\psi}{\partial x} = -ip_x\psi \;;\; \frac{\partial\psi}{\partial y} = -ip_y\psi$$
$$\frac{\partial\psi}{\partial z} = -ip_z\psi$$

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = 0$$

$$(-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)\, v(E, \vec{p})\, e^{-i(\vec{p}\cdot\vec{x} - Et)} = 0$$

$$-(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z + m)\, v(E, \vec{p}) = 0$$

$$(\gamma^\mu p_\mu + m)\, v(E, \vec{p}) = 0$$

By proceeding in a similar fashion one can determine v_1, v_2, v_3, and v_4.

v_1, v_2 = Antiparticle with +ve energy

v_3, v_4 = Particle (Anti-antiparticle) with −ve energy.

v_3, v_4 are discarded since particles are already described by u_1 and u_2.

Antiparticles are described by v_1 and v_2.

$$v_1 = N_1 \begin{pmatrix} (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}. \qquad v_2 = N_2 \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix}$$

<u>Normalization</u> <u>constant:</u>

Rel. normalization condition gives

$$\psi^\dagger \psi = 2E$$

$$\Rightarrow u_1^\dagger u_1 = 2E = u_2^\dagger u_2$$

$$v_1^\dagger v_1 = 2E = v_2^\dagger v_2$$

$u_1^\dagger u_1 = 2E$ gives

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix}; \quad u_1^\dagger = N_1^\dagger \left(1 \;\; 0 \;\; p_z/(E+m) \;\; (p_x - ip_y)/(E+m)\right)$$

$u_1^\dagger u_1 = 2E$ gives

$$|N_1|^2 \left[1 + \frac{p_z^2}{(E+m)^2} + \frac{(p_x - ip_y)(p_x+ip_y)}{(E+m)^2}\right] = 2E$$

$$|N_1|^2 \left[\frac{(E+m)^2 + p_z^2 + p_x^2 + p_y^2}{(E+m)^2}\right] = 2E$$

$$|N_1|^2 \left[E^2 + m^2 + 2Em + p^2\right] = 2E(E+m)^2$$

$$|N_1|^2 \left[2E^2 + 2Em\right] = 2E(E+m)^2 \Rightarrow N_1 = \sqrt{E+m}$$

$$u_1 = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} ; \quad u_2 = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

$$v_1 = \sqrt{E+m} \begin{pmatrix} (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix} ; \quad v_2 = \sqrt{E+m} \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix}$$

In general. u_1, u_2, v_1, v_2 are ~~not the~~ eigen states of the $S_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$.

For a particle which is at rest or moving along z-axis, u_1, u_2, v_1 and v_2 are eigen states of S_z.

$\underline{*}$ Determine u_1, u_2, v_1 & v_2 for $\vec{p} = \pm|p|\hat{z}$

Then show that

$$S_z u_1 = +\frac{1}{2} u_1 ; \quad S_z u_2 = -\frac{1}{2} u_2$$

$$S_z v_1 = +\frac{1}{2} v_1 ; \quad S_z v_2 = -\frac{1}{2} v_2$$

* Show that

(i) $\bar{u}u = 2m \Rightarrow \bar{u}_1 u_1 = 2m$

$\bar{u}_2 u_2 = 2m$

(ii) $\bar{v}v = -2m \Rightarrow \bar{v}_1 v_1 = -2m$

$\bar{v}_2 v_2 = -2m$

Particle solution $u(E,\vec{p})$ of Dirac Eqn. satisfies the Eqn.

$$(\gamma^\mu p_\mu - m)u = 0$$

* Show that

$$\bar{u}(\gamma^\mu p_\mu - m) = 0$$

Similarly antiparticle solution $v(E,\vec{p})$ of Dirac Eqn. satisfies the Eqn.

$$(\gamma^\mu p_\mu + m)v = 0$$

* Show that

$$\bar{v}(\gamma^\mu p_\mu + m) = 0$$

Proof: (i) $\bar{u}(\gamma^\mu p_\mu - m) = 0$

$$(\gamma^\mu p_\mu - m)u = 0$$

$$\gamma^\mu p_\mu u - mu = 0$$

Taking $\dagger$ of the above equation

$$(\gamma^\mu p_\mu u)^\dagger - mu^\dagger = 0$$

$$u^\dagger \gamma^{\mu\dagger} p_\mu - mu^\dagger = 0$$

$$u^\dagger \gamma^0 \gamma^\mu \gamma^0 p_\mu - mu^\dagger = 0$$

$$u^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^0 p_\mu - mu^\dagger \gamma^0 = 0$$

$$\bar{u}\gamma^\mu p_\mu - m\bar{u} = 0 \Rightarrow \bar{u}(\gamma^\mu p_\mu - m) = 0$$

(ii) $(\gamma^\mu p_\mu + m) v = 0$

$$\gamma^\mu p_\mu v + m v = 0$$

$$(\gamma^\mu p_\mu v)^\dagger + m v^\dagger = 0$$

$$v^\dagger \gamma^{\mu\dagger} p_\mu + m v^\dagger = 0$$

$$v^\dagger \gamma^0 \gamma^\mu \gamma^0 p_\mu + m v^\dagger = 0$$

$$v^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^0 p_\mu + m v^\dagger \gamma^0 = 0$$

$$\bar{v} \gamma^\mu p_\mu + m \bar{v} = 0$$

$$\Rightarrow \bar{v} (\gamma^\mu p_\mu + m) = 0$$

Show that

(i) $\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = \not{p} + m = \gamma^\mu p_\mu + m$

(ii) $\sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = \not{p} - m = \gamma^\mu p_\mu - m$

Above Eqns. are called as completeness relations. They are frequently used in the cross-section and decay rate calculation. Proofs are lengthy but straight forward.

Dirac - Pauli - Schrodinger - Pauli Eqn.:
(Dirac Eqn. describes spin 1/2 particles)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad - (1)$$

$$\psi = u(E, \vec{p})\, e^{i(\vec{p}\cdot\vec{x} - Et)} \quad \text{gives} \quad - (2)$$

$$(\gamma^\mu p_\mu - m) u = 0 \quad - (3)$$

$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$ gives

$$u_A = \frac{\sigma\cdot\vec{p}}{E-m}\, u_B \quad - (4)$$

$$u_B = \frac{\sigma\cdot\vec{p}}{E+m}\, u_A \quad - (5)$$

Combining (4) & (5), we get

$$(\sigma\cdot\vec{p})\, u_B = (E-m)\, u_A$$

$$(\sigma\cdot\vec{p})\frac{(\sigma\cdot\vec{p})}{E+m}\, u_A = (E-m)\, u_A$$

$$(\sigma\cdot\vec{p})(\sigma\cdot\vec{p})\, u_A = (E-m)(E+m)\, u_A \quad - (6)$$

Writing Eqn. (6) for an electron moving under the effect of em field. It can be written by using the replacements:

$$\vec{p} \longrightarrow \vec{p}+e\vec{A} \quad ; \quad E \rightarrow E+eA^0$$

$$(\hat{\sigma}\cdot\vec{p}+e\vec{A})(\hat{\sigma}\cdot\vec{p}+e\vec{A})\, u_A = (E-m+eA^0)(E+m+eA^0)\, u_A \quad - (7)$$

Using

$$(\hat{\sigma}\cdot\vec{A})(\hat{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\hat{\sigma}\cdot(\vec{A}\times\vec{B})$$

$$(\hat{\sigma}\cdot\vec{p}+e\vec{A})(\hat{\sigma}\cdot\vec{p}+e\vec{A}) = (\vec{p}+e\vec{A})^2 + i\hat{\sigma}\cdot[(\vec{p}+e\vec{A})\times(\vec{p}+e\vec{A})]$$

$$= (\vec{p}+e\vec{A})^2 + i\hat{\sigma}\cdot(e\vec{p}\times\vec{A} + e\,\vec{A}\times\vec{p})$$

$$= (\vec{p}+e\vec{A})^2 + ie\,\hat{\sigma}\cdot(\vec{p}\times\vec{A})$$

$$= (\vec{p}+e\vec{A})^2 + ie\,\hat{\sigma}\cdot(-i\vec{\nabla}\times\vec{A})$$

$$= (\vec{p}+e\vec{A})^2 + e(\hat{\sigma}\cdot\vec{B}) \quad - (8)$$

Substituting (8) into (7), we get

$$[(\vec{p}+e\vec{A})^2 + e(\hat{\sigma}\cdot\vec{B})]u_A = (E-m+eA^0)(E+m+eA^0)u_A$$

Taking the non-rel. limit of the above Eqn.

$$E-m = E_{kin} = E_{NR}$$

$$E-m+eA^0 = E_{NR}+eA^0$$

$$E+m \simeq 2m + E_{kin} \simeq 2m \Rightarrow E+m+eA^0 \simeq 2m+eA^0 \simeq 2m$$

$$\Rightarrow [(\vec{p}+e\vec{A})^2 + e(\hat{\sigma}\cdot\vec{B})]u_A = (E_{NR}+eA^0)\,2m\,u_A$$

$$\left[\frac{(\vec{p}+e\vec{A})^2}{2m} + \frac{e(\hat{\sigma}\cdot\vec{B})}{2m}\right]u_A = (E_{NR}+eA^0)u_A$$

$$\left[\frac{(\vec{p}+e\vec{A})^2}{2m} + \frac{e(\hat{S}\cdot\vec{B})}{m}\right]u_A = E_{NR}u_A$$

$$-eA^0$$

$$\left[\frac{(\vec{p}+e\vec{A})^2}{2m} + e\frac{(\hat{S}\cdot\vec{B})}{m} - eA^0\right]u_A = E_{NR}u_A \quad -(9)$$

Eqn. (9) is called the Schroodinger-Pauli Eqn.

$\vec{\mu}_S = \frac{e\vec{S}}{m}$ = magnetic mom. of spin 1/2 particle

$\vec{\mu}_S\cdot\vec{B}$ = corresponding energy

Eqn. (9) tells that Dirac Eqn. describes spin 1/2 particles.

<u>Helicity</u>: $[\hat{H}_D, \hat{S}] \neq 0$

In general u_1, u_2, v_1 & v_2 are not eigen states of S_z. $\Rightarrow$ motivation to think about a new quantity, called Helicity.

Helicity h commutes with H_D.
So, one can get simultaneous eigen states of H_D and h. h is defined by the projection of S on $\vec{p}$.

$$h = \vec{S}\cdot\hat{p} = \hat{S}\cdot\frac{\vec{p}}{|\vec{p}|} = \frac{\hat{\Sigma}\cdot\vec{p}}{2|\vec{p}|} \quad ; \hat{S} = \frac{\hat{\Sigma}}{2}$$

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \Rightarrow h = \frac{1}{2|\vec{p}|}\begin{pmatrix} \sigma\cdot\vec{p} & 0 \\ 0 & \sigma\cdot\vec{p} \end{pmatrix} \quad - (1)$$

$$\hat{h}\,u = \lambda\,u \quad \text{(eigen value equation)}$$

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\frac{1}{2|\vec{p}|}\begin{pmatrix} \sigma\cdot\vec{p} & 0 \\ 0 & \sigma\cdot\vec{p} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \lambda\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\Rightarrow (\sigma\cdot\vec{p})\,u_A = 2|\vec{p}|\lambda\,u_A \quad - (2)$$

$$(\sigma\cdot\vec{p})\,u_B = 2|\vec{p}|\lambda\,u_B \quad - (3)$$

multiplying (2) by $(\sigma\cdot\vec{p})$

$$(\sigma\cdot\vec{p})^2 u_A = 2|\vec{p}|\lambda\,(\sigma\cdot\vec{p})\,u_A$$

$$|\vec{p}|^2 u_A = 2|\vec{p}|\lambda\, 2|\vec{p}|\lambda\, u_A$$

$$\underline{\lambda = \pm 1/2}$$

$$\lambda = \frac{1}{2} \Rightarrow \underline{RH} \Rightarrow \underline{u\uparrow}$$

$$\lambda = \frac{1}{2} \Rightarrow \underline{LH} \Rightarrow \underline{u\downarrow}$$

$u\uparrow$ = Right handed helicity spinor

$u\downarrow$ = Left " " "

Similarly for antiparticles ; $\underline{v\uparrow}$, $\underline{v\downarrow}$.

Here we will directly write $u\uparrow$, $u\downarrow$, $v\uparrow$ and $v\downarrow$.
(Simultaneous eigenstates of h and H_D).
(Also solution of Dirac equation)

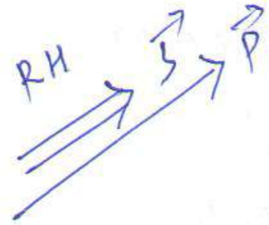
$$u\uparrow = N \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ p/(E+m)\cos\theta/2 \\ p/(E+m)\, e^{i\phi}\sin\theta/2 \end{pmatrix} \quad ; \quad N = \sqrt{E+m}$$

$$\cos\theta/2 = c \quad ; \quad \sin\theta/2 = s$$

$$u\uparrow = \sqrt{E+m}\,\begin{pmatrix} c \\ s e^{i\phi} \\ p/(E+m)\, c \\ p/(E+m)\, s e^{i\phi} \end{pmatrix} \qquad u\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ c e^{i\phi} \\ \frac{p}{E+m}\, s \\ -\frac{p}{E+m}\, c e^{i\phi} \end{pmatrix}$$

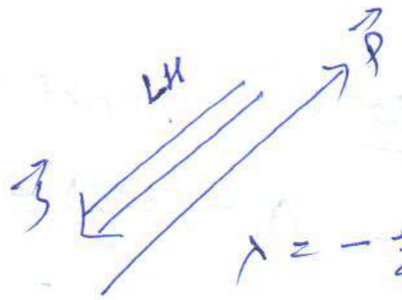
Particles

$u\uparrow$



$\lambda = +\frac{1}{2}$

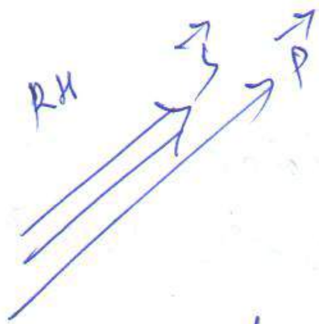
; $u\downarrow$



$\lambda = -\frac{1}{2}$

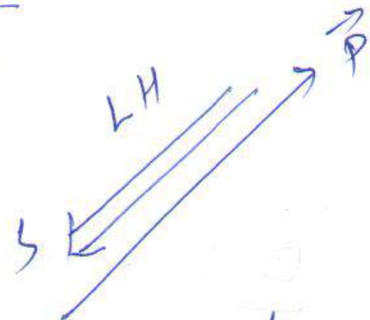
Antiparticles

$v\uparrow$



$\lambda = +\frac{1}{2}$

; $v\downarrow$



$\lambda = -\frac{1}{2}$

$$v\uparrow = \sqrt{E+m}\begin{pmatrix} p/(E+m)\, s \\ -p/(E+m)\, c e^{i\phi} \\ -s \\ c e^{i\phi} \end{pmatrix} \qquad v\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}\, c \\ \frac{p}{E+m}\, s e^{i\phi} \\ c \\ s e^{i\phi} \end{pmatrix}$$

Dirac spinor ψ under Lorentz Boo Transformation

$$\psi \longrightarrow \psi' = S_L \psi$$

where $S_L = a_+ + a_- \gamma^0\gamma^1 = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}$

$$a_\pm = \pm\sqrt{\tfrac{1}{2}(\gamma \pm 1)}$$

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

(i) $S_L^\dagger \gamma^0 S_L = \gamma^0$ (ii) $\gamma^5 S_L = S_L \gamma^5$ (iii) $S_L^\dagger S_L \neq 1$

$$(\psi^\dagger\psi)' = (\psi^\dagger)'\psi' = \psi'^\dagger \psi'$$
$$= (S_L\psi)^\dagger (S_L\psi)$$
$$= \psi^\dagger S_L^\dagger S_L \psi$$
$$\neq \psi^\dagger\psi \quad \text{since } S_L^\dagger S_L \neq 1$$

$\Rightarrow$ $\psi^\dagger\psi$ is not a Lorentz scalar.

$$(\bar\psi\psi)' = (\psi^\dagger\gamma^0\psi)' = \psi'^\dagger\gamma^0\psi'$$
$$= (S_L\psi)^\dagger\gamma^0 S_L\psi$$
$$= \psi^\dagger \underline{S_L^\dagger\gamma^0 S_L}\psi$$
$$= \psi^\dagger\gamma^0\psi$$
$$= \bar\psi\psi$$

$\Rightarrow$ $\bar\psi\psi$ is a Lorentz scalar.

$$\bar\psi\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$
$$= \text{Lorentz scalar}$$

Dirac spinor under parity transformation:

$$\psi(x, y, z, t) \rightarrow \psi'(x', y', z', t')$$

$$x' = -x; \quad y' = -y, \quad z' = -z, \quad t' = t$$

Let us write it as a transformation

$$\psi \rightarrow \psi' = P\psi \quad -(1)$$

$$\text{or} \quad P\psi' = P^2\psi = \psi \quad -(2)$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = 0$$

$$i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t} \quad -(3)$$

Parity transformed wavefunction $\psi'(x', y', z', t')$ must satisfies the Dirac equation:

$$i\gamma^1 \frac{\partial \psi'}{\partial x'} + i\gamma^2 \frac{\partial \psi'}{\partial y'} + i\gamma^3 \frac{\partial \psi'}{\partial z'} - m\psi' = -i\gamma^0 \frac{\partial \psi'}{\partial t'} \quad -(4)$$

Using $\psi = P\psi'$ in Eqn. (3)

$$i\gamma^1 P \frac{\partial \psi'}{\partial x} + i\gamma^2 P \frac{\partial \psi'}{\partial y} + i\gamma^3 P \frac{\partial \psi'}{\partial z} - mP\psi' = -i\gamma^0 P \frac{\partial \psi'}{\partial t}$$

$$x = -x', \quad y = -y', \quad z = -z', \quad t = t'$$

$$-i\gamma^1 P \frac{\partial \psi'}{\partial x'} - i\gamma^2 P \frac{\partial \psi'}{\partial y'} - i\gamma^3 P \frac{\partial \psi'}{\partial z'} - mP\psi' = -i\gamma^0 P \frac{\partial \psi'}{\partial t'}$$

Multiplying by γ^0 from left.

$$-i\gamma^0\gamma^1 P \frac{\partial \psi'}{\partial x'} - i\gamma^0\gamma^2 P \frac{\partial \psi'}{\partial y'} - i\gamma^0\gamma^3 P \frac{\partial \psi'}{\partial z'} - m\gamma^0 P\psi' = -i\gamma^0\gamma^0 P \frac{\partial \psi'}{\partial t'}$$

but $\gamma^0\gamma^1 = -\gamma^1\gamma^0$; $\gamma^0\gamma^2 = -\gamma^2\gamma^0$; $\gamma^0\gamma^3 = -\gamma^3\gamma^0$

$$i\gamma^1\gamma^0 P \frac{\partial \psi'}{\partial x'} + i\gamma^2\gamma^0 P \frac{\partial \psi'}{\partial y'} + i\gamma^3\gamma^0 P \frac{\partial \psi'}{\partial z'} - m\gamma^0 P\psi' = -i\gamma^0\gamma^0 P \frac{\partial \psi'}{\partial t'} \quad -(5)$$

Eqns. (4) and (5) must be the same.
Comparing (4) & (5), we get

$$\gamma^0 P = I \Rightarrow \underline{P = \gamma^0}$$

$$P^2 = I \Rightarrow P = +\gamma^0 \text{ or } -\gamma^0$$

$\underline{P = +\gamma^0}$ (Taken)

$$\psi \to \psi' = \gamma^0 \psi$$

$$\underline{\psi = \gamma^0 \psi'}$$

$$P u_1 = \gamma^0 u_1 = + u_1$$

$$P u_2 = \gamma^0 u_2 = + u_2$$

$$P v_1 = \gamma^0 v_1 = - v_1$$

$$P v_2 = \gamma^0 v_2 = - v_2$$

u_1, u_2, v_1, v_2
For particles at rest
$u_1 \to u_1(m, 0)$

$\Rightarrow$ Intrinsic parity of antiparticles are opposite to that of particles.

If $u_1 \to u_1(E, \vec{p})$

$$\hat{P}\, u_1(E, \vec{p}) = + u_1(E, -\vec{p})$$

$$P\, u_2(E, \vec{p}) = + u_2(E, -\vec{p})$$

<u>Bilinear</u> <u>Covariants</u>: $\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

By inserting above ~~ga~~ set of gamma matrices into $\bar{\psi}\, \cancel{\Gamma}\, \psi$, one can find 16 different quantities having definite properties (transformations) under parity and Lorentz transformation.

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

In other words,

$\bar{\psi}\Gamma\psi$ with Γ either one out of $\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$

$\bar{\psi}\psi$ $\quad \Gamma = 1$. Scalar (True scalar) – 1

$\bar{\psi}\gamma^5\psi$ $\quad \Gamma = \gamma^5$ Pseudoscalar – 1

$\bar{\psi}\gamma^\mu\psi$ $\quad \Gamma = \gamma^\mu$ Vector – 4

$\bar{\psi}\gamma^\mu\gamma^5\psi$ $\quad \Gamma = \gamma^\mu\gamma^5$ Pseudovector or Axial Vector – 4

$\bar{\psi}\sigma^{\mu\nu}\psi$ $\quad \Gamma = \sigma^{\mu\nu}$ Antisymmetric tensor – 6

Total 16 quantities

$\bar{\psi}\psi$ under Lorentz transformation:

$$(\bar{\psi}\psi)' = \bar{\psi}\psi$$

$\bar{\psi}\psi$ under parity transformation:

$$\begin{aligned}
(\bar{\psi}\psi)' &= (\psi^\dagger\gamma^0\psi)' \\
&= \psi'^\dagger\gamma^0\psi' = (\gamma^0\psi)^\dagger\gamma^0(\gamma^0\psi) \\
&= \psi^\dagger\gamma^{0\dagger}\gamma^0\gamma^0\psi \\
&= \psi^\dagger\gamma^0\gamma^0\gamma^0\psi \\
&= \bar{\psi}\psi
\end{aligned}$$

$\Rightarrow \bar{\psi}\psi$ = True scalar

$\bar{\psi}\gamma^5\psi$ under Lorentz Transformation:

$$\begin{aligned}
(\bar{\psi}\gamma^5\psi)' &= (\psi^\dagger\gamma^0\gamma^5\psi)' = \psi'^\dagger\gamma^0\gamma^5\psi' \\
&= (S_L\psi)^\dagger\gamma^0\gamma^5(S_L\psi) \\
&= \psi^\dagger S_L^\dagger\gamma^0\underline{\gamma^5 S_L}\psi \\
&= \psi^\dagger\underline{S_L^\dagger\gamma^0 S_L}\gamma^5\psi
\end{aligned}$$

$(\bar{\psi}\gamma^5\psi)' = \psi^\dagger \gamma^0 \gamma^5 \psi$; $\underline{S_L^\dagger \gamma^0 S_L = \gamma^0}$

$= \underline{\bar{\psi}\gamma^5\psi}$

$\Rightarrow \underline{\bar{\psi}\gamma^5\psi}$ = Lorentz scalar

$\bar{\psi}\gamma^5\psi$ under Parity Transformation:

$(\bar{\psi}\gamma^5\psi)' = (\psi^\dagger \gamma^0 \gamma^5 \psi)'$

$= \psi'^\dagger \gamma^0 \gamma^5 \psi'$

$= (\gamma^0\psi)^\dagger \gamma^0 \gamma^5 (\gamma^0 \psi)$

$= \psi^\dagger \gamma^{0\dagger} \gamma^0 \gamma^5 \gamma^0 \psi$

$= \psi^\dagger \gamma^0 \; \gamma^0 \gamma^5 \gamma^0 \psi$

$= \bar{\psi} \gamma^0 \gamma^5 \gamma^0 \psi$

$= - \bar{\psi} \gamma^5 \gamma^0 \gamma^0 \psi$; $\gamma^0\gamma^5 = -\gamma^5\gamma^0$

$= - \bar{\psi}\gamma^5\psi$

$\Rightarrow \underline{\bar{\psi}\gamma^5\psi}$ = $\underline{\text{Pseudoscalar}}$ $\underline{\text{(Parity)}}$

$(\bar{\psi}\gamma^5\psi)' = \psi^\dagger \gamma^0 \gamma^5 \psi$; $\underline{S_L^\dagger \gamma^0 S_L = \gamma^0}$

$= \underline{\bar{\psi}\gamma^5\psi}$

$\Rightarrow \underline{\bar{\psi}\gamma^5\psi}$ = Lorentz scalar

$\bar{\psi}\gamma^5\psi$ under Parity Transformation:

$$(\bar{\psi}\gamma^5\psi)' = (\psi^\dagger \gamma^0 \gamma^5 \psi)'$$
$$= \psi'^\dagger \gamma^0 \gamma^5 \psi'$$
$$= (\gamma^0\psi)^\dagger \gamma^0\gamma^5 (\gamma^0\psi)$$
$$= \psi^\dagger \gamma^{0\dagger} \gamma^0 \gamma^5 \gamma^0 \psi$$
$$= \psi^\dagger \gamma^0 \gamma^0 \gamma^5 \gamma^0 \psi$$
$$= \bar{\psi} \gamma^0 \gamma^5 \gamma^0 \psi$$
$$= -\bar{\psi} \gamma^5 \gamma^0 \gamma^0 \psi \quad ; \quad \gamma^0\gamma^5 = -\gamma^5\gamma^0$$
$$= -\bar{\psi}\gamma^5\psi$$

$\Rightarrow \underline{\bar{\psi}\gamma^5\psi}$ = Pseudoscalar (Parity)

Few identities related to Dirac γ-matrices and Dirac Trace Algebra:

Trace = Tr = sum of diag. elements.

1. $Tr(A+B) = Tr(A) + Tr(B)$
2. $Tr(\alpha A) = \alpha\, Tr(A)$
3. $Tr(AB) = Tr(BA)$
4. $Tr(ABC) = Tr(BCA) = Tr(CAB)$
5. $g_{\mu\nu} g^{\mu\nu} = \underline{4}$
6. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ $6' = \not{a}\not{b} + \not{b}\not{a} = 2a\cdot b$

Proof 5: $g_{\mu\nu} g^{\mu\nu} = g_{00} g^{00} + g_{11} g^{11} + g_{22} g^{22} + g_{33} g^{33}$

$= 1 \times 1 + (-1) \times (-1) + (-1) \times (-1) + (-1) \times (-1)$

$= 1 + 1 + 1 + 1 = \underline{4}$

Proof 6': 6 is just the combination of anticom. and square properties of Dirac gamma matrices

$\not{a} \not{b} + \not{b} \not{a} = \gamma^\mu a_\mu \gamma^\nu b_\nu + \gamma^\nu b_\nu \gamma^\mu a_\mu$

$= a_\mu b_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$

$= a_\mu b_\nu \cdot 2 g^{\mu\nu} = 2 a_\mu b^\mu$

$= 2 a \cdot b$

7 $\gamma_\mu \gamma^\mu = 4$

Proof 7: $\gamma_\mu \gamma^\mu = \gamma_0 \gamma^0 + \gamma_1 \gamma^1 + \gamma_2 \gamma^2 + \gamma_3 \gamma^3$

γ_μ is defined as

$\gamma_\mu = g_{\mu\nu} \gamma^\nu$ $\qquad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\gamma_0 = g_{0\nu} \gamma^\nu = g_{00} \gamma^0 + g_{01} \gamma^1 + g_{02} \gamma^2 + g_{03} \gamma^3$

$\underline{\gamma_0 = \gamma^0}$

$\gamma_1 = g_{1\nu} \gamma^\nu = g_{10} \gamma^0 + g_{11} \gamma^1 + g_{12} \gamma^2 + g_{13} \gamma^3$

$= (-1) \gamma^1 = -\gamma^1$

Similarly, $\gamma_2 = -\gamma^2$; $\gamma_3 = -\gamma^3$

$\gamma_\mu \gamma^\mu = \gamma^{0^2} - \gamma^{1^2} - \gamma^{2^2} - \gamma^{3^2}$

but $\gamma^{0^2} = I$, $\gamma^{1^2} = \gamma^{2^2} = \gamma^{3^2} = -I$

$\gamma_\mu \gamma^\mu = I + I + I + I = 4I$

8 $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$ **8'** $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

Proof 8: $\gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu)$
$$= 2\gamma^\nu - \gamma_\mu \gamma^\mu \gamma^\nu$$
$$= 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu$$

Proof 8': $\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu \gamma^\nu a_\nu \gamma^\mu$
$$= \gamma_\mu \gamma^\nu \gamma^\mu a_\nu$$
$$= -2\gamma^\nu a_\nu = -2\not{a}$$

9 $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$ **9'** $\gamma_\mu \not{a} \not{b} \gamma^\mu = 4a\cdot b$

Proof 9: $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = \gamma_\mu \gamma^\nu (2g^{\lambda\mu} - \gamma^\mu \gamma^\lambda)$
$$= \gamma_\mu \gamma^\nu 2g^{\lambda\mu} - \underline{\gamma_\mu \gamma^\nu \gamma^\mu} \gamma^\lambda$$
$$= 2\gamma^\lambda \gamma^\nu - (-2\gamma^\nu)\gamma^\lambda$$
$$= 2\gamma^\lambda \gamma^\nu + 2\gamma^\nu \gamma^\lambda$$
$$= 2(\gamma^\lambda \gamma^\nu + \gamma^\nu \gamma^\lambda)$$
$$= 4g^{\nu\lambda}$$

Proof 9': $\gamma_\mu \not{a} \not{b} \gamma^\mu = \gamma_\mu \gamma^\nu a_\nu \gamma^\lambda b_\lambda \gamma^\mu$
$$= \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu a_\nu b_\lambda$$
$$= 4g^{\nu\lambda} a_\nu b_\lambda$$
$$= 4a^\lambda b_\lambda = 4a\cdot b$$

10 $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu$ 10' $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$

Proof 10: $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = \gamma_\mu \gamma^\nu \gamma^\lambda (2g^{\sigma\mu} - \gamma^\mu \gamma^\sigma)$

$= \gamma_\mu \gamma^\nu \gamma^\lambda 2g^{\sigma\mu} - \underline{\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu} \gamma^\sigma$

$= \gamma_\mu \gamma^\nu \gamma^\lambda 2g^{\sigma\mu} - 4g^{\nu\lambda}\gamma^\sigma$

$= 2\gamma^\sigma \gamma^\nu \gamma^\lambda - 4g^{\nu\lambda}\gamma^\sigma$

$= 2\gamma^\sigma(\gamma^\nu \gamma^\lambda - 2g^{\nu\lambda})$

$= -2\gamma^\sigma \gamma^\lambda \gamma^\nu$

Proof 10': $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = \gamma_\mu \gamma^\nu a_\nu \gamma^\lambda b_\lambda \gamma^\sigma c_\sigma \gamma^\mu$

$= \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu a_\nu b_\lambda c_\sigma$

$= -2\gamma^\sigma \gamma^\lambda \gamma^\nu a_\nu b_\lambda c_\sigma$

$= -2\not{c}\not{b}\not{a}$

11 $Tr(\gamma^\mu \gamma^\nu \gamma^\lambda) = 0$ Tr(odd number of gamma mat) = 0

12 $Tr(\mathbb{1}) = 4$

13 $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$ 13' $Tr(\not{a}\not{b}) = 4a\cdot b$

14 $Tr(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4g^{\mu\nu}g^{\lambda\sigma} - 4g^{\mu\lambda}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\lambda}$

14' $Tr(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]$

Proof 11: $Tr(\gamma^\mu\gamma^\nu\gamma^\lambda) = Tr(\gamma^\mu\gamma^\nu\underline{\gamma^\lambda\gamma^5}\gamma^5)$; $\gamma^{5^2} = I$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\lambda) = -Tr(\gamma^\mu\underline{\gamma^\nu\gamma^5}\gamma^\lambda\gamma^5)$$

$$= +Tr(\underline{\gamma^\mu\gamma^5}\gamma^\nu\gamma^\lambda\gamma^5)$$

$$= -Tr(\underline{\gamma^5}\,\underline{\gamma^\mu\gamma^\nu\gamma^\lambda}\,\underline{\gamma^5})$$

$$= -Tr(\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda)$$

$$= -Tr(\gamma^\mu\gamma^\nu\gamma^\lambda)$$

$$\Rightarrow 2Tr(\gamma^\mu\gamma^\nu\gamma^\lambda) = 0 \Rightarrow Tr(\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$$

Proof 13: $Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

$$Tr(\gamma^\mu\gamma^\nu) = Tr(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)$$

$$= 2g^{\mu\nu}Tr(I) - Tr(\gamma^\nu\gamma^\mu)$$

$$= 8g^{\mu\nu} - Tr(\gamma^\mu\gamma^\nu)$$

$$2Tr(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu}$$

$$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

Proof 13': $Tr(\not{a}\not{b}) = 4a\cdot b$

$$LHS = Tr(\not{a}\not{b}) = Tr(\gamma^\mu a_\mu\gamma^\nu b_\nu)$$

$$= a_\mu b_\nu Tr(\gamma^\mu\gamma^\nu) = a_\mu b_\nu 4g^{\mu\nu}$$

$$= 4(a_\mu g^{\mu\nu})b_\nu$$

$$= 4a^\nu b_\nu$$

$$Tr(\not{a}\not{b}) = 4a\cdot b$$

Proof 14: $Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4g^{\mu\nu}g^{\lambda\sigma} - 4g^{\mu\lambda}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\lambda}$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu(2g^{\lambda\sigma} - \gamma^\sigma\gamma^\lambda)\right]$$

$$= \mathrm{Tr}(\gamma^\mu\gamma^\nu 2g^{\lambda\sigma}) - \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda)$$

$$= 2g^{\lambda\sigma}\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) - \mathrm{Tr}\left[\gamma^\mu(2g^{\nu\sigma} - \gamma^\sigma\gamma^\nu)\gamma^\lambda\right]$$

$$= 2g^{\lambda\sigma}\cdot 4g^{\mu\nu} - 2g^{\nu\sigma}\,\mathrm{Tr}(\gamma^\mu\gamma^\lambda) + \mathrm{Tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\lambda)$$

$$= 8g^{\lambda\sigma}g^{\mu\nu} - 8g^{\nu\sigma}g^{\mu\lambda} + \mathrm{Tr}\left[(2g^{\mu\sigma} - \gamma^\sigma\gamma^\mu)\gamma^\nu\gamma^\lambda\right]$$

$$= 8g^{\lambda\sigma}g^{\mu\nu} - 8g^{\nu\sigma}g^{\mu\lambda} + 2g^{\mu\sigma}\,\mathrm{Tr}(\gamma^\nu\gamma^\lambda) - \mathrm{Tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\lambda)$$

$$= 8g^{\lambda\sigma}g^{\mu\nu} - 8g^{\nu\sigma}g^{\mu\lambda} + 8g^{\mu\sigma}g^{\nu\lambda} - \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)$$

$$2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 8g^{\lambda\sigma}g^{\mu\nu} - 8g^{\nu\sigma}g^{\mu\lambda} + 8g^{\mu\sigma}g^{\nu\lambda}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4g^{\lambda\sigma}g^{\mu\nu} - 4g^{\nu\sigma}g^{\mu\lambda} + 4g^{\mu\sigma}g^{\nu\lambda}$$

Proof 14': $\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)]$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = \mathrm{Tr}(\gamma^\mu a_\mu\gamma^\nu b_\nu\gamma^\lambda c_\lambda\gamma^\sigma d_\sigma)$$

$$= a_\mu b_\nu c_\lambda d_\sigma\,\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)$$

$$= a_\mu b_\nu c_\lambda d_\sigma\left[4g^{\mu\nu}g^{\lambda\sigma} - 4g^{\mu\lambda}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\lambda}\right]$$

$$= 4(a_\mu g^{\mu\nu})b_\nu(c_\lambda g^{\lambda\sigma})d_\sigma - 4(a_\mu g^{\mu\lambda})c_\lambda(b_\nu g^{\nu\sigma})d_\sigma + 4(a_\mu g^{\mu\sigma})(b_\nu g^{\nu\lambda})c_\lambda d_\sigma)$$

$$= 4(a^\nu b_\nu)(c^\sigma d_\sigma) - 4(a^\lambda c_\lambda)(b^\sigma d_\sigma) + 4\,a^\sigma(b^\lambda c_\lambda)d_\sigma)$$

$$= 4\left[(a\cdot b)(c\cdot d) - 4(a\cdot c)(b\cdot d) + 4(a\cdot d)(b\cdot c)\right]$$

15 $Tr(\gamma^5) = 0$ $\quad\quad$ $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$\{\gamma^5, \gamma^\mu\} = 0$

16 $Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$ $\quad\quad$ 16′ $Tr(\gamma^5 \not{a} \not{b}) = 0$

17 $Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\varepsilon^{\mu\nu\lambda\sigma}$

17′ $Tr(\gamma^5 \not{a}\not{b}\not{c}\not{d}) = 4i\varepsilon^{\mu\nu\lambda\sigma} a_\mu b_\nu c_\lambda d_\sigma$

where $\varepsilon^{\mu\nu\lambda\sigma}$ = Levi-Civita Tensor of rank 4

$$\varepsilon^{\mu\nu\lambda\sigma} = \begin{cases} -1, & \text{if } \mu\nu\lambda\sigma \text{ is an even perm. of } 0123, \\ +1, & \text{if } \mu\nu\lambda\sigma \text{ is an odd perm. of } 0123, \\ 0, & \text{if any two indices are the same.} \end{cases}$$

15, 16, 16′, 17, 17′ are used to calculate cross-section and decay rate related to weak interactions.

Two Problems: Show that (i) $\not{p}\not{p} = p^2$ (ii) $(\sigma\cdot p)^2 = p^2$

Proof (i) $\not{p}\not{p} = \gamma^\mu p_\mu \gamma^\nu p_\nu = p_\mu p_\nu (\gamma^\mu\gamma^\nu)$

$= p_\mu p_\nu (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)$

$= 2(p_\mu g^{\mu\nu}) p_\nu - p_\mu p_\nu \gamma^\nu\gamma^\mu$

$= 2\, p^\nu p_\nu - (\gamma^\nu p_\nu)(\gamma^\mu p_\mu)$

$= 2p^2 - \not{p}\not{p}$

$2\not{p}\not{p} = 2p^2 \Rightarrow \underline{\not{p}\not{p} = p^2}$

Proof (ii) $(\sigma\cdot p)^2 = (\sigma\cdot p)(\sigma\cdot p) = (\sigma_x p_x + \sigma_y p_y + \sigma_z p_z)^2$

$= \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z\right]^2$

$= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}^2 = \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} = p^2 I = p^2$

1. QED Feynman Rules: (Electrodynamics of spin 1/2 Particles:

Dirac equation:

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

$$\psi = u(E,\vec{p})\, e^{-ip\cdot x} = u(E,\vec{p})\, e^{i(\vec{p}\cdot\vec{x} - Et)}$$

gives:

$$(\gamma^\mu p_\mu - m)\, u = 0$$

or $(\gamma_\mu p^\mu - m)\, u = 0$

or $(\gamma_\mu p^\mu - m)\, \psi = 0$ — ①

Writing Eqn. ① for an electron moving under the effect of an em field (represented by four pot. A^μ)

$$p^\mu \longrightarrow p^\mu + eA^\mu \quad \text{in ①}$$

$$[\gamma_\mu (p^\mu + eA^\mu) - m]\,\psi = 0$$

$$(\gamma_\mu p^\mu - m)\,\psi = -e\gamma_\mu A^\mu \psi \quad \text{— ②}$$

$$= \gamma^0 V \psi \quad \text{— (2)}'$$

where $\gamma^0 V = -e\gamma_\mu A^\mu$ — ③

[γ^0 is introduced in Eqn. (2)' and ③, so that V enters into the Dirac eqn. as $-eA^0 + \cdots$ [similar to the Schrodinger eqn in em field)]

$$\gamma^0 V = -e\gamma_\mu A^\mu$$

$$V = -e\gamma^0\gamma_\mu A^\mu$$

$$= -e\gamma^0\gamma_0 A^0 - e\gamma^0\gamma_1 A^1 - e\gamma^0\gamma_2 A^2 - e\gamma^0\gamma_3 A^3$$

Since $\gamma_0 = \gamma^0$, $\gamma_1 = -\gamma^1$, $\gamma_2 = -\gamma^2$, $\gamma_3 = -\gamma^3$

$$V = -eA^0 + e\gamma^0\gamma^1 A^1 + e\gamma^0\gamma^2 A^2 + e\gamma^0\gamma^3 A^3$$

$$= -eA^0 + \cdots\cdots$$

Here we would use

$V = -e\gamma^0 \gamma_\mu A^\mu$ comes from Eqn. ③

Now to determine the expression for invariant amplitude M, let us determine $\underline{T_{fi}}$

$$T_{fi} = -i\int \psi^*(x,t)\, V\, \psi(x,t)\, d4x$$

Since ψ has now (in Dirac eqn) become 4-comp. column vector, $* \longrightarrow \dagger$

$$T_{fi} = -i\int \psi_f^\dagger(x,t)\, V\, \psi_i(x,t)\, d4x$$

but $V = -e\gamma^0 \gamma_\mu A^\mu$

$$T_{fi} = -i\int \psi_f^\dagger(x,t)\,(-e\gamma^0\gamma_\mu A^\mu)\,\psi_i(x,t)\, d4x$$

$$= -i \times -e\int \psi_f^\dagger \gamma^0 \gamma_\mu A^\mu \psi_i(x,t)\, d4x$$

$$= ie\int \bar{\psi}_f \gamma_\mu A^\mu \psi_i\, d4x$$

$$= ie\int (\bar{\psi}_f \gamma_\mu \psi_i) A^\mu\, d4x \quad -④$$

$$T_{fi} = ie\int J_\mu A^\mu\, d4x$$

where $J_\mu = \bar{\psi}_f \gamma_\mu \psi_i \quad -⑤$

According to Pauli-Weisskopf prescription, J_μ is written as

$$J_\mu = \text{charge} \times \bar{\psi}_f \gamma_\mu \psi_i$$

For an electron, charge $= -e$

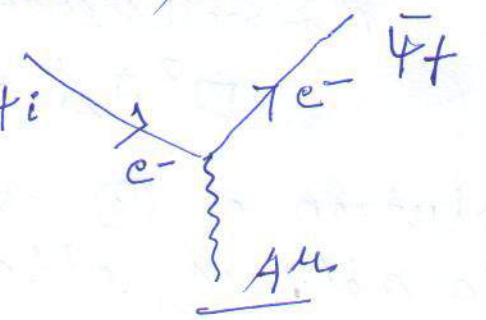
$$J_\mu = -e\,\bar{\psi}_f \gamma_\mu \psi_i \quad -⑥$$

Since J_μ is written as given by Eqn. (6), that is why Eqn. (4) is modified, to make it consistent with Eqn. (6).

$$T_{fi} = -i \int (-e \bar{\psi}_f \gamma_\mu \psi_i) A^\mu d^4x$$

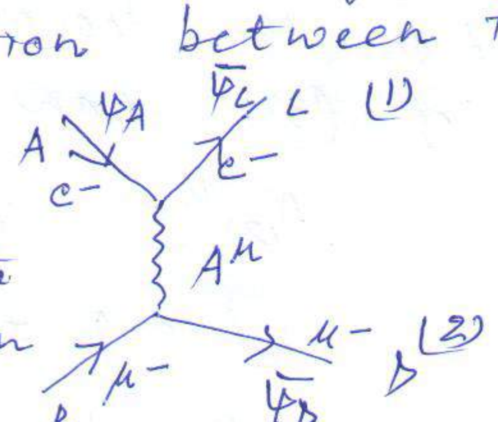
$$T_{fi} = -i \int J_\mu A^\mu d^4x. \quad -(7)$$

Eqn. (7) gives the expression for T_{fi} for an electron moving under the effect of A^μ. This process is described by the following Feynman diagram.



Now since in the particle physics, we generally talk about interaction between two particles (say e^- and μ^-), let us generalized Eqn. (7) for interaction between two particles.

Electron under the effect of A^μ constitute a current (four) given by (6)



$$J_\mu^{(1)} = -e \bar{\psi}_f \gamma_\mu \psi_i = -e \bar{\psi}_C \gamma_\mu \psi_A \quad (8)$$

$(1) \longrightarrow e^-$

Similarly for μ^-

$$J^\mu_{(2)} = -e \bar{\psi}_D \gamma^\mu \psi_B \quad -(9)$$

One can assume here that e^- is moving under the effect of A^μ created by μ^-. For this Eqn. (7), can be written as

$$J_\mu \rightarrow J_\mu^{(1)} \qquad \underline{(1) \rightarrow e^-}$$

$$A^\mu \rightarrow A^\mu_{(2)} \qquad (2) \rightarrow \mu^-$$

$$T_{fi} = -i \int J_\mu^{(1)} A^\mu_{(2)} \, d^4x$$

$A^\mu_{(2)}$ is due to $J^\mu_{(2)}$ and they are related by Maxwell's equation (written in terms of potentials)

$$\Box^2 A_{(2)}^\mu = J_{(2)}^\mu \qquad — (10)$$

Solution of (10) would give $A_{(2)}^\mu$ in terms of $J_{(2)}^\mu$. Its sol. is obtained by using the identity:

$$\Box^2 e^{iq\cdot x} = -q^2 e^{iq\cdot x} \qquad — (11)$$

Eqn. (11) tells that R.H.S of (11) $\times -\frac{1}{q^2} = e^{iq\cdot x}$ (ie sol. of 11).

Based on the learning (message) given by (11), Sol. of (10) can be written as

$$A_{(2)}^\mu = -\frac{1}{q^2} J_{(2)}^\mu \qquad - (12)$$

$$\underline{\text{where}} \; q = p_C - p_A$$

Putting $A^\mu_{(2)}$ from (12) into expression for T_{fi} (look at the third eqn. of this page).

$$T_{fi} = -i \int J_\mu^{(1)} \left(-\frac{1}{q^2}\right) J_{(2)}^\mu \, d^4x \qquad - (13)$$

Substituting

$$J_\mu^{(1)} = -e\, \bar{\psi}_C \gamma_\mu \psi_A$$

$$J_{(2)}^{\mu} = -e\, \bar{\psi}_D \gamma^\mu \psi_B$$

$$T_{fi} = -i \int \left(-e\, \bar{\psi}_C \gamma_\mu \psi_A\right)\left(-\frac{1}{q^2}\right)\left(-e\, \bar{\psi}_D \gamma^\mu \psi_B\right) d^4x \quad (14)$$

Writing ψ_A, ψ_B, $\bar{\psi}_C$ and $\bar{\psi}_D$ in terms of corresponding energy momentum spinors:

$$\left.\begin{aligned}
\psi_A &= N_A u_A\, e^{-i p_A\cdot x} = u_A e^{-i p_A\cdot x} \\
\psi_B &= N_B u_B\, e^{-i p_B\cdot x} = u_B e^{-i p_B\cdot x} \\
\psi_C &= N_C u_C\, e^{-i p_C\cdot x} = u_C e^{-i p_C\cdot x} \\
\bar{\psi}_C &= N_C \bar{u}_C\, e^{i p_C\cdot x} = \bar{u}_C e^{i p_C\cdot x} \\
\psi_D &= N_D u_D\, e^{-i p_D\cdot x} = u_D e^{-i p_D\cdot x} \\
\bar{\psi}_D &= N_D \bar{u}_D\, e^{+i p_D\cdot x} = \bar{u}_D e^{i p_D\cdot x}
\end{aligned}\right\} - (15)$$

$N_A = N_B = N_C = N_D = 1/\sqrt{V} = 1$ for $V=1$.

In view of above (15) Eqn. (14) becomes:

$$T_{fi} = -i \left(-e\, \bar{u}_C \gamma_\mu u_A\right)\left(-\frac{1}{q^2}\right)\left(-e\, \bar{u}_D \gamma^\mu u_B\right) \int e^{i(p_C+p_D-p_A-p_B)\cdot x}\, d^4x$$

$$T_{fi} = -i \left(-e\, \bar{u}_C \gamma_\mu u_A\right)\left(-\frac{1}{q^2}\right)\left(-e\, \bar{u}_D \gamma^\mu u_B\right) \cdot (2\pi)^4 \delta^4(p_C+p_D-p_A-p_B) \quad - (16)$$

where $\int e^{i(p_C+p_D-p_A-p_B)\cdot x}\, d^4x = (2\pi)^4 \delta^4(p_C+p_D-p_A-p_B)$: Integral representation of Dirac Delta function

In (16). Dirac delta function ensures four mom. cons. Coeff. of $(2\pi)^4\, \delta^4(p_C + p_D - p_A - p_B)$ in Eqn. (16) is defined as invariant amplitude $-iM$ for the given interaction process. Hence

$$-iM = \boxed{-i}\,(-e\,\bar{u}_C\,\gamma_\mu\,u_A)\left(\frac{1}{q^2}\right)(-e\,\bar{u}_D\,\gamma^\mu\,u_B)$$

$$-iM = (-e\,\bar{u}_C\,\gamma_\mu\,u_A)\left(\frac{-i}{q^2}\right)(-)\,(-e\,\bar{u}_D\,\gamma^\mu\,u_B)$$

In the <u>RHS</u>, $(-)(-)(-) = (-) \rightarrow$ <u>$i \times i$</u>

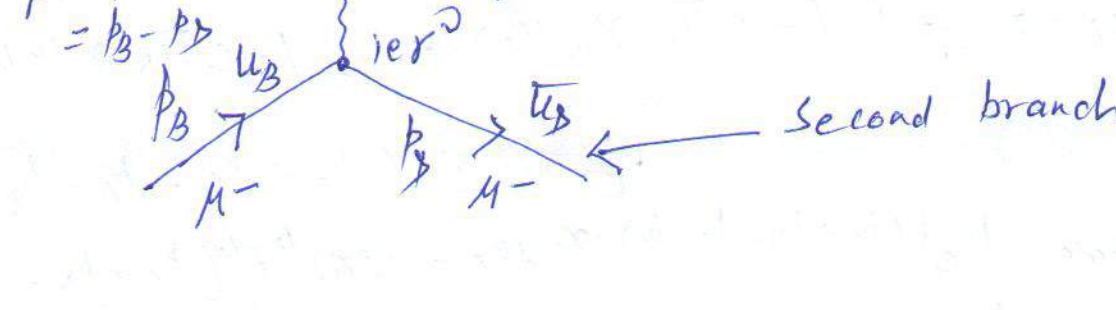
$$-iM = (ie\,\bar{u}_C\,\gamma_\mu\,u_A)\left(-\frac{i}{q^2}\right)(ie\,\bar{u}_D\,\gamma^\mu\,u_B)$$

$$\gamma_\mu\left(-\frac{i}{q^2}\right)\gamma^\mu \longrightarrow \gamma^\mu\left(\frac{-ig_{\mu\nu}}{q^2}\right)\gamma^\nu$$

Thus the final expression for $-iM$ is given by

$$-iM = (ie\,\bar{u}_C\,\gamma^\mu\,u_A)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(ie\,\bar{u}_D\,\gamma^\nu\,u_A) \quad - (17)$$

<u>Vertex of ext. lines</u> — Int. line → <u>Propagator</u> — Other branch: <u>Vertex of external lines</u>



Writing different terms of Eqn. (17) at the related parts of the Feynman diagram (corresponding), the following rules can be made to write $-iM$ directly from the Feynman diagram of any particular interaction process:

1 External lines: Incoming particles u
Outgoing particles $\bar{u}$
Incoming antiparticles $\bar{v}$
Outgoing antiparticles v

2 Vertices : $ie\gamma^\mu$ and $ie\gamma^\nu$

3 Internal lines: $\frac{-ig_{\mu\nu}}{q^2}$; q = four momentum of the photon (virtual)

$\frac{-ig_{\mu\nu}}{q^2}$ is called the propagator.

$$-iM = \underbrace{\begin{pmatrix}\text{Factors for first branch} \\ \text{of Feynman diagram}\end{pmatrix}}_{\text{Four current 1}} \begin{pmatrix}\text{Propagator}\end{pmatrix} \underbrace{\begin{pmatrix}\text{Factors for the} \\ \text{second branch} \\ \text{of Feynman diagram}\end{pmatrix}}_{\text{Four current 2}}$$

Above rules (1), (2) and (3) are called QED Feynman Rules (Almost the final destination of this course)

$$\text{Feynman diagram} \xrightarrow{\text{Weak Int. / QCD / QED Feynman Rules}} \text{Invariant Amplitude}$$

Gordon Decomposition:

$$\bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_f \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu} (p_f - p_i)_\nu \right] u_i$$

It is simply the decomposition of four current into two parts. First term represents its interaction due to its charge and second term (related to $\sigma^{\mu\nu}$) shows its interaction due to its intrinsic spin angular mom. (or spin magnetic moment).

Formal Statement of Gordon decomposition:
Physical electron interacts not only due to its charge but also due to its spin magnetic mom.

Proof:

$$\bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_f \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu} (p_f - p_i)_\nu \right] u_i \quad -(1)$$

(1) is equivalently written as

$$\bar{u}_f \, i\sigma^{\mu\nu} (p_f - p_i)_\nu u_i = 2m \bar{u}_f \gamma^\mu u_i - \bar{u}_f (p_f + p_i)^\mu u_i \quad -(2)$$

$$LHS = \bar{u}_f \, i\sigma^{\mu\nu} (p_f - p_i)_\nu u_i = \bar{u}_f \, i \frac{i}{2} (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) (p_{f\nu} - p_{i\nu}) u_i$$

$$LHS = -\frac{1}{2} \left[\bar{u}_f (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_{f\nu} - p_{i\nu}) u_i \right]$$

$$= -\frac{1}{2} \Big[\underbrace{\bar{u}_f \gamma^\mu\gamma^\nu p_{f\nu} u_i}_{\text{(1) term}} - \underbrace{\bar{u}_f \gamma^\mu\gamma^\nu p_{i\nu} u_i}_{\text{(2) term}} - \underbrace{\bar{u}_f \gamma^\nu\gamma^\mu p_{f\nu} u_i}_{\text{(3) term}} + \underbrace{\bar{u}_f \gamma^\nu\gamma^\mu p_{i\nu} u_i}_{\text{(4) term}} \Big] \quad -(3)$$

(2) term: $(\gamma^\nu p_\nu - m) u = 0$

$$u \to u_i \qquad p_\nu \to p_{i\nu}$$

$$\gamma^{\nu} p_{i\nu} u_i = m u_i$$

③ term: $\bar{u}(\gamma^{\nu} p_{\nu} - m) = 0$

$$\bar{u} \to \bar{u}_f \;;\; p_{\nu} \to p_{f\nu}$$

$$\bar{u}_f \gamma^{\nu} p_{f\nu} = m \bar{u}_f$$

In view of the above Eqn. ③ becomes:

$$LHS = -\frac{1}{2}\Big[\bar{u}_f \gamma^{\mu}\gamma^{\nu} p_{f\nu} u_i - m \bar{u}_f \gamma^{\mu} u_i - m \bar{u}_f \gamma^{\mu} u_i + \bar{u}_f \gamma^{\nu}\gamma^{\mu} p_{i\nu} u_i\Big]$$

① term

$$= -\frac{1}{2}\Big[\bar{u}_f \gamma^{\mu}\gamma^{\nu} p_{f\nu} u_i - 2m \bar{u}_f \gamma^{\mu} u_i + \bar{u}_f \gamma^{\nu}\gamma^{\mu} p_{i\nu} u_i\Big]$$

④ term

$$= -\frac{1}{2}\Big[\bar{u}_f (2g^{\mu\nu} - \gamma^{\nu}\gamma^{\mu}) p_{f\nu} u_i - 2m \bar{u}_f \gamma^{\mu} u_i + \bar{u}_f (2g^{\mu\nu} - \gamma^{\mu}\gamma^{\nu}) p_{i\nu} u_i\Big]$$

$$= -\frac{1}{2}\Big[2\bar{u}_f p_f^{\mu} u_i - \underline{\bar{u}_f \gamma^{\nu}} \gamma^{\mu} \underline{p_{f\nu}} u_i - 2m \bar{u}_f \gamma^{\mu} u_i + 2\bar{u}_f p_i^{\mu} u_i - \bar{u}_f \gamma^{\mu} \underline{\gamma^{\nu} p_{i\nu} u_i}\Big]$$

$$= -\frac{1}{2}\Big[2\bar{u}_f (p_f + p_i)^{\mu} u_i - 2m \bar{u}_f \gamma^{\mu} u_i - m \bar{u}_f \gamma^{\mu} u_i - m \bar{u}_f \gamma^{\mu} u_i\Big]$$

$$= -\frac{1}{2}\Big[2\bar{u}_f (p_f + p_i)^{\mu} u_i - 4m \bar{u}_f \gamma^{\mu} u_i\Big]$$

$$= 2m \bar{u}_f \gamma^{\mu} u_i - \bar{u}_f (p_f + p_i)^{\mu} u_i = \underline{RHS} \text{ of } ②$$

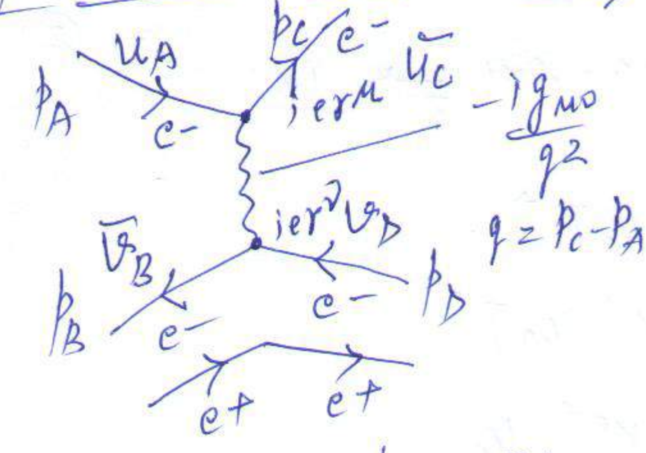
$\Rightarrow$ LHS = RHS ②

<u>Proved</u>

2 Bhabha Scattering: (Elastic Scattering)

$e^- e^+ \rightarrow e^- e^+$

* t-channel diagram $t \rightarrow$

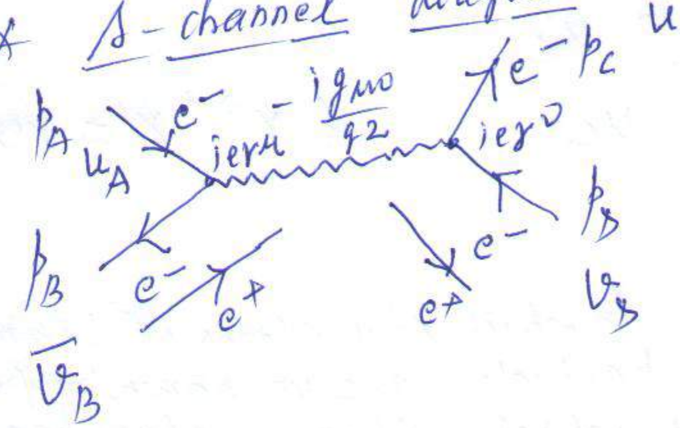


$$-iM_1 = [\bar{u}_C ie\gamma^\mu u_A]\left[\frac{-ig_{\mu\nu}}{q^2}\right][\bar{v}_B ie\gamma^\nu v_D]$$

$$M_1 = -\frac{e^2}{q^2}[\bar{u}_C \gamma^\mu u_A][\bar{v}_B \gamma_\mu v_D] \quad -(1)$$

$$q = p_C - p_A$$

* s-channel diagram:



$$-iM_2 = [\bar{v}_B ie\gamma^\mu u_A]\left[\frac{-ig_{\mu\nu}}{q^2}\right][\bar{u}_C ie\gamma^\nu v_D]$$

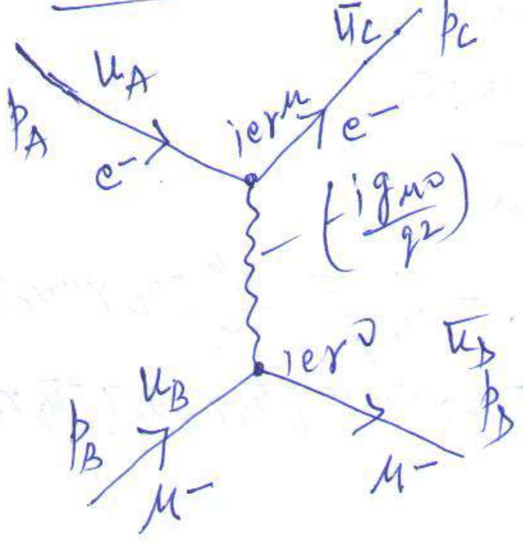
$$M_2 = -\frac{e^2}{q'^2}[\bar{v}_B \gamma^\mu u_A][\bar{u}_C \gamma_\mu v_D] \quad -(2)$$

$$q' = p_A + p_B \; (= p_C + p_D)$$

$$M = \underline{M_1 - M_2} = -\frac{e^2}{q^2}[\bar{u}_C \gamma^\mu u_A][\bar{v}_B \gamma_\mu v_D] + \frac{e^2}{q'^2}[\bar{v}_B \gamma^\mu u_A][\bar{u}_C \gamma_\mu v_D] \quad -(3)$$

3 $e^- \mu^- \rightarrow e^- \mu^-$ Scattering: (Elastic Scattering)

Only t-channel diagram



$$-iM = [\bar{u}_C ie\gamma^\mu u_A]\left[\frac{-ig_{\mu\nu}}{q^2}\right][\bar{u}_D ie\gamma^\nu u_B]$$

$$M = -\frac{e^2}{q^2}[\bar{u}_C \gamma^\mu u_A][\bar{u}_D \gamma_\mu u_B] \quad -(1)$$

$$q = p_C - p_A$$

Now we will calculate $|M|^2$ [as required by by cross-section formula]

$$|M|^2 = \frac{e^4}{q^4}[\bar{u}_C \gamma^\mu u_A][\bar{u}_C \gamma^\nu u_A]^*[\bar{u}_D \gamma_\mu u_B][\bar{u}_D \gamma_\nu u_B]^* \quad -(2)$$

$\bar{u}_C \gamma^\mu u_A \implies 1\times4 \cdot 4\times4 \cdot 4\times1 \implies \underline{1\times1}$

Hence $* \longrightarrow \dagger$

$$\begin{aligned}[\bar{u}_C \gamma^\nu u_A]^* &= [\bar{u}_C \gamma^\nu u_A]^\dagger \\ &= [u_C^\dagger \gamma^0 \gamma^\nu u_A]^\dagger \\ &= u_A^\dagger \gamma^{\nu\dagger} \gamma^{0\dagger} u_C \\ &= u_A^\dagger \gamma^{\nu\dagger}\gamma^0 u_C \\ &= u_A^\dagger \gamma^0 \gamma^\nu u_C \qquad \{\gamma^{\nu\dagger}\gamma^0 = \gamma^0\gamma^\nu\} \\ &= \bar{u}_A \gamma^\nu u_C\end{aligned}$$

* Unpolarized Scattering: In which spin states of beam particles are unknown/not fixed.
* Polarized Scattering: In which spin states of beam particles are known/fixed.

- Here we will take the case of unpolarized scatt. which is quite non-trivial and commonly used to relate theoretical prediction with experiments.

In such a case of unpolarized scattering, we calculate the $|M|^2$ with average over initial spins and sum over final spins. Thus

$$\langle |M|^2 \rangle = \frac{1}{(2s_A+1)}\frac{1}{(2s_B+1)}\sum_{\substack{\text{spin} \\ (e\ \&\ \mu\ \text{spins})}} \overline{|M|^2}$$

$s_A = s_B = 1/2$

$$\langle |M|^2 \rangle = \frac{1}{4}\sum_{\text{spin}} |M|^2 = \frac{1}{4}\sum_{\text{spins}}\frac{e^4}{q^4}[u_C \gamma^\mu u_A][u_C \gamma^\nu u_A]^*[\bar{u}_D \gamma_\mu u_B][\bar{u}_D \gamma_\nu u_B]^*$$

$$\langle |M|^2 \rangle = \frac{e^4}{q^4} \; \frac{1}{2} \sum_{e\text{-spins}} [\bar{u}_C \gamma^\mu u_A][\bar{u}_C \gamma^\nu u_A]^* \cdot \frac{1}{2} \sum_{\mu\text{-spins}} [\bar{u}_D \gamma_\mu u_B][\bar{u}_D \gamma_\nu u_B]^*$$

$$\langle |M|^2 \rangle = \frac{e^4}{q^4} \; L_{e^-}^{\mu\nu} \cdot L^{\mu^-}_{\mu\nu} \quad \text{— (3)}$$

where $L_{e^-}^{\mu\nu} = \frac{1}{2} \sum_{e\text{-spins}} [\bar{u}_C \gamma^\mu u_A][\bar{u}_C \gamma^\nu u_A]^*$ (4A)

$\frac{1}{2} \to \frac{1}{2s_A+1}$

and $L^{\mu^-}_{\mu\nu} = \frac{1}{2} \sum_{\mu\text{-spins}} [\bar{u}_D \gamma_\mu u_B][\bar{u}_D \gamma_\nu u_B]^*$ (4B)

$\frac{1}{2} \to \frac{1}{(2s_B+1)}$

Now we will solve (4A) & (4B) using Casimir's trick and Dirac trace algebra.

(4A): $L_{e^-}^{\mu\nu} = \frac{1}{2} \sum_{e\text{-spins}} [\bar{u}_C \gamma^\mu u_A][\bar{u}_C \gamma^\nu u_A]^*$

But $[\bar{u}_C \gamma^\nu u_A]^* = [\bar{u}_C \gamma^\nu u_A]^\dagger = [\bar{u}_A \gamma^\nu u_C]$

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$$L_{e^-}^{\mu\nu} = \frac{1}{2} \sum_{e\text{-spins}} [\bar{u}_C \gamma^\mu u_A][\bar{u}_A \gamma^\nu u_C]$$

$$= \frac{1}{2} \sum_{s_C} \bar{u}_{C\,\alpha}^{(s_C)} \gamma^\mu_{\alpha\beta} \underbrace{\sum_{s_A} u^{(s_A)}_{A\,\beta} \, \bar{u}^{(s_A)}_{A\,\gamma}} \gamma^\nu_{\gamma\delta} u^{(s_C)}_{C\,\delta}$$

$$= \frac{1}{2} \sum_{s_C} \bar{u}_{C\,\alpha}^{(s_C)} \gamma^\mu_{\alpha\beta} (\not{p}_A + m)_{\beta\gamma} \gamma^\nu_{\gamma\delta} u^{(s_C)}_{C\,\delta}$$

$$= \frac{1}{2} \gamma^\mu_{\alpha\beta} (\not{p}_A + m)_{\beta\gamma} \gamma^\nu_{\gamma\delta} \sum_{s_C} u^{(s_C)}_{C\,\delta} \bar{u}^{(s_C)}_{C\,\alpha}$$

$$= \frac{1}{2} Q_{\alpha\delta} (\not{p}_C + m)_{\delta\alpha} = \frac{1}{2} \sum_\alpha \{Q (\not{p}_C + m)\}_{\alpha\alpha}$$

$$= \frac{1}{2} \mathrm{Tr}[Q (\not{p}_C + m)]$$

$$L_{e^-}^{\mu\nu} = \frac{1}{2} \mathrm{Tr}[\gamma^\mu (\not{p}_A + m) \gamma^\nu (\not{p}_C + m)] \quad \text{— (5)}$$

(4B) : Following the procedure similar to used in (4A). $L^{\mu^-}_{\mu\nu}$ can be written as

$$L^{\mu^-}_{\mu\nu} = \frac{1}{2} Tr[\gamma_\mu (\not{p}_B + M) \gamma_\nu (\not{p}_D + M)] \quad -(6)$$

~~Writing (3) using (5) & (6), we get~~

$$\langle |M|^2 \rangle = \frac{e^4}{4q^4} Tr[\gamma^\mu (\not{p}_A + m) \gamma^\nu (\not{p}_C + m)] Tr[\gamma_\mu (\not{p}_B + M) \gamma_\nu (\not{p}_D + M)]$$

$$= \frac{e^4}{4q^4} Tr[(\gamma^\mu \not{p}_A + \gamma^\mu)(\gamma^\nu \not{p}_C + \gamma^\nu] Tr[(\gamma_\mu \not{p}_B + \gamma_\mu M)(\gamma_\nu \not{p}_D + \gamma_\nu M)]$$

(5) :

$$L_e^{\mu\nu} = \frac{1}{2} Tr[(\gamma^\mu \not{p}_A + m\gamma^\mu)(\gamma^\nu \not{p}_C + m\gamma^\nu)]$$

$$= \frac{1}{2} Tr[\gamma^\mu \not{p}_A \gamma^\nu \not{p}_C] + \frac{1}{2} m^2 Tr(\gamma^\mu \gamma^\nu)$$

$$= 2[p_A^\mu p_C^\nu + p_A^\nu p_C^\mu - p_A \cdot p_C g^{\mu\nu} + m^2 g^{\mu\nu}] \quad -(7)$$

Similarly

(6) : $$L^{\mu^-}_{\mu\nu} = 2[p_{B\mu} p_{D\nu} + p_{B\nu} p_{D\mu} - p_B \cdot p_D g_{\mu\nu} + M^2 g_{\mu\nu}] \quad -(8)$$

Substituting (7) & (8) into (3). we get

$$\langle |M|^2 \rangle = \frac{4e^4}{q^4} [p_A^\mu p_C^\nu + p_A^\nu p_C^\mu - p_A \cdot p_C g^{\mu\nu} + m^2 g^{\mu\nu}] [p_{B\mu} p_{D\nu} + p_{B\nu} p_{D\mu} - p_B \cdot p_D g_{\mu\nu} + M^2 g_{\mu\nu}]$$

$$= \frac{8e^4}{q^4} [(p_C \cdot p_D)(p_A \cdot p_B) + (p_C \cdot p_B)(p_A \cdot p_D) - m^2 p_D \cdot p_B - M^2 (p_C \cdot p_A) + 2m^2 M^2]$$

In the extreme relativistic limit, we can neglect mass terms

$$\langle |M|^2 \rangle = \frac{8e^4}{q^4} [(p_C \cdot p_D)(p_A \cdot p_B) + (p_C \cdot p_B)(p_A \cdot p_D)] \quad -(9)$$

Now

$$s = (p_A + p_B)^2 = p_A^2 + p_B^2 + 2 p_A \cdot p_B = (p_C + p_D)^2$$
$$= \underline{m^2} + \underline{M^2} + 2 p_A \cdot p_B$$
$$\simeq 2 p_A \cdot p_B$$
$$\Rightarrow p_A \cdot p_B = s/2 \;;\; p_C \cdot p_D = s/2$$

$$t = (p_A - p_C)^2 = p_A^2 + p_C^2 - 2 p_A \cdot p_C$$
$$= m^2 + m^2 - 2 p_A \cdot p_C$$
$$p_A \cdot p_C = t/(-2) = -t/2$$

$$u = (p_A - p_D)^2 = p_A^2 + p_D^2 - 2 p_A \cdot p_D = (p_C - p_B)^2$$
$$= m^2 + M^2 - 2 p_A \cdot p_D$$
$$p_A \cdot p_D = u/2 = p_C \cdot p_B$$

$$\langle |M|^2 \rangle = 2e^4 \left(\frac{s^2 + u^2}{t^2} \right)$$

Using $\sqrt{s} = 2E \Rightarrow s = \underline{16 E^4}$

$t = -2k^2(1 - \cos\theta)$; $k = |\vec{k}_i| = |\vec{k}_f|$

$u = -2k^2(1 + \cos\theta)$ $\quad = |\vec{p}_i| = |\vec{p}_f|$

$$\langle |M|^2 \rangle = \frac{16E^4 + 4k^4(1+\cos\theta)^2}{4k^4(1-\cos\theta)^2} \cdot 2e^4$$

$$= \frac{16E^4 + 4k^4 \cdot 4\cos^4\theta/2}{4k^4 \cdot 4\sin^2\theta/2} \cdot 2e^4$$

$$= \frac{2e^4(1 + \cos^4\theta/2)}{\sin^4\theta/2} \qquad = \frac{e^4}{E^2}\frac{1}{128\pi^2}\frac{(1+\cos^4\theta/2)}{\sin^4\theta/2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}\langle |M|^2 \rangle = \frac{2e^4(1+\cos^4\theta/2)}{\sin^4\theta/2}\frac{1}{64\pi^2 \cdot 4E^2}$$